A Binomial Model for Valuing Arithmetic Average Options When the Averaging Period is Fixed

ABSTRACT

This paper derives the algorithms that can be used to price arithmetic average options or "Asian" options when the averaging period is fixed. The pricing of this class of options using numerical analysis as well as analytic methods has not been possible up to now. It is disclosed that the price curve of American Asian options is humped as the averaging period increases. That is, ceteris paribus, American Asian options have a larger price than that of standard options within the range of some averaging periods.

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I. INTRODUCTION

Recently, the trading volume in over-the-counter markets has grown rapidly and surpassed the volume at the exchanges around the world by a large margin. Especially, average price or "Asian" options have attracted much attention and become one of the standard products in over-the-counter markets. Payoffs of Asian options depend on average prices of the underlying asset during at least some parts of the life of the option. Thus, their payoffs are generally less volatile than those of standard options. The Asian options are one special kind of path-dependent option because the valuation of an Asian option depends on the past prices of the underlying asset rather than the current price. This feature makes the valuation problem difficult in the sense that closed-form solutions for arithmetic average options do not exist.

Several approximation procedures have been discussed by Kemna and Vorst (1990), Levy and Turnbull (1992), and Carverhill and Clewlow (1997). Hull and White (1990) have described how binomial approaches can be extended to value some types of path-dependent options, especially European and American options on arithmetic average prices over historical prices of an underlying asset. However, their procedures do not work for American Asian options where the payoff at time t [1] depends on the average price of the asset over the previous fixed periods, rather than the whole period from the starting date (time 0).

This paper extends the binomial approach to value American Asian options as well as European Asian options when the payoff at time t depends on the previous fixed periods. No numerical procedures have so far been available for American Asian options over the fixed averaging periods. Hull and White's (1993) procedure considers the case where the averaging period at time t is from the starting date (time 0) to time t. In contrast, our procedures consider the case where the averaging period at time t is from some date before t (i.e., t-τ) to t for a fixed averaging period τ. Therefore, Hull and White's procedure can be considered as a special case of our procedure when τ = t.

The remainder of this article is organized as follows: Section II presents a pricing framework for Asian options. Section III consecutively presents binomial models for pricing European and American Asian options. In section IV, the accuracy of the model is tested and an important characteristic of American Asian options is shown. Section V concludes the paper.

II. A Pricing Framework for Asian Options

For the binomial model, the notation is as follows:

s : price of the underlying asset at present time

r : risk-free interest rate (assumed constant)

T: life of the derivative security

τ : fixed averaging period ($0 \le \tau \le T$)

n : time steps of length Δt into which the life of the option is divided ($n = \frac{T}{\Delta t}$)

m : time steps into which the fixed averaging period is divided ($m = \frac{\tau}{\Delta t}$)

x : strike price

σ : stock price volatility.

In time Δt the stock price moves up to s $\cdot u$ with probability p and down to s $\cdot d$ with probability (1-p). We assume that the world is risk neutral and suppose that, for convenience, the underlying asset is a non-dividend-paying stock.

There are two methods to evaluate a European Asian option when the average prices over the whole period (i.e., from the initial date to maturity) are used to calculate the final payoffs at the maturity. One is Hull and White (1993) and the other is Cho and Lee (1997). To evaluate a European Asian option with the binomial model, we need to keep track of every possible path from time 0 to the maturity date (i.e., time n). The number of possible paths is 2^n which makes the pricing of European Asian options nearly impossible when n is large, since the number of paths is too large to be processed in an ordinary computer. To overcome this problem, we need to approximate 2^n possible paths with far fewer number. Hull & White and Cho & Lee differ in terms of two aspects in respect to how to approximate 2^n possible paths with a manageable number of paths without deviating too much from the exact pricing. The first aspect of approximation is how to determine the maximum and the minimum path among all possible 2^n paths. The second aspect is how many paths they assign between the maximum path and the minimum path to approximate the interim paths between the two extreme paths. As a general rule, the more interim paths they assign, the more accurate pricing they have, but the more time it takes for them to evaluate. Hull and White determine the maximum path at time i$\cdot \Delta t$ with the highest path going to time i$\cdot \Delta t$ in the binomial tree. The minimum path is determined in a similar manner. Referring to Figure 1, at time i$\cdot \Delta t$, the maximum path is OAU and the minimum path is OBD.

Figure 1 inserted here.

Hull and White assign the same maximum path and minimum path to all nodes at time $i\cdot\Delta t$. The arithmetic averages, denoted by $HW_{max}(i,j)$ and $HW_{min}(i,j)$, for the maximum and minimum path at a particular time-state node Z(i, j) are as follows:

$$HW_{max}(i,j) = s(1+u+u^2+\cdots+u^i)/(i+1)$$

$$HW_{min}(i,j) = s(1+d+d^2+\cdots+d^i)/(i+1)$$

$$0 \le i \le n,\ 0 \le j \le i. \qquad (1)$$

They determine the representative interim paths as follows:

$$HW_{min}(i,j) \le S\cdot e^{k\cdot h} \le HW_{max}(i,j) \qquad (2)$$

where k is an integer and h is a constant.

However, Cho and Lee take a different approach to determine the maximum and the minimum path at time $i\cdot\Delta t$. First, unlike Hull and White, they take a different maximum and minimum path depending on where the nodes are located at time $i\cdot\Delta t$. For example, referring to figure 1, at node Z, they take OAZ as the maximum path going to Z rather than OAU. The arithmetic averages, denoted by $CL_{max}(i,j)$ and $CL_{min}(i,j)$, for the maximum and minimum path at a particular node Z(i, j) are as follows:

$$CL_{max}(i,j) = s(1+u+\cdots+u^j+u^jd+\cdots+u^jd^{i-j})/(i+1)$$

$$CL_{\min}(i,j) = s(1 + d + \cdots + d^{i-j} + ud^{i-j} + \cdots + u^{j}d^{i-j})/(i+1). \qquad (3)$$

Cho and Lee assign the interim paths between the maximum and the minimum path with equal spacing from each other.

Comparing the two methods, we have noticed that Cho and Lee's method has smaller interim paths because of the smaller distance between the maximum and the minimum path. Therefore, it requires less computing time than Hull and White's method. As we have shown in Table 1, if we assign the same maximum and the same minimum path to Hull and White as to Cho and Lee, similar results are produced.

Table 1 inserted here

For this reason, we follow Cho and Lee's method to determine the maximum and the minimum path, while Hull and White's method is applied to determine the interim paths between the maximum and the minimum path.

III. Asian Options Pricing

We can classify Asian options into 4 groups for the purpose of pricing; first, we divide them into American Asian options and European Asian options. Second, we further divide the American or European Asian options depending on whether the averaging period is the option's time-to-maturity or less than the option's time-to-maturity. For example, suppose that starting date of an Asian option is today and the maturity date is 3 months hence. Then the averaging period might be 3 months if the averaging period is the time-to-maturity, or the averaging period might be 2 months which is less than the time-to-maturity. We are going to explain how to evaluate these 4 cases.

1. European Asian options when the averaging period is the time-to-maturity

In the previous section, we have explained how to determine the maximum path and the minimum path for a certain node. To evaluate a European Asian option when the averaging period is the time-to-maturity, first of all, we will determine the European option value at a certain node at time T($n \cdot \Delta t$).

Figure 2 inserted here.

In Figure 2, since we have all possible paths leading from node O to node P, we can determine the average prices over all possible paths. After that, based on the average prices, we determine the European Asian option values by subtracting the exercise price from the average prices if the average prices are greater than the exercise price. If the average prices are less than the exercise price, the European Asian option value is determined to be zero. Then, we have a pair of the average prices and the corresponding option values at node P and similarly at node Q. In the same manner, we determine the average prices at node Z.

Since it is well-known fact that there is a relationship among option values of Z, P, and Q if no arbitrage condition holds (i.e., there is no free lunch in security markets), all we have to do is to connect the average prices at node Z and the average prices at nodes P or Q. Since node P can be reached by an upward movement from node Z, and node Q can be reached by a downward movement from node Z, we can create a path which goes through Z and arrives at P (a path going through OZP) and the other path which goes through Z and arrives at Q (a path going through OZQ) by adding stock prices at nodes P or Q to paths at node A. For those two paths, we can determine the average prices over the T periods. Note that the average price at node Z is calculated over the T-Δt periods. The average prices of the path OZP and the path OZQ will be located somewhere between the average prices calculated based on the maximum path, the minimum path, and interim paths at nodes P and Q. Using the interpolation method, the option value of path OZP or OZQ is determined proportionally in terms of how much the average prices differ compared to all possible paths at nodes P or Q. Once the option values for a certain path OZP or OZQ going through node Z is determined, we can easily determine the option value at node Z. Repeating the same procedure with each path among all possible paths arriving at node Z, we have a pair of the arithmetic average prices and the corresponding option values at node Z.

2. European Asian options when the averaging period is less than the time-to-maturity

When the averaging period is less than the option's time-to-maturity, a path-dependent part is a portion of the binomial tree. For example, in Figure 3, t=0 represents the present time and t=T represents the maturity date. τ is the averaging period.

Figure 3 inserted here

Payoffs at time T depend on price history over τ periods before T. Therefore, the path-dependent part is UDHL, not the whole triangle OHL. Since OUD is a path-independent part, we can apply the standard option valuation model once option values at every node along UD are determined by backward substitution from time T. As a stock price process starting from O can follow OUH if the stock price goes up at every period (i.e., the maximum path), a stock price starting at node A at time T-τ can arrive at node B at time T along the line AB as a maximum path. Similarly, the stock at node A can arrive at node C along the line AC as a minimum path. In other words, the option value at node A can be determined by all possible paths which are generated in the triangle ABC. In the last section, we have shown how to evaluate the option value at node O when the averaging period is the time-to-maturity. If we repeat the same procedure at the triangle ABC, we can determine the option value at node A. Here, the only difference is that we treat node A as if it were the starting node and BC is all possible states at maturity date. For each node along the line UD, we have a different triangle depending on the location of starting node at time T-τ.

In summary, when the averaging period is the time-to-maturity, a path-dependent portion is the whole triangle (i.e., OHL) and we apply the same procedure as explained in section III.1. However, when the averaging period is less than the time-to-maturity, a path-dependent part is a portion of the whole triangle depending on where a stock price starting at node O intersects along UD (for example, when the stock price cuts across UD at node A, the triangle of the path-dependent part is ABC), and we apply the same procedure as we apply for the case where the averaging period is the time-to-maturity at node O.

3. American Asian options when the averaging period is the time-to-maturity

If the option can be exercised at all nodes in Figure 2, the option values at the nodes in section III.1 must be compared with the early exercise value. In section II, we have shown how to evaluate the arithmetic averages for the representative interim paths at a particular node Z. The early exercise value for a representative path at node Z in the case of call options is MAX[the arithmetic average - x, 0]. Thus the market value of the American Asian option at node Z is the greater of the option value in section II.1 and the early exercise value.

4. American Asian options when the averaging period is less than the time-to-maturity

The prices of American Asian options, when the averaging period is the time-to-maturity, can be obtained by applying the pricing of European Asian options and only by considering the early exercise values. However, the pricing of American Asian options when the averaging period is less than the time-to-maturity has a problem that must be considered, which occurs because of the early exercise value. For example, in Figure 3, the early exercise value at time T-Δt uses the arithmetic average of the representative paths between time T-τ -Δt and time T-Δt. However, in the case of the pricing of American Asian options when the averaging period is the time-to-maturity, the early exercise value at time T-Δt uses the arithmetic average between time 0 and time T-Δt. That is, the starting time when the averaging period is the time-to-maturity is fixed at time 0, but the starting time when the averaging period is less than the time-to-maturity is not fixed. Therefore, we need to maintain the starting time for the pricing of American options when the averaging period is less than the time-to-maturity.

Figure 4 inserted here

In Figure 4, the paths which start from the node A(i,j) arrive at between the node B(i+m,j+m) and node C(i+m,j) after τ time. We denote that k is the number of upstate movements during the time. If $Ave_{\max}(i,j,k)$ and $Ave_{\min}(i,j,k)$ are the maximum and the minimum arithmetic averages for the paths which start from the node A(i,j) and arrive at the node D(i+m,j+k), then

$$Ave_{\max}(i,j,k) = su^j d^{i-j}(1+u+\cdots+u^k+u^k d+\cdots+u^k d^{m-k})/(m+1)$$

$$Ave_{\min}(i,j,k) = su^j d^{i-j}(1+d+\cdots+d^{m-k}+ud^{m-k}+\cdots+u^k d^{m-k})/(m+1). \qquad (4)$$

The other arithmetic averages for the paths must have values between the maximum and the minimum arithmetic averages. We set the number of representative paths using Hull and White's method. If we denote the arithmetic averages for the representative interim paths by $Ave(i,j,k,l)$, then

$$Ave(i,j,k,l) = su^j d^{i-j} e^{l\cdot h} \qquad (5)$$

where l is an integer which satisfies

$$Ave_{\min}(i,j,k) \le Ave(i,j,k,l) \le Ave_{\max}(i,j,k). \quad (6)$$

Now we consider the recursive pricing procedure of the algorithm. The time interval [i,i+m] changes into the time interval [i+1,i+m+1] after one time step. Accordingly, the node A(i,j) can move to the node G(i+1,j+1) or the node H(i+1,j), and the node D(i+m,j+k) can move to the node P(i+m+1,j+k+1) or the node Q(i+m+1,j+k) because we adopt the binomial model. The l th arithmetic average, $Ave(i,j,k,l)$, between the node A(i,j) and the node D(i+m,j+k) evolves into two values, $Ave_{up}(i,j,k,l)$ and $Ave_{dn}(i,j,k,l)$. $Ave_{up}(i,j,k,l)$ is the arithmetic average if upstate movement occurs during the time interval [i+m,i+m+1] and $Ave_{dn}(i,j,k,l)$ is the arithmetic average if downstate movement occurs during the time interval [i+m,i+m+1]. Then

$$Ave_{up}(i,j,k,l) = (Ave(i,j,k,l)\cdot(m+1) - s\cdot u^{j}d^{i-j} + s\cdot u^{j+k+1}d^{i+m-j-k})/(m+1)$$
$$Ave_{dn}(i,j,k,l) = (Ave(i,j,k,l)\cdot(m+1) - s\cdot u^{j}d^{i-j} + s\cdot u^{j+k}d^{i+m-j-k+1})/(m+1). \quad (7)$$

We denote that $C_{up}(i,j,k,Ave(i,j,k,l))$ and $C_{dn}(i,j,k,Ave(i,j,k,l))$ are the prices of the Asian option at the node D(i+m, j+k) when the node (i, j) moves to the node (i+1,j+1) and the node (i+1,j), respectively. Also, we denote that $C(i,j,k,Ave(i,j,k,l))$ is the price of the Asian option at the node D(i+m,j+k) when the arithmetic average for an interim path between the node (i,j) and the node (i+m,j+k) is $Ave(i,j,k,l)$. For the triangle GVX and the triangle HWY applying the same procedures as explained in section III.1, $C_{up}(i,j,k,Ave(i,j,k,l))$ and $C_{dn}(i,j,k, Ave(i,j,k,l))$ are determined respectively as follows:

$$C_{up}(i,j,k,Ave(i,j,k,l)) = [p \cdot C(i+1,j+1,k,Ave_{up}(i,j,k,l))$$
$$+(1-p)\cdot C(i+1,j+1,k-1,Ave_{dn}(i,j,k,l))]e^{-r\cdot\Delta t}$$
$$C_{dn}(i,j,k,Ave(i,j,k,l)) = [p \cdot C(i+1,j,k+1,Ave_{up}(i,j,k,l))$$
$$+(1-p)\cdot C(i+1,j,k,Ave_{dn}(i,j,k,l))]e^{-r\cdot\Delta t}. \quad (8)$$

In the above equation, as in Hull and White's method, $C(i+1,j+1,k,Ave_{up}\ (i,j,k,l))$ can be determined by interpolating between $C(i+1,j+1,k,Ave(i+1,\ j+1,k,l_u))$ and $C(i+1,j+1,k,\ Ave(i+1,j+1,k,l_d))$, where l_u and l_d are chosen so that $Ave(i+1,j+1,k,l_u)$ and $Ave(i+1,j+1,k,l_d)$ are the closest values of $Ave(i+1,j+1,k,l)$ to $Ave_{up}(i,j,k,l)$ and are such that

$$Ave(i+1,j+1,k,l_d) \le Ave_{up}(i,j,k,l) \le Ave(i+1,j+1,k,l_u). \quad (9)$$

We can similarly determine $C(i+1,j+1,k-1,Ave_{dn}(i,j,k,l))$, $C(i+1,j,k+1,\ Ave_{up}(i,j,k,l))$ and $C(i+1,j,k,Ave_{dn}(i,j,k,l))$.

Now we have to calculate $C(i,j,k,Ave(i,j,k,l))$ using $C_{up}(i,j,k,Ave(\ i,j,k,l))$ and $C_{dn}(i,j,k,Ave(i,j,k,l))$. A path between the node (i,j) and the node (i+m,j+k) must go through the node (i+1,j+1) or the node (i+1,j). We let G, H, and D denote, respectively, the event that the node A(i,j) arrives at the node G(i+1,j+1), the node H(i+1,j), and the node D(i+m,j+k). Then, the posterior probabilities, $\Pr(G|D)$ and $\Pr(H|D)$, that G and H occurs respectively given that D has occurred, are obtained as in Appendix. Using these conditional probabilities, we can calculate $C(i,j,k,Ave\ (i,j,k,l))$ as follows:

$$
\begin{aligned}
C(i,j,k,Ave(i,j,k,l)) &= C_{up}(i,j,k,Ave(i,j,k,l))\cdot \Pr(G|D) + \\
&\quad C_{dn}(i,j,k,Ave(i,j,k,l))\cdot \Pr(H|D) \\
&= C_{up}(i,j,k,Ave(i,j,k,l))\cdot \frac{k}{m} + \\
&\quad C_{up}(i,j,k,Ave(i,j,k,l))\cdot \frac{m-k}{m}\ . \qquad (10)
\end{aligned}
$$

When early exercise is taken into account, this option price must be compared with the option's exercise value, and we obtain

$$C(i,j,k,Ave(i,j,k,l)) = MAX\{Ave(i,j,k,l) - x,\ C(i,j,k,Ave(i,j,k,l)\}. \qquad (11)$$

Figure 5 inserted here.

Figure 5 shows the tree obtained with four steps ($n=4$). Suppose that T is one year (therefore $\Delta t = 3$ months), $\tau = 9$ months ($m=3$), s is 50, u is 1.1224, for convenience $p=0.5$, and $r=$ 10% per year.[2] We aim to obtain the price of the American option having above conditions at the node (3,2) for a representative path which starts from the node (0,0) and arrives at the node (3,2).

Figure 6 inserted here.

Figure 6 illustrates a part of the binomial tree in Figure 5 to calculate the value of the American Asian option. We suppose that h is 0.1 and the strike price, x, is 50. In the box of the node (4,3) in Figure 6, the two small boxes on the middle row show the averages when the starting nodes are the nodes (1,1) and (1,0), respectively. In the case of the small box on the left, the maximum and minimum averages achievable are

$$Ave_{\max}(1,1,2) = (56.12+62.99+70.70+62.99)/4 = 63.20,$$

$$Ave_{\min}(1,1,2) = (44.55+50.00+56.12+62.99)/4 = 56.31.$$

So $l_{\min} = 1$ and $l_{\max} = 3$. Therefore, the number of representative averages is three and the values are 55.26, 61.07, and 67.49. Similarly, at the node (3,2) the averages are 50.00, 55.26, and 61.07.

We illustrate how to calculate the value of the option for the representative path of which the average is 55.26 at the node (3,2). The arithmetic averages if the node (3,2) goes up to the node (4,3) or goes down to the node (4,2) are as follows:

$$AVE_{up}(0,0,2,2) = (55.26\times4-50.00+62.99)/4 = 58.51,$$

$$AVE_{dn}(0,0,2,2) = (55.26\times4-50.00+50.00)/4 = 55.26.$$

Using linear interpolation, the values of the option at the node (4,3) and the node (4,2) for moving up and down when the starting node is the node (1,1) are 8.51 and 5.26, respectively. So $C_{up}(0,0,2,55.26)$ is $(0.5\times8.51+0.5\times5.26)\, e^{-0.1\times0.25} = 6.72$. Also the values of the option at the node (4,3) and the node (4,2) for moving up and down when the starting node is the node (1,0) are 8.51 and 5.26, respectively. So $C_{dn}(0,0,2,55.26)$ is 6.72. By considering the posterior probabilities with m=3 and k=2 and exercise value, the value of the option at the node (3,2) when the average is 55.26 is

$$C(0,0,2,55.26) = \text{MAX}\{55.26\text{-}50.0,\ 2/3\times 6.72+1/3\times 6.72\} = 6.72.$$

IV. Numerical Results

In this section, the accuracy of the algorithm for the pricing of arithmetic average options is tested. We assume that the initial price of stock is 50, the risk-free interest rate is 10% per year, and the volatility of stock price is 30% per year.

Table 2 inserted here.

Table 2 shows the results of using the above approaches to value European and American Asian options when the averaging period is less than the time-to-maturity. The European results for the algorithm of this paper and for Monte Carlo simulation are very close. It can be said that the approximation method adopted in this paper works well. For verifying the accuracy of our approach in the case of American options, there are no benchmark models except the binomial model that considers all possible paths. So, we compare, in Table 3, the option price from our approximation method with that from the binomial model which considers all possible paths; but the number of time steps in the latter is constrained by our computer memory size and processing speed.[3] Consequently, the only alternative is to use 24 time steps as a maximum level. If the averaging period in our procedures is zero, the price of the standard option is obtained. Also the price of the option of which the averaging period is the time-to-maturity can be obtained when the averaging period is equal to the time-to-maturity in our procedures. Therefore, our procedures is a general model which includes Hull and White's method and Cho and Lee's method.

Table 3 inserted here.

Table 4 inserted here.

Table 5 inserted here.

Figure 7 inserted here.

We usually know that the price of a standard option is greater than that of an Asian option because the volatility of the former is greater than that of the latter. Up to now this fact is realized only when the averaging period is the time-to-maturity. However, as in Table 3, 4, 5, and Figure 7, when the averaging period is less than the time-to-maturity, it remains true for European Asian options, but it is not true for American Asian options. In the case of European options, the option price is always less than that of a standard option regardless of the length of the averaging period, but American options can have a larger price than that of standard options as the averaging period changes. For example, when the maturity is 1 year, in Table 4 the price of the standard option is 8.348. But the price of the American option whose averaging period is 1/8 year is 9.609. Also the price of the option whose averaging period is a half year is 7.453. That is, as shown in Figure 7, the shape of prices of American Asian options is humped as the averaging period changes.

Table 6 inserted here.

Table 6 compares convergence of our approach by varying h and the number of time steps for the option. From the Table, we can observe that the option price converges as the number of time steps increases.

V. CONCLUSION

The numerical pricing methods of Arithmetic Average options that have been developed up to now are for the options of which the averaging period is the time-to-maturity as in Hull and White (1993)'s method and Cho and Lee (1997)'s method. We have explored a method that can be used for the valuation of Asian options when the averaging period is less than the time-to-maturity.

European Asian options are priced easily by extending Hull and White's procedures. That is, for the options with maturity T and the averaging period τ, each node at time T-τ is the starting node and the nodes at time T that the starting node can arrive at are the ending nodes. Hull and White's procedures for each triangle which is composed of these nodes can be applied for getting the price at the starting nodes. The price of the European Asian options at present time can be calculated by applying the pricing method for the standard option and using the option prices at time T-τ. However, the pricing of American Asian options is more complicated. For example, the early exercise value at time T-)t when the averaging period is the time-to-maturity uses the arithmetic average between time 0 and time T-)t, but the value when the averaging period is less than the time-to-maturity uses the arithmetic average between time T-τ-)t and time T-)t. That is, the starting time when the averaging period is the time-to-maturity is fixed at time 0 but the starting time when the averaging period is less than the time-to-maturity is not fixed. We have used posterior probabilities in Bayes' Formula for settling this problem.

To verify our approximation model in the case of European Asian options, we compare our model with the Monte Carlo methods; and in the case of American Asian options we compare our model with the Binomial model which considers all possible paths. The numerical results show that the algorithm is acceptable for pricing the options. Especially, it is interesting to note that the American Asian options can have a higher price than the standard options when the averaging period is short relatively, and a lower price can be produced when the averaging period is long relatively. That is, as the averaging period increases the curve of the option price is humped. We expect the further researches making this characteristic clear.

APPENDIX

In Figure 4, using the definition of the binomial model, the probabilities for the events G, H, or D that the node A reaches the nodes G, H, or D are respectively

$$Pr(G) = p, \quad Pr(H) = 1 - p, \quad Pr(D) = {}_{m}C_{k} p^{k} (1 - p)^{m-k}.$$

Also the conditional probabilities for the events that D occurs given that G or H has occurred are respectively

$$\Pr(D|G) = {}_{m-1}C_{k-1} p^{k-1} (1 - p)^{m-k}$$

$$\Pr(D|H) = {}_{m-1}C_{k} p^{k} (1 - p)^{m-k-1}.$$

By using Bayes' Formula, the posterior probabilities are

$$\Pr(G|D) = \frac{\Pr(G) \cdot \Pr(D|G)}{\Pr(D)} = \frac{k}{m}$$

$$\Pr(H|D) = \frac{\Pr(H) \cdot \Pr(D|H)}{\Pr(D)} = \frac{m-k}{m}.$$

NOTES

1. Present time is denoted by 0 and future time is denoted by t. For example, t is a time point t periods from now.
2. In a risk-neutral world, originally, we can approximate u, d, and p as follows: $u = e^{\sigma\sqrt{\Delta t}}$, $d = \frac{1}{u}$, $p = \frac{e^{r \cdot \Delta t} - d}{u - d}$.
3. We used a Hewlett Packard 9000/827s computer and FORTRAN 77 as the programming language.

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Table 1
The number of representative paths for the algorithms of Hull and White, Cho and Lee, and Revised Hull and White at all nodes at time 20

Node	H-W	C-L	Revised H-W	Node	H-W	C-L	Revised H-W
0	281	1	3	11	281	100	92
1	281	20	20	12	281	97	89
2	281	37	35	13	281	92	85
3	281	52	49	14	281	85	79
4	281	65	61	15	281	76	71
5	281	76	71	16	281	65	61
6	281	85	79	17	281	52	49
7	281	92	84	18	281	37	36
8	281	97	90	19	281	20	20
9	281	100	92	20	281	1	3
10	281	101	93				

Notes: The stock price volatility : 30%, the maturity of the option : 2 years, the value for h : 0.007, time steps = 40.

H-W : algorithm of Hull and White,

C-L : algorithm of Cho and Lee,

Revised H-W : revised algorithm of Hull and White.

Table 2
Value of Asian Options
when the averaging period is less than the time-to-maturity

Life of Option (Years)		Strike Price				
		40	45	50	55	60
0.5	Amer/Our	13.749	8.908	4.996	2.675	1.320
	Euro/Our	11.547	7.491	4.339	2.239	1.037
	Euro/MC	11.547	7.487	4.331	2.232	1.031
		(0.005)	(0.008)	(0.010)	(0.009)	(0.007)

1.0	Amer/Our	15.588	10.998	7.453	4.996	3.258
	Euro/Our	13.130	9.514	6.565	4.328	2.739
	Euro/MC	13.129	9.511	6.558	4.320	2.731
		(0.010)	(0.013)	(0.014)	(0.014)	(0.013)
1.5	Amer/Our	17.066	12.764	9.441	6.963	5.069
	Euro/Our	14.543	11.200	8.389	6.129	4.382
	Euro/MC	14.541	11.196	8.381	6.120	4.373
		(0.015)	(0.017)	(0.018)	(0.019)	(0.018)
2.0	Amer/Our	18.336	14.308	11.155	8.697	6.735
	Euro/Our	15.803	12.668	9.975	7.734	5.918
	Euro/MC	15.799	12.662	9.967	7.725	5.907
		(0.018)	(0.020)	(0.022)	(0.022)	(0.022)

Notes: Averaging period is half of the life of the option. Tree calculations are based on eighty time steps and a value for h equals to 0.007. The Monte Carlo simulations are based on eighty time steps and 100,000 trials using the antithetic variable technique.

Amer/Our = American option valued using our algorithm.

Euro/Our = European option valued using our algorithm.

Euro/MC = European option valued using Monte Carlo simulation.

The standard errors of the Monte Carlo estimates are shown in parentheses.

Table 3
Value of American Asian Options
when the averaging period is less than the time-to-maturity

Averaging period (Years)	0	0.042	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
Amer/Our	8.305	8.858	9.062	8.450	7.785	7.145	6.499	5.871	5.301	4.839
Amer/Bin	8.305	8.857	9.056	8.591	7.923	7.187	6.504	5.872	5.305	4.837

Notes: Amer/Our and Amer/Bin are based on 24 time steps. Strike price is $50, the maturity is one year, and the stock price volatility is 30%. The value for h in Amer/Our equals to 0.007.

Amer/Our = American option valued using our algorithm.

Amer/Bin = American option valued using the binomial model which considers all possible paths.

Table 4
Value of Asian Options
as the Fixed Averaging Period and the Maturity Change

Maturity (years)	Option Type	Ratio of Fixed Averaging Period (τ / T) 0	1/8	1/4	3/8	1/2	5/8	3/4	7/8	1
1.0	American	8.348	9.609	9.109	8.318	7.453	6.657	5.972	5.399	4.948
	European	8.348	7.924	7.481	7.028	6.565	6.088	5.595	5.079	4.532
1.25	American	9.610	11.016	10.388	9.479	8.489	7.582	6.801	6.147	5.633
	European	9.610	9.109	8.585	8.054	7.513	6.959	6.388	5.794	5.169
1.5	American	10.790	12.327	11.581	10.547	9.441	8.432	7.562	6.834	6.264
	European	10.790	10.213	9.612	9.005	8.389	7.761	7.117	6.451	5.753

Notes:
The strike price is $50. Tree calculations are based on eighty time steps and a value for h equals to 0.007. The averaging period, τ, of standard options is zero.

Table 5
Value of Options on the Average Price as the Fixed Averaging Period and the Strike Price change

Strike Price	Fixed Averaging Period (years)						
	0	1/40	1/16	1/8	1/4	1/2	1
40	14.721	16.025	17.217	17.653	17.266	15.588	13.293
50	8.348	9.221	9.698	9.609	9.109	7.453	4.948
60	4.308	4.747	4.840	4.669	4.429	3.258	1.229

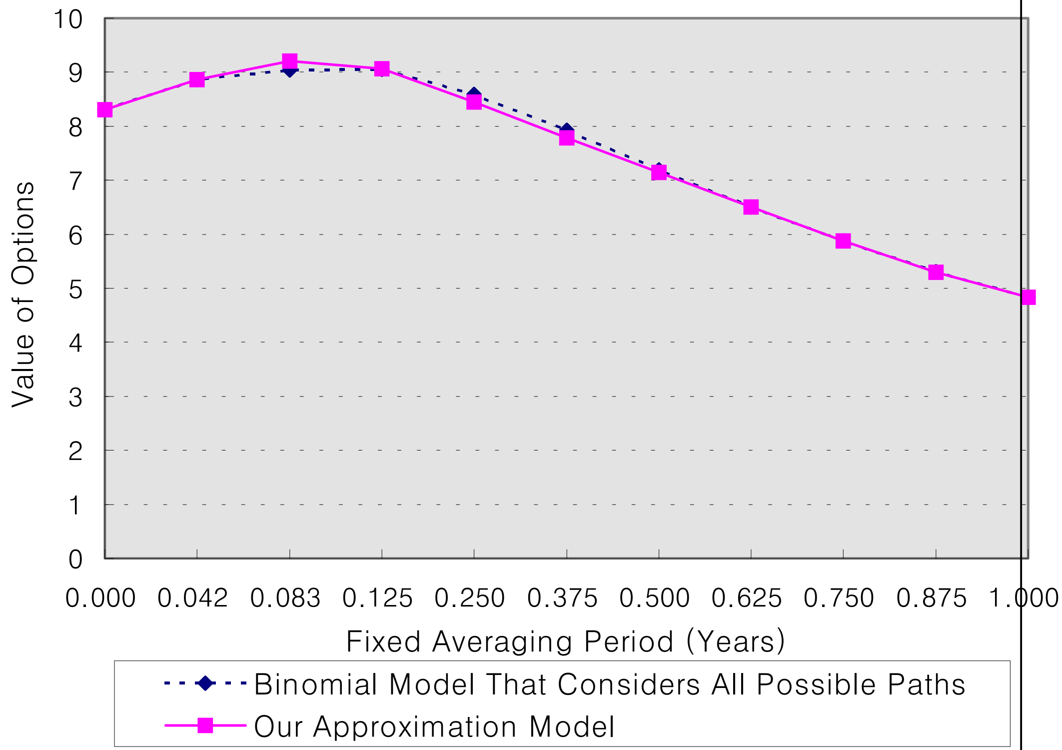
Notes: The maturity is one year and option type is American. Tree calculations are based on eighty time steps and a value for h equals to 0.007.

Table 6
Value of Asian Options
as the Number of Time Steps and the Value of h change

Option h	Type	Number of Time Steps 40	60	80	100
0.009	American	7.322	7.416	7.464	7.492
	European	6.562	6.565	6.568	6.569
0.007	American	7.317	7.411	7.453	7.480
	European	6.560	6.563	6.565	6.566
0.005	American	7.315	7.406	7.448	7.473
	European	6.561	6.562	6.563	6.564
0.004	American	7.315	7.404	7.445	7.470
	European	6.560	6.562	6.562	6.563

Notes: Strike price is $50, the maturity is one year, and averaging period is half year.

Figure 7
Shape of Price Curve for an American Asian Option as the Averaging Period Changes



The maturity is one year and option type is American. Our approximation model and the binomial model are based on 24 time steps. A value for h equals to 0.007 and the strike price is $50.