Only Average Transaction Costs for Large Trades Matter: The Fundamental Theorem of Asset Pricing

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Abstract

The paper establishes the fundamental theorem of asset pricing (FTAP) in markets where transaction costs need not be convex and are allowed to jump. Specifically, it contributes to the literature in three ways. The first finding is that only average costs for large transactions matter to asset pricing in markets where transaction costs are not convex possibly due to the fixed cost component, indivisibility of assets or shifts in the cost structure. Remarkably, no matter how complex the non-convex transaction cost functions are, the pricing rules are characterized in a simple, concrete form as in the case with proportional transaction costs. Second, the results of the paper are differentiated from equilibrium theory which usually requires the continuity and convexity of transaction cost functions. Finally, the notion of arbitrage used here is appropriate to explaining asset prices. Specifically, the no arbitrage condition is equivalent to viability of asset prices. The consequence vindicates the coherence of arbitrage as a conceptual framework for equilibrium analysis. Moreover, the pricing rules can be characterized by minimal information on the nature of transaction costs.

KEYWORDS: The fundamental theorem of asset pricing, arbitrage, fixed transaction costs, non-convex transaction costs.

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I. Introduction

Borrowing and lending rates differ in the real world. The spread between them is ascribed to financial intermediation, which is costly due to market frictions. This is an easy example where the law of one price is violated in the face of market frictions.[1] The arbitrage pricing theory which does not take market frictions into account may be unable to characterize asset prices in economies which are far from being ideal. The effect of market frictions on asset pricing must be properly understood to make an asset pricing theory come closer to reality.

The paper establishes the fundamental theorem of asset pricing (FTAP) in markets where transaction costs need not be convex and are allowed to jump. Specifically, it contributes to the literature in three ways. The first finding is that only average costs for large transactions matter to asset pricing in markets where transaction costs are not convex possibly due to the fixed cost component, indivisibility of assets or shifts in the cost structure. Remarkably, no matter how complex the non-convex transaction cost functions are, the pricing rules are characterized in a simple, concrete form as in the case with proportional transaction costs. Therefore, asset valuation can be free from the intractability of fixed and non-convex transaction costs. Second, the results of the paper are differentiated from equilibrium theory which usually requires the continuity and convexity of transaction cost functions. It is also worth noting that marginal transaction costs are irrelevant to the determination of the pricing rules as far as they differ from average costs for large transactions. The advantages of arbitrage pricing theory of the paper over equilibrium theory are not shared with the literature which deals with frictionless markets or with proportional transaction costs. Finally, the notion of arbitrage used here is appropriate to explaining asset prices. Specifically, the no arbitrage condition is equivalent to viability of asset prices.[2] The consequence vindicates the coherence of arbitrage as a conceptual framework for equilibrium analysis. 'Arbitrage' pricing theory would be almost vacuous if it undergos serious failure in the viability test.[3] Moreover, the pricing rules can be characterized

[1] Evidences of mispricing are abundant in the literature; stock index futures (Canina and Figlewski (1995)), primes and scores (Jarrow and O'Hara (1989)), closed-end funds (Pontiff (1996)), stock options (Conrad (1989)) among others.

[2] Informally speaking, asset prices are viable if they allow agents to make an optimal choice in asset markets.

[3] As shown later, the well-known notions of arbitrage do not pass the viability test. Won and Hahn (2004) also illustrate the failure of viability from a different perspective.

by minimal information on the nature of transaction costs. What is required to capture the pricing rules is information on the average cost for large transactions which is independent of the local behavior of transaction cost functions.

The last point deserves remarks. The amount of information on the nature of market frictions which is necessary to describe the arbitrage pricing rules is determined by the definition of arbitrage. Broadly speaking, the notions of arbitrage with frictional markets can be classified as 'local arbitrage' and 'global arbitrage'.[4] Local arbitrage is introduced to describe the conditions for a price to allow no costless financial improvement in all contingencies from the given position. Thus the no local arbitrage condition depends on the initial position. On the other hand, global arbitrage is useful in describing the conditions which are satisfied with arbitrary equilibrium prices by keeping minimal the information on unobservable data of the markets such as risk preferences, the initial wealth and optimal choices of individuals. This paper takes the latter approach.[5] Local arbitrage is used in most literature with convex transaction costs and taxation schedules.[6] Both notions of arbitrage usually lead to the same consequence of asset pricing in the case with proportional transaction costs. This is not the case, however, with non proportional transaction costs or taxes.[7] As illustrated in the main text, asset pricing by local arbitrage tends to extremely underestimate the multiplicity of the pricing rules when transaction costs are convex and extremely overestimate it when they are non-convex.

Moreover, both notions of arbitrage drastically diverge in informational requirement to characterize the pricing rules. The marginal transaction costs or tax rates are placed in the pricing kernel in the presence of transaction costs and taxation. Thus they are indispensable information to capture the pricing rules. If transaction costs or capital income taxes are proportional to the size of transactions, the marginal transaction cost or tax rate is constant over all positions.[8] If transaction cost functions are nonlinear, however, the marginal frictional cost depends upon

[4]'Global' is used to contrast the latter to the former.

[5]Dammon and Green (1987) take the same line of research to investigate the existence of equilibrium with progressive taxation.

[6]Ross (1987), and Dybvig and Ross (1986), Prisman (1986), Dermody and Prisman (1993) among others.

[7]Ross (1987), and Dybvig and Ross (1986), Prisman (1986), Dermody and Prisman (1993) adopt local arbitrage to examine the effect of convex transaction costs or convex taxation schedules on arbitrage pricing. In this case, the implications of the two approaches to asset pricing diverge as shown below.

[8]Garman and Ohlson (1981), Boyle and Vorst (1992), Jouini and Kallal (1995), Kabanov (1999), Kabanov and Stricker (2001), Delbaen, Kabanov and Valkeila (200), Zhang, Xu and Deng (2002), Schachermayer (2004) among others examine the effect of proportional transaction costs on asset pricing.

the functional form of market frictions as well as the position to be concerned about. In theory, local arbitrage leads to sharper results than global arbitrage in the case with nonlinear cost functions. In reality, however, it is hard to pick out the pricing rules which meet the status quo of the markets under the unobservable pricing kernel. For example, Ross (1987) and Dybvig and Ross (1986) introduce local tax arbitrage to address arbitrage pricing theory with progressive taxation. In this case, the marginal tax rate is calculable on the basis of the knowledge of both the tax schedule and the current portfolio position of individuals, which do not belong to public domain of information in general.

II. The Model

Asset markets are assumed to persist over finite time periods, $t = 0, 1, \ldots, T$. Let $\Omega = \{1, 2, \ldots, S\}$ denote a finite partition of states of nature. The revelation of information is described by a collection of partitions of Ω, $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}$, where $\mathcal{F}_t$ is finer than $\mathcal{F}_{t-1}$ (i.e. $\sigma \in \mathcal{F}_t$ and $\sigma' \in \mathcal{F}_{t-1}$ imply that $\sigma \subset \sigma'$ or $\sigma \cap \sigma' = \emptyset$) for all $t = 1, \ldots, T$.[9] We assume that $\mathcal{F}_0 = \{\Omega\}$. The information available at time $t = 0, 1, \ldots, T$ is described by the set $\sigma \in \mathcal{F}_t$ of the states of nature. We set $D = \bigcup_{t=0}^{T} \mathcal{F}_t$ and $D_{-T} = \bigcup_{t=0}^{T-1} \mathcal{F}_t$. An element in D is called a node or an event and D is called an event tree. In particular, σ_t in D denotes an event in $\mathcal{F}_t$. For each $\sigma_t \in \mathcal{F}_t$, let σ_t^- denote the event which immediately precedes σ_t, σ_t^+ the set of events which immediately succeed σ_t, and D_{σ_t} the set of events which consist of σ_t and all the events succeeding σ_t. The set D_{σ_t} is a subtree at σ_t. For some positive integer n, let $\mathcal{L}(D_{-T}, \mathbb{R}^n)$ denote the collection of all $\mathbb{R}^n$-valued functions on D_{-T}. For brevity, $\mathcal{L}^n$ will be used instead of $\mathcal{L}(D_{-T}, \mathbb{R}^n)$. Let $\#D$ and $\#D_{-T}$ denote the number of elements in D and $\#D_{-T}$, respectively. Then $\mathcal{L}^n$ is the Euclidean space of dimension $(\#D_{-T}) \times n$. Let L denote the set of all real-valued functions defined on D. We set $L_+ = \{x \in L : x(\sigma) \geq 0, \sigma \in D\}$ and $L_{++} = \{x \in L : x(\sigma) > 0, \sigma \in D\}$.

There are J long-lived assets issued at time 0 and traded in each state of time $t = 0, \ldots, T - 1$. Allowing for some notational abuse, we also denote the set of assets by J. A price process of asset j is a function $q_j : D_{-T} \to \mathbb{R}$ and a trading strategy is a function $\theta : D_{-T} \to \mathbb{R}^J$. Thus,

[9] For more details on the stochastic economy, see Magill and Shafer (1991) or Magill and Quinzii (1996).

$q = (q_1, \ldots, q_J)$ and θ are a point in $\mathcal{L}^J$. More specifically, $q^j(\sigma)$ and $\theta^j(\sigma)$ denote a price and a position of asset j, and $q(\sigma) \in \mathbb{R}^J$ and $\theta(\sigma) \in \mathbb{R}^J$ denote prices and positions of J assets at the node $\sigma \in D$. For a price-event pair (q, σ) in $\mathcal{L}^J \times D$, let $R(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ denote the net return schedule which is derived from deducting transaction costs from the gross return. Specifically, if a trading strategy $\theta \in \mathcal{L}^J$ is chosen at the price q, the net return $R(\theta, q; \sigma)$ will be delivered to the investor in the event σ. For a price $q \in \mathcal{L}^J$, let $R(\cdot, q)$ denote the function which assigns each $\sigma \in D$ to $R(\cdot, q; \sigma)$. Thus, for a trading strategy $\theta \in \mathcal{L}^J$, $R(\theta, q)$ is a $\#D$-dimensional net return vector.

Transaction costs are incurred in buying and selling assets. They can be decomposed as two parts, one which is quite independent of variations in the size of the positions and the other which is responsive to them. The former includes fixed transaction costs and the latter corresponds to variable transaction costs. For a pair $(v, q) \in \mathcal{L}^J \times \mathcal{L}^J$ of a net trade and a price, let $C(v(\sigma), q(\sigma); \sigma)$ denote a transaction cost function in the event $\sigma \in D$. We assume that there exist functions $F(\cdot, q(\sigma); \sigma) : \mathcal{L}^J \to \mathbb{R}$ and $C^j_\sigma(\cdot, q^j(\sigma)) : \mathbb{R} \to \mathbb{R}$ for each $j \in J$ such that for all $v \in \mathcal{L}^J$,[10]

$$C(v(\sigma), q(\sigma); \sigma) = F(v(\sigma), q(\sigma); \sigma) + \sum_{j \in J} C^j_\sigma(v^j(\sigma), q^j(\sigma)).$$

The function $F(v(\sigma), q(\sigma); \sigma)$ can be interpreted as the fixed transaction cost component and each $C^j_\sigma(v^j(\sigma), q^j(\sigma))$ as the variable transaction cost for the change of the position $v^j(\sigma)$ for asset j in the event σ.[11] For each $\sigma \in D$, let $R_\sigma \in \mathbb{R}^J$ denote the gross returns of assets which are available before transaction costs are deducted. Then for each $(\theta, q) \in \mathcal{L}^J \times \mathcal{L}^J$, the net return process $R(\theta, q; \sigma)$ is represented by

$$R(\theta, q; \sigma) = \begin{cases} -q(\sigma) \cdot \theta(\sigma) - C(\theta(\sigma), q(\sigma); \sigma), & \sigma = \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-) - q(\sigma) \cdot (\theta(\sigma) - \theta(\sigma^-)) - C(\theta(\sigma) - \theta(\sigma^-), q(\sigma); \sigma), & \sigma \in D_{-T} \setminus \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-), & \sigma \in \mathcal{F}_T \end{cases}$$

[10] The decomposition of C need not be unique but it can be uniquely determined by putting in F all the costs which vanish on the average as net trades go to infinity.

[11] Dermody and Prisman (1993) indicate that transaction costs on trading each individual asset is a function of the number of shares traded, and transaction costs on a trade is the sum of the transaction costs on trading each individual stock.

For each $v \in \mathcal{L}^J, q\mathcal{L}^J$ and $\sigma \in D$, we define the function

$$\widetilde{C}(v(\sigma), q(\sigma); \sigma) = \sum_{j \in J} C^j_\sigma(v^j(\sigma), q^j(\sigma)).$$

The function $\widetilde{C}(v(\sigma), q(\sigma); \sigma)$ denotes the variable transaction cost in the event σ. For each $\theta \in \mathcal{L}^J, q\mathcal{L}^J$ and $\sigma \in D$, we define the return function net the variable cost $\widetilde{C}(v(\sigma), q(\sigma); \sigma)$ by

$$\widetilde{R}(\theta, q; \sigma) = \begin{cases} -q(\sigma) \cdot \theta(\sigma) - \widetilde{C}(\theta(\sigma), q(\sigma); \sigma), & \sigma = \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-) - q(\sigma) \cdot (\theta(\sigma) - \theta(\sigma^-)) - \widetilde{C}(\theta(\sigma) - \theta(\sigma^-), q(\sigma); \sigma), & \sigma \in D_{-T} \setminus \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-), & \sigma \in \mathcal{F}_T \end{cases}$$

III. Examples

Four transaction cost functions are presented in the first example which look quite distinct but give the same average cost for large transactions. As shown later, they give the same consequence in terms of asset pricing. It is illustrated in the second example that the no arbitrage condition of of Dermody and Prisman (1993) may be far from being a necessary condition for viability when the transaction cost function is convex, piece-wise linear and differentiable at zero. Specifically, the no arbitrage condition of of Dermody and Prisman (1993) explains only a 'small' part of viable prices.[12]

Example 1. Transaction cost functions are given which locally differ but behave asymptotically in the same manner. As shown later, they are indistinguishable in terms of arbitrage pricing. Let q denote the price of an asset. For each $i = 1, \ldots, 4$, we define $C_i(\cdot, q) : \mathbb{R} \to \mathbb{R}$ as the transaction cost function for the asset. They are depicted in Figure 1.

<Figure 1>

The function $C_1(\cdot, q)$ is piecewise linear with kinks at $-1/q$ and $1/q$, and flat at zero while $C_2(\cdot, q)$ is strictly convex and differentiable. The function $C_3(\cdot, q)$ jumps regularly and represents transaction costs with indivisible assets. In contrast to the first three functions which are

[12]This result is true in general when transaction cost functions are convex and differentiable at zero.

free from fixed cost component, $C_4(\cdot, q)$ has the fixed transaction cost denoted by K. If K is ignored in $C_4(\cdot, q)$, it is the same as $C_3(\cdot, q)$.

Despite their local difference, they have the same asymptotic property. We set $\overline{C}_i(\theta, q) = \lim_{\lambda\to\infty} C_i(\lambda\theta, q)/\lambda$. It is easy to see that for each $i = 1, \ldots, 4$, $\overline{C}_i(\theta, q)$ is equal to the function $C(\theta, q)$ defined by

$$C_p(\theta, q) = \begin{cases} \frac{q}{20}\theta, & \text{if } \theta \geq 0 \\ \frac{-q}{30}\theta, & \text{if } \theta < 0 \end{cases}$$

This function represents proportional transaction costs in such a way that $q/20$ is a transaction cost for buying one unit of the asset and $q/30$ is a transaction cost for selling one unit of the asset. The average transaction cost of the four cases for large transactions is all the same as that of the proportional transaction cost function $C_p(\cdot, q)$.

<Figure 2>

Example 2. A two-asset one-state economy with convex transaction costs is considered to illustrate that the set of prices which satisfy the no arbitrage condition of Dermody and Prisman (1993) constitutes only a small part of viable prices. This means that the no arbitrage condition of Dermody and Prisman (1993) may extremely underestimate the multiplicity of viable prices. Both assets pay one dollar in the state of the next period. Thus the gross return structure is represented by the 1×2 matrix $R = [1 \;\; 1]$. Let q^2 denote the price of the second asset. We assume that the transaction cost function for trading the second asset takes the form

$$C_f(\theta^2, q^2) = \begin{cases} (q^2\theta^2 - 1)/20, & \text{if } \theta^2 \geq 1/q^2 \\ 0, & \text{if } -1/q^2 \leq \theta^2 < 1/q^2 \\ -(q^2\theta^2 + 1)/30, & \text{if } \theta^2 < -1/q^2 \end{cases}$$

This function is depicted in the first diagram of Figure 1. Clearly, $C_f(0, q^2) = 0$ and $C_f(\theta^2, q^2)$ is piece-wise linear, continuous and convex. The following is an adaptation of the notion of arbitrage introduced in Dermody and Prisman (1993) to the current example.

Definition DP: The price (q^1, q^2) admits *no arbitrage* if it satisfies

$\max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0$ and
$\bar{\theta}^1 + \bar{\theta}^2 = 0$ for all optimal solution $(\bar{\theta}^1, \bar{\theta}^2)$.

We check the validity of the above notion of arbitrage as a conceptual framework for equilibrium analysis. This can be done by investigating how the no arbitrage condition is appropriate to explaining the viability of asset prices.

Let Λ_{DP} denote the set of prices which admit no arbitrage in the sense of Definition DP. We claim that

$$\Lambda_{DP} = \{(q^1, q^2) \in \mathbb{R}^2_{++} : q^1 = q^2\}.$$

Let $(\bar{\theta}^1, \bar{\theta}^2)$ denote the solution to $\max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0$. Suppose that $q^1 = q^2$. Then it is easy to see that $\bar{\theta}^1 + \bar{\theta}^2 = 0$ and

$$\max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} = 0.$$

Therefore, (q^1, q^2) satisfies the no arbitrage condition of Definition DP.

We show that any (q^1, q^2) with $q^1 \neq q^2$ admits an arbitrage in the sense of Definition DP. Suppose that $q^1 > q^2$. We set $\eta^1 = -1/2q^2$ and $\eta^2 = 1/2q^2$. Then $-1/q^2 \leq \eta^2 < 1/q^2$ and $\eta^1 + \eta^2 \geq 0$. Moreover,

$$\begin{aligned} -\eta^1 q^1 - \eta^2 q^2 - C_f(\eta^2, q^2) &= -\eta^1 q^1 - \eta^2 q^2 \\ &= -\tfrac{1}{2}\left(1 - \tfrac{q^1}{q^2}\right) > 0 \end{aligned}$$

Suppose that $q^1 < q^2$. We set $\eta^1 = 1/2q^2$ and $\eta^2 = -1/2q^2$. Then $-1/q^2 \leq \eta^2 < 1/q^2$ and $\eta^1 + \eta^2 \geq 0$. Moreover,

$$\begin{aligned} -\eta^1 q^1 - \eta^2 q^2 - C_f(\eta^2, q^2) &= -\eta^1 q^1 - \eta^2 q^2 \\ &= \tfrac{1}{2}\left(1 - \tfrac{q^1}{q^2}\right) > 0 \end{aligned}$$

It follows that for any (q^1, q^2) with $q^1 \neq q^2$, $\max\{-\theta^1 q^1 - \theta^2 q^2 - C_f(\theta^2, q^2) : \theta^1 + \theta^2 \geq 0\} > 0$. Thus, whenever $q^1 \neq q^2$, (q^1, q^2) does not satisfies the no arbitrage condition of Definition DP.

To do a viability test with prices in Λ_{DP}, we introduce the set Λ_V of viable prices. Any prices in Λ_V would allow agents with monotonic preferences to have an optimal choice in asset markets. We claim that

$$\Lambda_V = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (29/30)q^2 \leq q^1 \leq (21/20)q^2\}.$$

(This claim is verified in the Appendix.) Prices in Λ_V are viable but any $(q^1, q^2) \in \Lambda_V$ with $q^1 \neq q^2$ does not satisfy the no arbitrage condition of Dermody and Prisman (1993). Thus, the no arbitrage condition of Dermody and Prisman (1993) explains a very small part of viable prices.

IV. Transaction Costs in the Large

Transaction cost functions are characterized in terms of average costs for large transactions. Specifically, the artificial proportional transaction costs are constructed which turn out to share the set of pricing rules with the original nonlinear transaction cost functions. The proportional function is determined independently of the non-convexity and local behavior of the original functions such as fixed transaction costs. The following conditions are imposed on the transaction cost functions.

Assumption 1: For all $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, the following hold.

i) $F(0, q; \sigma) = 0$ and $C^j_\sigma(0, q^j(\sigma)) = 0$.

ii) There exists $f \geq 0$ such that $F(v, q; \sigma) \leq f$ for all $v \in \mathcal{L}^J$.

The first condition of Assumption 1 requires that no transaction costs nothing. The second condition indicates that the cost function is bounded.[13] In particular, it subsumes transaction cost functions with the fixed cost component.

For each $q \in \mathcal{L}^J, j \in J$ and $\sigma \in D$, we set

$$b^j(q^j(\sigma); \sigma) = \lim_{z \to \infty} \frac{C^j_\sigma(z, q^j(\sigma))}{z} \text{ and } s^j(q^j(\sigma); \sigma) = \lim_{z \to -\infty} \frac{C^j_\sigma(z, q^j(\sigma))}{z}.$$

Then $b^j(q^j(\sigma); \sigma)$ and $s^j(q^j(\sigma); \sigma)$ are the average transaction cost for large long positions and short positions in asset j, respectively. We assume that for each $q \in \mathcal{L}^J, j \in J$ and $\sigma \in D$,

$$-\infty < s^j(q^j(\sigma); \sigma) \leq b^j(q^j(\sigma); \sigma) < \infty.$$

[13] Allowing for some analytical complications, this condition can be replaced by the more general condition that $\lim_{\lambda \to \infty} F(\lambda v, q; \sigma)/\lambda = 0$ for all $v \in \mathcal{L}^J$.

We define the function

$$\overline{C}^j_\sigma(z, q^j(\sigma)) = \begin{cases} b^j(q^j(\sigma);\sigma)z, & z \geq 0 \\ s^j(q^j(\sigma);\sigma)z, & z < 0 \end{cases}$$

This function can be considered a proportional transaction cost for the change of the position z for asset j in markets where $b^j(q^j(\sigma);\sigma)$ and $s^j(q^j(\sigma);\sigma)$ are charged as the unit transaction cost for long and short positions, respectively. In fact, $\overline{C}^j_\sigma(\cdot, q^j(\sigma))$ describes the asymptotic behavior of transaction costs for $C^j_\sigma(\cdot, q^j(\sigma))$; for all $z \in \mathbb{R}$,

$$\overline{C}^j_\sigma(z, q^j(\sigma)) = \lim_{\lambda\to\infty} \frac{C^j_\sigma(\lambda z, q^j(\sigma))}{\lambda}.$$

This is because $C^j_\sigma(0, q^j(\sigma)) = \overline{C}^j_\sigma(0, q^j(\sigma)) = 0$ and for a price $q^j(\sigma)$ and a nonzero position z for asset $j \in J$,

$$\begin{aligned} \lim_{\lambda\to\infty} \tfrac{C^j_\sigma(\lambda z, q^j(\sigma))}{\lambda} &= \lim_{\lambda\to\infty} \tfrac{C^j_\sigma(\lambda z, q^j(\sigma))}{\lambda z} z \\ &= \left\{ \begin{array}{ll} b^j(q^j(\sigma);\sigma)z, & z > 0 \\ s^j(q^j(\sigma);\sigma)z, & z < 0 \end{array} \right\} \\ &= \overline{C}^j_\sigma(z, q^j(\sigma)) \end{aligned}$$

Thus, $\overline{C}^j_\sigma(z, q^j(\sigma))$ has the same average cost for large transactions as $C^j_\sigma(\cdot, q^j(\sigma))$. For each each $v \in \mathcal{L}^J$, we set

$$\overline{C}(v(\sigma), q(\sigma);\sigma) = \lim_{\lambda\to\infty} \frac{C(\lambda v(\sigma), q(\sigma);\sigma)}{\lambda}.$$

Since by Assumption 1, $\lim_{\lambda\to\infty} F(\lambda v(\sigma), q(\sigma);\sigma)/\lambda = 0$, it follows that that for each $v \in \mathcal{L}^J$,

$$\overline{C}(v(\sigma), q(\sigma);\sigma) = \sum_{j\in J} \overline{C}^j_\sigma(v^j(\sigma), q^j(\sigma)).$$

We define the following notion of transaction cost function.

Definition 4.1 For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $\overline{C}^j_\sigma(\cdot, q^j(\sigma)) : \mathbb{R} \to \mathbb{R}$ and $\overline{C}(\cdot, q(\sigma);\sigma) : \mathbb{R}^J \to \mathbb{R}$ are the *transaction cost function in the large (LTC function)* for asset $j \in J$ and for the asset structure J, respectively.

Clearly, the LTC function is convex and positively homogeneous.

Lemma 4.1 For each $q \in \mathcal{L}^J, j \in J$ and $\sigma \in D$, $\overline{C}^j_\sigma(\cdot, q^j(\sigma))$ is convex and $\overline{C}^j_\sigma(\lambda z, q^j(\sigma)) = \lambda \overline{C}^j_\sigma(z, q^j(\sigma))$ for all $z \in \mathbb{R}$ and $\lambda \geq 0$.

If transaction cost functions are proportional, they coincide with the LTC function.[14] Let $\bar{b}^j(\sigma) \in [0, 1)$ and $\bar{s}^j(\sigma) \in [0, 1)$ denote the transaction cost rate for buying and selling the asset j at σ, respectively. Clearly, $\bar{b}^j(\sigma)q^j(\sigma) = b^j(q^j(\sigma); \sigma)$ and $\bar{s}^j(\sigma)q^j(\sigma) = -s^j(q^j(\sigma); \sigma)$ for all $q \in \mathcal{L}^J, j \in J$ and $\sigma \in D$. In this case, the LTC function has the form

$$\overline{C}^j_\sigma(z, q^j(\sigma)) = \begin{cases} \bar{b}^j(\sigma)q^j(\sigma)z, & z \geq 0 \\ -\bar{s}^j(\sigma)q^j(\sigma)z, & z < 0 \end{cases}$$

We assume that the transaction cost and LTC functions satisfy the following condition..

Assumption 2: For all $q \in \mathcal{L}^J$, $v \in \mathcal{L}^J$, $\lambda > 0$ and $\sigma \in D$, there exists a number $\bar{c} \geq 0$ such that $|C(v(\sigma), q(\sigma); \sigma) - \overline{C}(v(\sigma), q(\sigma); \sigma)| \leq \bar{c}$ and $\widetilde{C}(\lambda v(\sigma), q(\sigma); \sigma) + \bar{c} \geq \lambda \widetilde{C}(v(\sigma), q(\sigma); \sigma)$.

Assumption 2 states that the difference between the LTC function and the original transaction cost function is uniformly bounded from above by $\bar{c}$, and the variable cost function $\widetilde{C}$ shows the decreasing return to scale of transaction when it is shifted upward by the positive number $\bar{c}$.

Remarkably, no conventional conditions like continuity and convexity are imposed on the transaction cost functions. Thus Assumption 1-2 enable us to examine the effect of fixed transaction costs as well as non-convex transaction costs on the existence and the form of pricing rules. In particular, they are satisfied with transaction cost functions with jumps due to the fixed cost component and the indivisibility of assets. For example, they hold in the third and fourth diagrams of Figure 1.

For each q, θ in $\mathcal{L}^J$ and $\sigma \in D$, we define the set

$$G(\theta, q; \sigma) = \{v \in \mathcal{L}^J : \exists\, \lambda_\sigma > 0 \text{ such that } \widetilde{R}(\theta + \lambda v, q; \sigma) \geq \widetilde{R}(\theta, q; \sigma) \;\; \forall \lambda \geq \lambda_\sigma\}.$$

The set $G(\theta, q; \sigma)$ contains portfolios corresponding to a direction in which the current position θ can be changed to a large extent without reducing the return net the variable transaction

[14]The effect of proportional transaction costs on asset valuation is examined in Garman and Ohlson (1981), and Zhang, Xu and Deng (2002) among others.

costs in the event σ. This set will be very useful in characterizing arbitrage opportunities with transaction costs. Clearly, $G(\theta, q; \sigma)$ is a cone. If $\widetilde{R}(\cdot, q; \sigma)$ is concave, then it coincides with the recession cone of the level set $\{\theta \in \mathcal{L}^J : \widetilde{R}(\theta, q; \sigma) \geq c\}$ of $\widetilde{R}(\cdot, q; \sigma)$.[15] For notational ease, we set $G(q; \sigma) = G(0, q; \sigma)$ for each $q \in \mathcal{L}^J$. We make the following assumption.

Assumption 3. For each $q \in \mathcal{L}^J$ and $\sigma \in D$, the following hold.

i) $G(\theta, q; \sigma) = G(q; \sigma)$ for each $\theta \in \mathcal{L}^J$.

ii) If $v \in G(q; \sigma)$ and $\widetilde{R}(v, q; \sigma) > 0$, then $\widetilde{R}(\lambda v, q; \sigma) \to \infty$ as $\lambda \to \infty$.

The first condition of Assumption 3 states that if a portfolio as a change of the position generates additional income in the large at some position, then it does at all other positions. The two conditions of Assumption 3 are useful in examining the effect of transaction costs on the availability of arbitrage from all the positions. In particular, they hold true when $\widetilde{R}(\cdot, q; \sigma)$ is concave.[16] We set $G(q) = \bigcap_{\sigma \in D} G(q; \sigma)$. A nonzero $v \in G(q)$ represents a change of the position which adds nonnegative income to any positions in each state. Clearly, $G(q)$ is a cone.

For each $q \in \mathcal{L}^J$ and $\sigma \in D$, we consider the function $V(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ defined by

$$V(\theta, q; \sigma) = \begin{cases} -q(\sigma) \cdot \theta(\sigma) - \overline{C}(\theta(\sigma), q(\sigma); \sigma), & \sigma = \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-) - q(\sigma) \cdot (\theta(\sigma) - \theta(\sigma^-)) - \overline{C}(\theta(\sigma) - \theta(\sigma^-), q(\sigma); \sigma), & \sigma \in D_{-T} \setminus \sigma_0 \\ R_\sigma \cdot \theta(\sigma^-), & \sigma \in \mathcal{F}_T \end{cases}$$

Recalling that $\overline{C}(v(\sigma), q(\sigma); \sigma) = \lim_{\lambda \to \infty} C(\lambda v(\sigma), q(\sigma); \sigma)/\lambda$ for each $v \in \mathcal{L}^J$ and $\sigma \in D$, we see that for all $\theta \in \mathcal{L}^J$,

$$V(\theta, q; \sigma) = \lim_{\lambda \to \infty} \frac{R(\lambda\theta, q; \sigma)}{\lambda} = \lim_{\lambda \to \infty} \frac{\widetilde{R}(\lambda\theta, q; \sigma)}{\lambda}.$$

The function $V(\cdot, q; \sigma)$ represents the net return schedule at the price q in the event σ if transaction costs were to be charged according to the LTC function $\overline{C}(\cdot, q(\sigma); \sigma)$.

[15] For a convex set C in a Euclidean space E, $v \in E$ is a direction of recession of C if $v + x \in C$ for all $x \in C$. The recession cone of C is a set of directions of recession of C. For details, see Rockafellar (1970).

[16] For details on this point, see Theorem 8.7 of Rockafellar (1970).

Definition 4.2 For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $V(\cdot, q; \sigma)$ is the *return function in the large (L-return function)* for the asset structure J.

The following properties of the L-return function $V(\cdot, q; \sigma)$ are immediate from Lemma 4.1.

Lemma 4.2 For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $V(\cdot, q; \sigma)$ is concave and $V(\lambda\theta, q; \sigma) = \lambda V(\theta, q; \sigma)$ for all $\theta \in \mathcal{L}^J$ and $\lambda \geq 0$.

V. Arbitrage with Transaction Costs

As indicated in Jouini, Kallal and Napp (2001), fixed transaction costs may affect the accessibility of arbitrage opportunities. Consider a two-period economy, i.e., $T = 1$. Suppose that for some $q \in \mathbb{R}^J$, there exists $\theta \in \mathbb{R}^J$ such that $q \cdot \theta = 0$, $R_\sigma \cdot \theta \geq 0$ for all $\sigma \in D \setminus \{\sigma_0\}$ and $R_\sigma \cdot \theta > 0$ for some $\sigma \in D \setminus \{\sigma_0\}$. We assume that preferences follow the expected utility hypothesis, the utility function is strictly increasing, and the probability that each $\sigma \in D \setminus \{\sigma_0\}$ occurs is positive. Clearly, θ is an arbitrage opportunity in markets with no transaction costs. If fixed cost is charged for transactions in the markets, the arbitrage opportunity would be available only to an agent with sufficiently large initial wealth to cover it. Let e denote his initial wealth and K the fixed transaction cost to be paid for participating in the asset markets. If $K > e$, the agent can increase income to be delivered in the event σ with $R_\sigma \cdot \theta > 0$ and therefore, his utility as much as possible by exploiting the arbitrage opportunity at the cost of K. This is impossible, however, for agents with $e < K$ because they are not able to finance market participation.

We will assume the presence of agents with sufficiently large initial wealth to cover the fixed transaction costs F in each $\sigma \in D_{-T}$.[17] Under this assumption, the notion arbitrage can be freed from fixed transaction cost components. As verified later, the notion of arbitrage which ignores the effect of F on the net returns is appropriate to studying the effect of transaction costs on asset pricing and equilibrium.[18]

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[17] A typical example of such an agent is institutional investors.

[18] The notion of arbitrage used here is comparable to that of Jouini, Kallal and Napp (2001) who aim to examine arbitrage pricing theory in the absence of agents with large initial wealth. As illustrated above, the no arbitrage

Definition 5.1: An asset price $q \in \mathcal{L}^J$ admits *no arbitrage opportunities* if there is no $\theta \in G(q)$ which satisfies $\widetilde{R}(\theta, q) > 0$.

The above notion of arbitrage has several desired properties. First, it allows us to characterize as easily the pricing rules in markets with non-proportional transaction costs as in markets with proportional transaction costs. In particular, the cost functions need not be convex and are allowed to have the fixed cost component. Second, it turns out to exactly match viability of asset prices. This is one of the virtues that no arbitrage conditions must satisfy as a conceptual framework for equilibrium analysis. Third, the no arbitrage condition does not depend on the initial position. This property is particularly useful in characterizing the pricing rules when marginal transaction costs are an information not to be observed. If transaction costs is non-proportional, information on the cost function and the initial position must be provided to calculate marginal transaction costs. But such information is specific to individuals and does not belong to the public domain of information. Finally, it subsumes as a special case the existing notions of arbitrage with proportional transaction costs used in Garman and Ohlson (1981) and Zhang, Xu and Deng (2002) among others.

Let Λ denote the set of no arbitrage prices for agent i. Then we see that

$$\Lambda = \{q \in \mathcal{L}^J : \widetilde{R}(v, q) \not> 0 \text{ for all } v \in G(q)\},$$

where $\not>$ denotes the negation of the vector inequality $>$.[19]

We show that the no arbitrage condition can be characterized by the L-return function.

Proposition 5.1: Under Assumption 1-3, $q \in \Lambda$ if and only if there exists no nonzero $v \in \mathcal{L}^J$ which satisfies $V(v, q) > 0$.

PROOF : Suppose that there exists a nonzero $v \in \mathcal{L}^J$ such that $V(v, q) > 0$. We define the set

$$D^+ = \{\sigma \in D : V(v, q; \sigma) > 0\}.$$

condition of Jouini, Kallal and Napp (2001) is not compatible with viability in general in markets where some agent has sufficiently large initial wealth in each $\sigma \in D_{-T}$ to cover fixed transaction costs.

[19]Let x and x' be vectors in a Euclidean space. Then $x \geq x'$ implies x is greater than or equal to x' in a component-wise manner; $x > x'$ implies that $x \geq x'$ and $x \neq x'$; $x \gg x'$ implies that each component of x is greater than the counterpart of x'.

Then $D^+ \neq \emptyset$. For each $\sigma \in D \setminus D^+$, $V(v, q; \sigma) = 0$. We consider the case with $\sigma = \sigma_0$. By Assumption 2, we see that for all $\lambda > o$,

$$\begin{aligned} 0 &= -\lambda q(\sigma) \cdot v(\sigma) - \overline{C}(\lambda v(\sigma), q(\sigma); \sigma) \\ &\leq -\lambda q(\sigma) \cdot v(\sigma) - C(\lambda v(\sigma), q(\sigma); \sigma) + \bar{c} \\ &\leq -\lambda q(\sigma) \cdot v(\sigma) - \lambda \widetilde{C}(v(\sigma), q(\sigma); \sigma) - F(\lambda v(\sigma), q(\sigma); \sigma) + 2\bar{c} \end{aligned}$$

By Assumption 1, we have $\lim_{\lambda \to \infty} F(\lambda v(\sigma), q(\sigma); \sigma)/\lambda = 0$. It follows by dividing both of the above inequality by λ and letting $\lambda \to \infty$ that

$$\widetilde{R}(v, q; \sigma) = q(\sigma) \cdot v(\sigma) - \widetilde{C}(v(\sigma), q(\sigma); \sigma) \geq 0.$$

By applying the same arguments to the case with $\sigma \in D \setminus \{\sigma_0\}$, we can show that

$$\widetilde{R}(v, q; \sigma) = R_\sigma \cdot v(\sigma^-) - q(\sigma) \cdot (v(\sigma) - v(\sigma^-)) - \widetilde{C}(v(\sigma) - v(\sigma^-), q(\sigma); \sigma) \geq 0.$$

On the other hand, $V(v, q; \sigma) > 0$ for each $\sigma \in D^+$. Since $V(v, q; \sigma) = \lim_{\lambda \to \infty} \widetilde{R}(\lambda v, q; \sigma)/\lambda$, there exists $\lambda_\sigma > 0$ such that $\widetilde{R}(\lambda v, q; \sigma) > 0$ for all $\lambda \geq \lambda_\sigma$ and therefore $v \in G(q; \sigma)$. Thus, we conclude that there exists $v' \in G(q)$ such that $R(v', q) > 0$.

Suppose that there exists a nonzero $v \in G(q)$ such that $\widetilde{R}(v, q) > 0$. Since $v \in G_i(q)$ and $\widetilde{R}_i(0, q; \sigma) = 0$ for each $\sigma \in D$, there exists $\lambda_\sigma > 0$ such that $\widetilde{R}(\lambda v, q; \sigma) \geq 0$ for all $\lambda \geq \lambda_\sigma$. This implies that for all $\sigma \in D$,

$$V(v, q; \sigma) = \lim_{\lambda \to \infty} \widetilde{R}_i(\lambda v, q; \sigma)/\lambda \geq 0.$$

We show that $V(v, q) > 0$. Since $V(v, q) \geq 0$, we have only to show that there exists $\sigma \in D$ such that $V(v, q; \sigma) > 0$. Recalling that $\widetilde{R}(v, q) > 0$, we can pick $\bar{\sigma} \in D$ such that $\widetilde{R}_i(v, q; \bar{\sigma}) > 0$. If $\bar{\sigma} \in \mathcal{F}_T$, we are done because

$$V(v, q; \bar{\sigma}) = \widetilde{R}(v, q; \bar{\sigma}) = R_{\bar{\sigma}} \cdot v(\bar{\sigma}^-).$$

Thus without loss of generality, we can assume that $\bar{\sigma} \in D \setminus \mathcal{F}_T$. Suppose that $\bar{\sigma} = \sigma_0$. Since $v \in G_i(q)$ and $\widetilde{R}_i(v, q; \sigma_0) > 0$, it follows by (ii) of Assumption 3 that $\widetilde{R}_i(\mu' v, q; \sigma_0) \to \infty$ as $\mu' \to \infty$. Thus by (ii) of Assumption 1, there exists $\mu > 0$ such that

$$\widetilde{R}(\mu v, q; \sigma_0) - F(\mu v(\sigma_0), q(\sigma_0); \sigma_0) - \bar{c} > 0.$$

This implies that

$$-\mu q(\sigma_0) \cdot v(\sigma_0) - C(\mu v(\sigma_0), q(\sigma_0); \sigma_0) - \bar{c} > 0.$$

Since by Assumption 2, $C(\mu v(\sigma_0), q(\sigma_0); \sigma_0) + \bar{c} \geq \overline{C}(\mu v(\sigma_0), q(\sigma_0); \sigma_0)$, we have

$$\begin{aligned} -\mu q(\sigma_0) \cdot v(\sigma_0) - C(\mu v(\sigma_0), q(\sigma_0); \sigma_0) - \bar{c} &\leq -\mu q(\sigma_0) \cdot v(\sigma_0) - \overline{C}(\mu v(\sigma_0), q(\sigma_0); \sigma_0) \\ &= \mu V(v, q; \sigma_0). \end{aligned}$$

Thus, we have $V(v, q; \sigma_0) > 0$. By applying the same argument to the case with $\bar{\sigma} \in D_{-T} \backslash \{\sigma_0\}$, we can show that $V(v, q; \bar{\sigma}) > 0$. Therefore, we conclude that $V(v, q) > 0$. $\square$

Suppose that the transaction cost function $\widetilde{R}(\cdot, q; \sigma)$ is convex for each $q \in \mathcal{L}^J$ and $\sigma \in D$. Then each $\widetilde{R}(\cdot, q; \sigma)$ is concave. Thus, the following corollary is immediate from Proposition 5.1.

Corollary 5.1: Suppose that $\widetilde{C}(\cdot, q(\sigma); \sigma)$ is convex for all $q \in L^J$ and $\sigma \in D$. Then under Assumption 1-2, $q \in \Lambda$ if and only if there exists no nonzero $v \in \mathcal{L}^J$ such that $V(v, q) > 0$

By Proposition 5.1, the notion of arbitrage in Definition 5.1 is equivalent to the following in the case with non-convex transaction costs.

Definition 5.1′: An asset price $q \in \mathcal{L}^J$ admits *no arbitrage opportunities* if there is no $\theta \in \mathcal{L}^J$ which satisfies $V(v, q) > 0$.

The set of no arbitrage prices is expressed as

$$\Lambda = \left\{ q \in \mathcal{L}^J : V(v, q) \not> 0 \text{ for all } v \in \mathcal{L}^J \right\}.$$

It is worth noting that the effect of transaction costs on $V(v, q)$ is determined by the LTC function which describes the behavior of average costs for large transactions. Thus, transaction costs affect asset pricing only via the LTC function. Neither fixed transaction costs nor any costs attributed to small transactions have no influence on the existence and the form of pricing rules.

VI. Arbitrage and the Existence of Pricing Rules

The no arbitrage condition of Definition 5.1 allows us to extend the fundamental theorem of asset pricing to the case with fixed and non-proportional transaction costs.[20] In this section, we show the equivalence between the no arbitrage condition and the existence of pricing rules. The viability test for Definition 5.1 is given in the next section.

Theorem 6.1: Under Assumption 1-3, $q \in \Lambda$ if and only if there exists $\pi \in L_{++}$ such that $\pi \cdot V(v,q) \leq 0$ for all $v \in \mathcal{L}^J$.

PROOF : Suppose that $q \notin \Lambda$. Then by Proposition 5.1 there exists $v \in \mathcal{L}^J$ such that $V(v,q) > 0$. Since $\pi \in L_{++}$, this implies that $\pi \cdot V(v,q) > 0$, which leads to a contradiction.

Suppose that $q \in \Lambda$. We define the set

$$Z(q) = \{y \in L : y \leq V(v,q) \ \text{ for some } \ v \in \mathcal{L}^J\}.$$

First we show that $Z(q)$ is a closed, convex cone. By Lemma 4.2, $Z(q)$ is a convex cone. The following lemma shows that $Z(q)$ is closed.

Lemma 6.1: For all $q \in \mathcal{L}^J$, the set $Z(q)$ is closed.

PROOF : See the appendix.

Let Δ denote the set $\{y \in L_+ : \sum_{\sigma \in D} y(\sigma) = 1\}$. Clearly, Δ is compact and convex. Then $q \in \Lambda$ is equivalent to the condition that $Z(q) \cap (L_+ \setminus \{0\}) = \emptyset$ or $Z(q) \cap \Delta = \emptyset$. Since $Z(q)$ is a closed, convex cone, by the separating hyperplane theorem there exists a nonzero $\pi \in L$ such that

$$\sup_{v \in \mathcal{L}^J} \pi \cdot V(v,q) < \inf_{y \in \Delta} \pi \cdot y.$$

In particular, we see that

$$0 = \pi \cdot V(0,q) \leq \sup_{v \in \mathcal{L}^J} \pi \cdot V(v,q) < \inf_{y \in \Delta} \pi \cdot y.$$

[20] Magill and Quinzii (1996), and Dybvig and Ross (1989) are a great reference to the fundamental theorem of asset pricing in a frictionless market.

Thus we have $\inf_{y\in\Delta} \pi \cdot y > 0$, which implies that $\pi \in L_{++}$. Let $v \in \mathcal{L}^J$. Then for each $\lambda > 0$, we have $\pi \cdot V(\lambda v, q) = \lambda\pi \cdot V(v, q) < \inf_{y\in\Delta} \pi \cdot y$, or

$$\pi \cdot V(v, q) < \inf_{y\in\Delta} (\pi \cdot y)/\lambda,$$

By letting $\lambda \to \infty$, we have $\pi \cdot V(v, q) \leq 0$

□

The pricing rules which satisfy the no arbitrage condition are characterized in a concrete form as follows.

Theorem 6.2: Under Assumption 1-3, $q \in \Lambda$ if and only if there exists $\pi \in L_{++}$ such that for each $\sigma \in D$ and $j \in J$,

$$\pi(\sigma)[q^j(\sigma) + s^j(q^j(\sigma);\sigma)] \leq \sum_{\hat{\sigma}\in D_\sigma\setminus\{\sigma\}} \pi(\hat{\sigma})R^j_{\hat{\sigma}} \leq \pi(\sigma)[q^j(\sigma) + b^j(q^j(\sigma);\sigma)].$$

PROOF : For each $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, the LTC function $\overline{C}^j_\sigma(\cdot, q^j(\sigma))$ in $V(\cdot, q;\sigma)$ is proportional to the parameter which is equal to $b^j(q^j(\sigma);\sigma)$ if the changes of position are nonnegative and to $s^j(q^j(\sigma);\sigma)$ if the changes of position are negative. Thus the proof of the theorem can be done by applying the same argument made in the proof of Theorem 3.1 and 3.2 of Zhang, Xu and Deng (2002) which assume that the original transaction cost function $C^j_\sigma(\cdot, q^j(\sigma))$ is proportional to the unit cost of transactions. □

The results of Theorem 6.1 and 6.2 deserve several remarks. Neither convexity nor continuity are imposed on the transaction cost functions. Specifically, the theorems apply to the case where transaction cost functions are non-convex due to the fixed cost component or indivisibility of assets. They are differentiated from equilibrium theory which requires the convexity and continuity of cost functions. It is worth noting that the pricing rules with non-convex transaction costs are characterized in a simple, concrete form as in the case with proportional transaction costs. This fact makes asset valuation free from the intractability of fixed and non-convex transaction costs. Moreover, the pricing rules characterized in Theorem 6.2 are viable under additional assumptions as shown below.

The following examples show that compared to Definition 5.1, the notion of arbitrage used in Dermody and Prisman (1993) underestimates the multiplicity of the pricing rules when transaction cost functions are convex and overestimates it when they are non-convex.

Example 3. We consider a two-asset one-state economy which is the same as in Example 2 except for the transaction cost functions. Both assets pay one dollar in the state. Then the return function is a 1×2 matrix $R = [1 \;\; 1]$. Let q^2 denote the price of the second asset. We assume that the transaction cost function for trading the second asset is one of the four one of the four $C_i(\cdot, q^2)$'s depicted in Figure 1. For each $i = 1, \ldots, 4$, let Λ_i denote the set of no arbitrage prices with C_i. Then it follows by Theorem 6.2 that for each $i = 1, \ldots, 4$,

$$\Lambda_i = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (29/30)q^2 \leq q^1 \leq (21/20)q^2\}.$$

It is worth noting that in the case with C_1 and C_2, Λ_{DP} defined in Example 2 is equal to $\{(q^1, q^2) \in \mathbb{R}^2_{++} : q^1 = q^2\}$ and therefore, is much smaller than Λ_1. Thus the no arbitrage condition of Dermody and Prisman (1993) extremely underestimates the multiplicity of the pricing rules which are shown to be viable in the next section.[21]

Example 4. We consider a two-asset one-state economy which is the same as in Example 2 except that the transaction cost function is replaced by the following function

$$C_{nc}(\theta^2, q^2) = \begin{cases} (q^2/2)|\theta|, & \text{if } |\theta^2| < 1 \\ (q^2/3)|\theta|, & \text{if } 1 \leq |\theta^2| < 2 \\ (q^2/4)|\theta|, & \text{if } |\theta^2| \geq 3 \end{cases}$$

This function is continuous and locally non-convex. We set $\overline{C}_{nc}(\theta^2, q^2) = \lim_{\lambda\to\infty} C_{nc}(\lambda\theta^2, q^2)/\lambda$ and $\bar{c} = 2q^2$. Clearly, $\overline{C}_{nc}(\theta^2, q^2) = (q^2/4)|\theta^2|$. It is easy to see that for all $\theta \in \mathbb{R}$,

$$\overline{C}_{nc}(\theta^2, q^2) - \bar{c} < C_{nc}(\theta^2, q^2) < \overline{C}_{nc}(\theta^2, q^2) + \bar{c}.$$

Thus Assumption 1-2 hold true here.

[21] Such underestimation is a general phenomenon when transaction cost functions are convex and smooth at the origin.

Let Λ denote the set of no arbitrage prices. By Theorem 6.2, we have

$$\Lambda = \{(q^1, q^2) \in \mathbb{R}^2_{++} : q^2/2 \leq q^1 \leq (3/2)q^2\}.$$

It is worth noting that $\Lambda_{DP} = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (3/4)q^2 \leq q^1 \leq (5/4)q^2\}$ and therefore, $\Lambda_{DP} \subset \Lambda$. Thus the no arbitrage condition of Dermody and Prisman (1993) overestimates the multiplicity of the pricing rules.[22]

VII. Arbitrage and Viability

Most literature on asset valuation by arbitrage focuses on verifying the equivalence between the no arbitrage conditions and the existence of pricing functionals. If the notions of arbitrage do not pass viability test, however, they fail to exactly characterize asset pricing in equilibrium. We shows that the no arbitrage condition of Definition 5.1 is equivalent to viability. Thus the no arbitrage condition of Definition 5.1 provides a coherent conceptual framework for studying asset pricing, portfolio choice problem, or equilibrium in markets with general transaction cost structures.

To examine viability of arbitrage-free prices, we introduce an agent who has the endowment of consumptions $e \in L_+$ and preferences represented by a utility function $u : L_+ \to \mathbb{R}$.[23] For a price $q \in \mathcal{L}^J$, the agent chooses $(x^*, \theta^*) \in L_+ \times \mathcal{L}^J$ which solves the optimization problem:

$$\max_{(x,\theta)} u(x)$$

subject to the budget set

$$\mathcal{B}(q) = \{(x, \theta) \in L_+ \times \mathcal{L}^J : x - e \leq R(\theta, q)\}.$$

The demand correspondence $\xi(q)$ is the set of optimal choices in $L_+ \times \mathcal{L}^J$ which solve the above optimization problem.

Definition 7.1: An asset price $q \in \mathcal{L}^J$ is *viable* if $\xi(q) \neq \emptyset$.

[22] Such overestimation is a general phenomenon when transaction cost functions are not convex and smooth at the origin.

[23] It is implicitly assumed that a single consumption good is available in each state of the economy.

To investigate the relationship between the no arbitrage condition and viability, we make the following assumptions.

Assumption 4: u is continuous, strictly increasing and quasiconcave.

Assumption 5: For each $\{x^n\}$ in L_+ with $\lim_{n\to\infty} \|x^n\| \to \infty$, $\lim_{n\to\infty} u(x^n) = \infty$.

Assumption 6: For a price $q \in \Lambda$, the following set is closed.

$$X(q) = \{x \in L_+ : x - e \leq R(\theta, q) \quad \text{for some } \theta \in \mathcal{L}^J\}.$$

Assumption 4 is standard. Assumption 5 holds in the case where preferences follow the expected utility hypothesis and the expected utility goes to infinity as income increases indefinitely in some states of the world. Assumption 6 requires that the set of income transfers be closed. If $V(\cdot, q) = R(\cdot, q)$ for all $q \in \mathcal{L}^J$, then Lemma 6.1 shows that Assumption 6 holds true. By the same argument used in proving Lemma 6.1, we can show that Assumption 6 is satisfied in the case where $R(\cdot, q)$ is piece-wise linear with finitely many kinks. It is also satisfied with transaction cost functions with fixed cost component as in the fourth diagram of Figure 4.[24]

Assumption 7: For each $q \in \mathcal{L}^J$, $j \in J$ and $\sigma \in D$, there exists a differentiable function $D^j_\sigma(\cdot, q^j(\sigma)) : \mathbb{R} \to \mathbb{R}$ such that $D^j_\sigma(0, q^j(\sigma)) = 0$ and for all $\theta \in \mathcal{L}^J$,

$$\lim_{\lambda\to\infty} \frac{D^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))}{\lambda} = \lim_{\lambda\to\infty} \frac{C^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))}{\lambda}$$

Assumption 7 states that there exists a differentiable function $D^j_\sigma(\cdot, q^j(\sigma))$ which displays the same average cost for large transactions as $C^j_\sigma(\cdot, q^j(\sigma))$, i.e., for any $\theta^j(\sigma)$,

$$\lim_{\lambda\to\infty} \frac{D^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))}{\lambda\theta^j(\sigma)} = \overline{C}^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma)).$$

[24] Assumption 6 holds true with transaction cost functions which is lower-semicontinuous and piece-wise linear with finitely many jumps. This is the case with transaction cost functions with fixed cost component. Assumption 6, however, may be violated with indivisible assets. Transaction costs with indivisible assets are usually upper-semicontinuous as shown in the third diagram of Figure 1, implying that the net return functions are lower-semicontinuous. In this case, $X(q)$ is not closed in general for a price $q \in \mathcal{L}^J$.

As shown earlier, the effect of transaction costs on the pricing rules are determined by the LTC function. This implies that each $C^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))$ can be replaced by the differentiable function $D^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))$ without distorting the effect of transaction costs on asset pricing. This observation is useful in analyzing market frictions which cause the finitely many kinks or jumps of $C^j_\sigma(\lambda\theta^j(\sigma), q^j(\sigma))$.

The following result shows that the no arbitrage condition is fully compatible with viability of the pricing rules.

Theorem 7.1: Under Assumption 1-7, $q \in \Lambda$ if and only if $\xi(q) \neq \emptyset$.

PROOF : ($\Leftarrow$) Let q be a price in $\mathcal{L}^J$ such that $\xi(q) \neq \emptyset$. Then there exists $(x, \theta) \in \xi(q)$. Suppose that there exists a nonzero $v \in G(q)$ such that $\widetilde{R}(v, q) > 0$. By Proposition 5.1, we have $V(v, q) > 0$. In particular, there exists $\sigma \in D$ such that $V(v, q; \sigma) > 0$. We claim that

$$V(v, q; \sigma) = \lim_{\lambda \to \infty} \frac{R(\theta + \lambda v, q; \sigma) - R(\theta, q; \sigma)}{\lambda}.$$

By Assumption 7, we see that

$$\lim_{\lambda \to \infty} \frac{D^j_\sigma(\theta^j(\sigma) + \lambda v^j(\sigma), q^j(\sigma))}{\lambda} = \lim_{\lambda \to \infty} \frac{C^j_\sigma(\theta^j(\sigma) + \lambda v^j(\sigma), q^j(\sigma))}{\lambda}$$

On the other hand, L'Hôpital's rule gives

$$\begin{aligned}
\lim_{\lambda \to \infty} \tfrac{D^j_\sigma(\theta^j(\sigma) + \lambda v^j(\sigma), q^j(\sigma))}{\lambda} &= \lim_{\lambda \to \infty} D^{j'}_\sigma(\theta^j(\sigma) + \lambda v^j(\sigma), q^j(\sigma)) v^j(\sigma) \\
&= \lim_{\lambda \to \infty} D^{j'}_\sigma(\lambda v^j(\sigma), q^j(\sigma)) v^j(\sigma) \\
&= \lim_{\lambda \to \infty} \tfrac{D^j_\sigma(\lambda v^j(\sigma), q^j(\sigma))}{\lambda}.
\end{aligned}$$

It follows that

$$\lim_{\lambda \to \infty} \frac{C^j_\sigma(\lambda v^j(\sigma), q^j(\sigma))}{\lambda} = \lim_{\lambda \to \infty} \frac{C^j_\sigma(\theta^j(\sigma) + \lambda v^j(\sigma), q^j(\sigma))}{\lambda}.$$

Therefore, we have

$$V(v, q; \sigma) = \lim_{\lambda \to \infty} \frac{\widetilde{R}(\theta + \lambda v, q; \sigma)}{\lambda} = \lim_{\lambda \to \infty} \frac{\widetilde{R}(\theta + \lambda v, q; \sigma) - \widetilde{R}(\theta, q; \sigma)}{\lambda}.$$

Since by Assumption 1 $F(\theta(\sigma) + \lambda v(\sigma), q(\sigma); \sigma) < f$ for all $\lambda > 0$ and $\sigma \in D$, it follows that

$$V(v, q; \sigma) = \lim_{\lambda \to \infty} \frac{R(\theta + \lambda v, q; \sigma)}{\lambda} = \lim_{\lambda \to \infty} \frac{R(\theta + \lambda v, q; \sigma) - R(\theta, q; \sigma)}{\lambda}.$$

Clearly, $V(v,q;\sigma) > 0$ implies that $\lim_{\lambda\to\infty}[R(\theta+\lambda v,q;\sigma) - R(\theta,q;\sigma)] = \infty$. Thus $\theta+\lambda v$ generates an indefinite amount of income in the event σ as $\lambda \to \infty$. By Assumption 5, utility goes to infinity as $\lambda \to \infty$, which contradicts the optimality of (x,θ) in $B(q)$.

($\Rightarrow$) Let q be a price in Λ. We set

$$X(q) = \{x \in L_+ : x - e \leq R(\theta, q) \quad \text{for some } \theta \in \mathcal{L}^J\}.$$

Suppose that $X(q)$ is compact. Since u is continuous and $X(q)$ is compact, there exists $x \in L_+$ which satisfies $u(x) \geq z'$ for all $z \in X(q)$, and therefore, $\theta \in \mathcal{L}^J$ such that $(x,\theta) \in \xi(q)$. Thus we have only to show that $X(q)$ is compact.

By Assumption 6, $X(q)$ is closed. Now we show that it is bounded. Suppose otherwise. Then there exists a sequence $\{x^n\}$ in $X(q)$ such $\|x^n\| \to \infty$. For each n, we can choose $\theta^n \in \mathcal{L}^J$ such that $(x^n,\theta^n) \in \mathcal{B}(q)$. We set $b^n = 1/\|x^n\|$. Now by multiplying both sides of the budget constraints by b^n, we have $b^n x^n - b^n e \leq b^n R(\theta^n, q)$ for each n. Clearly, $\{b^n x^n\}$ is bounded and therefore, has a subsequence convergent to a point y. Clearly $y \geq 0$ and $\|y\| = 1$ and therefore, $y > 0$.

We set

$$Z(q) = \{y \in L : y \leq V(\theta, q) \quad \text{for some} \quad \theta \in \mathcal{L}^J\}.$$

By Lemma 6.1, $Z(q)$ is closed. For each $\sigma \in D_{-T}$, we claim that for all $v \in \mathcal{L}^J$,

$$V(v,q;\sigma) \geq R(v,q;\sigma) + \bar{c}.$$

First we consider the case with $\sigma = \sigma_0$. By Assumption 2, we see that $\overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) + \bar{c} \geq C(v(\sigma_0), q(\sigma_0); \sigma_0)$. It follows that

$$\begin{aligned} V(v,q;\sigma_0) &= -q(\sigma_0)\cdot v(\sigma_0) - \overline{C}(v(\sigma_0), q(\sigma_0); \sigma_0) \\ &\geq -q(\sigma_0)\cdot v(\sigma_0) - C(v(\sigma_0), q(\sigma_0); \sigma_0) + \bar{c} \\ &= R(v,q;\sigma_0) + \bar{c}. \end{aligned}$$

By applying the same argument to the case with $\sigma \in D_{-T}\setminus\{\sigma_0\}$, we can show that $V(v,q;\sigma) \geq R(v,q;\sigma) + \bar{c}$.

By the positive homogeneity of $V(\cdot, q)$, we have

$$\begin{aligned} b^n x^n - b^n e &\leq b^n R(\theta^n, q) \\ &\leq b^n V(\theta^n, q) - b^n \bar{c}\mathbf{1}_{-T} \\ &= V(b^n\theta^n, q) - b^n \bar{c}\mathbf{1}_{-T} \end{aligned}$$

where $\mathbf{1}_{-T}$ denotes a vector in L with 1 in the coordinate corresponding to each $\sigma \in D_{-T}$ and with 0 otherwise. This implies that $b^n x^n - b^n e + b^n \bar{c}\mathbf{1}_{-T} \leq V(b^n\theta^n, q)$ and therefore, $b^n x^n - b^n e + b^n \bar{c}\mathbf{1}_{-T} \in Z(q)$ for all n. Since $b^n x^n - b^n e + b^n \bar{c}\mathbf{1}_{-T} \to y$ and $Z(q)$ is closed, y is in $Z(q)$. Thus there exists $v \in \mathcal{L}^J$ such that $V(v, q) \geq y$. Since $y > 0$, we have $V(v, q) > 0$. By Proposition 5.1 we must have $q \notin \Lambda$, which is impossible. Thus, $X(q)$ is bounded. □

Theorem 7.1 shows the equivalence between the no arbitrage condition and viability of asset prices. Thus, Theorem 6.2 and 7.1 lead to an extension of the fundamental theorem of asset pricing stated in Harrison and Kreps (1979) and Dybvig and Ross (1989) to the case with transaction costs.

Theorem 7.2: Under Assumption 1-7, the following statements are equivalent.

(i) $q \in \Lambda$.

(ii) There exists $\pi \in L_{++}$ such that for each $\sigma \in D$ and $j \in J$,

$$\pi(\sigma)[q^j(\sigma) + s^j(q^j(\sigma);\sigma)] \leq \sum_{\hat{\sigma} \in D_\sigma \setminus \{\sigma\}} \pi(\hat{\sigma}) R^j_{\hat{\sigma}} \leq \pi(\sigma)[q^j(\sigma) + b^j(q^j(\sigma);\sigma)].$$

(iii) $\xi(q) \neq \emptyset$.

Appendix

Proof of the Claim in Example 2: We show that

$$\Lambda_V = \{(q^1, q^2) \in \mathbb{R}^2_{++} : (29/30)q^2 \leq q^1 \leq (21/20)q^2\}.$$

We consider an agent who has preferences represented by a utility function $u(x^0, x^1) = \sqrt{x^0} + \sqrt{x^1}$ and the initial endowment of consumption goods $(1, 1)$.[25] Then he faces the following

[25]The arguments below do not depend on the form of utility functions and the size of the endowments as far as the utility functions are monotonic.

optimization problem

$$\max\left\{\sqrt{x^0}+\sqrt{x^1}\;\middle|\;\begin{array}{l} x^0-1\le -\theta^1 q^1-\theta^2 q^2-C(\theta^2,q^2)\\ x^1-1\le \theta^1+\theta^2\end{array}\right\}$$

The above maximization problem is reduced to the following

$$\max\sqrt{1-\theta^1 q^1-\theta^2 q^2-C(\theta^2,q^2)}+\sqrt{1+\theta^1+\theta^2}.$$

Let $(\hat{\theta}^1,\hat{\theta}^2)$ denote the solution to the maximization problem.

i) $21q^1=20q^2$.

Clearly, we have

$$(\hat{\theta}^1,\hat{\theta}^2)\in\{(\theta^1,\theta^2):\theta^1+\theta^2=\frac{21/20-(q^1)^2}{(q^1)^2+q^1},\theta^2\ge\frac{1}{q^2}\}.$$

ii) $30q^1=29q^2$.

Similarly, we have

$$(\hat{\theta}^1,\hat{\theta}^2)\in\{(\theta^1,\theta^2):\theta^1+\theta^2=\frac{31/30-(q^1)^2}{(q^1)^2+q^1},\theta^2<\frac{-1}{q^2}\}.$$

(iii) $(29/30)q^2<q^1<(21/20)q^2$.

The maximization problem has a solution because the following set is compact.

$$\{(\theta^1,\theta^2)\in\mathbb{R}^2:\theta^1 q^1+\theta^2 q^2+C(\theta^2,q^2)\le 1,\theta^1+\theta^2\ge -1\}.$$

□

Proof of Lemma 6.1: Let q be a price in Λ. First we show that the following set is closed.

$$Y(q)=\{y\in L:y=V(v,q),v\in\mathcal{L}^J\}.$$

Let v be a point in $\mathcal{L}^J$. For some $j\in J$ and $\sigma\in D$, it follows that if $v^j(\sigma)-v^j(\sigma^-)\ge 0$, then

$$\begin{aligned}\overline{C}^j_\sigma(v^j(\sigma)-v^j(\sigma^-),q^j(\sigma)) &= b^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-))\\ \overline{C}^j_\sigma(v^j(\sigma^-)-v^j(\sigma),q^j(\sigma)) &= s^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-)),\end{aligned}$$

and if $v^j(\sigma)-v^j(\sigma^-)<0$, then

$$\begin{aligned}\overline{C}^j_\sigma(v^j(\sigma)-v^j(\sigma^-),q^j(\sigma)) &= s^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-))\\ \overline{C}^j_\sigma(v^j(\sigma^-)-v^j(\sigma),q^j(\sigma)) &= b^j(q^j(\sigma);\sigma)(v^j(\sigma)-v^j(\sigma^-)).\end{aligned}$$

(In the above, we follow the notational convention that $v^j(\sigma_0^-) = 0$ for all $j \in J$.)

Let $\{y^n\}$ be a sequence in $Y(q)$ which converges to a point y. Since $0 \in Y(q)$, without loss of generality we may assume that $y \neq 0$. Then $y^n \neq 0$ for sufficiently large n. For each n we choose v^n in $\mathcal{L}^J$ such that $y^n = V(v^n, q)$. Since D is finite, there exists a subsequence $\{v^m\}$ such that $\{v^{jm}(\sigma) - v^{jm}(\sigma^-)\}$ has the same sign for a given pair $(j, \sigma) \in J \times D$. Thus there exists a $(\#D) \times [J \times (\#D_{-T})]$ matrix Ψ such that $V(v^m, q) = \Psi \cdot v^m$.

We define the sets

$$\Theta^+(q) = \left\{ v \in \mathcal{L}^J \,|v^j(\sigma) - v^j(\sigma^-) \geq 0 \text{ for each } (j,\sigma) \text{ with } v^{jm}(\sigma) - v^{jm}(\sigma^-) \geq 0 \text{ for all } m \right\}$$
$$\Theta^-(q) = \left\{ v \in \mathcal{L}^J \,|v^j(\sigma) - v^j(\sigma^-) \leq 0 \text{ for each } (j,\sigma) \text{ with } v^{jm}(\sigma) - v^{jm}(\sigma^-) < 0 \text{ for all } m \right\}$$

Since $J \times D$ consists of finitely many elements, $\Theta^+(q)$ and $\Theta^-(q)$ are the intersection of finitely many closed half spaces which contain the origin on the boundary, and therefore, they are a polyhedral cone. By construction, $\{v^m\}$ is in $\Theta^+(q) \cap \Theta^-(q)$. We set

$$Y^{\pm}(q) = \{y \in L : y = V(v,q) \ \text{ for some } \ v \in \Theta^+(q) \cap \Theta^-(q)\}.$$

Since $\Psi \cdot v = V(v,q)$ for any $v \in \Theta^+(q) \cap \Theta^-(q)$, by Theorem 19.3 of Rockafellar (1970) the set $Y^{\pm}(q)$ is a polyhedral cone. In particular, it is closed. Since $\{y^m\}$ is in $Y^{\pm}(q)$, y is in $Y^{\pm}(q)$. Noting that $Y^{\pm}(q) \subset Y(q)$, y is in in $Y(q)$. Thus, $Y(q)$ is closed.

Now show that the set $Z(q) = \{y \in L : y \leq V(v,q), v \in \mathcal{L}^J\}$. Let $\{y^n\}$ be a sequence in $Z(q)$ which converges to a point y. For each n we choose v^n in $\mathcal{L}^J$ such that $y^n \leq V(v^n, q)$. For each n, we set $z^n = V(v^n, q)$. We claim that $\{z^n\}$ is bounded. Suppose that $\|z^n\| \to \infty$. By positive homogeneity of $V(\cdot, q)$, we have

$$z^n/\|z^n\| = V(v^n, q)/\|z^n\| = V(v^n/\|z^n\|, q).$$

This implies that $z^n/\|z^n\| \in Y(q)$ for each n. Clearly, $\{z^n/\|z^n\|\}$ is bounded. Thus, it has a subsequence convergent to a point $\dot{z}$. Since $Y(q)$ is closed, $\dot{z}$ is in $Y(q)$. Thus there exists $v \in \mathcal{L}^J$ such that $V(v,q) = \dot{z}$. On the other hand, we have $y^n/\|z^n\| \leq z^n/\|z^n\|$. Since $y^n \to y$ and $\|z^n\| \to \infty$, $y^n/\|z^n\| \to 0$. By passing to the limit we have $\dot{z} \geq 0$. Recalling that $\dot{z} \neq 0$, we must have $V(v,q) > 0$, which contradicts the fact that $q \in \Lambda$. Since $\{z^n\}$ is bounded, it has a subsequence convergent to a point z in $Y(q)$. Recalling that $y^n \leq z^n$ for each n, we have $y \leq z$. Thus $y \in Z(q)$. □

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