Transaction Cost Can Have A First-Order Effect on Liquidity Premium*

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Abstract

Constantinides (1986) finds that although transaction cost alters trading strategy significantly, it only has a second-order effect on the liquidity premia. We show that his conclusion depends crucially on his assumption of constant investment opportunity set. In contrast, in a stochastic regime-switching model with transaction cost, we find that transaction cost can have a first-order effect on the liquidity premia. In addition, we show that concerns over a potential liquidity crash, no matter how unlikely it is, can dramatically reduce investment in stock even when the current market is perfectly liquid and the equity premium is high. This suggests that the existence of liquidity risk may largely explain the Equity-Premium Puzzle.

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1 Introduction

Transaction costs are prevalent in almost all financial markets. Extensive research has been conducted on the optimal consumption and investment policy in the presence of transaction costs (e.g., Constantinides (1986), Davis and Norman (1990), Koo (1992a), Liu and Loewenstein (2002), Liu (2004)). As shown in these studies, the presence of transaction costs significantly changes the optimal consumption and optimal investment strategy. For example, an investor no longer trades continuously and even a small transaction cost can dramatically decrease the frequency of trading to save transaction costs. However, the utility loss is found to be small by most of the existing literature. In particular, in his seminal paper Constantinides (1986) finds that the liquidity premium (i.e., the maximum expected return an investor is willing to give up in exchange for zero transaction cost) is small relative to the transaction cost, even for a suboptimal trading strategy and thus concludes that transaction costs are of second-order effect for asset pricing.

One of the common assumptions of the existing literature on optimal consumption and investment with transaction costs is that the investment opportunity set is constant. For example, Constantinides (1986), Davis and Norman (1990), Liu and Loewenstein (2002), and Liu (2004) all assume that not only the expected stock return, the return volatility but also the liquidity (transaction cost) are constant throughout the investment horizon. Intuitively, with a constant investment opportunity set, an investor does not need to trade much and thus the transaction costs incurred is small. Empirical research, however, documented a great deal of evidence against the constant investment opportunity set hypothesis. For example, Campbell (1991) and Lewellen (2003) find that expected returns on equities change over time. Schwert (1989) and Campbell and Hentschel (1992) conclude the volatilities of stock returns also vary substantially over time. Fama and French (1988) and Poterba and Summers (1988) conclude that there is a mean reversion component in stock prices. In addition, large liquidity shock may also appear from time to time (e.g., 1987 crash, 1998 LTCM event).

Taking into account the stochastic nature of the investment opportunity set may qualitatively change the well-known conclusion of Constantinides (1986) that the transaction costs are of second-order effect. This conjecture follows from a simple intuition that if the investment opportunity set changes stochastically, an investor would need to rebalance more often and thus would incur higher transaction costs. Following this intuition, in this paper we build a model similar to Constantinides (1986) and Davis and Norman (1990), but with a regime switching for fundamental parameters that include expected return, volatility, and liquidity. Specifically, we consider the optimal consumption and investment problem for a small investor (i.e., no price impact) who derives constant relative risk averse (CRRA) utility from intertemporal consumption and bequest.[1] The investor can invest in one stock and one risk-free asset. In contrast to most of the existing literature, we assume that the investment opportunity set is not constant and there are two regimes with different fundamental parameters. One regime switches to the other regime at the first jump

[1]The bequest can also be interpreted as an exogenous need to cash.

time of an independent, regime dependent Poisson process.[2]

Our extensive numerical analysis demonstrates that in contrast to the now standard conclusion that transaction costs only have a second-order effect and consistent with our intuition, transaction costs may have a first-order effect if investment opportunity set varies stochastically. Specifically, the liquidity premium to transaction cost ratio could be well above 1 for a wide range of parameters similar to those used in Constantinides (1986). This is in sharp contrast with the results in Constantinides (1986). The consideration of time-varying investment opportunity set makes this ratio almost 10 times higher (recall that Constantinides (1986) used a suboptimal consumption policy to emphasize how small the liquidity premium is). In addition, we find that the liquidity premium to transaction cost (LPTC) ratio increases with expected return and decreases with return volatility. Intuitively, as expected return increases or volatility decreases, an investor invests more in the stock, trades more often and thus incurs higher transaction costs. In addition, as the transaction cost in one regime increases, the LPTC ratio in the other regime increases, while the ratio in this regime decreases. This dichotomy is caused by the investor's cross-regime smoothing behavior. For example, as the bull regime transaction cost increases, the investor trades less often in the bull regime because it is more costly and trades more in the bear regime because it becomes relatively cheaper to trade in this regime. This example also shows a general result that changes in the parameters of one regime affect the optimal policies not only in that regime but also in the other regime, due to the cross regime smoothing behavior. For another example, as the expected return in the bull regime increases, the investor increases consumption not only in the bull regime but also in the bear regime.

In addition, our model also has important implications for the resolution of the Equity-Premium Puzzle. We show that concerns over a potential liquidity crash, no matter how unlikely it is, can dramatically reduce investment in stock even when the current market is perfectly liquid and the expected excess return is high. Intuitively, if an investor has to liquidate her stock position at an exogenously given random time due to death or other sudden need, the sheer possibility of a liquidity crash, no matter how small the probability is, would make leverage suboptimal no matter how high the equity premium is or how liquid the current market is. This suggests that the existence of liquidity risk may largely explain the Equity-Premium Puzzle. In contrast to the existing literature, this liquidity risk explanation does not require high risk aversion (e.g., Mehra and Prescott (1985)) or the separation of the risk aversion and the intertemporal rate of substitution (e.g., Epstein and Zin (1989)) or habit formation (e.g., Constantinides (1990)).

Extending the two-regime model into a model with more regimes is straightforward but requires significantly more intensive computation. More importantly, the qualitative results obtained in our paper would stay the same as long as the transaction cost is small relative to the changes in the optimal portfolio target through these regimes. Intuitively, when the transaction cost is small, the investor trades more frequently to stay close to the

[2]The investor we consider in this paper can be an institutional investor who does not have any price impact and updates the estimates of fundamental parameters from time to time.

optimal target and thus incurs high transaction costs, which makes it a first order effect. This intuition also applies to the case with the investment opportunity set dependent on a continuous state variable. Therefore jumps in the fundamental parameters in the financial market is critical for our results and employed only for tractability.

The rest of the paper is organized as follows. Section 2 presents the model with transaction cost and regime switching. Section 3 derives the steady-state distribution for the stock investment and several measures of liquidity premium. Numerical and graphical analysis is presented in Section 4. Section 5 closes the paper. All of the proofs are in the Appendix.

2 Optimal Consumption and Investment

2.1 The Basic Model

Throughout this paper we assume a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the model is generated by a standard one dimensional Brownian motion w and the regime switch risk. We will assume that w_t is adapted.

There are two assets an investor can trade. The first asset ("the bond") is a money market account and the second asset ("the stock") is a risky investment. There exist two regimes: "Bull" (regime B) and "bear" (regime b) and the fundamental parameters in the financial market may be regime dependent. In addition, we assume that regime i switches into regime j at the first jump time of an independent Poisson process with intensity λ_i, for $i, j \in \{B, b\}$. In regime i, the risk-free interest rate is r_i and the investor can buy the stock at the ask price $S_t^A = (1+\theta_i)S_t$ and sell the stock at the bid price $S_t^B = (1-\alpha_i)S_t$, where $0 \leq \theta_i, \alpha_i < 1$ represents the proportional transaction cost rates and S_t satisfies

$$\frac{dS_t}{S_t} = \mu_i dt + \sigma_i dw_t, \tag{1}$$

where we assume all parameters are positive constants and $\mu_i > r_i$.

As in Constantinides (1986), we consider a constant relative risk aversion (CRRA) investor who derives von Neumann-Morgenstern time additive utility from intertemporal consumption c with weight $1-k$ and bequest with weight k at death with time discount rate of ρ. We assume the mortality rate δ is constant for simplicity; that is,

$$U(c) \equiv E\Big[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt\Big].$$

In regime $i \in \{B, b\}$, when $\theta_i + \alpha_i > 0$, the above model gives rise to equations governing the evolution of the amount invested in the bond, x_t, and the amount invested in the stock, y_t:

$$dx_t = r_i x_t dt - (1+\theta_i)dI_t + (1-\alpha_i)dD_t - c_t dt, \tag{2}$$

$$dy_t = \mu_i y_t dt + \sigma_i y_t dw_t + dI_t - dD_t, \tag{3}$$

where the processes D and I represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing, right continuous adapted processes with $D(0) = I(0) = 0$. Let x_0 and y_0 be the given initial positions in the bond and the stock respectively. We let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (c, D, I) such that (2) and (3) are satisfied, $\int_0^t c_s ds < \infty$ for all t, and the investor is always solvent, i.e.,

$$x_t + (1 - \alpha_i)y_t \geq 0, \forall t \geq 0 \text{ and } i \in \{B, b\}. \tag{4}$$

Then the investor solves

$$\max_{(c,D,I)\in\Theta(x_0,y_0)} U(c). \tag{5}$$

2.2 Optimal Policies with No Transaction Costs

In this section we solve the optimal consumption and portfolio selection problem in the absence of transaction costs, i.e. $\theta_i = \alpha_i = 0$, under the regime-switching model presented in the previous section. The results in this section can serve as a benchmark for the subsequent analysis.

In this case, the cumulative purchases and sales of the stock can be of infinite variation. Let τ_i be the first jump time since the beginning of regime i. The investor's problem in regime $i \in \{B, b\}$ can be rewritten as

$$V_i(W) = \sup_{\{y_t : t\geq 0\}} E\Big[\int_0^{\tau_i} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{W_t^{1-\gamma}}{1-\gamma}\right)dt + e^{-(\rho+\delta)\tau_i}V_j(W_{\tau_i})\Big], \tag{6}$$

subject to

$$dW_t = r_i W_t dt + (\mu_i - r_i)y_t dt + \sigma_i y_t dw_t, \forall i \in \{B, b\}, \tag{7}$$

where $W_t \equiv x_t + y_t \geq 0$ and $V_j(x, y)$ is the value function in regime j.

Under regularity conditions on V_i and V_j, the Hamilton-Jacobi-Bellman(HJB) equations take the form

$$\begin{aligned}\sup_{(c_i,y_i)} \Big\{\frac{1}{2}\sigma_i^2 y_i^2 V_{iWW} + r_i W V_{iW} + (\mu_i - r_i)y_i V_{iW} - c_i V_{iW} - (\rho + \delta + \lambda_i)V_i + \lambda_i V_j \\ +(1-k)\frac{c_i^{1-\gamma}}{1-\gamma} + k\delta\frac{W_t^{1-\gamma}}{1-\gamma}\Big\} = 0,\end{aligned} \tag{8}$$

where $i, j \in \{B, b\},\ i \neq j$. We conjecture

$$V_i(W) = M_i\frac{W^{1-\gamma}}{1-\gamma} \quad \text{for} \quad M_i > 0, i \in \{B, b\}. \tag{9}$$

By the first-order conditions we have

$$c_i = \left(\frac{V_{iW}}{1-k}\right)^{-\frac{1}{\gamma}} \quad \text{and} \quad y_i = -\frac{(\mu_i - r_i)V_{iW}}{\sigma_i^2 V_{iWW}}.$$

Then plugging (9) into (8), we obtain that M_i and M_j satisfy the system of equations

$$-(\eta_i + \lambda_i)M_i + \gamma(1-k)^{1/\gamma} M_i^{1-1/\gamma} + \lambda_i M_j + k\delta = 0, \tag{10}$$

where

$$\eta_i = \rho + \delta - (1-\gamma)\Big(r_i + \frac{\kappa_i^2}{2\gamma}\Big), \quad \kappa_i = \frac{\mu_i - r}{\sigma_i}.$$

To ensure the existence of optimal solution, we adopt the following assumption similar to Merton (1971).

Assumption 1. $\eta_i > 0, \forall i \in \{B, b\}$.

Lemma 2.1 *Under Assumption 1, there is a unique solution (M_B, M_b) to (10). In addition, there exist finite constants $\bar{M}$ and $\underline{M}$ such that*

$$\bar{M} \geq M_i \geq \underline{M}.$$

Proof: see Appendix.

The following verification theorem shows that indeed our conjecture is correct.

Theorem 2.2 *Under Assumption 1 for regime $i \in \{B, b\}$, V_i defined in (9) is the value function defined in (6). In addition, the optimal consumption policy is $c_i^* = \left(\frac{M_i}{1-k}\right)^{-\frac{1}{\gamma}} W$ and the optimal fraction of wealth invested in the stock is $\pi_i^* = \dfrac{\mu_i - r_i}{\gamma\sigma_i^2}$.*

Proof: This follows from a similar and simpler argument presented in the proof of Theorem 2.3 that is provided in the Appendix of Davis and Norman (1990).

Theorem 2.2 implies that both the optimal consumption and the optimal dollar amount invested in the stock is a constant fraction of the investor's wealth in each regime. However, the investor does smooth consumption across regimes. This is reflected in the fact that M_i and M_j are jointly determined by equations (10). In contrast the optimal investment policy is myopic in the sense that π_i^* only depends on the current regime parameters. Intuitively, the risk of regime switching is unhedgable using the existing securities.

2.3 Optimal Policies with Transaction Costs

Now suppose there are transaction costs in each regime, i.e., $\theta_i + \alpha_i > 0, i \in \{B, b\}$. Using the same notations as in the previous section, we can rewrite the investor's problem as

$$\begin{aligned} v_i(x,y) \quad = \quad & \sup_{(c,D,I)} E\Big[\int_0^{\tau_i} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt \\ & + e^{-(\rho+\delta)\tau_i} v_j(x_{\tau_i}, y_{\tau_i})\Big], \end{aligned} \tag{11}$$

subject to (2)-(4).

It can be easily verified that the value functions v_B and v_b are concave and homogeneous of degree $1-\gamma$ in (x, y).(cf. Fleming and Soner(1993), Lemma VIII.3.2). As in Davis and Norman(1990), Liu and Loewenstein (2002) the solvency region $\mathcal{S}_i$ splits into three regions, a 'no-trading'(NT_i) region, a 'buy'(B_i) region and a 'sell'(S_i) region. The homogeneity of v_i implies that the transaction boundaries are straight lines (see Figure 1). In addition, there exists an interval $[\underline{z}_i, \bar{z}_i]$ such that in regime i the investor trades only the minimum amount to keep the ratio

$$z = \frac{x}{y}$$

inside the interval. We depict this analysis in Figures 2 and 3. Figure 2 shows the solvency region when the two no-transaction regions (NT_B and NT_b) are separated. Intuitively, this case occurs when the difference between the two regimes is large and the transition probability is low (e.g, the expected return on stock in the bull regime is sufficiently larger than that in the bear regime). As the transition probability increases, the no-transaction regions move toward each other and eventually they become overlapped, as shown in Figure 3.

The above optimal transaction policy implies that the HJB equation takes the following form

$$\frac{1}{2}\sigma_i^2 y^2 v_{iyy} + r_i x v_{ix} + \mu_i y v_{iy} + \frac{\gamma}{1-\gamma}(1-k)^{1/\gamma} v_{ix}^{1-1/\gamma} - (\rho+\delta) v_i + k\delta \frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma} + \lambda_i(v_j - v_i) = 0, \tag{12}$$

for $j \neq i$ in NT^i. In the sell region, the investor transacts immediately to the sell boundary. Therefore,

$$v_i(x, y) = A_i \frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma}, \tag{13}$$

where A_i is also a positive constant to be determined. Similarly, in the buy region, the investor transacts immediately to the buy boundary. Therefore,

$$v_i(x, y) = B_i \frac{(x+(1+\theta_i)y)^{1-\gamma}}{1-\gamma}, \tag{14}$$

where B_i is a positive constant to be determined.

By the homogeneity of the value functions, there exists a function $\psi_i : (\underline{z}_i, \bar{z}_i) \to \mathbb{R}$ in regime i satisfying

$$v_i(x, y) \equiv y^{1-\gamma} \psi_i\Big(\frac{x}{y}\Big). \tag{15}$$

This implies that

$$\psi_i(z) = \begin{cases} A_i \frac{(z+(1-\alpha_i))^{1-\gamma}}{1-\gamma} & z < \underline{z}_i. \\ B_i \frac{(z+(1+\theta_i))^{1-\gamma}}{1-\gamma} & z > \bar{z}_i \end{cases} \tag{16}$$

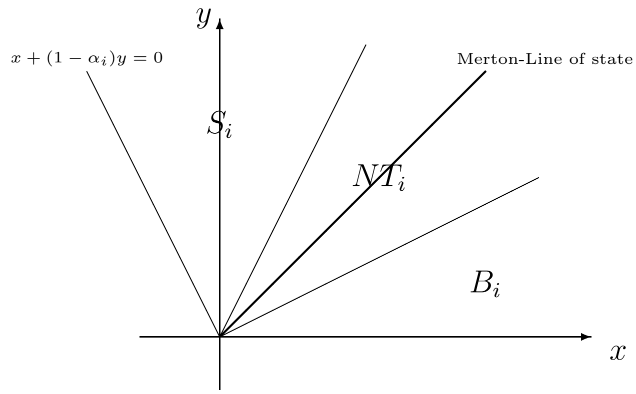

Figure 1: The solvency region splits into the regions $B_i,\ S_i,\ NT_i$

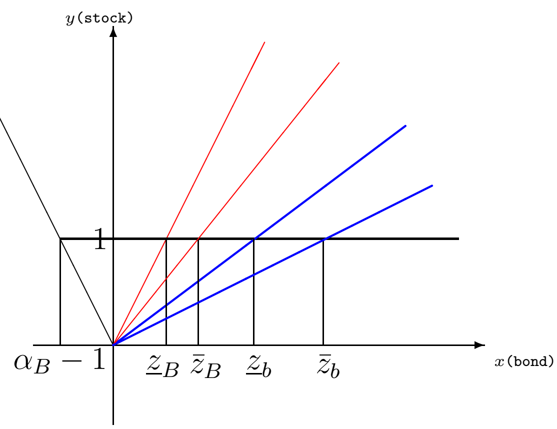

Figure 2: The solvency region of a separated case when $\alpha_B = \alpha_b$

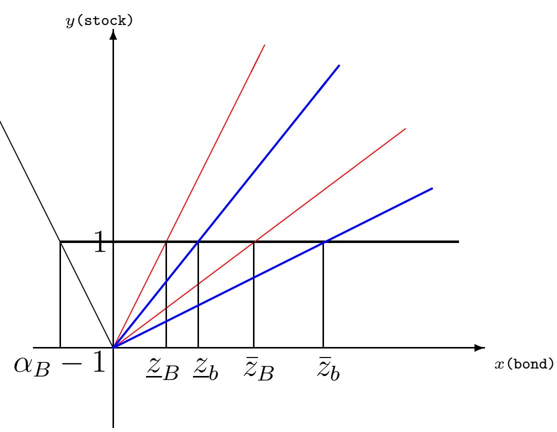

Figure 3: The solvency region of an overlapped case when $\alpha_B = \alpha_b$

Using Equation (15) and the ratio z, we can simplify the partial differential equation(PDE) to get the following ordinary differential equation (ODE) in NT_i:

$$\beta_2^i z^2 \psi_i''(z) + \beta_1^i z \psi_i'(z) + \beta_0^i \psi_i(z) + \frac{\gamma}{1-\gamma}(1-k)^{1/\gamma}\psi_i'(z)^{1-1/\gamma} + k\delta\frac{(z+(1-\alpha_i))^{1-\gamma}}{1-\gamma} + \lambda_i \psi_j(z) = 0, \tag{17}$$

$i \neq j$ for $z \in (\underline{z}_i, \bar{z}_i)$, where $\beta_2^i = \frac{1}{2}\sigma_i^2,\ \beta_1^i = \gamma\sigma_i^2 - (\mu_i - r_i),\ \beta_0^i = \frac{1}{2}\sigma_i^2\gamma(\gamma-1) + (1-\gamma)\mu_i - \rho - \delta - \lambda_i$.

Theorem 2.3 *For $i, j \in \{B, b\}$ and $j \neq i$, suppose we have concave, increasing, and homothetic $C^{2,2}$ solutions to*

$$\frac{1}{2}\sigma_i^2 y^2 v_{iyy} + r_i x v_{ix} + \mu_i y v_{iy} + \frac{\gamma}{1-\gamma}(1-k)^{1/\gamma} v_{ix}^{1-\frac{1}{\gamma}} - (\rho+\delta)v_i + k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma} + \lambda_i(v_j - v_i) \leq 0 \tag{18}$$

with equality for $\frac{x}{y} \in [\underline{z}_i, \bar{z}_i]$,which satisfy

$$(1+\theta_i)v_{ix} \geq v_{iy} \tag{19}$$

with equality for $\frac{x}{y} > \bar{z}_i$, and

$$(1-\alpha_i)v_{ix} \leq v_{iy} \tag{20}$$

with equality for $\frac{x}{y} < \underline{z}_i$. Then v_i is the value function, the optimal consumption is given by

$$c^* = \left(\frac{v_{ix}}{1-k}\right)^{-\frac{1}{\gamma}}, \tag{21}$$

and the optimal transaction policy is to transact the minimum amount so as to keep $\frac{x}{y}$ between $\underline{z}_i$ and $\bar{z}_i$.

Proof: See Appendix.

The following propositions give closed-form solutions of the free boundary problem with $k = 1$, that is, the case that the investor derives the utility only from bequest.

Proposition 2.4 *Suppose $k = 1$ and all the conditions in Theorem 2.3 are satisfied by $v_i(x, y)$ defined in (15) and (16) and $0 < \underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b < \infty$. Then*

$$\psi_B(z) = C_{1B} z^{\xi_{1B}} + C_{2B} z^{\xi_{2B}} + \psi_B^p(z) \tag{22}$$

and

$$\psi_b(z) = C_{1b} z^{\xi_{1b}} + C_{2b} z^{\xi_{2b}} + \psi_b^p(z), \tag{23}$$

where

$$\psi_B^p(z) \equiv \int_{\underline{z}_B}^{z} \frac{t^{\xi_{1B}} z^{\xi_{2B}} - z^{\xi_{1B}} t^{\xi_{2B}}}{\beta_2^B(\xi_{1B} - \xi_{2B}) t^{\xi_{1B}+\xi_{2B}+1}} \left[\delta\frac{(t+1-\alpha_B)^{1-\gamma}}{1-\gamma} + \lambda_B A_b \frac{(t+1-\alpha_b)^{1-\gamma}}{1-\gamma}\right] dt, \tag{24}$$

$$\psi_b^p(z) \equiv \int_{\underline{z}_b}^{z} \frac{t^{\xi_{1b}} z^{\xi_{2b}} - z^{\xi_{1b}} t^{\xi_{2b}}}{\beta_2^b(\xi_{1b} - \xi_{2b}) t^{\xi_{1b}+\xi_{2b}+1}} \Big[\delta \frac{(t+1-\alpha_b)^{1-\gamma}}{1-\gamma} + \lambda_b B_B \frac{(t+1+\theta_B)^{1-\gamma}}{1-\gamma}\Big] dt, \tag{25}$$

$$\xi_{1i} = \frac{(\beta_2^i - \beta_1^i) + \sqrt{(\beta_2^i - \beta_1^i)^2 - 4\beta_0^i \beta_2^i}}{2}, \quad \xi_{2i} = \frac{(\beta_2^i - \beta_1^i) - \sqrt{(\beta_2^i - \beta_1^i)^2 - 4\beta_0^i \beta_2^i}}{2},$$

and constants C_{1i} and C_{2i} are determined together with $\underline{z}_i$, $\bar{z}_i$, A_i, and B_i $(i \in \{B, b\})$ by the twelve smooth pasting conditions.

Proof: See Appendix.

Proposition 2.5 *Suppose $k = 1$, $\lambda_B > 0$, and $\lambda_b > 0$. If all the conditions in Theorem 2.3 are satisfied by $v_i(x, y)$ defined in (15) and (16) and $0 < \underline{z}_B < \underline{z}_b < \bar{z}_B < \bar{z}_b < \infty$, then*

1. *if $\underline{z}_B \le z \le \underline{z}_b$, $\psi_B(z)$ is of the same form with (22),*
2. *if $\bar{z}_B \le z \le \bar{z}_b$, $\psi_b(z)$ is of the same form with (23),*
3. *if $\underline{z}_b \le z \le \bar{z}_B$, then*

$$\begin{pmatrix} \psi_B(z) \\ \psi_b(z) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^4 z^{N_j}(D_j + u_j(z)) \\ \sum_{j=1}^4 a_j z^{N_j}(D_j + u_j(z)) \end{pmatrix}, \tag{26}$$

where N_j and a_j are as specified in (84) and (85) and u_j is determined by (88), (92), and (94),

where constants $C_{1i}, C_{2i}, D_1,\ D_2,\ D_3$ and D_4 together with $\underline{z}_i,\ \bar{z}_i,\ A_i,\ B_i(i \in \{B, b\})$ are determined by sixteen smooth pasting conditions.

Proof: See Appendix.

One can directly extend these results to the case that at least one of NT region contains the y-axis, i.e., $z = 0$.

There is a large literature on the explanation of the Equity-Premium Puzzle (Risk-free Rate Puzzle) first identified by Mehra and Prescott (1985), including using separation of the risk aversion and the intertemporal rate of substitution (e.g., Epstein and Zin (1989)) or using habit formation (e.g., Constantinides (1990)). Our model suggests that this puzzle may be resolved by the existence of liquidity risk.

To make clear the basic intuition why our model can potentially explain the puzzle, suppose there are two liquidity regimes where *only* liquidity changes across the regimes and the investor has to liquidate her stock position at the first jump time of a Poisson process with intensity δ (i.e., $k\delta > 0$). To really drive home this basic intuition, suppose further that in the "Liquid" regime the market is perfectly liquid (i.e., no transaction cost) and in the "Illiquid" regime the market is perfectly illiquid (i.e., $\alpha_I = 1$). At any point in time, there is a positive (but maybe very small) probability of switching from the Liquid regime to the Illiquid regime (i.e., the Liquid regime Poisson intensity $\lambda_L > 0$). Then it can be shown that in the Liquid regime the value function v_L takes the following form:

$$v_L(x,y) = A_L \frac{(x+y)^{1-\gamma}}{1-\gamma}, \tag{27}$$

where A_L is a constant. This implies that in the Illiquid regime the value function ψ_I satisfies

$$\beta_2^I z^2 \psi_I''(z) + \beta_1^I z \psi_I'(z) + \beta_0^I \psi_I(z) + \frac{\gamma}{1-\gamma}(1-k)^{1/\gamma} \psi_I'(z)^{1-1/\gamma} + k\delta \frac{z^{1-\gamma}}{1-\gamma} + \lambda_I A_L \frac{(z+1)^{1-\gamma}}{1-\gamma} = 0, \tag{28}$$

where β_2^I, β_1^I, β_0^I, and λ_I are the corresponding parameters in Theorem 2.3 after relabelling. This equation implies in particular that in the no-transaction region $z > 0$, i.e., the investor does not lever up in the Illiquid regime and

$$\lim_{z \to 0} \psi_I'(z) = \infty.$$

Back to the Liquid regime, as in the Merton case, the investor invests a constant fraction $1/(z^* + 1)$ of wealth in the stock due to the absence of transaction cost in this regime. The value function satisfies the following equation at z^*:

$$\begin{aligned} & -A_L \gamma \beta_2^L z^{*2} (z^*+1)^{-\gamma-1} + \beta_1^L z^* (z^*+1)^{-\gamma} + \beta_0^L A_L \frac{(z^*+1)^{1-\gamma}}{1-\gamma} \\ & + \frac{\gamma}{1-\gamma}(1-k)^{1/\gamma} A_L^{1-1/\gamma} (z^*+1)^{1-\gamma} + k\delta \frac{(z^*+1)^{1-\gamma}}{1-\gamma} + \lambda_L \psi_I(z^*) = 0, \end{aligned} \tag{29}$$

where β_2^L, β_1^L, β_0^L, and λ_L are the corresponding parameters in Theorem 2.3 after relabelling. Since the marginal utility $\psi_I'(z)$ at $z = 0$ is infinity, the investor would never lever up in the Liquid market either, no matter how large the risk premium is. This is because there is always a positive probability of the liquidity crash and there is a positive probability the investor has to liquidate her stock position in the Illiquid regime.

Similar intuition implies that even when the liquidity shock is not as extreme as above, the investor would reduce significantly her stock investment even when the market is perfectly liquid.[3] This shows that concerns over a potential liquidity crash, no matter how unlikely it is, can dramatically reduce investment in stock even when the current market is perfectly liquid and the expected excess return is high. This suggests that the existence of liquidity risk may largely explain the Equity-Premium Puzzle.

In this special case, as the transaction cost for a sale α_I in the illiquid regime increases, the sell boundary $\underline{z}_I$ approaches 0, i.e., the investor becomes more and more unlikely to sell in the illiquid regime. The following proposition provides the limit of the value functions and the transaction boundaries, which serves as a good approximation for the solution when α_I is high.

[3] Our ongoing, preliminary numerical analysis (to be included in the next version) also supports this claim.

Proposition 2.6 *Suppose $k = 1$, $0 < \alpha_I < 1$, and $\alpha_L = \theta_L = 0$. Then as $\underline{z}_I \to 0$, the value functions in the liquid and illiquid regimes approach*

$$\psi_L(z) = A_L \frac{(z+1)^{1-\gamma}}{1-\gamma} \tag{30}$$

and

$$\psi_I(z) = \begin{cases} C_1 z^m + C_2 z^\xi - \frac{1}{\beta_2^I} \int_z^{\bar{z}_I} \frac{t^m z^\xi - t^\xi z^m}{(m-\xi) t^{m+\xi+1}} \left(\delta \frac{(t+1-\alpha_I)^{1-\gamma}}{1-\gamma} + \lambda_I A_L \frac{(t+1)^{1-\gamma}}{1-\gamma} \right) dt & z \le \bar{z}_I \\ B_I \frac{(z+1+\theta_I)^{1-\gamma}}{1-\gamma} & z > \bar{z}_I, \end{cases} \tag{31}$$

respectively, where

$$m = \frac{(\beta_2^I - \beta_1^I) + \sqrt{(\beta_2^I - \beta_1^I)^2 - 4\beta_0^I \beta_2^I}}{2\beta_2^I}, \ \xi = \frac{(\beta_2^I - \beta_1^I) - \sqrt{(\beta_2^I - \beta_1^I)^2 - 4\beta_0^I \beta_2^I}}{2\beta_2^I}, \tag{32}$$

and in the liquid regime it is optimal to keep z_t constantly at z_L^*, while in the illiquid regime it is optimal to buy only when $z_t > \bar{z}_I$, where constants A_L, B_I, $\bar{z}_I$, z_L^*, C_1, and C_2 solve equations (97)-(105) in the Appendix.

Proof: See Appendix

3 Steady-state Distribution, Liquidity Premium and Expected Transaction Costs

To facilitate subsequent analysis we also derive the steady-state distribution of z_t. It can be verified that

$$dz_t = \mu_{zi}(z_t)dt - \sigma_i z_t dw_t, \tag{33}$$

where

$$\mu_{zi}(z) = (r_i - \mu_i + \sigma_i^2)z - \left(\frac{\psi_i'(z)}{1-k} \right)^{-1/\gamma}.$$

For simplicity, we focus on the case with separated no-transaction regions. The corresponding results for other cases can be derived using the same method.

Proposition 3.1 *Suppose $0 < \underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b$. Let $\phi(z)$ be the steady-state density function. Then we have*

$$\phi(z) = \begin{cases} \phi_B(z) & \underline{z}_B < z < \bar{z}_B \\ \phi_b(z) & \underline{z}_b < z < \bar{z}_b \\ 0 & \text{otherwise}, \end{cases} \tag{34}$$

where $\phi_B(z)$ and $\phi_b(z)$ solve

$$\frac{1}{2}\sigma_B^2 z^2 \phi_B''(z) - (\mu_{zB}(z) - 2\sigma_B^2 z)\phi_B'(z) - (\lambda_B - \sigma_B^2 + \mu_{zB}'(z))\phi_B(z) = 0,$$

subject to

$$\frac{1}{2}\sigma_B^2 \underline{z}_B^2 \phi_B'(\underline{z}_B) - (\mu_{zB}(\underline{z}_B) - \sigma_B^2 \underline{z}_B)\phi_B(\underline{z}_B) = 0,$$

$$\frac{1}{2}\sigma_B^2 \bar{z}_B^2 \phi_B'(\bar{z}_B) - (\mu_{zB}(\bar{z}_B) - \sigma_B^2 \bar{z}_B)\phi_B(\bar{z}_B) - \frac{\lambda_b \lambda_B}{\lambda_b + \lambda_B} = 0$$

and

$$\frac{1}{2}\sigma_b^2 z^2 \phi_b''(z) - (\mu_{zb}(z) - 2\sigma_b^2 z)\phi_b'(z) - (\lambda_b - \sigma_b^2 + \mu_{zb}'(z))\phi_b(z) = 0,$$

subject to

$$\frac{1}{2}\sigma_b^2 \underline{z}_b^2 \phi_b'(\underline{z}_b) - (\mu_{zb}(\underline{z}_b) - \sigma_b^2 \underline{z}_b)\phi_b(\underline{z}_b) + \frac{\lambda_b \lambda_B}{\lambda_b + \lambda_B} = 0,$$

$$\frac{1}{2}\sigma_b^2 \bar{z}_b^2 \phi_b'(\bar{z}_b) - (\mu_{zb}(\bar{z}_b) - \sigma_b^2 \bar{z}_b)\phi_b(\bar{z}_b) = 0.$$

Proof: See Appendix.

Note that Proposition 3.1 implies that

$$\int_{\underline{z}_B}^{\bar{z}_B} \phi_B(z)dz = \frac{\lambda_b}{\lambda_b + \lambda_B} \quad \text{and} \quad \int_{\underline{z}_b}^{\bar{z}_b} \phi_b(z)dz = \frac{\lambda_B}{\lambda_b + \lambda_B}.$$

In the seminal paper, Constantinides (1986) introduces the concept of liquidity premia. He defines the liquidity premium to be the decrease in the expected return which combined with the elimination of the transaction costs that makes the investor obtain the same utility. He concludes that transaction costs have only second-order effect on investors' utility, i.e., the ratio of the liquidity premium to the transaction cost rate is much smaller than 1. However, he assumes that an investor's investment opportunity set is constant all the time. This assumption tends to decrease the effect of transaction cost because an investor trades infrequently. In contrast to his model, investment opportunity set is stochastically changing in our model. This time-varying opportunity set may induce an investor to trade more frequently and may produce a first-order effect of the transaction cost.

To investigate this possibility, we will use two alternative measures to compute the liquidity premium. For the first measure, similar to Constantinides (1986), we take the liquidity premium to be the maximum expected return that an investor is willing to give up in both regimes to get rid of the transaction costs.

Definition 3.2 *Let $\Delta_i(x_0, y_0)$ be the liquidity premium in regime i at (x_0, y_0) for $i \in \{B, b\}$. Then Δ_i is such that*

$$v_i(x_0, y_0) = M_i \frac{(x_0 + y_0)^{1-\gamma}}{1-\gamma}, \tag{35}$$

where M_i is the unique solution of (10) with

$$\kappa_i = \frac{(\mu_i - \Delta_i - r_i)^+}{\sigma_i}, \kappa_j = \frac{(\mu_j - \Delta_i - r_j)^+}{\sigma_j}, \tag{36}$$

and $j \neq i$.

In this measure, we take the positive parts because an investor's utility is minimized when the risk premium is zero. For the second measure, we use the steady-state distribution of z_t to compute the average liquidity premium. This can also be interpreted as the cross sectional liquidity premium average for different investors. Let $\Delta_i(x_0, y_0)$ be the liquidity premium in regime i at (x_0, y_0) for $i \in \{B, b\}$, as defined in Definition 3.2. Then the average liquidity premium $\bar{\Delta}_i$ in regime i for $i \in \{B, b\}$ is

$$\bar{\Delta}_i = \int_{\underline{z}_i}^{\bar{z}_i} \Delta_i(z, 1) \frac{\lambda_B + \lambda_b}{\lambda_b} \phi_i(z) dz. \tag{37}$$

The average liquidity premium across both regimes is

$$\bar{\Delta} = \frac{\lambda_b}{\lambda_B + \lambda_b} \bar{\Delta}_B + \frac{\lambda_B}{\lambda_B + \lambda_b} \bar{\Delta}_b. \tag{38}$$

Another measure of the effect of transaction costs is the expected discounted transaction costs an investor expects to pay over the entire investment horizon. The following proposition shows one way of computing these costs for the case with separated no-transaction regions.

Proposition 3.3 *Suppose* $\nu > \max_i[-\lambda_i + r_i + (\mu_i - r_i)/(\underline{z}_B + 1)]$ *and* $0 < \underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b$. *Let* $C_i(x_t, y_t)$ *be the expected discounted transaction costs in regime i starting from* (x_t, y_t). *Then given the optimal polity* (I_t^*, D_t^*),

$$C_i(x, y) \equiv E\left[\int_0^{\tau_i} e^{-\nu t}(\theta_i dI_t^* + \alpha_i dD_t^*) + e^{-\nu \tau_i} C_j(x_{\tau_i}, y_{\tau_i})\right] = y g_i(x/y),$$

where for $\underline{z}_i \le z \le \bar{z}_i$, $g_i(\cdot)$ *solves*

$$\frac{1}{2}\sigma^2 z^2 g_i''(z) - \left((\mu_i - r_i)z + \left(\frac{\psi'(z)}{1-k}\right)^{-1/\gamma}\right) g_i'(z) - (\nu + \lambda_i - \mu_i) g_i(z) + \lambda_i g_j(z) = 0, \tag{39}$$

subject to

$$g_i(\bar{z}_i) - (\bar{z}_i + 1 + \theta_i) g_i'(\bar{z}_i) + \theta_i = 0 \tag{40}$$

and

$$g_i(\underline{z}_i) - (\underline{z}_i + 1 - \alpha_i) g_i'(\underline{z}_i) - \alpha_i = 0, \tag{41}$$

$$g_b(z) = \alpha_b \frac{\underline{z}_b - z}{\underline{z}_b + 1 - \alpha_b} + g_b(\underline{z}_b) \tag{42}$$

for $z < \underline{z}_b$, *and*

$$g_B(z) = \theta_B \frac{z - \bar{z}_B}{\bar{z}_B + 1 + \theta_B} + g_B(\bar{z}_B) \tag{43}$$

for $z > \bar{z}_B$.

Proof: See Appendix.

4 Numerical Analysis

In this section we conduct an extensive numerical analysis on the optimal investment, optimal consumption, and liquidity premium. In the subsequent analysis we use the following default parameters: $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, $\theta_b = \alpha_b = 0.01$, $k = 0$, and $\delta = 0$.[4] These parameters are chosen such that the the average (across regimes) parameter value is the same as that in Constantinides (1986). In particular, we set $k = 0$ and $\delta = 0$ so that the utility is only from consumption as in Constantinides (1986). The Poisson jump intensities are chosen to be consistent with the typical pattern that bull regime lasts longer than a bear regime. The higher volatility in the bear regime is supported by many empirical studies (e.g., Ang and Bekaert (1999), Bekaert and Wu (2000)).

We use an iterative method to solve the coupled free boundary value problem described in equations (16) and (17).

4.1 Changes in Transaction Costs

Figure 4 plots the fraction of liquidation wealth invested in the stock in the Bull regime against the transaction cost in the Bull regime. Similar to Liu and Loewenstein (2002), as the transaction cost increases, the buy boundary goes down and the sell boundary goes up. The expansion of the no-transaction region decreases the trading frequency and thus the transaction costs incurred. Figure 5 shows that the change in the Bull regime transaction cost also has a significant impact on the no-transaction region in the bear regime. As the Bull regime transaction cost increases, the investor trades less often in the Bull regime and invests more in the stock in the bear regime.

Figure 6 plots the consumption to the liquidation wealth ratio in the both regimes against the transaction cost in the Bull regime. This figure shows that as the Bull regime transaction cost increases, both consumption in not only the bull regime but also in the bear regime decreases.

Figure 7 plots the steady-state average liquidity premia against the transaction cost in the Bull regime. This figure shows that when the transaction cost is small, in both regimes transaction cost has a first-order effect on the liquidity premium, i.e., the liquidity premium to the transaction cost ratio is greater than 1. Even when the transaction cost becomes large, the ratio is still much greater than what Constantinides (1986) find. This suggests that time-varying investment opportunity set is important in affecting the liquidity premium.

Although we focus on the case with jumps in the parameters across regimes in this paper, the existence of jumps is not critical for our central result that transaction costs can have first order effect on liquidity premium. To see this, consider the case where the investor has a log preference and only the expected return changes continuously through

[4] It is worth noting that although the risk-free rate appears high, what matters to our analysis is the risk premium.

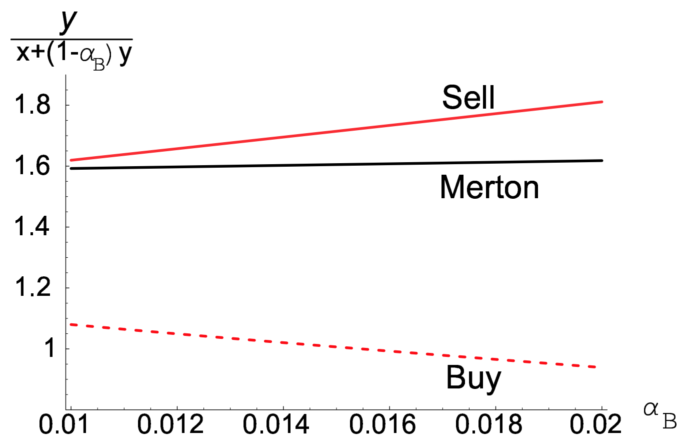

Figure 4: The fraction of wealth invested in the stock in the Bull regime, as a function of the transaction cost in the Bull regime for parameters: $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

time. To illustrate the fundamental intuition, suppose that the expected return changes seasonally through a year. In this case, without transaction cost, then the investor's optimal fraction of wealth in stock also varies seasonally through time, as depicted in Figure 8 along the middle curve. When there is a small transaction cost, the investor would form a small no-transaction interval around the optimal target (represented by the upper and the lower curves in Figure 8). Therefore as time passes, the investor would trade to keep the fraction close to the target. In addition, since the transaction cost is proportional, the investor is basically earning an effective risk premium that is equal to the true risk premium minus the transaction cost rate. Therefore, transaction cost can have a first order effect on the liquidity premium whether the parameters vary discretely or not, as long as they vary significantly and the transaction cost is small.

4.2 Changes in the Expected Return

Figure 9 plots the fraction of liquidation wealth invested in the stock in the Bull regime against the expected return in the Bull regime. Similar to Liu and Loewenstein (2002) and Liu (2004), as the expected return increases, both the buy boundary and the sell boundary go up. This basically follows the Merton line (the target line in the absence of transaction costs). Figure 10 shows that the change in the Bull regime expected return also has a significant impact on the no-transaction region in the bear regime. As the Bull regime expected return increases, the sell boundary in the bear regime moves down significantly, reflecting the tendency to reduce holding in stock in the bear regime given that the investment opportunity becomes better in the bull regime. On the other hand the

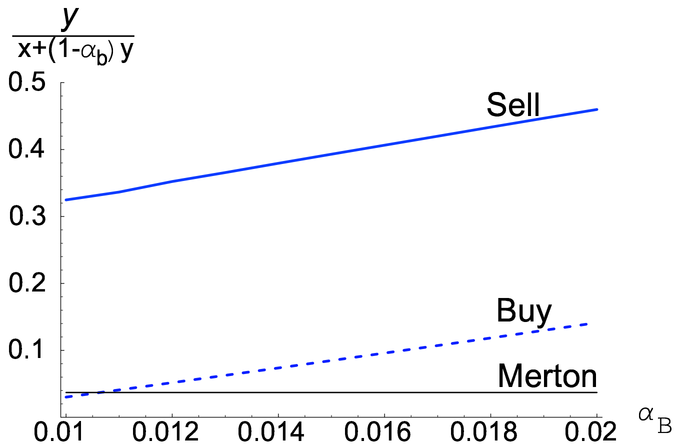

Figure 5: The fraction of wealth invested in the stock in the bear regime, as a function of the transaction cost in the Bull regime for parameters: $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$**.**

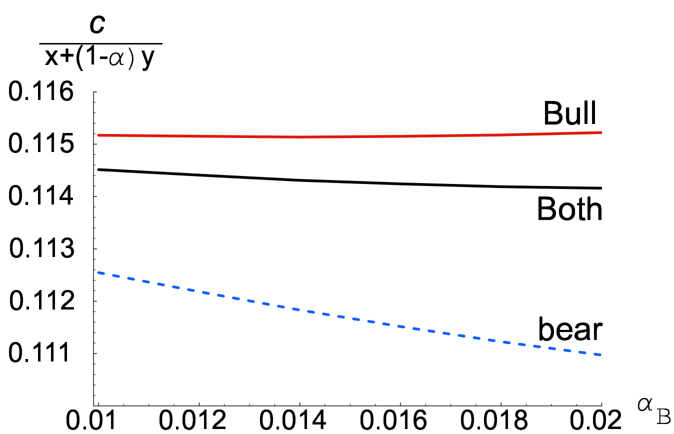

Figure 6: The consumption to wealth ratio, as a function of the transaction cost for parameters: $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$**.**

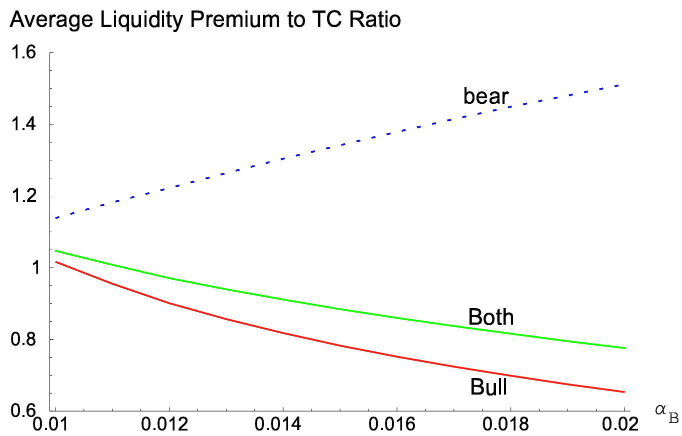

Figure 7: The liquidity premium to transaction cost ratio, as a function of the transaction cost for parameters: $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

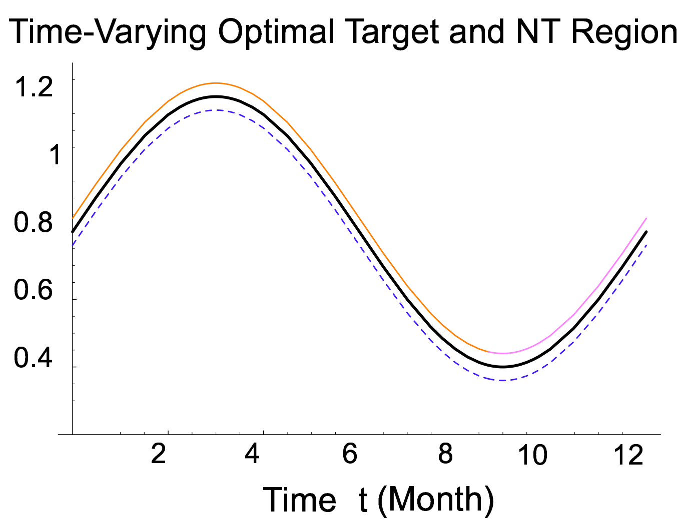

Figure 8: Continuously varying optimal target and NT region: an illustration.

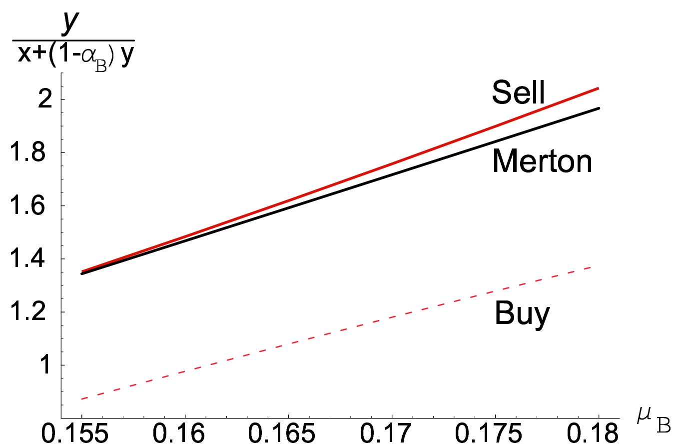

Figure 9: The fraction of wealth invested in the stock in the Bull regime, as a function of the expected return in the Bull regime for parameters: $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

buy boundary is less sensitive to the increase of the expected return in the bull regime.

Figure 11 plots the consumption to the liquidation wealth ratio in the both regimes against the expected return in the Bull regime. This figure shows that as the Bull regime expected return increases, not only the consumption in the bull regime but also the consumption in the bear regime increases because of the expected increase in the market returns and consumption smoothing.

Figure 12 plots the steady-state average liquidity premia against the expected return in the Bull regime. This figure shows that when the expected return increases, the liquidity premium becomes even greater. This is because that as the two regimes become further separated, an investor needs to pay more transaction costs at the regime switching time.

4.3 Changes in the Volatility of Stock

Figure 13 plots the fraction of liquidation wealth invested in the stock in the Bull regime against the return volatility in the Bull regime. As the expected return increases, both the buy boundary and the sell boundary go down. Similar to the case with changing expected return, this pattern basically follows the Merton line. In the bear regime, we obtain a similar pattern.

Figure 14 plots the consumption to the liquidation wealth ratio in the both regimes against the return volatility in the Bull regime. This figure shows that as the Bull regime return volatility increases, not only the consumption in the bull regime but also the consumption in the bear regime decreases because of the increase in the riskiness of the investment and thus the present value of future wealth.

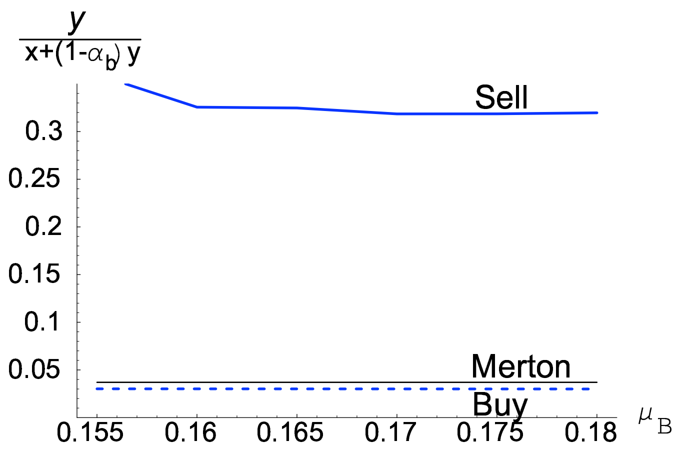

Figure 10: The fraction of wealth invested in the stock in the bear regime, as a function of the expected return in the Bull regime for parameters: $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

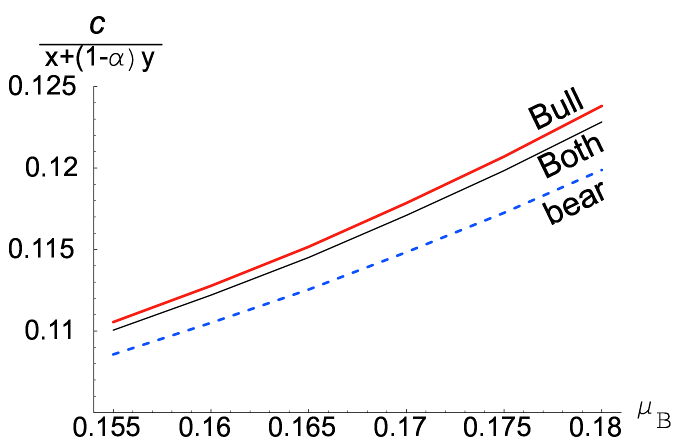

Figure 11: The consumption to wealth ratio, as a function of the expected return in the Bull regime for parameters: $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

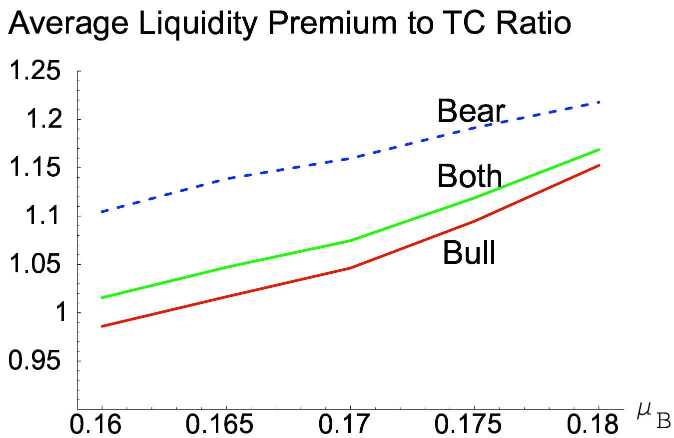

Figure 12: The liquidity premium to transaction cost ratio, as a function of the expected return in the Bull regime for parameters: $\mu_b = 0.105$**,** $\sigma_B = 0.144$**,** $\sigma_b = 0.26$**,** $\lambda_B = 0.6$**,** $\lambda_b = 1.8$**,** $\rho = 0.1$**,** $\gamma = 2.0$**,** $r_B = r_b = 0.10$**,** $\theta_B = \alpha_B = 0.01$**,** $\theta_b = \alpha_b = 0.01$**,** $k = 0$**, and** $\delta = 0$**.**

Figure 15 plots the steady-state average liquidity premia against the return volatility in the Bull regime. This figure shows that when the return volatility increases, the liquidity premium becomes smaller, but still a first-order effect. The decrease in the liquidity premium is because the increase in the riskiness decreases the investment in the stock and thus the transaction costs.

4.4 Changes in Other Parameters

Table 1 shows the comparative statics with respect to other parameters. It shows the intuitive results that the liquidity premium ratio increases as risk aversion decreases, as the probability of switching to the bear regime increases, or as the volatility in the bear regime increases. It is also worth noting that the optimal trading strategy is insensitive to the changes in the parameters in the bear regime.

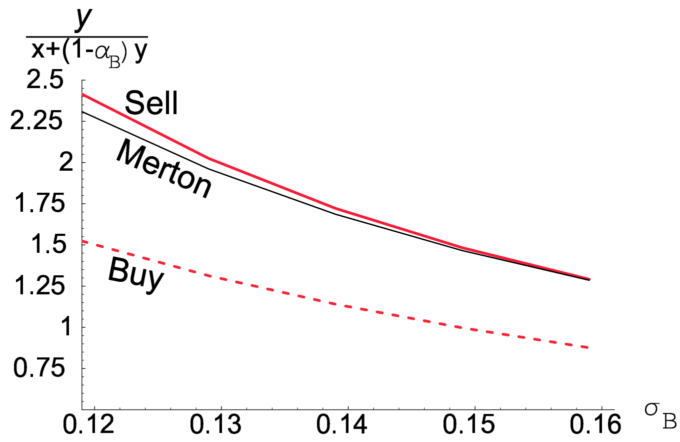

Figure 13: The fraction of wealth invested in the stock in the Bull regime, as a function of the volatility in the Bull regime for parameters: $\mu_B = 0.105$, $\mu_b = 0.105$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

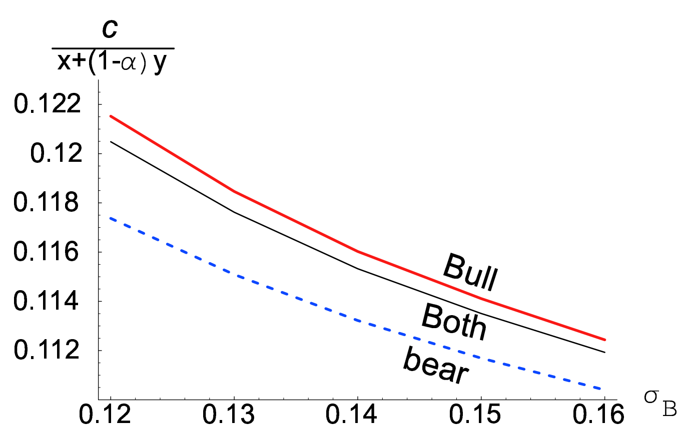

Figure 14: The consumption to wealth ratio, as a function of the volatility in the Bull regime for parameters: $\mu_B = 0.105$, $\mu_b = 0.105$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

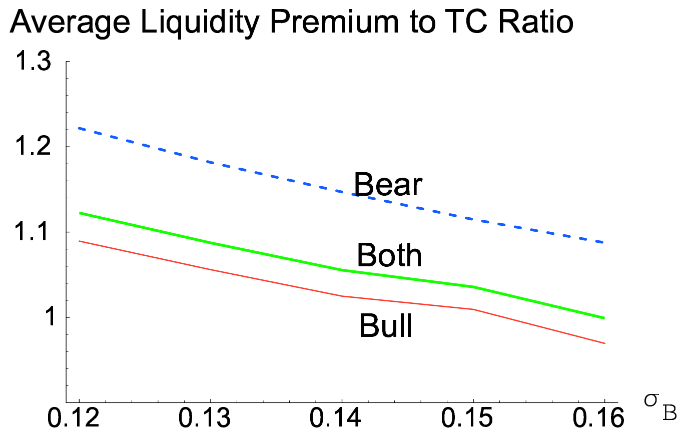

Figure 15: The liquidity premium to transaction cost ratio, as a function of the volatility in the Bull regime for parameters: $\mu_B = 0.105$, $\mu_b = 0.105$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.10$, $\theta_B = \alpha_B = 0.01$, , $\theta_b = \alpha_b = 0.01$, $k = 0$, **and** $\delta = 0$.

5 Conclusion

Constantinides (1986) finds that although transaction cost alters trading strategy significantly, it only has second-order effect on the liquidity premia implied by the equilibrium asset returns. In this paper, we show that his conclusion depends crucially on his assumption of constant investment opportunity set. In contrast, in a stochastic regime-switching model with transaction cost, we find that transaction cost can have a first-order effect on the liquidity premia for a wide range of parameters. This suggests that transaction costs may be very important for asset pricing if investment opportunity set is stochastic.

In addition, our model also has important implications for the resolution of the Equity-Premium Puzzle. We show that concerns over a potential liquidity crash, no matter how unlikely it is, can dramatically reduce investment in stock even when the current market is perfectly liquid and the expected excess return is high. This suggests that the existence of liquidity risk may largely explain the Equity-Premium Puzzle.

Parameters		c_B^*/W at $\underline{z}_B$	$\bar{z}_B$	c_b^*/W at $\underline{z}_b$	$\bar{z}_b$	y_B^*/W at $\underline{z}_B$	$\bar{z}_B$	y_b^*/W at $\underline{z}_b$	$\bar{z}_b$	Average LPTC Ratio $\bar{\Delta}_B/\alpha_B$	$\bar{\Delta}_b/\alpha_b$	$\bar{\Delta}/\alpha$
Benchmark		0.113	0.114	0.112	0.112	1.594	1.068	0.324	0.030	1.017	1.139	1.047
γ	$\times 1.05$	0.113	0.114	0.112	0.112	1.512	1.018	0.310	0.029	1.012	1.134	1.043
	$\times 0.95$	0.113	0.114	0.112	0.112	1.685	1.123	0.365	0.032	1.012	1.137	1.043
δ	$\times 1.05$	0.115	0.117	0.115	0.115	1.593	1.065	0.329	0.030	1.015	1.140	1.047
	$\times 0.95$	0.110	0.112	0.110	0.110	1.594	1.071	0.323	0.030	1.016	1.135	1.046
λ_B	$\times 1.05$	0.112	0.114	0.112	0.112	1.594	1.061	0.303	0.030	1.060	1.181	1.092
	$\times 0.95$	0.113	0.115	0.112	0.113	1.594	1.075	0.351	0.030	0.967	1.091	0.997
σ_b	$\times 1.05$	0.113	0.114	0.112	0.112	1.594	1.068	0.304	0.027	1.029	1.153	1.060
	$\times 0.95$	0.113	0.114	0.112	0.113	1.594	1.068	0.381	0.034	0.994	1.119	1.025
θ_b	$\times 1.02$	0.113	0.114	0.112	0.112	1.590	1.066	0.414	0.028	1.028	1.143	1.058
	$\times 0.98$	0.113	0.114	0.112	0.112	1.597	1.071	0.311	0.032	1.007	1.150	1.042

Table 1: The benchmark case is with parameters $\mu_B = 0.165$, $\mu_b = 0.105$, $\sigma_B = 0.144$, $\sigma_b = 0.26$, $\lambda_B = 0.6$, $\lambda_b = 1.8$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.1$, , $\theta_B = \alpha_B = \theta_b = \alpha_b = 0.01$, $k = 0$, and $\delta = 0$. Here, $\bar{\Delta}_i$ represents the average liquidity premium for the whole inaction interval of regime i, $\bar{\Delta}$ is the average liquidity premium across both regimes and $\alpha = (\lambda_b\alpha_B + \lambda_B\alpha_b)/(\lambda_B + \lambda_b)$.

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Appendix

In this Appendix, we provide proofs for the results.

Proof of Lemma 2.1 We prove for the case with $\eta_i < \eta_j$. The other case can be proved using a similar argument. (10) implies that

$$M_j = f(M_i) = \frac{\eta_i + \lambda_i}{\lambda_i} M_i - \frac{\gamma}{\lambda_i}(1-k)^{1/\gamma} M_i^{1-1/\gamma} - \frac{k\delta}{\lambda_i}.$$

For $i \in \{B, b\}$, let $\bar{M}_i > 0$ be the unique solution of

$$-\eta_i M_i + \gamma(1-k)^{1/\gamma}(M_i)^{1-1/\gamma} + k\delta = 0$$

and let $\underline{M}_i > 0$ be the unique solution of

$$-(\eta_i + \lambda_i) M_i + \gamma(1-k)^{1/\gamma}(M_i)^{1-1/\gamma} + k\delta = 0.$$

Define

$$g(M_i) = -(\eta_j + \lambda_j) f(M_i) + \gamma(1-k)^{1/\gamma} f(M_i)^{1-1/\gamma} + \lambda_j M_i + k\delta.$$

After simplification, we have that

$$g(\bar{M}_i) = (\eta_i - \eta_j)\bar{M}_i < 0.$$

In addition, it can be easily verified that $g(\underline{M}_i) = \lambda_j \underline{M}_i + k\delta \geq 0$. By continuity, there exists a M_i such that $g(M_i) = 0$. This implies that $\bar{M}_i \geq M_i \geq \underline{M}_i$. Finally, if $\gamma < 1$, it is obvious that $g'(x) > 0$, $\forall x > 0$ and therefore the solution is unique. If $\gamma > 1$, direct computation reveals that $g''(x) < 0$, $g'(0) > 0$, and $g(\underline{M}_i) > 0$, which also implies that the solution is unique.

Proof of Theorem 2.3. We first show state some properties of the candidate value function that satisfies the conditions in Theorem 2.3.

Lemma 5.1 *Suppose $v_i(x, y)$ and $v_j(x, y)$ are as in Theorem 2.3. Then we have*

1. $v_i(x, y) \geq v_i(x + (1 - \alpha_i)y, 0)$
2. *There exist k_i and K_j such that*

$$k_i(x + y)^{1-\gamma} \leq v_i(x, y) \leq K_i(x + y)^{1-\gamma} \tag{44}$$

3. $\frac{y v_{iy}}{v_i}$ *is bounded for* $\frac{x}{y} > \alpha_i - 1$.
4. $\frac{v_{ix}^{\frac{\gamma-1}{\gamma}}}{v_i}$ *is bounded for* $\frac{x}{y} > \alpha_i - 1$.

Proof of Lemma 5.1. Part 1 follows from the well known inequality for concave functions U

$$\nabla U(z_1)\cdot(z_1-z_2)\le U(z_1)-U(z_2). \tag{45}$$

$$[v_{ix}(x,y)\ \ v_{iy}(x,y)][x-(x+(1-\alpha_i)y)\ \ y]^\top \le v_i(x,y)-v_i(x+(1-\alpha_i)y,0). \tag{46}$$

Using condition (20), we have

$$0\le y(v_{iy}(x,y)-(1-\alpha_i)v_{ix}(x,y))\le v_i(x,y)-v_i(x+(1-\alpha_i)y,0), \tag{47}$$

which proves 1.

Part 2 follows from the fact that $v_i(x,y)=(x+y)^{1-\gamma}v_i(\frac{x}{x+y},\frac{y}{x+y})$.

Part 3 follows from $v_i(x,y)=y^{1-\gamma}\phi(\frac{x}{y})$, $v_{iy}=(1-\gamma)y^{-\gamma}\phi(\frac{x}{y})-y^{-\gamma}\frac{x}{y}\phi'(\frac{x}{y})$ so $\frac{yv_{iy}}{v_i}=(1-\gamma)-\frac{\frac{x}{y}\phi'(\frac{x}{y})}{\phi(\frac{x}{y})}$, which is bounded.

Part 4 follows from $v_{ix}=y^{-\gamma}\phi'(\frac{x}{y})$, $v_{ix}^{1-\frac{1}{\gamma}}=y^{1-\gamma}(\phi'(\frac{x}{y}))^{1-\frac{1}{\gamma}}$ so $\frac{v_{ix}^{\frac{\gamma-1}{\gamma}}}{v_i}=\frac{(\phi'(\frac{x}{y}))^{1-\frac{1}{\gamma}}}{\phi(\frac{x}{y})}$, which is bounded. □

Now we are ready to prove Theorem 2.3.

The proof relies on results first proved in Davis and Norman(1990). We repeat many of their arguments here adapting for our particular setting. Applying Itô's lemma to $\log[e^{-(\rho+\delta+\lambda_i)t}(1-\gamma)v_i(x(t),y(t))]$ leads to:

$$\begin{aligned}
& e^{-(\rho+\delta+\lambda_i)t}v_i(x_t,y_t)\\
= \;& v_i(x,y)\exp\Big(\int_0^t\frac{1}{v_i}(Gv-(1-k)\frac{c_s^{1-\gamma}}{1-\gamma}-k\delta\frac{(x_s+(1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma}-\lambda_i v_j)ds\\
+ \;& \int_0^t\frac{1}{v_i}[(v_{iy}-(1+\theta_i)v_{ix})dI_s+((1-\alpha_i)v_{ix}-v_{iy})dD_s]\\
+ \;& \int_0^t\frac{1}{v_i}v_{iy}y\sigma dw_s-\frac{1}{2}\int_0^t\frac{v_{iy}^2}{v_i^2}\sigma^2y^2ds\Big)
\end{aligned} \tag{48}$$

where $Gv\equiv\frac{1}{2}\sigma^2y^2v_{yy}+r_ixv_{ix}-cv_{ix}+\mu_iyv_{iy}-(\rho+\delta)v_i+\lambda_i(v_j-v_i)+(1-k)\frac{c^{1-\gamma}}{1-\gamma}+k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma}$.

In particular, setting $c=c^*\equiv\left(\frac{v_{ix}}{1-k}\right)^{-\frac{1}{\gamma}}$ and following the candidate transaction policy, we have $Gv\equiv 0$ in the no transaction cost region, the terms involving dI and dD are 0 and $\frac{1}{v_i}((1-k)\frac{c^{1-\gamma}}{1-\gamma}+k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma}+\lambda_iv_j)$ is a positive bounded function bounded away from 0. Moreover, $\frac{v_{iy}y}{v_i}$ is a bounded function for the candidate transaction policy. These properties are proved in Lemma 5.1. Notice that using Itô's Lemma we also have for a sequence of stopping times $\tau_n\to\infty$

$$\begin{aligned}
v_i(x,y) \;=\; & E\Big[\int_0^{\tau_n\wedge t}e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{*1-\gamma}}{1-\gamma}+k\delta\frac{(x_s^*+(1-\alpha_i)y_s^*)^{1-\gamma}}{1-\gamma}+\lambda_iv_j(x_s^*,y_s^*))ds\\
& +e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t}v_i(x^*_{\tau_n\wedge t},y^*_{\tau_n\wedge t})\Big].
\end{aligned} \tag{49}$$

From (48) we see if $\gamma > 1$

$$0 \geq v_i(x^*_{\tau_n \wedge t}, y^*_{\tau_n \wedge t}) \geq v_i(x,y) \exp(\int_0^{\tau_n \wedge t} \frac{1}{v_i} v_{iy} y^*_s \sigma dw_s - \frac{1}{2}\int_0^{\tau_n \wedge t} \frac{v_{iy}^2}{v_i^2}\sigma^2 y_s^{*2} ds) e^{(\rho+\delta+\lambda_i)t} \quad (50)$$

while if $0 < \gamma < 1$

$$0 \leq v_i(x^*_{\tau_n \wedge t}, y^*_{\tau_n \wedge t}) \leq v_i(x,y) \exp(\int_0^{\tau_n \wedge t} \frac{1}{v_i} v_{iy} y^*_s \sigma dw_s - \frac{1}{2}\int_0^{\tau_n \wedge t} \frac{v_{iy}^2}{v_i^2}\sigma^2 y_s^{*2} ds) e^{(\delta+\lambda_i)t} \quad (51)$$

We remind the reader that the exponential local martingales in Equations (50) and (51) are in fact Class D martingales since $\frac{1}{v_i} v_{iy} y$ is bounded. Letting $n \to \infty$ in Equation (49), observe that random variables $v_i(x^*_{\tau_n \wedge t}, y^*_{\tau_n \wedge t})$ are bounded by uniformly integrable random variables and using the dominated convergence theorem (Ash)

$$\begin{aligned} v_i(x,y) &= E[\int_0^t e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} + k\delta \frac{(x^*_s + (1-\alpha_i)y^*_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x^*_s, y^*_s))ds \\ &\quad + e^{-(\rho+\delta+\lambda_i)t} v_i(x^*_t, y^*_t)] \end{aligned} \quad (52)$$

We also have, using (48)

$$\begin{aligned} & e^{-(\rho+\delta+\lambda_i)t} v_i(x_t, y_t) \\ = \ & v_i(x,y) \exp(\int_0^t \frac{1}{v_i}(-(1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} - k\delta \frac{(x^*_s + (1-\alpha_i)y^*_s)^{1-\gamma}}{1-\gamma} - \lambda_i v_j) ds) \\ \times \ & \exp(\int_0^t \frac{1}{v_i} v_{iy} y \sigma dw_s - \frac{1}{2}\int_0^t \frac{v_{iy}^2}{v_i^2} \sigma^2 y^2 ds) \end{aligned} \quad (53)$$

Since $\frac{1}{v_i}(-(1-k)\frac{-(1-k)c_s^{1-\gamma}}{1-\gamma} - k\delta \frac{(x_s+(1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} - \lambda_i v_j)$ is a negative function bounded away from 0, and the exponential local martingale is a Class D Martingale, we have $\lim_{t\to\infty} e^{(-\delta+\lambda_i)t} E[v_i(x^*(t), y^*(t))] = 0$. As a result,

$$v_i(x,y) = E[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} + k\delta \frac{(x^*_s + (1-\alpha_i)y^*_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x^*_s, y^*_s))ds] \quad (54)$$

Next we show that given v_j, v_i is the value function and vice versa. We start by considering the case $\gamma > 1$. Consider trading strategies which start with $(x+\epsilon, y)$, and follow an admissible consumption and trading strategy for initial endowments (x,y), say $(c, \hat{x}, \hat{y}) \in \Theta(x,y)$ plus maintain ϵe^{rt} in the risk-free account. For these strategies a simple application of Itô's lemma for a set of stopping times $\tau_n \to \infty$ lead to

$$\begin{aligned} & v_i(x+\epsilon, y) \\ \geq \ & E[\int_0^{\tau_n \wedge t} e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(\hat{x}_s + \epsilon e^{rs}, \hat{y}_s))ds \\ & + e^{-(\rho+\delta+\lambda_i)\tau_n \wedge t} v_i(\hat{x}_{\tau_n \wedge t} + \epsilon e^{r\tau_n \wedge t}, \hat{y}_{\tau_n \wedge t})] \end{aligned} \quad (55)$$

From monotonicity and Part 1 of Lemma 5.1,

$$\begin{aligned} 0 &\geq e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i(\hat{x}_{\tau_n\wedge t} + \epsilon e^{r\tau_n\wedge t}, \hat{y}_{\tau_n\wedge t}) \geq e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i(\hat{x}_{\tau_n\wedge t} - (1-\alpha_i)\hat{y}_{\tau_n\wedge t} + \epsilon e^{r\tau_n\wedge t}, 0) \\ &\geq e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i(\epsilon, 0) \end{aligned}$$

and so by the dominated convergence theorem, we can let $n \to \infty$ to get

$$\begin{aligned} v_i(x+\epsilon, y) &\geq E[\int_0^t e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(\hat{x}_s + \epsilon e^{rs}, \hat{y}_s))ds \\ &\quad + e^{-(\rho+\delta+\lambda_i)t} v_i(\hat{x}_t + \epsilon e^{rt}, \hat{y}_t)]. \end{aligned} \tag{56}$$

Letting $t \to \infty$ we have $0 \geq e^{-(\rho+\delta+\lambda_i)t} v_i(\hat{x}_t + \epsilon e^{rt}, \hat{y}_t) \geq e^{-(\rho+\delta+\lambda_i)t} v_i(\epsilon, 0) \to 0$. Using the monotone convergence theorem, we then have

$$v_i(x+\epsilon, y) \geq E[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda_i v_j(\hat{x}_s + \epsilon e^{rs}, \hat{y}_s))ds]. \tag{57}$$

Next, letting $\epsilon \downarrow 0$ using the continuity of v_i and the monotone convergence theorem, we have

$$v_i(x, y) \geq E[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} - k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_s, y_s))ds] \tag{58}$$

for all feasible consumption trading strategies in $\Theta(x, y)$. This implies v is the value function given v_j.

In the case $0 < \gamma < 1$, we have

$$\begin{aligned} v_i(x, y) &\geq E[\int_0^{\tau_n\wedge t} e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_s, y_s))ds \\ &\quad + e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i(x_{\tau_n\wedge t}, y_{\tau_n\wedge t})] \end{aligned} \tag{59}$$

and $v_i(x_t, y_t) \geq 0$. This leads immediately to the conclusion

$$v_i(x, y) \geq E[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_s, y_s))ds] \tag{60}$$

for all feasible consumption trading strategies from initial position (x, y). This implies v_i is the value function given v_j. Similar argument shows that v_j is the value function given v_i.

Finally, we show that v_i and v_j are the values in Regimes i and j respectively. First, we define

$$v_i(x, y) = \sup_{(c,D,I)\in\Theta(x,y)} E[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt] \tag{61}$$

Let $v_n(x,y) = v_j(x,y)$ if n is odd and $v_n(x,y) = v_i(x,y)$ if n is even. Let τ_n be the time of the nth regime change. The above proof implies

$$v_i(x,y) = E[\int_0^{\tau_1} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-(\rho+\delta)\tau_1} v_1(x^*_{\tau_1}, y^*_{\tau_1})] \tag{62}$$

and for all feasible consumption-investment strategies

$$v_i(x,y) \geq E[\int_0^{\tau_1} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-(\rho+\delta)\tau_1} v_1(x_{\tau_1}, y_{\tau_1})] \tag{63}$$

Given time τ_1 information, we also know

$$\begin{aligned} v_1(x_{\tau_1}, y_{\tau_1}) \;\; = \;\; & E_{\tau_1}[\int_{\tau_1}^{\tau_2} e^{-(\rho+\delta)(t-\tau_1)} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{(x^*_t+(1-\alpha_i)y^*_t)^{1-\gamma}}{1-\gamma}\right) dt \\ & + e^{-(\rho+\delta)(\tau_2-\tau_1)} v_2(x^*_{\tau_2}, y^*_{\tau_2})] \end{aligned} \tag{64}$$

and for all feasible consumption-investment strategies

$$\begin{aligned} v_1(x_{\tau_1}, y_{\tau_1}) \;\; \geq \;\; & E[\int_{\tau_1}^{\tau_2} e^{-(\rho+\delta)(t-\tau_1)} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right) dt \\ & + e^{-(\rho+\delta)(\tau_2-\tau_1)} v_2(x_{\tau_2}, y_{\tau_2})]. \end{aligned} \tag{65}$$

Inserting these expressions into Equations (62) and (63) yields

$$v_i(x,y) = E[\int_0^{\tau_2} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-(\rho+\delta)\tau_2} v_2(x^*_{\tau_2}, y^*_{\tau_2})] \tag{66}$$

and

$$v_i(x,y) \geq E[\int_0^{\tau_2} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-(\rho+\delta)\tau_2} v_2(x_{\tau_2}, y_{\tau_2})] \tag{67}$$

Continuing in this manner we have

$$v_i(x,y) = E[\int_0^{\tau_n} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{(x^*_t+(1-\alpha_i)y^*_t)^{1-\gamma}}{1-\gamma}\right) dt + e^{-(\rho+\delta)\tau_n} v_n(x^*_{\tau_n}, y^*_{\tau_n})] \tag{68}$$

and

$$v_i(x,y) \geq E[\int_0^{\tau_n} e^{-(\rho+\delta)t} \left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s+(1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma}\right) ds + e^{-(\rho+\delta)\tau_n} v_n(x_{\tau_n}, y_{\tau_n})]. \tag{69}$$

We now consider the case $\gamma > 1$. As in the previous theorem consider strategies which start with initial position $(x+\epsilon, y)$, but follow a feasible consumption and trading strategy for initial position (x, y) and always maintain ϵe^{rt} in the riskless account. Very similar arguments to those in Theorem 2.1 of Davis and Norman (1990) lead to the conclusion

$$v_i(x+\epsilon, y) \geq E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_s + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma} \right) dt] \tag{70}$$

and letting $\epsilon \to 0$ we have

$$v_i(x, y) \geq E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma} \right) dt] \tag{71}$$

for all feasible consumption and trading strategies.

Since $v_n < 0$ from Equation (68) it follows

$$v_i(x, y) \leq E[\int_0^{\tau_n} e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma} \right) dt] \tag{72}$$

and since $\tau_n \to \infty$ almost surely as $n \to \infty$ we have from the monotone convergence theorem,

$$v_i(x, y) \leq E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta \frac{(x_t^* + (1-\alpha_i)y_t^*)^{1-\gamma}}{1-\gamma} \right) dt] \tag{73}$$

Thus from Equation (71) and Equation (73) we have

$$v_i(x, y) = E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta \frac{(x_t^* + (1-\alpha_i)y_t^*)^{1-\gamma}}{1-\gamma} \right) dt] \tag{74}$$

and

$$v_i(x, y) \geq E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma} \right) dt] \tag{75}$$

for all feasible trading and consumption strategies. We have proved the result for $\gamma > 1$.

When $0 < \gamma < 1$, $v_n > 0$. From Equation (69) we have

$$v_i(x, y) \geq E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta \frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma} \right) dt] \tag{76}$$

for all feasible consumption and trading strategies. To conclude the proof, we need to show that

$$\lim_{n\to\infty} E[e^{-(\rho+\delta)\tau_n} v_n(x_{\tau_n}^*, y_{\tau_n}^*)] = 0 \tag{77}$$

since this and Equation (68) would imply

$$v_i(x, y) = E[\int_0^\infty e^{-(\rho+\delta)t} \left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta \frac{(x_t^* + (1-\alpha_i)y_t^*)^{1-\gamma}}{1-\gamma} \right) dt] \tag{78}$$

Equation (77) follows from the observation that

$$\lim_{\tau_n\to\infty} E[\int_{\tau_n}^{\infty} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma}+k\delta\frac{(x_t^*+(1-\alpha_i)y_t^*)^{1-\gamma}}{1-\gamma}\right)dt]\to 0,$$

so

$$\lim_{n\to\infty} E[e^{-(\rho+\delta)\tau_n}\left((1-k)\frac{c_{\tau_n}^{*1-\gamma}}{1-\gamma}+k\delta\frac{(x_{\tau_n}^*+(1-\alpha_i)y_{\tau_n}^*)^{1-\gamma}}{1-\gamma}\right)]=0$$

and

$$0\le v_n(x,y)\le K\left((1-k)\frac{c_{\tau_n}^{*1-\gamma}}{1-\gamma}+k\delta\frac{(x_{\tau_n}^*+(1-\alpha_i)y_{\tau_n}^*)^{1-\gamma}}{1-\gamma}\right) \tag{79}$$

for a suitable constant K which can be derived from the homotheticity properties of v_i and v_j (implied by Parts 2 and 4 in Lemma 5.1).

Proof of Proposition 2.4 We split up the solvency region in these five(see Figure 2). (a) For $\alpha_B\wedge\alpha_b-1<z<\underline{z}_B$,

$$\psi_B(z)=\frac{A_B}{1-\gamma}(z+1-\alpha_B)^{1-\gamma},\quad \psi_b(z)=\frac{A_b}{1-\gamma}(z+1-\alpha_b)^{1-\gamma}.$$

(b) For $\underline{z}_B\le z\le\bar{z}_B$, ψ_B satisfies the HJB equation

$$\beta_2^B z^2\psi_B''(z)+\beta_1^B z\psi_B'(z)+\beta_0^B\psi_B(z)+\frac{\delta(z+1-\alpha_B)^{1-\gamma}}{1-\gamma}+\lambda_B\psi_b(z)=0,$$

and

$$\psi_b(z)=\frac{A_b}{1-\gamma}(z+1-\alpha_b)^{1-\gamma}.$$

Therefore, by using the method of variation of parameters we can obtain that,

$$\psi_B(z)=C_{1B}z^{\xi_{1B}}+C_{2B}z^{\xi_{2B}}+\psi_B^p(z).$$

(c) For $\bar{z}_B<z<\underline{z}_b$,

$$\psi_B(z)=\frac{B_B}{1-\gamma}(z+1+\theta_B)^{1-\gamma},\quad \psi_b(z)=\frac{A_b}{1-\gamma}(z+1-\alpha_b)^{1-\gamma}.$$

(d) For $\underline{z}_b\le z\le\bar{z}_b$,

$$\psi_B(z)=\frac{B_B}{1-\gamma}(z+1+\theta_B)^{1-\gamma},$$

and ψ_b satisfies the HJB equation

$$\beta_2^b z^2\psi_b''(z)+\beta_1^b z\psi_b'(z)+\beta_0^b\psi_b(z)+\frac{\delta(z+1-\alpha_b)^{1-\gamma}}{1-\gamma}+\lambda_b\psi_B(z)=0.$$

Using the method of variation of parameters yields that, for some constants $C_{1b},\ C_{2b}$,

$$\psi_b(z) = C_{1b} z^{\xi_{1b}} + C_{2b} z^{\xi_{2b}} + \psi_b^p(z).$$

(e) Finally, for $\bar{z}_b < z < \infty$,

$$\psi_B(z) = \frac{B_B}{1-\gamma}(z+1+\theta_B)^{1-\gamma}, \quad \psi_b(z) = \frac{B_b}{1-\gamma}(z+1+\theta_b)^{1-\gamma}.$$

The smooth pasting conditions(which are twelve) on $\underline{z}_i$ and $\bar{z}_i$ make twelve constants $\underline{z}_i,\ \bar{z}_i,\ C_{1i},\ C_{2i},\ A_i$, and B_i $(i \in \{B, b\})$ determined uniquely.

Proof of Proposition 2.5 Using the argument similar to the proof Proposition 2.4, we can prove the first two statements. Thus we assume $\underline{z}_b \le z \le \bar{z}_B$.

In this region, ψ_i $(i \in \{B, b\})$ satisfies

$$\beta_2^i z^2 \psi_i''(z) + \beta_1^i z \psi_i'(z) + \beta_0^i \psi_i(z) + \delta \frac{(z + (1-\alpha_i))^{1-\gamma}}{1-\gamma} + \lambda_i \psi_j(z) = 0. \tag{80}$$

First we find the homogeneous solutions of (80) with the conjecture of the following homogeneous solutions

$$\psi_B^h(z) = z^N \quad \text{for} \quad \psi_B(z), \quad \text{and} \quad \psi_b^h(z) = a z^N \quad \text{for} \quad \psi_b(z),$$

for some constants N and a. Then the the system (80) becomes

$$\begin{cases} \beta_2^B N(N-1) z^N + \beta_1^B N z^N + \beta_0^B z^N + b\lambda_B z^N + \delta \dfrac{(z+1-\alpha_B)^{1-\gamma}}{1-\gamma} = 0 \\ \beta_2^b b N(N-1) z^N + \beta_1^b b N z^N + \beta_0^b b z^N + \lambda_b z^N + \delta \dfrac{(z+1-\alpha_b)^{1-\gamma}}{1-\gamma} = 0. \end{cases} \tag{81}$$

If we consider only the homogeneous parts of the system (81), we can determine the constants N and b from the following two equations:

$$\beta_2^B N^2 + (\beta_1^B - \beta_2^B) N + \beta_0^B + a\lambda_B = 0 \tag{82}$$
$$a(\beta_2^b N^2 + (\beta_1^b - \beta_2^b) N + \beta_0^b) + \lambda_b = 0. \tag{83}$$

After solving $b\lambda_B$ from (82) and plugging into (83), we can obtain a 4th order polynomial equation for N:

$$-(\beta_2^B N^2 + (\beta_1^B - \beta_2^B) N + \beta_0^B)(\beta_2^b N^2 + (\beta_1^b - \beta_2^b) N + \beta_0^b) + \lambda_b \lambda_B = 0. \tag{84}$$

Let N_j $(j = 1, 2, 3, 4)$ be the four solutions to (84) and accordingly

$$a_j = -\frac{1}{\lambda_B}(\beta_2^B N_j^2 + (\beta_1^B - \beta_2^B) N_j + \beta_0^B). \tag{85}$$

Then the homogeneous solution can be represented as

$$\begin{pmatrix} \psi_B^h(z) \\ \psi_b^h(z) \end{pmatrix} = \sum_{j=1}^{4} D_j \begin{pmatrix} 1 \\ a_j \end{pmatrix} z^{N_j}, \tag{86}$$

for some constants D_j's ($j = 1, 2, 3, 4$).

In order to find the particular solution we use the method of *variation of parameter.*

From (80), we derive the matrix equation

$$X'(z) = P(z)X(z) + G(z), \tag{87}$$

where, for the particular solutions ψ_B^p for ψ_B and ψ_b^p for ψ_b,

$$X(z) = \begin{pmatrix} \psi_B^p(z) \\ \psi_b^p(z) \\ \psi_B^{p\,\prime}(z) \\ \psi_b^{p\prime}(z) \end{pmatrix}, \quad G(z) = \begin{pmatrix} 0 \\ 0 \\ -\frac{\lambda}{\beta_2^B} \frac{(z+1-\alpha_B)^{1-\gamma}}{(1-\gamma)z^2} \\ -\frac{\lambda}{\beta_2^b} \frac{(z+1-\alpha_b)^{1-\gamma}}{(1-\gamma)z^2} \end{pmatrix} \tag{88}$$

and

$$P(z) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{\beta_0^B}{\beta_2^B z^2} & \frac{\lambda_B}{\beta_2^B z^2} & \frac{\beta_1^B}{\beta_2^B z} & 0 \\ \frac{p_b}{\beta_2^b z^2} & \frac{\beta_0^b}{\beta_2^b z^2} & 0 & \frac{\beta_1^b}{\beta_2^b z} \end{pmatrix}. \tag{89}$$

Also set the *fundamental solution* $\Phi(x)$ to be

$$\Phi(z) = \begin{pmatrix} z^{N_1} & z^{N_2} & z^{N_3} & z^{N_4} \\ a_1 z^{N_1} & a_2 z^{N_2} & a_3 z^{N_3} & a_4 z^{N_4} \\ N_1 z^{N_1-1} & N_2 z^{N_2-1} & N_3 z^{N_3-1} & N_4 z^{N_4-1} \\ a_1 N_1 z^{N_1-1} & a_2 N_2 z^{N_2-1} & a_3 N_3 z^{N_3-1} & a_4 N_4 z^{N_4-1} \end{pmatrix}. \tag{90}$$

Note that the fundamental solution satisfies the equation

$$\Phi'(z) = P(z)\Phi(z). \tag{91}$$

Assume $\Phi(z)$ *is invertible for all* $\underline{z}_b \leq z \leq \bar{z}_B$.

The method of variation of parameter is started by letting

$$X(z) \equiv \Phi(z)U(z) \equiv \Phi(z)(u_1(z),\ u_2(z),\ u_3(z),\ u_4(z))^\top \tag{92}$$

From (87) and (91), we obtain

$$\Phi(z)U'(z) = G(z), \tag{93}$$

which implies that

$$U(z) = \int_{\underline{z}_B}^{z} \Phi^{-1}(t)G(t)dt. \tag{94}$$

Thus the particular solution is

$$\begin{pmatrix} \psi_B^p(z) \\ \psi_b^p(z) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^4 u_j(z) z^{N_j} \\ \sum_{j=1}^4 a_j u_j(z) z^{N_j} \end{pmatrix}. \tag{95}$$

Consequently,

$$\begin{pmatrix} \psi_B(z) \\ \psi_b(z) \end{pmatrix} = \begin{pmatrix} \psi_B^h(z) \\ \psi_b^h(z) \end{pmatrix} + \begin{pmatrix} \psi_B^p(z) \\ \psi_b^p(z) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^4 (D_j + u_j(z)) z^{N_j} \\ \sum_{j=1}^4 a_j (D_j + u_j(z)) z^{N_j} \end{pmatrix}. \tag{96}$$

Proof of Proposition 2.6 Note that, in the liquid regime, $\psi_L(z)$ must be of the form

$$\psi_L(z) = A_L \frac{(z+1)^{1-\gamma}}{1-\gamma},$$

for some constant A_L. Thus in the NT region of the illiquid regime $\psi_L(z)$ satisfies the HJB equation

$$\beta_2^I z^2 \psi_I''(z) + \beta_1^I z \psi_I'(z) + \beta_0^I \psi_I(z) + g(z) = 0$$

where $g(z) = \delta \frac{(z+1-\alpha_I)^{1-\gamma}}{1-\gamma} + \lambda_I A_L \frac{(z+1)^{1-\gamma}}{1-\gamma}$. By using the method of variation of parameters we derive that $\psi_I(z)$ is of the form

$$\psi_I(z) = C_1 z^m + C_2 z^\xi - \frac{1}{\beta_2^I} \int_z^{\bar{z}_I} \frac{t^m z^\xi - t^\xi z^m}{(m-\xi) t^{m+\xi+1}} g(t) dt,$$

where $m = \frac{(\beta_2^I - \beta_1^I) + \sqrt{(\beta_2^I - \beta_1^I)^2 - 4\beta_0^I \beta_2^I}}{2\beta_2^I} > 0$ and $\xi = \frac{(\beta_2^I - \beta_1^I) - \sqrt{(\beta_2^I - \beta_1^I)^2 - 4\beta_0^I \beta_2^I}}{2\beta_2^I} < 0$. Since $\underline{z}_I = 0$ and $\xi < 0$, we must have

$$C_2 = \int_0^{\bar{z}_I} \frac{g(t)}{\beta_2^I (m-\xi) t^{\xi+1}} dt. \tag{97}$$

The smooth pasting conditions at $\bar{z}_I$ yields

$$C_1 \bar{z}_I^m + C_2 \bar{z}_I^\xi = \frac{B_I}{1-\gamma} (\bar{z}_I + 1 + \theta_I)^{1-\gamma}, \tag{98}$$

$$C_1 m \bar{z}_I^{m-1} + C_2 \xi \bar{z}_I^{\xi-1} = B_I (\bar{z}_I + 1 + \theta_I)^{-\gamma}, \tag{99}$$

and

$$C_1 m(m-1) \bar{z}_I^{m-2} + C_2 \xi(\xi-1) \bar{z}_I^{\xi-2} - \frac{g(\bar{z}_I)}{\beta_2^I \bar{z}_I^2} = -\gamma B_I (\bar{z}_I + 1 + \theta_I)^{-\gamma-1}, \tag{100}$$

Solving (98) and (99), we have

$$C_1 = C_2 \frac{(\xi - 1 + \gamma) \bar{z}_I^\xi + \xi(1+\theta_I) \bar{z}_I^{\xi-1}}{(1-\gamma-m) \bar{z}_I^m - m(1+\theta_I) \bar{z}_I^{m-1}} \tag{101}$$

$$B_I = C_2 \frac{(1-\gamma)(m-\xi)\bar{z}_I^{\xi}}{(\bar{z}_I + 1 + \theta_I)^{-\gamma}((m-1+\gamma)\bar{z}_I + m(1+\theta_I))}. \tag{102}$$

Plugging (97), (101), and (102) into (100), we can solve A_L in terms of $\bar{z}_I$:

$$A_L = \frac{\delta[\bar{z}_I^{\xi} h(\bar{z}_I) g_1 - (m-\xi)(\bar{z}_I + 1 - \alpha)^{1-\gamma}]}{\lambda_I[(m-\xi)(\bar{z}_I+1)^{1-\gamma} - \bar{z}_I^{\xi} h(\bar{z}_I) g_2]}, \tag{103}$$

where

$$g_1 = \int_0^{\bar{z}_I} t^{-\xi-1}(t+1-\alpha_I)^{1-\gamma} dt, \quad g_2 = \int_0^{\bar{z}_I} t^{-\xi-1}(t+1)^{1-\gamma} dt$$

and

$$h(z) = \frac{m(m-1)[(\xi-1+\gamma)z + \xi(1+\theta_I)] - \gamma(1-\gamma)(m-\xi)z^2/(z+1+\theta_I)}{(1-\gamma-m)z - m(1+\theta_I)} + \xi(\xi-1).$$

In the liquid regime, the HJB must hold at z_L^*:

$$-A_L\gamma\beta_2^L z_L^{*2}(z_L^*+1)^{-\gamma-1} + A_L\beta_1^L z_L^*(z_L^*+1)^{-\gamma} + (A_L\beta_0^L + \delta)\frac{(z_L^*+1)^{1-\gamma}}{1-\gamma} + \lambda_L\psi_I(z_L^*) = 0. \tag{104}$$

The optimality of z_L^* implies that

$$A_L((\mu_L - r_L)(z_L^*+1) - \gamma\sigma_L^2) = \lambda_L(z_L^*+1)^{\gamma+2}\psi_I'(z_L^*) - \lambda_L(1-\gamma)(z_L^*+1)^{\gamma+1}\psi_I(z_L^*). \tag{105}$$

We can reduce equations (97)-(105) to two equations in $\bar{z}_I$ and z_L^*, which can be solved numerically.

Proof of Proposition 3.1 Given the optimal transaction policy, any steady-state density function ϕ of z_t must have the form of (34). In addition, for any C^2 functions $f(z,B)$ and $f(z,b)$ such that $f'(\underline{z}_B, B) = f'(\bar{z}_B, B) = f'(\underline{z}_b, b) = f'(\bar{z}_b, b) = 0$, we must have,

$$\int_{\underline{z}_B}^{\bar{z}_B} (\frac{1}{2}\sigma_B^2 z^2 f''(z,B) + \mu_{zB}(z) f'(z,B) + \lambda_B(f(\underline{z}_b, b) - f(z,B)))\phi_B(z)dz$$
$$+ \quad \int_{\underline{z}_b}^{\bar{z}_b} (\frac{1}{2}\sigma_b^2 z^2 f''(z,b) + \mu_{zb}(z) f'(z,b) + \lambda_b(f(\bar{z}_B, B) - f(z,b)))\phi_b(z)dz = 0.$$

By the property of the continuous time Markov chain, we must have

$$\int_{\underline{z}_B}^{\bar{z}_B} \phi_B(z)dz = \frac{\lambda_b}{\lambda_b + \lambda_B} \quad \text{and} \quad \int_{\underline{z}_b}^{\bar{z}_b} \phi_b(z)dz = \frac{\lambda_B}{\lambda_b + \lambda_B}.$$

Then by integration by parts, we have

$$\int_{\underline{z}_B}^{\bar{z}_B} \Big[\frac{1}{2}\sigma_B^2 (z^2\phi_B(z))'' - (\mu_{zB}(z)\phi_B(z))' - \lambda_B\phi_B(z)\Big] f(z,B) dz$$

$$
\begin{aligned}
&+ \int_{\underline{z}_b}^{\bar{z}_b} \Big[\frac{1}{2}\sigma_b^2(z^2\phi_b(z))'' - (\mu_{zb}(z)\phi_b(z))' - \lambda_b\phi_b(z)\Big] f(z,b)dz \\
&+ f(\underline{z}_B, B)\Big[\frac{1}{2}\sigma_B^2(\underline{z}_B^2\phi_B'(\underline{z}_B) + 2\underline{z}_B\phi_B(\underline{z}_B)) - \mu_{zB}(\underline{z}_B)\phi_B(\underline{z}_B)\Big] \\
&- f(\bar{z}_B, B)\Big[\frac{1}{2}\sigma_B^2(\bar{z}_B^2\phi_B'(\bar{z}_B) + 2\bar{z}_B\phi_B(\bar{z}_B)) - \mu_{zB}(\bar{z}_B)\phi_B(\bar{z}_B) - \frac{\lambda_b\lambda_B}{\lambda_b+\lambda_B}\Big] \\
&+ f(\underline{z}_b, b)\Big[\frac{1}{2}\sigma_b^2(\underline{z}_b^2\phi_b'(\underline{z}_b) + 2\underline{z}_b\phi_b(\underline{z}_b)) - \mu_{zb}(\underline{z}_b)\phi_b(\underline{z}_b) + \frac{\lambda_b\lambda_B}{\lambda_b+\lambda_B}\Big] \\
&- f(\bar{z}_b, b)\Big[\frac{1}{2}\sigma_b^2(\bar{z}_b^2\phi_b'(\bar{z}_b) + 2\bar{z}_b\phi_b(\bar{z}_b)) - \mu_{zb}(\bar{z}_b)\phi_b(\bar{z}_b)\Big] \\
&= 0.
\end{aligned}
$$

Since $f(z,B)$ and $f(z,b)$ are arbitrary C^2 functions (only need to satisfy $f'(\underline{z}_B, B) = f'(\bar{z}_B, B) = f'(\underline{z}_b, b) = f'(\bar{z}_b, b) = 0$), we must have that each bracketed term is equal to zero, which implies that Proposition 1 holds after some simplification.

Proof of Proposition 3.3 Define $W_t \equiv x_t + y_t$ and $\pi_t \equiv \frac{y_t}{x_t+y_t}$. Direct application of Itô's lemma to $e^{-(\nu+\lambda_i)t}C_i(x_t, y_t)$ yields that

$$
\begin{aligned}
& e^{-(\nu+\lambda_i)s}C_i(x_s, y_s)) - C_i(x,y)) \\
= & \int_0^s e^{-(\nu+\lambda_i)t}\left(\frac{1}{2}\sigma^2 z^2 g_i''(z) - \left((\mu_i - r_i)z + \left(\frac{\psi'(z)}{1-k}\right)^{-1/\gamma}\right) g_i'(z) - (\nu+\lambda_i-\mu_i)g_i(z)\right) y_t dt \\
+ & \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - (z_t + 1 + \theta_i)g_i'(z_t))dI_t^* + \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - (z_t + 1 - \alpha_i)g_i'(z_t))dD_t^* \\
+ & \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - z_t g_i'(z_t))\sigma y_t dw_t \\
= & -\int_0^s e^{-(\nu+\lambda_i)t}[\theta_i dI_t^* + \alpha_i dD_t^* + \lambda_i C_j(x_t, y_t)dt] + \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - z_t g_i'(z_t))\sigma\pi_t W_t dw_t,
\end{aligned} \tag{106}
$$

where the second equality follows from (39)-(41). Then we have

$$
dW_t = r_i W_t dt + (\mu_i - r_i)\pi_t W_t dt + \sigma\pi_t W_t dw_t - \alpha_i dD_t^* - \theta_i dI_t^*. \tag{107}
$$

We then have

$$
W_t \le W_0 e^{\int_0^t (r_i + (\mu_i - r_i)\pi_s)ds} N_t, \tag{108}
$$

where

$$
N_t \equiv e^{-\frac{1}{2}\int_0^t (\pi_s\sigma)^2 ds + \int_0^t \pi_s\sigma dw_s}
$$

is a martingale. Thus

$$
\begin{aligned}
0 \le E[e^{-(\nu+\lambda_i)t}C_i(x_t, y_t)] &= E[e^{-(\nu+\lambda_i)t}\pi_t W_t g_i(x_t/y_t)] \\
&\le E[e^{-(\nu+\lambda_i)t} M W_t]
\end{aligned}
$$

$$\begin{aligned} &\leq ME[e^{-(\nu+\lambda_i)t}e^{\int_0^t (r_i+(\mu_i-r_i)\pi_s-\nu-\lambda_i)ds}N_t] \\ &\leq Me^{-(\nu+\lambda_i-r_i-(\mu_i-r_i)/(\underline{z}_B+1))t} \to 0, \quad as \quad t\to\infty, \end{aligned} \tag{109}$$

where M is a constant, the first inequality follows from the boundedness of the optimal π_t and $g(.)$, the second inequality follows from (108), the third inequality holds because the optimal π_t is bounded below by $1/(\underline{z}_B+1)$ and $E[N_t]=1$, and the convergence follows from the first assumption in the proposition.

In addition, (108) also implies that the last term in (106) is a martingale. Therefore, taking expectation and limit as $t\to\infty$ and using (106) and (109), we have that

$$yg_i(x/y) = E\left[\int_0^{\tau_i} e^{-\nu t}(\theta_i dI_t^* + \alpha_i dD_t^*) + e^{-\nu\tau_i}C_j(x_{\tau_i}, y_{\tau_i})\right].$$

The expressions in (42) and (42) follow from the fact that in these transaction regions the investor immediately transacts to the corresponding boundary, incurring the costs represented by the first terms. This completes the proof.