Disutility, Optimal Retirement, and Portfolio Selection

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Abstract

We study an optimal retirement and consumption/investment choice of an economic agent. The agent is infinitely-lived and has general von Neumann-Morgenstern utility. A particular aspect of our problem is that the agent has an option to retire from his work and avoid a utility loss: before retirement the agent receives labor income but suffers a utility loss due to work, however, by deciding to retire from work, he saves the utility loss but gives up labor income. We show that the agent retires optimally if his wealth exceeds a certain critical level. We also show that the agent consumes less and invests more in risky assets when he has an option to retire than he would in the absence of such an option.

An explicit solution can be provided by solving a free boundary value problem. In particular, the critical wealth level and the optimal consumption and portfolio policy are provided in explicit forms.

Keywords : Consumption, portfolio selection, retirement, disutility, labor income.

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1 Introduction

We study an optimal retirement and consumption/investment choice of an economic agent. A particular aspect of our problem is that, when the agent works as a wage earner he suffers a utility loss, but after retirement, he no longer experiences the utility loss. A retired person, however, does not have labor income and therefore must live on invested wealth.

The problem is modelled as a mixture of an optimal consumption/investment choice with two control variables $(\mathbf{c}, \boldsymbol{\pi})$ and an optimal choice of a stopping time τ. Thus, our problem is more realistic in the economic point of view and rather general in the mathematical point of view than the classical consumption and portfolio selection problems. By solving a free boundary value problem we find an explicit solution and characterize optimal polices in a continuous-time framework with an infinite horizon.

We obtain closed forms for the optimal retirement policy as well as for the optimal consumption and portfolio policy under a fairly general assumption that the agent has time-separable von Neumann-Morgenstern utility. We show that it is optimal to retire if and only if the agent's wealth exceeds a certain critical level. A wage earner retires from his work as soon as he becomes sufficiently wealthy, an intuitively appealing result. We also compare optimal consumption and investment policy with that for the case the agent does not have a retirement option(i.e., he is forced to work forever) and show that the agent consumes less and invests more in risky assets when he has an option to retire than he would in the absence of such an option. Intuitively, the agent tries to reach the critical wealth level and avoids a utility loss from labor as soon as possible by reducing consumption and investing more in high-return assets and thereby increasing the growth rate of wealth. In the case where the agent has CRRA utility, a particular aspect of the optimal portfolio policy is that a proportion of wealth invested in risky assets is fairly higher near by critical wealth level than that of at the low level wealth. (See Figure 5.2 and 5.4.)

There has been extensive research in consumption and portfolio selection after Merton's pioneering study (Merton 1969, 1971). Bodie, Merton, and Samuelson (1992) have studied an optimal consumption and investment problem of an economic agent who has flexibility in his labor supply and shown that flexibility in labor supply tends to increase the agent's risk taking in market securities. Bodie, Detemple, Otruba, and Walter (2004) have studied a similar problem in the context of optimal retirement planning, i.e., there is a fixed retirement time and the agent chooses consumption and investment in preparation for a scheduled retirement. However, these authors have not solved for an agent's optimal choice of retirement time as we have done here.

Karatzas and Wang (2000) first studied a discretionary stopping problem by using a martingale method. Choi, Koo, and Kwak (2003) have extended Karatzas and Wang's results to the case where an economic agent has stochastic differential utility. Choi and Koo (2003) have studied the effect of a preference change around a discretionary stopping time. These papers on the mixture of optimal stopping and optimal consumption and portfolio selection problem has relied on the martingale method, but in this paper we extend a dynamic programming method as in Karatzas, Lehoczky, Sethi, and Shreve (1986) to obtain an explicit solution. Also, the particular feature of our problem is that the model allows the agent still has the investment horizon after the discretionary retirement time, which is related to an interesting open problem suggested in Appendix B of Karatzas and Wang (2000).

Jeanblanc and Lakner (2004) have solved a problem of an agent under obligation to pay a debt at a fixed rate who can declare bankruptcy by using dynamic programming method. In their work, the optimal bankruptcy time is nontrivially determined by the assumption that the agent can keep only a fraction of his or her wealth minus a fixed cost at the bankruptcy time. However, in our problem the optimal retirement time is determined by the trade-off between distuility and income. In the similar framework to this paper, the companion paper Choi, Koo, and Shim (2004) have studied an optimal choice problem of a wage earner who wants to enlarge his or her investment opportunity in the financial market by retiring from the current job.

In this paper there is no risk in wage income and we do not consider effects of uninsurable income risk. Uninsurable income risk has been investigated by Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998) and Cuoco (1999), etc. Here we focus on studying the optimal consumption/investment problem of an agent who has the option to retire and avoid a utility loss.

The rest of the paper proceeds as follows. Section 2 sets up the mixture of optimal retirement and optimal consumption/investment problem. Section 3 presents a general solution to the problem and section 4 investigates properties of optimal policies. Section 5 studies the special case where the agent has constant relative risk aversion (CRRA) utility. Section 6 concludes. All the proofs in this paper are contained in Appendix.

2 An investment problem

We consider a market in which there are a riskless asset and m risky assets. We assume that the risk-free rate is a constant $r > 0$ and the price $p_0(t)$ of the riskless asset follows a deterministic process

$$dp_0(t) = p_0(t) r dt, \quad p_0(0) = p_0.$$

The price $p_j(t)$ of the j-th risky asset, as in Karatzas, Lehoczky, Sethi, and Shreve (1986) and Merton (1969, 1971), follows geometric Brownian motion

$$dp_j(t) = p_j(t)\big\{\alpha_j dt + \sum_{k=1}^{m} \sigma_{jk} dw_k(t)\big\}, \quad p_j(0) = p_j, \quad j = 1, \dots, m,$$

where $\mathbf{w}(t) = (w_1(t), \dots, w_m(t))$ is a m dimensional standard Brownian motion defined on the underlying probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t=0}^{\infty}$, the augmentation under P of the natural filtration generated by the standard Brownian motion $(\mathbf{w}(t))_{t=0}^{\infty}$. The market parameters, α_j's and σ_{jk}'s for $j, k = 1, \dots, m$, are assumed to be constants. We assume that the matrix $D = (\sigma_{ij})_{i,j=1}^{m}$ is nonsingular, i.e., there is no redundant asset among the m risky assets. Hence $\Sigma \equiv DD^T$ is positive definite.

Let $\boldsymbol{\pi}_t = (\pi_{1,t}, \dots, \pi_{m,t})$ be the row vector of amount of money invested in the risky assets at time t, c_t be the consumption rate at time t and τ be the time of retirement from labor. τ is a $\mathcal{F}_t$-stopping time, the consumption rate process $\mathbf{c} \equiv (c_t)_{t=0}^{\infty}$ is a nonnegative process adapted to $\mathcal{F}_t$ and satisfies

$$\int_0^t c_s \, ds < \infty,$$

for all $t \geq 0$, a.s., and the portfolio process $\boldsymbol{\pi} \equiv (\boldsymbol{\pi}_t)_{t=0}^{\infty}$ is a $\mathcal{F}_t$ measurable adapted process such that

$$\int_0^t \|\boldsymbol{\pi}_s\|^2 \, ds < \infty,$$

for all $t \geq 0$, a.s.

The agent receives labor income at a constant rate $\epsilon > 0$ until retirement. Therefore the investor's wealth process x_t with initial wealth $x_0 = x$ evolves according to

$$dx_t = (\boldsymbol{\alpha} - r\mathbf{1}_m)\boldsymbol{\pi}_t^T \, dt + (rx_t - c_t + \epsilon \, \mathbf{1}_{\{t \leq \tau\}}) \, dt + \boldsymbol{\pi}_t D \, d\mathbf{w}^T(t), \quad 0 \leq t < \infty, \tag{2.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ is the row vector of returns of risky assets and $\mathbf{1}_m = (1, \dots, 1)$ the row vector of m ones. The superscript T denotes the transpose of a matrix or a vector. Since the present value of the future income stream is $\frac{\epsilon}{r}$ we let

$$x_0 = x > -\frac{\epsilon}{r}$$

and we assume that if the investor's wealth level touches $-\frac{\epsilon}{r}$ at some time before retirement, then from then on he can neither consume nor invest and is under obligation to use all his wage income to repay the debt of amount $\frac{\epsilon}{r}$ without retirement. After retirement the agent faces a nonnegative wealth constraint

$$x_t \geq 0, \quad \text{for all } t \geq \tau \text{ a.s.} \tag{2.2}$$

In particular, the agent's wealth must be nonnegative at the time of retirement if it occurs.

We call a triple of control $(\tau, \mathbf{c}, \boldsymbol{\pi})$ satisfying above conditions with $x_0 = x > -\frac{\epsilon}{r}$ admissible at x. Let $A(x)$ denote the set of admissible controls at x.

Our optimization problem is to maximize the following time-separable von Neumann-Morgenstern utility:

$$V_{(\tau,\mathbf{c},\boldsymbol{\pi})}(x) \equiv E_x \int_0^\infty \exp(-\beta t)[U(c_t) - l\mathbf{1}_{\{t<\tau\}}]\, dt \tag{2.3}$$

over all admissible policies $(\tau, \mathbf{c}, \boldsymbol{\pi}) \in A(x)$ such that

$$E_x \int_0^\infty \exp(-\beta t) U^-(c_t)\, dt < \infty, \tag{2.4}$$

where E_x denotes the expectation operator conditioned on $x_0 = x$. U, called a utility function, is real-valued on $(0, \infty)$, $l > 0$ is a constant representing disutility (or a utility loss) due to labor, and $\beta > 0$ is a subjective discount rate. We make the following assumption:

Assumption 2.1. *U is strictly increasing, strictly concave and three times continuously differentiable, and* $\lim_{c\to\infty} U'(c) = 0$.

For later use, we let $I(\cdot)$ be the inverse function of $U'(\cdot)$. Put

$$\kappa \equiv \frac{1}{2}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}(\boldsymbol{\alpha} - r\mathbf{1}_m)^T.$$

If we assume that $\kappa > 0$, then the quadratic equation of λ

$$\kappa\lambda^2 - (r - \beta - \kappa)\lambda - r = 0 \tag{2.5}$$

has two distinct solutions $\lambda_- < -1$ and $\lambda_+ > 0$. For $x \geq 0$, let $\bar{V}(x)$ be the optimal value function when the investor is forced to choose $\tau = 0$, i.e., he must retire at time 0. As is shown in Karatzas, Lehoczky, Sethi, and Shreve (KLSS, 1986), $\bar{V}(x)$ is finite and attainable by a strategy for all $x > 0$ under the following assumption

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \tag{2.6}$$

for all $c > 0$. In this case, we denote by $\bar{C}(x_t)$ and $\bar{\Pi}(x_t)$ the feedback form for the optimal consumption and investment in the risky assets, respectively. Similarly, it can be shown that when the retirement τ is forced to be infinite, that is, when the investor has no option to retire, the optimal value at x, say $V_0(x)$, is finite and attainable by a strategy for all $x > -\frac{\epsilon}{r}$ if condition (2.6) is valid. If the utility function is given by $U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad 0 < \gamma \neq 1$ for $c > 0$, condition (2.6) is equivalent to $-\gamma\lambda_- > 1$, which is again equivalent to

$$K > 0, \tag{2.7}$$

where

$$K \equiv r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\kappa, \tag{2.8}$$

since λ_- is the negative solution of the equation (2.5). Condition (2.7) is equivalent to condition (40) in Merton (1969). If the utility function is given by $U(c) = \log c$ or $U(c) = -\exp(-ac), \quad a > 0$ for $c > 0$, condition (2.6) is automatically satisfied. Thus, we assume

Assumption 2.2.

$$\kappa > 0 \quad \textit{and} \quad \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty, \quad \forall c > 0.$$

An intuitively obvious fact is that after retirement an optimizing investor will follow the optimal consumption and investment policies $\bar{C}(x_t)$ and $\bar{\Pi}(x_t)$, thus we are only interested in the optimal retirement time τ^* and the optimal consumption and investment policies $(\mathbf{c}^*, \boldsymbol{\pi}^*)$ (for our original optimization problem) up to τ^*.

Definition 2.1. *We denote by $A_1(x) \subset A(x)$ the class of admissible controls satisfying (2.4) and*

$$(c_t, \boldsymbol{\pi}_t) = (\bar{C}(x_t), \bar{\Pi}(x_t)) \qquad \textit{for all} \quad \tau \le t < \infty.$$

By the above argument, it is sufficient to maximize (2.3) over the class $A_1(x)$. We let

$$V^*(x) \equiv \sup \{V_{(\tau,\mathbf{c},\boldsymbol{\pi})}(x) : (\tau, \mathbf{c}, \boldsymbol{\pi}) \in A_1(x)\} \tag{2.9}$$

be the optimal value of expected utility at wealth $x > -\frac{\epsilon}{r}$.

To solve the problem we assume

Assumption 2.3.

$$1 - \frac{\beta\lambda_-}{r(1+\lambda_-)} \le 0.$$

Remark 2.1. *A sufficient condition for assumption 2.3 is $\beta \ge r$.*

3 A solution under a general utility class

In this section we find a solution to the optimal retirement and consumption/investment problem under a general utility class. The HJB (Hamilton-Jaccobi-Bellman) equation for $t < \tau$ is given by

(3.1)

$$\beta V(x) = \max_{c \ge 0, \boldsymbol{\pi}} \left\{ (\boldsymbol{\alpha} - r\mathbf{1}_m)\boldsymbol{\pi}^T V'(x) + (rx - c + \epsilon)V'(x) + \frac{1}{2}\boldsymbol{\pi}\Sigma\boldsymbol{\pi}^T V''(x) + U(c) - l \right\},$$

for $x > -\frac{\epsilon}{r}$.

We proceed to obtain a solution as follows: first, we conjecture that there is a critical wealth level z^* such that if the agent's wealth reaches this level then he retires, second, we also conjecture that the agent's value function satisfies the HJB equation (3.1) for $x < z^*$ and is equal to $\bar{V}(x)$ for $x \geq z$ and smoothly pasted (namely, continuously differentialbe) at $x = z^*$, and finally we give a formal proof that the above conjecture is correct.

3.1 The Case where $U'(0) = \infty$

We first consider the case where $U'(0) = \infty$. For this case, we need the following Lemma.

Lemma 3.1. *If $U'(0) = \infty$, then*

$$\lim_{c \downarrow 0} \frac{U(c)}{U'(c)} = 0, \tag{3.2}$$

$$\lim_{c \downarrow 0} (U'(c))^{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} = 0 \tag{3.3}$$

and

$$\lim_{c \downarrow 0} (U'(c))^{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} = 0. \tag{3.4}$$

We would like to provide intuition behind our solution, before proceeding to give its formal derivation and proof. Borrowing an idea from KLSS (1986), the HJB equation (3.1) can be linearized by introducing a function $X(c)$ which is equal to the agent's financial wealth expressed as a function of consumption. That is, the HJB equation can be transformed into the the following equation

$$\kappa X''(c) = \big\{(r - \beta - 2\kappa)\frac{U''(c)}{U'(c)} + \kappa \frac{U'''(c)}{U''(c)}\big\} X'(c) + \big\{\frac{U''(c)}{U'(c)}\big\}^2 (rX(c) - c + \epsilon), \quad c > 0. \tag{3.5}$$

A general solution to the above equation can be expressed as a particular solution and a general solution to a corresponding homogenous solution. A particular solution will be provided as follows:

$$X_0(c) = \frac{c}{r} - \frac{1}{\kappa(\lambda_+ - \lambda_-)} \Big\{ \frac{(U'(c))^{\lambda_+}}{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{(U'(c))^{\lambda_-}}{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \Big\} - \frac{\epsilon}{r},$$

and our conjectured solution is given by

$$X(c;\hat{B}) = \hat{B}(U'(c))^{\lambda_-} + X_0(c) \tag{3.6}$$

for $c > 0$ with a certain constant $\hat{B}$ which will be determined later. It will be shown that $X(c;\hat{B})$ is one-to-one and maps maps $[0,\infty)$ onto $[-\frac{\epsilon}{r},\infty)$ so that its inverse function $C(\cdot;\hat{B})$ exists and maps $[-\frac{\epsilon}{r},\infty)$ onto $[0,\infty)$. The candidate value function $V:(-\frac{\epsilon}{r},\infty)\to R$ that satisfies the HJB equation can now be obtained as $V:(-\frac{\epsilon}{r},\infty)\to R$ of the form

$$V(x) \equiv J(C(x;\hat{B});\hat{A}), \quad -\frac{\epsilon}{r} < x < z^*, \tag{3.7}$$

and

$$V(x) \equiv \bar{V}(x), \quad x \geq z^*, \tag{3.8}$$

where

$$J(c;\hat{A}) = \hat{A}(U'(c))^{\rho_-} + J_0(c), \tag{3.9}$$

$\hat{A} = \frac{\lambda_-}{\rho_-}\hat{B}$, and

$$J_0(c) = \frac{U(c)-l}{\beta} - \frac{1}{\kappa(\rho_+-\rho_-)}\{\frac{(U'(c))^{\rho_+}}{\rho_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{(U'(c))^{\rho_-}}{\rho_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\}.$$

The smooth-pasting condition at $x = z^*$ implies that

$$X(\bar{C}(z^*);\hat{B}) = z^* \tag{3.10}$$

where $\bar{C}(x)$ is the agent's optimal consumption expressed as a function of financial wealth for $x \geq z^*$ whose formal definition will be given later. From (3.10) and definitions of $X(\cdot)$ and $J(\cdot)$ in (3.6) and (3.9) we have

$$X(\bar{C}(z^*);\hat{B}) = \hat{B}(U'(\bar{C}(z^*)))^{\lambda_-} + X_0(\bar{C}(z^*)) = z^*, \tag{3.11}$$

and

$$J(\bar{C}(z^*);\hat{A}) = \hat{A}(U'(\bar{C}(z^*)))^{\rho_-} + J_0(\bar{C}(z^*)) = \bar{V}(z^*). \tag{3.12}$$

Multiplying $\lambda_- U'(\bar{C}(z))$ and ρ_- in both sides of equations (3.11) and (3.12) respectively, and subtracting (3.12) from (3.11) and using the fact that $\hat{A} = \frac{\lambda_-}{\rho_-}\hat{B}$, we get

$$\lambda_- U'(\bar{C}(z^*))\{X_0(\bar{C}(z^*)) - z^*\} - \rho_-\{J_0(\bar{C}(z^*)) - \bar{V}(z^*)\} = 0. \tag{3.13}$$

Define a function $G : (0, \infty) \to R$ by

$$G(z) \equiv \lambda_- U'(\bar{C}(z))\{X_0(\bar{C}(z)) - z\} - \rho_-\{J_0(\bar{C}(z)) - \bar{V}(z)\}. \tag{3.14}$$

Then, by equation (3.13) we have

$$G(z^*) = 0. \tag{3.15}$$

Namely, the threshold z^* is equal to a solution to the equation $G(z) = 0$.

We now proceed to formal derivation and proof of the conjectured solution.

For $c > 0$ let us define the following functions:

$$\bar{X}(c) = \frac{c}{r} - \frac{1}{\kappa(\lambda_+ - \lambda_-)}\Big\{\frac{(U'(c))^{\lambda_+}}{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{(U'(c))^{\lambda_-}}{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Big\},$$

$$\bar{J}(c) = \frac{U(c)}{\beta} - \frac{1}{\kappa(\rho_+ - \rho_-)}\{\frac{(U'(c))^{\rho_+}}{\rho_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{(U'(c))^{\rho_-}}{\rho_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\},$$

where $\rho_+ = 1 + \lambda_+$ and $\rho_- = 1 + \lambda_-$.

By (3.3) and (3.4), we have

$$X_0(0) \equiv \lim_{c \downarrow 0} X_0(c) = -\frac{\epsilon}{r} \quad \text{and} \quad \bar{X}(0) \equiv \lim_{c \downarrow 0} \bar{X}(c) = 0$$

if $U'(0) = \infty$. As in (6.11) of KLSS (1986), $\lim_{c \uparrow \infty} X_0(c) = \lim_{c \uparrow \infty} \bar{X}(c) = \infty$. Using the relation $\lambda_+\lambda_- = -\frac{r}{\kappa_1}$, we have

$$\begin{aligned} X_0'(c) &= \bar{X}'(c) \\ &= -\frac{U''(c)}{\kappa_1(\lambda_+ - \lambda_-)}\Big\{(U'(c))^{\lambda_+ - 1}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + (U'(c))^{\lambda_- - 1}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Big\}. \end{aligned}$$

Since $U(\cdot)$ is strictly concave, $X_0'(c) = \bar{X}'(c) > 0$ for all $c > 0$. Hence $X_0(\cdot)$ is strictly increasing and maps $[0, \infty)$ onto $[-\frac{\epsilon}{r}, \infty)$ so that its inverse function $C_0(\cdot)$ exists and is also strictly increasing and maps $[-\frac{\epsilon}{r}, \infty)$ onto $[0, \infty)$. Likewise, $\bar{X}(\cdot)$ is strictly increasing and maps $[0, \infty)$ onto $[0, \infty)$ so that its inverse function $\bar{C}(\cdot)$ exists and is also strictly increasing and maps $[0, \infty)$ onto $[0, \infty)$. $\bar{C}(\cdot)$ is the same function as the one introduced in Section 2 (with the same notation) as an optimal feedback consumption policy in the case where τ is forced to be zero. As is shown in KLSS (1986), it holds that

$$\bar{V}(x) = \bar{J}(\bar{C}(x))$$

for all $x \geq 0$. Similarly it can be shown that

$$V_0(x) = J_0(C_0(x))$$

for all $x \geq -\frac{\epsilon}{r}$. A simple calculation shows that

$$G(z) = -\frac{\lambda_- \epsilon}{r} U'(\bar{C}(z)) + \frac{\rho_- l}{\beta}. \tag{3.16}$$

Thus, the function $G(\cdot)$ is strictly decreasing, since $\bar{C}(\cdot)$ is strictly increasing and $U'(\cdot)$ is strictly decreasing. Since $U'(0) = \infty$ and $\lim_{c \to \infty} U'(c) = 0$ by assumption (2.1), it holds that $\lim_{z \downarrow 0} G(z) = \infty$ and that $\lim_{z \uparrow \infty} G(z) = \frac{\rho_- l}{\beta} < 0$. Letting

$$z^* \equiv \bar{X}(I(\frac{\rho_- r l}{\lambda_- \beta \epsilon})), \tag{3.17}$$

then z^* is positive and satisfies

$$G(z^*) = 0. \tag{3.18}$$

We determine the constant $\hat{B}$ by

$$\begin{aligned}
\hat{B} &\equiv (U'(\bar{C}(z^*)))^{-\lambda_-} \{z^* - X_0(\bar{C}(z^*))\} && (3.19)\\
&= (U'(\bar{C}(z^*)))^{-\lambda_-} \{z^* - (\bar{X}(\bar{C}(z^*)) - \frac{\epsilon}{r})\} \\
&= (U'(\bar{C}(z^*)))^{-\lambda_-} \frac{\epsilon}{r} && (3.20)\\
&= (\frac{\rho_- r l}{\lambda_- \beta \epsilon})^{-\lambda_-} \frac{\epsilon}{r} > 0. && (3.21)
\end{aligned}$$

With this $\hat{B} > 0$, we have

$$X(\bar{C}(z^*); \hat{B}) = z^*. \tag{3.22}$$

By (3.3) and (3.4), we have

$$X(0; \hat{B}) \equiv \lim_{c \downarrow 0} X(c; \hat{B}) = -\frac{\epsilon}{r}$$

if $U'(0) = \infty$. As in (6.11) of KLSS (1986), $\lim_{c \uparrow \infty} X(c; \hat{B}) = \infty$. Using the relation $\lambda_+ \lambda_- = -\frac{r}{\kappa_1}$, we have

$$\begin{aligned} X'(c; \hat{B}) &= \lambda_- \hat{B}(U'(c))^{\lambda_- - 1} U''(c) - \\ & \frac{U''(c)}{\kappa_1(\lambda_+ - \lambda_-)} \Big\{ (U'(c))^{\lambda_+ - 1} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + (U'(c))^{\lambda_- - 1} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \Big\}. \end{aligned}$$

Since $U(\cdot)$ is strictly concave, $X'(c; \hat{B}) > 0$ for all $c > 0$. Hence $X(\cdot; \hat{B})$ is strictly increasing and maps $[0, \infty)$ onto $[-\frac{\epsilon}{r}, \infty)$ so that its inverse function $C(\cdot; \hat{B})$ exists and is also strictly increasing and maps $[-\frac{\epsilon}{r}, \infty)$ onto $[0, \infty)$.

Now, with z^* and $\hat{B}$ determined in (3.17) and (3.19), define a function $V : (-\frac{\epsilon}{r}, \infty) \to R$ by

$$V(x) \equiv J(C(x; \hat{B}); \frac{\lambda_-}{\rho_-} \hat{B}), \quad -\frac{\epsilon}{r} < x < z^*, \tag{3.23}$$

and

$$V(x) \equiv \bar{V}(x), \quad x \geq z^*. \tag{3.24}$$

As in Lemma 8.7 of KLSS (1986), we have $\lim_{x \downarrow -\frac{\epsilon}{r}} V(x) = \frac{U(0) - l}{\beta}$. By (3.22) we have

$$\lim_{x \uparrow z^*} C(x; \hat{B}) = C(z^*; \hat{B}) = \bar{C}(z^*), \tag{3.25}$$

so that

$$\lim_{x \uparrow z^*} V(x) = J(\bar{C}(z^*); \frac{\lambda_-}{\rho_-} \hat{B}). \tag{3.26}$$

By (3.14), (3.18) and (3.19) we have

$$\begin{aligned}\bar{V}(z^*) &= -\frac{\lambda_-}{\rho_-}U'(\bar{C}(z^*))\{X_0(\bar{C}(z^*)) - z^*\} + J_0(\bar{C}(z^*)) \\ &= \frac{\lambda_-}{\rho_-}\hat{B}(U'(\bar{C}(z^*)))^{\rho_-} + J_0(\bar{C}(z^*)) \\ &= J(\bar{C}(z^*); \frac{\lambda_-}{\rho_-}\hat{B}).\end{aligned}$$

Hence by (3.26), we get

$$\lim_{x\uparrow z^*} V(x) = \bar{V}(z^*). \tag{3.27}$$

We have the following lemma.

Lemma 3.2. *The function $V(x)$ defined by (3.23) and (3.24)is strictly increasing for $x > -\frac{\epsilon}{r}$, strictly concave and satisfies the HJB equation (3.1) for $-\frac{\epsilon}{r} < x < z^*$. Furthermore, $V(\cdot) \in C^1(-\frac{\epsilon}{r}, \infty) \cap C^2((-\frac{\epsilon}{r}, z^*) \cup (z^*, \infty))$, $\lim_{x\to z^*+} V''(x)$ and $\lim_{x\to z^*-} V''(x)$ exist and finite.*

Let's consider the strategy

$$\tau = \infty, \quad c_t = C(x_t; \hat{B}), \quad \boldsymbol{\pi}_t = -\frac{V'(x_t)}{V''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad t \geq 0.$$

As in equation (7.4) in KLSS(1986), the stochastic differential equation for $\{c_t \equiv C(x_t; \hat{B}), t \geq 0\}$ becomes

$$dy_t = -(r-\beta)y_t dt - y_t(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}Dd\mathbf{w}^T(t), \tag{3.28}$$

where $y_t \equiv U'(c_t)$. Hence

$$U'(c_t) = y_t = U'(c_0)\exp[-(r-\beta+\kappa)t - (\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}D\mathbf{w}^T(t)], \quad t \geq 0,$$

so that we get

$$c_t = I(U'(c_0)\exp[-(r-\beta+\kappa)t - (\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}D\mathbf{w}^T(t)]), \quad t \geq 0. \tag{3.29}$$

Therefore, if $U'(0) = \infty$, then

(3.30)

$$\inf\{t \geq 0 : x_t = -\frac{\epsilon}{r}\} = \inf\{t \geq 0 : c_t = 0\} = \inf\{t \geq 0 : y_t = \infty\} = \infty, a.s.$$

Let's define the following notation

$$T^{\xi} \equiv \inf\{t \geq 0 : x_t \geq \xi\}.$$

Now we give a solution to the problem when $U'(0) = \infty$ in the following theorem

.

Theorem 3.1. *When $U'(0) = \infty$ the optimal value function is $V(x)$ defined by (3.23) and (3.24), and an optimal strategy is given by $(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)$:*

$$\tau^* = T^{z^*}, \tag{3.31}$$

$$c_t^* = C(x_t; \hat{B}), \quad \boldsymbol{\pi}_t^* = -\frac{V'(x_t)}{V''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad 0 \leq t < \tau^*, \tag{3.32}$$

and

$$c_t^* = \bar{C}(x_t), \quad \boldsymbol{\pi}_t^* = \bar{\Pi}(x_t), \quad t \geq \tau^*. \tag{3.33}$$

3.2 The Case where $U'(0) < \infty$

We now consider the case where $U'(0)$ is finite so that $U(0)$ is also finite.

In the preceding subsection we have used consumption as an intermediate variable to define $X(c; \hat{B})$ and $J(c; \frac{\lambda_-}{\rho_-}\hat{B})$. However, in this subsection we can not use consumption as an intermediate variable since it turns out not to be a one-to-one function of wealth. Instead, $y = V'(x)$ plays a role of intermediate variable. Then, the process of finding a solution is similar to that of the preceding subsection.

Recall $I : (0, U'(0)] \to [0, \infty)$ is the inverse of U'. We extended I by setting $I \equiv 0$ on $[U'(0), \infty)$. If V is C^2, strictly increasing, and strictly concave, then the HJB equation (3.1) for $t < \tau$ becomes

$$\beta V(x) = -\kappa \frac{(V'(x))^2}{V''(x)} + [rx - I(V'(x)) + \epsilon] V'(x) + U(I(V'(x))) - l, \tag{3.34}$$

for $x > -\frac{\epsilon}{r}$. Let's define the following functions:

$$\mathcal{X}_0(y) = \frac{I(y)}{r} - \frac{1}{\kappa(\lambda_+ - \lambda_-)}[\frac{y^{\lambda_+}}{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\lambda_-}}{\lambda_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}] - \frac{\epsilon}{r}, \quad y > 0,$$

$$\mathcal{J}_0(y) = \frac{U(I(y)) - l}{\beta} - \frac{1}{\kappa(\rho_+ - \rho_-)}[\frac{y^{\rho_+}}{\rho_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\rho_-}}{\rho_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}], \quad y > 0,$$

$$\bar{\mathcal{X}}(y) = \frac{I(y)}{r} - \frac{1}{\kappa(\lambda_+ - \lambda_-)}[\frac{y^{\lambda_+}}{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\lambda_-}}{\lambda_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}], \quad y > 0,$$

$$\bar{\mathcal{J}}(y) = \frac{U(I(y))}{\beta} - \frac{1}{\kappa(\rho_+ - \rho_-)}[\frac{y^{\rho_+}}{\rho_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\rho_-}}{\rho_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}], \quad y > 0.$$

Then, as is shown in KLSS (1986), $\bar{\mathcal{X}}(\cdot)$ is strictly decreasing, maps $(0, \infty)$ onto itself and has an inverse function $\bar{\mathcal{Y}}(\cdot)$. Similarly, $\mathcal{X}_0(\cdot)$ is strictly decreasing, maps $(0, \infty)$ onto $(-\frac{\epsilon}{r}, \infty)$ and has an inverse function $\mathcal{Y}_0(\cdot)$. As is shown in KLSS (1986), it holds that

$$\bar{V}(x) = \bar{\mathcal{J}}(\bar{\mathcal{Y}}(x))$$

for all $x \geq 0$. Similarly, it can be shown that

$$V_0(x) = \mathcal{J}_0(\mathcal{Y}_0(x))$$

for all $x \geq -\frac{\epsilon}{r}$.

Define a function $F : (0, \infty) \to R$ by

$$\begin{aligned}
F(z) &\equiv \lambda_- \bar{\mathcal{Y}}(z)\{\mathcal{X}_0(\bar{\mathcal{Y}}(z)) - z\} - \rho_-\{\mathcal{J}_0(\bar{\mathcal{Y}}(z)) - \bar{V}(z)\} && (3.35)\\
&= \lambda_- \bar{\mathcal{Y}}(z)\{\bar{\mathcal{X}}(\bar{\mathcal{Y}}(z)) - \frac{\epsilon}{r} - z\} - \rho_-\{\bar{\mathcal{J}}(\bar{\mathcal{Y}}(z)) - \frac{l}{\beta} - \bar{V}(z)\} && (3.36)\\
&= -\frac{\lambda_- \epsilon}{r}\bar{\mathcal{Y}}(z) + \frac{\rho_- l}{\beta}. && (3.37)
\end{aligned}$$

Then the function $F(\cdot)$ is strictly decreasing since $\bar{\mathcal{Y}}(\cdot)$ is strictly decreasing, and it holds that $\lim_{z\downarrow 0} F(z) = \infty$ and that $\lim_{z\uparrow\infty} F(z) = \frac{\rho_- l}{\beta} < 0$. Letting

$$z^* \equiv \bar{\mathcal{X}}(\frac{\rho_- rl}{\lambda_- \beta\epsilon}), \tag{3.38}$$

then z^* is positive and satisfies

$$F(z^*) = 0. \tag{3.39}$$

We define a constant $\hat{B}$ by

$$\hat{B} \equiv (\bar{\mathcal{Y}}(z^*))^{-\lambda_-}\{z^* - \mathcal{X}_0(\bar{\mathcal{Y}}(z^*))\} \tag{3.40}$$

$$= (\frac{\rho_- rl}{\lambda_- \beta\epsilon})^{-\lambda_-}\{z^* - (\bar{\mathcal{X}}(\bar{\mathcal{Y}}(z^*)) - \frac{\epsilon}{r})\} \tag{3.41}$$

$$= (\frac{\rho_- rl}{\lambda_- \beta\epsilon})^{-\lambda_-}\frac{\epsilon}{r} > 0, \tag{3.42}$$

which is the same constant as the one defined for the case $U'(0) = \infty$. With this $\hat{B} > 0$, we define a function

$$\mathcal{X}(y;\hat{B}) = \hat{B}y^{\lambda_-} + \mathcal{X}_0(y), \tag{3.43}$$

for $y > 0$, then we have

$$\mathcal{X}(\bar{\mathcal{Y}}(z^*);\hat{B}) = z^* \tag{3.44}$$

For $c \geq 0$, we have $c = I(U'(c))$, hence

$$\mathcal{X}(U'(c);\hat{B}) = X(c;\hat{B}).$$

For $y > 0$ and $y \neq U'(0)$, using the relation $\lambda_+\lambda_- = -\frac{r}{\kappa_1}$, we get

$$\begin{aligned}\mathcal{X}'(y) &= \hat{B}\lambda_- y^{\lambda_- -1} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}[y^{\lambda_+}\int_0^{I(y)}\frac{d\theta}{(U'(\theta))^{\lambda_+}} + y^{\lambda_-}\int_{I(y)}^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}}] \\ &< 0.\end{aligned}$$

Hence $\mathcal{X}(\cdot;\hat{B})$ is strictly decreasing. Furthermore,

$$\begin{aligned}\lim_{y\downarrow 0}\mathcal{X}(y;\hat{B}) &= \lim_{c\uparrow\infty}\mathcal{X}(U'(c);\hat{B}) \\ &= \lim_{c\uparrow\infty}X(c;\hat{B}) \\ &= \infty\end{aligned}$$

and

$$\begin{aligned}\lim_{y\uparrow\infty}\mathcal{X}(y;\hat{B}) &= \lim_{y\uparrow\infty}[\hat{B}y^{\lambda_-} - \frac{1}{\kappa_1(\lambda_+-\lambda_-)}\frac{y^{\lambda_-}}{\lambda_-}\int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} - \frac{\epsilon}{r}] \\ &= -\frac{\epsilon}{r}.\end{aligned}$$

Therefore $\mathcal{X}(\cdot;\hat{B})$ maps $(0,\infty)$ onto $(-\frac{\epsilon}{r},\infty)$ and has an inverse function $\mathcal{Y}(\cdot;\hat{B}):$ $(-\frac{\epsilon}{r},\infty)\to(0,\infty)$. For $\hat{A}\geq 0$, we define

$$\mathcal{J}(y;\hat{A}) = \hat{A}y^{\rho_-} + \mathcal{J}_0(y), \tag{3.45}$$

for $y>0$. Now, define a function $\mathcal{V}:(-\frac{\epsilon}{r},\infty)\to R$ by

$$\mathcal{V}(x)\equiv\mathcal{J}(\mathcal{Y}(x;\hat{B});\frac{\lambda_-}{\rho_-}\hat{B}),\quad -\frac{\epsilon}{r}<x<z^*, \tag{3.46}$$

and

$$\mathcal{V}(x)\equiv\bar{V}(x),\quad x\geq z^*. \tag{3.47}$$

Then, we have

$$\begin{aligned}\lim_{x\downarrow-\frac{\epsilon}{r}}\mathcal{V}(x) &= \lim_{y\uparrow\infty}\mathcal{J}(y;\frac{\lambda_-}{\rho_-}\hat{B}) \\ &= \lim_{y\uparrow\infty}[\frac{\lambda_-}{\rho_-}\hat{B}y^{\rho_-} + \frac{U(0)-l}{\beta} - \frac{1}{\kappa_1(\rho_+-\rho_-)}\frac{y^{\rho_-}}{\rho_-}\int_0^\infty\frac{d\theta}{(U'(\theta))^{\lambda_-}}] \\ &= \frac{U(0)-l}{\beta}.\end{aligned}$$

By (3.44) we have

$$\lim_{x\uparrow z^*}\mathcal{Y}(x;\hat{B}) = \mathcal{Y}(z^*;\hat{B}) = \bar{\mathcal{Y}}(z^*),$$

so that

$$\lim_{x\uparrow z^*}\mathcal{V}(x) = \mathcal{J}(\bar{\mathcal{Y}}(z^*);\frac{\lambda_-}{\rho_-}\hat{B}).$$

Using (3.35), (3.39) and (3.40) we get

$$\lim_{x\uparrow z^*}\mathcal{V}(x) = \bar{V}(z^*).$$

We have the following lemma which can be proved similarly to Lemma 3.2

Lemma 3.3. *The function* $\mathcal{V}(x)$ *defined by (3.46) and (3.47) is strictly increasing for* $x > -\frac{\epsilon}{r}$*, strictly concave and satisfies the HJB equation (3.1) for* $-\frac{\epsilon}{r} < x < z^*$*.*

We get the following theorem which can be proved using an argument similar to that for the case $U'(0) = \infty$.

Theorem 3.2. *If* $U'(0)$ *is finite, then the optimal value function is* $\mathcal{V}(x)$ *defined by (3.46) and (3.47), and an optimal strategy is given by the following strategy* $(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)$*:*

$$\tau^* = T_{z^*},$$

$$c_t^* = I(\mathcal{V}'(x_t)), \quad \boldsymbol{\pi}_t^* = -\frac{\mathcal{V}'(x_t)}{\mathcal{V}''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad 0 \leq t < \tau^*,$$

and

$$c_t^* = \bar{C}(x_t), \quad \boldsymbol{\pi}_t^* = \bar{\Pi}(x_t), \quad t \geq \tau^*.$$

4 Properties of Optimal Policies

In this section we study properties of optimal policies found in Section 3.

The following intuitively clear observation is easily checked.

Observation 4.1. *The wealth level* z^* *at the optimal retirement time in (3.17) and (3.38) is decreasing in* l *and increasing in* ϵ*. Furthermore, it satisfies*

$$\lim_{l \uparrow \infty} z^* = 0, \quad \lim_{l \downarrow 0} z^* = \infty, \quad \lim_{\epsilon \uparrow \infty} z^* = \infty, \quad \lim_{\epsilon \downarrow 0} z^* = 0.$$

If the agent does not have an option to retire from labor, that is, if we restrict τ to be infinite, then as in KLSS (1986) an optimal strategy takes the following form: When $U'(0) = \infty$,

$$c_t = C_0(x_t), \quad \boldsymbol{\pi}_t = -\frac{V_0'(x_t)}{V_0''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \tag{4.1}$$

for $t \geq 0$. When $U'(0) < \infty$,

$$c_t = I(V_0'(x_t)), \quad \boldsymbol{\pi}_t = -\frac{V_0'(x_t)}{V_0''(x_t)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \tag{4.2}$$

for $t \geq 0$.

The following two propositions illustrate effects of retirement option. Proposition 4.1 states that the agent consumes less if the investor has a retirement option than he does if he does not have such an option. Intuitively, he tries to accumulate his wealth fast enough to reach the wealth level at which he retires and stops incurring a utility loss due to labor.

Proposion 4.1.

$$C(x;\hat{B}) < C_0(x) \tag{4.3}$$

for $-\frac{\epsilon}{r} < x < z^*$, *where* $C(x;\hat{B})$ *is given in Theorem 3.1 and* $C_0(x)$ *in (4.1). If* $\mathcal{X}_0(U'(0)) < z^*$, *then*

$$I(\mathcal{V}'(x)) = I(V_0'(x)) = 0 \tag{4.4}$$

for $x \leq \mathcal{X}_0(U'(0))$ *and*

$$I(\mathcal{V}'(x)) < I(V_0'(x)) \tag{4.5}$$

for $\mathcal{X}_0(U'(0)) < x < z^*$, *where* $I(\mathcal{V}'(x))$ *is given in Theorem 3.2 and* $I(V_0'(x))$ *in (4.2).*

The agent takes more risk in risky assets if he has a retirement option than he does if he does not have such a retirement option. Intuitively, the agent tries to increase the expected growth rate of his wealth to reach the wealth level fast enough at which the he retires and stops incurring a utility loss due to labor. This is summarized in Proposition 4.2.

Proposion 4.2. *In Theorem 3.1*

$$\frac{V'(x)}{-V''(x)} > \frac{V_0'(x)}{-V_0''(x)} \tag{4.6}$$

for $-\frac{\epsilon}{r} < x < z^*$, *and in Theorem 3.2*

$$\frac{\mathcal{V}'(x)}{-\mathcal{V}''(x)} > \frac{V_0'(x)}{-V_0''(x)} \tag{4.7}$$

for $-\frac{\epsilon}{r} < x < z^*$.

5 A Solution under the CRRA utility class

In this section we find the value function and optimal policy in the special case where the utility function is in the CRRA class.

We first consider the case where

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad 0 < \gamma \neq 1 \tag{5.1}$$

for $c > 0$, which means that the agent's coefficient of relative risk aversion is constant and equal to γ. In this case $U'(c) = c^{-\gamma}$ so that $U'(0) = \infty$. By calculation we have

$$X_0(c) = \frac{c}{K} - \frac{\epsilon}{r}, \quad c > 0,$$

$$J_0(c) = \frac{1}{(1-\gamma)K} c^{1-\gamma} - \frac{l}{\beta}, \quad c > 0,$$

$$\bar{X}(c) = \frac{c}{K}, \quad c > 0,$$

$$\bar{J}(c) = \frac{1}{(1-\gamma)K} c^{1-\gamma}, \quad c > 0,$$

where K is given in (2.8). Therefore

$$C_0(x) = K(x + \frac{\epsilon}{r}), \quad x > -\frac{\epsilon}{r},$$

$$\bar{C}(x) = Kx, \quad x > 0,$$

$$V_0(x) = \frac{K^{-\gamma}}{(1-\gamma)}(x + \frac{\epsilon}{r})^{1-\gamma} - \frac{l}{\beta}, \quad x > -\frac{\epsilon}{r},$$

and

$$\bar{V}(x) = \frac{K^{-\gamma}}{(1-\gamma)} x^{1-\gamma}, \quad x > 0.$$

As is shown in Karatzas, Lehoczky, Sethi and Shreve (1986)

$$\begin{aligned} \bar{\Pi}(x) &= \frac{\bar{V}'(x)}{-\bar{V}''(x)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1} \\ &= \frac{x}{\gamma}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad x > 0. \end{aligned}$$

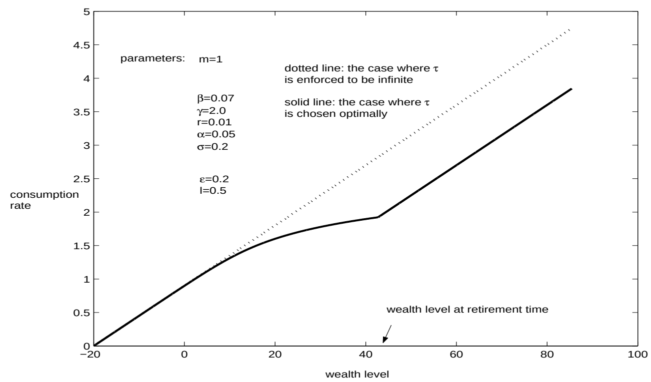

Figure 5.1: comparison of consumption rates when $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

The wealth level at the optimal retirement time in (3.17) becomes

$$z^* = \frac{1}{K}(\frac{\rho_- rl}{\lambda_- \beta\epsilon})^{-\frac{1}{\gamma}}$$

and the function (3.6) becomes

$$X(c;\hat{B}) = \hat{B}c^{-\gamma\lambda_-} + \frac{c}{K} - \frac{\epsilon}{r}$$

where $\hat{B}$ is given by (3.21).

The value function $V : (-\frac{\epsilon}{r}, \infty) \to R$ defined in (3.23) and (3.24) becomes

$$V(x) = \frac{\lambda_-}{\rho_-}\hat{B}(C(x;\hat{B}))^{-\gamma\rho_-} + \frac{1}{(1-\gamma)K}(C(x;\hat{B}))^{1-\gamma} - \frac{l}{\beta}, \quad -\frac{\epsilon}{r} < x < z^*,$$

and

$$V(x) = \frac{K^{-\gamma}}{1-\gamma}x^{1-\gamma}, \quad x \geq z^*.$$

The optimal policy $(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)$ in Theorem 3.1 becomes

$$\tau^* = T_{z^*},$$

$$c_t^* = C(x_t;\hat{B}), \quad \boldsymbol{\pi}_t^* = (-\lambda_-\hat{B}c_t^{*-\gamma\lambda_-} + \frac{1}{\gamma K}c_t^*)(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad 0 \leq t < \tau^*,$$

and

$$c_t^* = Kx_t, \quad \boldsymbol{\pi}_t^* = \frac{x_t}{\gamma}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad t \geq \tau^*.$$

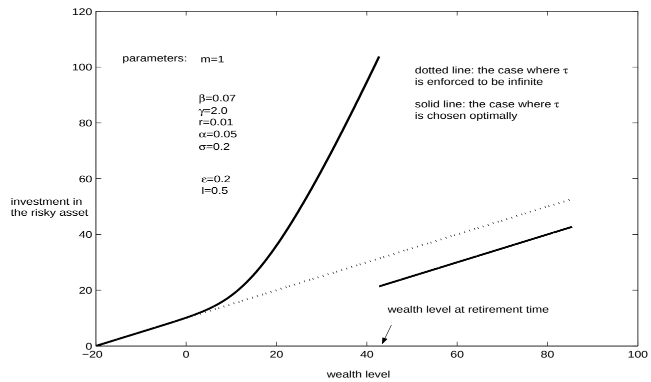

Figure 5.2: comparison of amount of wealth invested in the risky asset when $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

Figure 5.1 compares the rates of consumption for the two cases: (1) τ is enforced to be infinite and (2) the agent has an option to retire. As explained in Proposition 4.1, the figure shows that the wage earner consumes less before touching the critical wealth level in the latter case than in the former case.

Figure 5.2 compares amount of wealth invested in the risky asset in the two cases. As is shown in the figure, before retirement the agent invests more in the risky asset in the latter case than in the former case.

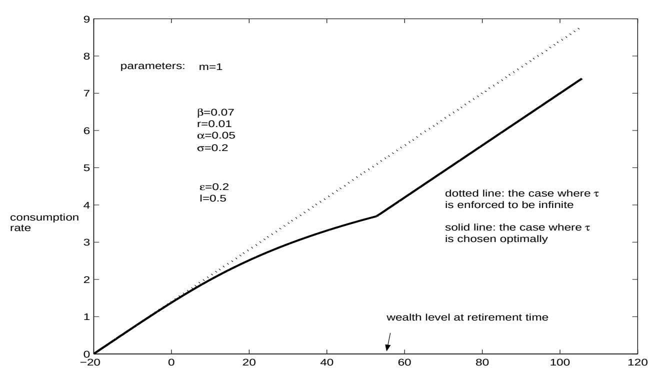

Figure 5.3: comparison of consumption rates when $U(c) = \log c$.

Now consider the case where

$$U(c) = \log c \tag{5.2}$$

for $c > 0$, which means that the agent's coefficient of relative risk aversion is constant and equal to $\gamma = 1$. In this case $U'(c) = c^{-1}$ so that $U'(0) = \infty$. By calculation we have:

$$X_0(c) = \frac{c}{\beta} - \frac{\epsilon}{r}, \quad c > 0,$$

$$J_0(c) = \frac{\beta(\log c - l) + \kappa + r - \beta}{\beta^2}, \quad c > 0,$$

$$\bar{X}(c) = \frac{c}{\beta}, \quad c > 0,$$

$$\bar{J}(c) = \frac{\beta \log c + \kappa + r - \beta}{\beta^2}, \quad c > 0.$$

Therefore

$$C_0(x) = \beta(x + \frac{\epsilon}{r}), \quad x > -\frac{\epsilon}{r},$$

$$\bar{C}(x) = \beta x, \quad x > 0,$$

$$V_0(x) = \frac{\beta(\log(\beta(x + \frac{\epsilon}{r})) - l) + \kappa + r - \beta}{\beta^2}, \quad x > -\frac{\epsilon}{r},$$

and

$$\bar{V}(x) = \frac{\beta \log \beta x + \kappa + r - \beta}{\beta^2}, \quad x > 0.$$

As is shown in Karatzas, Lehoczky, Sethi and Shreve (1986)

$$\begin{aligned} \bar{\Pi}(x) &= \frac{\bar{V}'(x)}{-\bar{V}''(x)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1} \\ &= x(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad x > 0. \end{aligned}$$

The wealth level at the optimal retirement time in (3.17) becomes

$$z^* = \frac{1}{\beta}(\frac{\rho_- r l}{\lambda_- \beta \epsilon})^{-1}$$

and the function (3.6) becomes

$$X(c; \hat{B}) = \hat{B}c^{-\lambda_-} + \frac{c}{\beta} - \frac{\epsilon}{r}$$

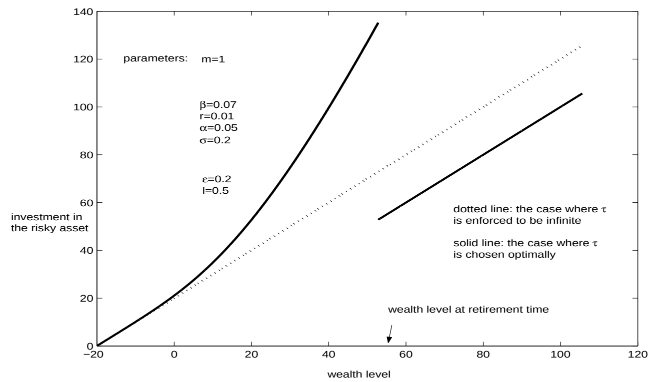

Figure 5.4: comparison of amount of wealth invested in the risky asset when $U(c) = \log c$.

where $\hat{B}$ is given by (3.21).

The value function $V : (-\frac{\epsilon}{r}, \infty) \to R$ defined in (3.23) and (3.24) becomes

$$V(x) = \frac{\lambda_-}{\rho_-}\hat{B}(C(x;\hat{B}))^{-\rho_-} + \frac{\beta(\log(C(x;\hat{B})) - l) + \kappa + r - \beta}{\beta^2}, \quad -\frac{\epsilon}{r} < x < z^*,$$

and

$$V(x) = \frac{\beta \log(\beta x) + \kappa + r - \beta}{\beta^2}, \quad x \geq z^*.$$

The optimal policy $(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)$ in Theorem 3.1 becomes

$$\tau^* = T_{z^*},$$

$$c_t^* = C(x_t;\hat{B}), \quad \boldsymbol{\pi}_t^* = (-\lambda_-\hat{B}{c_t^*}^{-\lambda_-} + \frac{1}{\beta}c_t^*)(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad 0 \leq t < \tau^*,$$

and

$$c_t^* = \beta x_t, \quad \boldsymbol{\pi}_t^* = x_t(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}, \quad t \geq \tau^*.$$

Figure 5.3 compares the rates of consumption for the two cases when the utility function is given by (5.2): (1) τ is enforced to be infinite and (2) the agent has an option to retire. As is shown in the Figure, the wage earner consume less before retirement in the latter case than in the former case.

Figure 5.4 compares amounts of wealth invested in the risky asset in the two cases. As is shown in the figure, the agent invests more in the risky asset before retirement in the latter case than in the former case.

6 Conclusion

In this paper we have studied an optimal retirement and consumption/portfolio decision problem of a wage earner. We have obtained a solution for the case where the wage earner has general von Neuman-Morgenstern time-separable utility. We have shown that the wage earner retires from his work as soon as his wealth exceeds a critical wealth level that is obtained from a free boundary value problem.

We have not considered uninsurable income risk in this paper, which in reality has an effect on optimal retirement and consumption portfolio selection. We leave the study of this effect of uninsurable income risk as future research.

Appendix

A Proof of Lemma 3.1

When $U(0)$ is finite, (3.2) trivially holds. When $U(0) = -\infty$, $\limsup_{c\downarrow 0} \frac{U(c)}{U'(c)} \leq 0$. For every $\delta > 0$ and $0 < c < \delta$, $U(c) \geq U(\delta) - U'(c)(\delta - c)$. Therefore,

$$\liminf_{c\downarrow 0} \frac{U(c)}{U'(c)} \geq \liminf_{c\downarrow 0} (\frac{U(\delta)}{U'(c)} - \delta + c) = -\delta.$$

Since $\delta > 0$ is arbitrary,

$$\liminf_{c\downarrow 0} \frac{U(c)}{U'(c)} \geq 0.$$

Hence (3.2) holds.

Since

$$\begin{aligned} 0 &\leq \liminf_{c\downarrow 0} (U'(c))^{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \\ &\leq \limsup_{c\downarrow 0} (U'(c))^{\lambda_+} \int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} \\ &\leq \limsup_{c\downarrow 0} (U'(c))^{\lambda_+} \int_0^c \frac{d\theta}{(U'(c))^{\lambda_+}} \\ &= \limsup_{c\downarrow 0} c = 0, \end{aligned}$$

(3.3) holds. Finally, since, for every $\delta > 0$,

$$\begin{aligned} 0 &\leq \liminf_{c\downarrow 0} (U'(c))^{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\ &\leq \limsup_{c\downarrow 0} (U'(c))^{\lambda_-} \int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\ &\leq \limsup_{c\downarrow 0} \int_c^\delta (\frac{U'(c)}{U'(\theta)})^{\lambda_-} d\theta + \limsup_{c\downarrow 0} (U'(c))^{\lambda_-} \int_\delta^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} \\ &\leq \limsup_{c\downarrow 0} (\delta - c) = \delta, \end{aligned}$$

(3.4) holds.

B Proof of Lemma 3.2

By calculation, we have

$$\begin{aligned}\frac{\partial}{\partial x} J(C(x;\hat{B}); \frac{\lambda_-}{\rho_-}\hat{B}) &= \frac{J'(C(x;\hat{B}); \frac{\lambda_-}{\rho_-}\hat{B})}{X'(C(x;\hat{B});\hat{B})} \\ &= U'(C(x;\hat{B})) > 0, \quad x > -\frac{\epsilon}{r}, \end{aligned} \tag{B.1}$$

and

$$\bar{V}'(x) = U'(\bar{C}(x)) > 0, \quad x > 0. \tag{B.2}$$

Therefore, by (3.27), $V(x)$ is strictly increasing for $x > -\frac{\epsilon}{r}$. For $-\frac{\epsilon}{r} < x < z^*$,

$$V''(x) \;=\; U''(C(x;\hat{B}))C'(x;\hat{B}) < 0, \quad -\frac{\epsilon}{r} < x < z^*. \tag{B.3}$$

Thus, $V(\cdot)$ is strictly concave for $-\frac{\epsilon}{r} < x < z^*$. Hence for $-\frac{\epsilon}{r} < x < z^*$ applying this $V(\cdot)$ in equation (3.1) and maximizing over investments in risky assets gives $\boldsymbol{\pi} = -\frac{V'(x)}{V''(x)}(\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}$. Hence the HJB equation (3.1) becomes

$$\beta V(x) = -\kappa\frac{(V'(x))^2}{V''(x)} + \max_{c\geq 0}\{(rx - c + \epsilon)V'(x) + U(c) - l\}.$$

By (B.1) and (B.3), this takes the form

$$\beta V(x) = -\kappa\frac{(U'(C(x;\hat{B})))^2 X'(C(x;\hat{B});\hat{B})}{U''(C(x;\hat{B}))} + (rx - C(x;\hat{B}) + \epsilon)V'(x) + U(C(x;\hat{B})) - l$$

for $-\frac{\epsilon}{r} < x < z^*$, which is equivalent to

$$\beta J(c; \frac{\lambda_-}{\rho_-}\hat{B}) = -\kappa\frac{(U'(c))^2 X'(c;\hat{B})}{U''(c)} + (rX(c;\hat{B}) - c + \epsilon)U'(c) + U(c) - l$$

for $0 < c < \bar{C}(z^*)$ by (3.25). By calculation and using the relation $\rho_+\rho_- = -\frac{\beta}{\kappa_1}$, the above equation can be shown to hold for $0 < c < \bar{C}(z^*)$. Hence $V(\cdot)$ satisfies equation (3.1) for $-\frac{\epsilon}{r} < x < z^*$.

By (B.3), we have that $V \in C^2(-\frac{\epsilon}{r}, z^*)$, $\lim_{x\to z^*+} V''(x)$ exists and finite. As is shown in KLSS (1986), $\bar{V} \in C^2(0,\infty)$. Hence by (B.1), (B.2) and (3.25), $V(\cdot) \in C^1(-\frac{\epsilon}{r},\infty) \cap C^2((-\frac{\epsilon}{r}, z^*) \cup (z^*,\infty))$, $\lim_{x\to z^*+} V''(x)$ and $\lim_{x\to z^*-} V''(x)$ exist and finite.

C Proof of Theorem 3.1

With the strategy the wealth process does not touch $-\frac{\epsilon}{r}$ before retirement by (3.30). For $c_0^* > 0$ (or equivalently $x > -\frac{\epsilon}{r}$), let

$$\text{(C.1)} \qquad H(c_0^*) \equiv V_{(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)}(x) = E_x \int_0^\infty \exp(-\beta t)[U(c_t^*) - l\mathbf{1}_{\{t<\tau^*\}}]\, dt.$$

If $c_0^* \geq \bar{C}(z^*)$ (or equivalently $x \geq z^*$), then $\tau^* = 0$. Thus

$$\text{(C.2)} \qquad H(c_0^*) = \bar{V}(x), \quad c_0^* \geq \bar{C}(z^*)(\text{or equivalently} x \geq z^*).$$

For $0 < c_0^* < \bar{C}(z^*)$ (or equivalently $-\frac{\epsilon}{r} < x < z^*$), (C.1) be rewritten as

$$H(c_0^*) \equiv V_{(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)}(x) = E_x[\int_0^{\tau^*} \exp(-\beta t)(U(c_t^*) - l)dt + \exp(-\beta\tau^*)\bar{V}(z^*)],$$

where the equality comes from the strong Markov property. Note that since $C_0(\cdot)$ is strictly increasing and maps $(-\frac{\epsilon}{r}, \infty)$ onto $(0, \infty)$, there exists $\hat{x} > -\frac{\epsilon}{r}$ such that

$$\text{(C.3)} \qquad c_0^* = C_0(\hat{x}).$$

When retirement time τ is forced to be infinite, that is, when the investor has no option to retire, it can be shown similarly to KLSS (1986) that the optimal consumption strategy $(\hat{c}_t)_{\{t\geq 0\}}$with initial wealth $\hat{x} > -\frac{\epsilon}{r}$ satisfies

$$\hat{c}_t = I(U'(C_0(\hat{x}))\exp[-(r-\beta+\kappa)t - (\boldsymbol{\alpha} - r\mathbf{1}_m)\Sigma^{-1}D\mathbf{w}^T(t)])$$

for $t \geq 0$. Thus $V_0(\hat{x}) = E_x[\int_0^\infty \exp(-\beta t)U(\hat{c}_t)dt]$, which is well-defined and finite. By (C.3) and (3.29), $\hat{c}_t = c_t^*$ for all $0 \leq t \leq \tau^*$. Hence it follows that $H(c_0^*)$ is well defined and finite for $0 < c_0^* < \bar{C}(z^*)$ (or equivalently $-\frac{\epsilon}{r} < x < z^*$). In particular condition (2.4) holds with the strategy defined by (3.31), (3.32) and (3.33). Define

$$\Psi(y_0) \equiv H(I(y_0)) = E_x[\int_0^{\tau^*} \exp(-\beta t)(U(I(y_t)) - l)dt + \exp(-\beta\tau^*)\bar{V}(z^*)]$$

for $U'(\bar{C}(z^*)) < y_0 < U'(0) = \infty$ where $y_t = U'(c_t^*)$ so that y_t satisfies the stochastic differentiable equation (3.28) for $0 \le t \le \tau^*$ with $y_0 = U'(c_0^*)$. By Theorem 13.16 of Dynkin (1965)(Feynman-Kac formula), Ψ is C^2 on $(U'(\bar{C}(z^*)), \infty)$ and satisfies

$$\beta\Psi(y) = -(r-\beta)y\Psi'(y) + \kappa y^2\Psi''(y) + U(I(y)) - l$$

for $U'(\bar{C}(z^*)) < y_0 < \infty$ with $\lim_{y \downarrow U'(\bar{C}(z^*))} \Psi(y) = \bar{V}(z^*)$. Hence H is C^2 on $(0, \bar{C}(z^*))$ and satisfies

$$\text{(C.4)}\quad \beta H(c) = -\frac{U'(c)}{U''(c)}[r-\beta+\kappa\frac{U'(c)U'''(c)}{(U''(c))^2}]H'(c)+\kappa(\frac{U'(c)}{U''(c)})^2H''(c)+U(c)-l$$

for $0 < c < \bar{C}(z^*)$ with $\lim_{c \uparrow \bar{C}(z^*)} H(c) = \bar{V}(z^*)$. A general solution to equation (C.4) takes the following form $A(U'(c))^{\rho_+} + J(c;\hat{A})$ for $0 < c < \bar{C}(z^*)$. Hence for $0 < c < \bar{C}(z^*)$, $H(c) = A(U'(c))^{\rho_+} + J(c;\hat{A})$ for some A and $\hat{A}$ such that $\lim_{c \uparrow \bar{C}(z^*)} H(c) = A(U'(\bar{C}(z^*)))^{\rho_+} + J(\bar{C}(z^*);\hat{A}) = \bar{V}(z^*)$. As in Theorem 8.8 of KLSS (1986), it is shown that $A = 0$ when $U'(0) = \infty$ so that for $0 < c < \bar{C}(z^*)$,

$$H(c) = J(c;\hat{A})$$

for some $\hat{A}$ such that

$$\text{(C.5)}\qquad \lim_{c \uparrow \bar{C}(z^*)} H(c) = J(\bar{C}(z^*);\hat{A}) = \bar{V}(z^*).$$

Using (3.18), (3.22) and (C.5), we get $\hat{A} = \frac{\lambda_-}{\rho_-}\hat{B}$ so that $H(c_0^*) = J(c_0^*;\frac{\lambda_-}{\rho_-}\hat{B}) = J(C(x;\hat{B});\frac{\lambda_-}{\rho_-}\hat{B})$ for $0 < c_0^* < \bar{C}(z^*)$ (or equivalently $-\frac{\epsilon}{r} < x < z^*$). This equality and (C.2) imply

$$\text{(C.6)}\qquad H(c_0^*) = V_{(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*)}(x) = V(x), \quad x > -\frac{\epsilon}{r}.$$

If $0 \le x \le z^*$, then we have

$$\begin{aligned} X(\bar{C}(x);\hat{B}) &= \hat{B}(U'(\bar{C}(x)))^{\lambda_-} + X_0(\bar{C}(x)) \\ &= \hat{B}(U'(\bar{C}(x)))^{\lambda_-} + x - \frac{\epsilon}{r} \\ &= (\frac{U'(\bar{C}(x))}{U'(\bar{C}(z^*))})^{\lambda_-}\frac{\epsilon}{r} - \frac{\epsilon}{r} + x \\ &\le x, \end{aligned}$$

where the the third equality follows from (3.20) and the fourth inequality follows from the fact that $(\frac{U'(\bar{C}(x))}{U'(\bar{C}(z^*))})^{\lambda_-} \leq 1$ for $0 \leq x \leq z^*$. Hence $\bar{C}(x) \leq C(x;\hat{B}), \quad 0 \leq x \leq z^*$, so that $U'(\bar{C}(x)) \geq U'(C(x;\hat{B})), \quad 0 \leq x \leq z^*$. By (B.1) and (B.2), this inequality implies that $\bar{V}'(x) \geq V'(x), \quad 0 \leq x \leq z^*$. Using this and the fact that $\bar{V}(z^*) = V(z^*)$, we get

(C.7) $$\bar{V}(x) \leq V(x), \quad 0 \leq x \leq z^*.$$

As is shown in KLSS (1986), $\bar{V}(\cdot)$ satisfies

(C.8)
$$\beta\bar{V}(x) = \max_{c\geq 0,\boldsymbol{\pi}} \left\{(\boldsymbol{\alpha} - r\mathbf{1}_m)\boldsymbol{\pi}^T\bar{V}'(x) + (rx - c)\bar{V}'(x) + \frac{1}{2}\boldsymbol{\pi}\Sigma\boldsymbol{\pi}^T\bar{V}''(x) + U(c)\right\}$$

for $x > 0$. If $x \geq z^*$ then $G(x) \leq 0$. Therefore by (3.16) it holds that

$$-l \leq -\frac{\epsilon\beta\lambda_-}{r\rho_-}U'(\bar{C}(x)) = -\frac{\epsilon\beta\lambda_-}{r\rho_-}\bar{V}'(x), \quad x \geq z^*,$$

where the second equality comes from (B.2). Using this and Assumption 2.3 we get

(C.9) $$\epsilon\bar{V}'(x) - l \leq \epsilon\bar{V}'(x)(1 - \frac{\beta\lambda_-}{r\rho_-}) \leq 0, \quad x \geq z^*.$$

Using (C.9) and (C.8) we get

(C.10)
$$\beta\bar{V}(x) \geq \max_{c\geq 0,\boldsymbol{\pi}} \left\{(\boldsymbol{\alpha} - r\mathbf{1}_m)\boldsymbol{\pi}^T\bar{V}'(x) + (rx - c + \epsilon)\bar{V}'(x) + \frac{1}{2}\boldsymbol{\pi}\Sigma\boldsymbol{\pi}^T\bar{V}''(x) + U(c) - l\right\}$$

for $x \geq z^*$.

Fix $x > -\frac{\epsilon}{r}$. Let $(\tau, \mathbf{c}, \boldsymbol{\pi}) \in A_1(x)$ arbitrary. Choose $x < \xi < \infty$ and define $S_n = \inf\{t \geq 0 : \int_0^t \|\pi_s\|^2\, ds = n\}$. Put

$$\tau_n \equiv T^\xi \wedge S_n \wedge \tau \wedge n$$

so that $\tau_n \to \tau$ as $\xi \uparrow \infty$ and $n \uparrow \infty$. By the strong Markovian property,

$$\begin{aligned} V_{(\tau,\mathbf{c},\boldsymbol{\pi})}(x) &= E_x \int_0^\infty \exp(-\beta t)(U(c_t) - l\mathbf{1}_{\{t<\tau\}})dt \\ &= E_x[\int_0^\tau \exp(-\beta t)(U(c_t) - l)]dt \\ &+ E_x[\exp(-\beta\tau)\bar{V}(x_\tau)\mathbf{1}_{\{\tau<\infty\}}]. \end{aligned}$$

With a $\delta > 0$ let $z_t \equiv x_t + \delta$ for $t \geq 0$. From equation (C.10) and the fact that $V(x)$, defined by (3.23) and (3.24), satisfies the HJB equation (3.1) for $-\frac{\epsilon}{r} < x < z^*$, we get by using generalized Itô's rule

$$\begin{aligned}
E_x \int_0^{\tau_n} \exp(-\beta t)(U(c_t) - l)dt &\leq E_x[\int_0^{\tau_n} \exp(-\beta t)[\beta V(z_t) - (\boldsymbol{\alpha} - r\mathbf{1}_m)\boldsymbol{\pi}_t^T V'(z_t) \\
&\quad -(rz_t - c_t + \epsilon)V'(z_t) - \frac{1}{2}\boldsymbol{\pi}_t \Sigma \boldsymbol{\pi}_t^T V''(z_t)]dt] \\
&= E_x[\int_0^{\tau_n} [-d(\exp(-\beta t)V(z_t)) + \exp(-\beta t)V'(z_t)\boldsymbol{\pi}_t D d\mathbf{w}^T(t)]] \\
&\quad +E_x[\int_0^{\tau_n} -r\delta \exp(-\beta t)V'(z_t)dt] \\
&= -E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)] + V(x + \delta) \\
&\quad +E_x[\int_0^{\tau_n} -r\delta \exp(-\beta t)V'(x_t + \delta)dt] \\
&\leq -E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)] + V(x + \delta).
\end{aligned}$$

Hence

$$\begin{aligned}
V(x + \delta) &\geq E_x[\int_0^{\tau_n} \exp(-\beta t)(U(c_t) - l)dt] + E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)] \\
&= E_x[\int_0^{\tau_n} \exp(-\beta t)(U(c_t) - l)dt] + E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)\mathbf{1}_{\{\tau<\infty\}}] \\
&+ E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)\mathbf{1}_{\{\tau=\infty\}}]
\end{aligned}$$

By (2.4) and applying the monotone convergence theorem to $E_x \int_0^{\tau_n} \exp(-\beta t)U^{\pm}(c_t)dt$ and $E_x \int_0^{\tau_n} \exp(-\beta t)ldt$, we get

$$E_x \int_0^{\tau_n} \exp(-\beta t)(U(c_t) - l)dt \to E_x \int_0^{\tau} \exp(-\beta t)(U(c_t) - l)dt$$

as $\xi \uparrow \infty$ and $n \uparrow \infty$. Since $V(x(\tau_n)+\delta) \geq V(-\frac{\epsilon}{r}+\delta) > -\infty$, by Fatou's lemma,

$$\begin{aligned}
&\liminf_{\xi\uparrow\infty, n\uparrow\infty} E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)\mathbf{1}_{\{\tau<\infty\}}] \\
&\geq E_x[\exp(-\beta\tau)V(x_\tau + \delta)\mathbf{1}_{\{\tau<\infty\}}]
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{\xi_2\uparrow\infty, n\uparrow\infty} E_x[\exp(-\beta\tau_n)V(x(\tau_n) + \delta)\mathbf{1}_{\{\tau=\infty\}}] \\
&\geq V(-\frac{\epsilon}{r} + \delta)E_x[\lim_{\xi\uparrow\infty, n\uparrow\infty} \exp(-\beta\tau_n)\mathbf{1}_{\{\tau=\infty\}}] = 0.
\end{aligned}$$

Therefore, we get

$$\begin{aligned} V(x+\delta) &\geq E_x \int_0^\tau \exp(-\beta t)(U(c_t)-l)dt + E_x[\exp(-\beta\tau)V(x_\tau+\delta)\mathbf{1}_{\{\tau<\infty\}}] \\ &\geq E_x \int_0^\tau \exp(-\beta t)(U(c_t)-l)dt + E_x[\exp(-\beta\tau)V(x_\tau)\mathbf{1}_{\{\tau<\infty\}}] \\ &\geq E_x \int_0^\tau \exp(-\beta t)(U(c_t)-l)dt + E_x[\exp(-\beta\tau)\bar{V}(x_\tau)\mathbf{1}_{\{\tau<\infty\}}], \end{aligned}$$

where the third inequality comes from (C.7) and the fact that $\bar{V}(x) = V(x)$ for $x \geq z^*$. Letting $\delta \downarrow 0$ we get

$$V(x) \geq V_{(\tau,\mathbf{c},\boldsymbol{\pi})}(x).$$

Since $(\tau, \mathbf{c}, \boldsymbol{\pi}) \in A_1(x)$ is arbitrary, we get

$$V(x) \geq V^*(x).$$

Since $(\tau^*, \mathbf{c}^*, \boldsymbol{\pi}^*) \in A_1(x)$ and (C.6) holds, we have

$$V(x) = V^*(x).$$

D Proofs of Proposition 4.1 and Proposition 4.2

Proof of Proposition 4.1 We first prove (4.3) corresponding to the case where $U'(0) = \infty$. Since $\hat{B} > 0$, $X(c;\hat{B}) > X_0(c)$ for all $c > 0$. Hence their inverse functions satisfy $C(x;\hat{B}) < C_0(x)$ for all $x > -\frac{\epsilon}{r}$ since $X(\cdot;\hat{B})$ and $X_0(\cdot)$ are increasing functions.

Now let us prove (4.4) and (4.5) corresponding to the case where $U'(0)$ is finite. Since $\hat{B} > 0$, $\mathcal{X}(y;\hat{B}) > \mathcal{X}_0(y)$ for all $y > 0$. Hence their inverse functions satisfy $\mathcal{Y}(x;\hat{B}) > \mathcal{Y}_0(x)$ for all $x > -\frac{\epsilon}{r}$ since $\mathcal{X}(\cdot;\hat{B})$ and $\mathcal{X}_0(\cdot)$ are decreasing functions. It is easily checked that $\mathcal{V}'(x) = \mathcal{Y}(x;\hat{B})$ and $V_0'(x) = \mathcal{Y}_0(x)$. If $x \leq \mathcal{X}_0(U'(0))$, then $\mathcal{Y}(x;\hat{B}) > \mathcal{Y}_0(x) \geq U'(0)$. Therefore $I(\mathcal{V}'(x)) = I(V_0'(x)) = 0$ for $x \leq \mathcal{X}_0(U'(0))$ since $I \equiv 0$ on $[U'(0), \infty)$. If $\mathcal{X}_0(U'(0)) < x \leq \mathcal{X}(U'(0);\hat{B})$, then $\mathcal{Y}_0(x) < U'(0)$ and $\mathcal{Y}(x;\hat{B}) \geq U'(0)$. Hence $I(V_0'(x)) > 0$ and $I(\mathcal{V}'(x)) = 0$

for $\mathcal{X}_0(U'(0)) < x \leq \mathcal{X}(U'(0);\hat{B})$ since $I(y) > 0$ for $0 < y < U'(0)$ and $I \equiv 0$ on $[U'(0),\infty)$. If $x > \mathcal{X}(U'(0);\hat{B})$, then $0 < \mathcal{Y}_0(x) < \mathcal{Y}(x;\hat{B}) < U'(0)$. Hence $I(\mathcal{V}'(x)) < I(V_0'(x))$ for $x > \mathcal{X}(U'(0);\hat{B})$ since $I(\cdot)$ is strictly decreasing for $0 < y < U'(0)$.

Proof of Proposition 4.2. We first prove (4.6), that is, we consider the case where $U'(0) = \infty$:

Using the fact that $V'(x) = U'(C(x;\hat{B})$ and $V_0'(x) = U'(C_0(x))$ for $-\frac{\epsilon}{r} < x < z^*$, some calculation gives

$$\frac{V'(x)}{-V''(x)} = -\lambda_-\{x - \frac{C(x;\hat{B})}{r} + \frac{1}{r}(U'(C(x;\hat{B})))^{\lambda_+}\int_0^{C(x;\hat{B})}\frac{d\theta}{(U'(\theta))^{\lambda_+}}\} - \frac{\lambda_-\epsilon}{r}$$

and

$$\frac{V_0'(x)}{-V_0''(x)} = -\lambda_-\{x - \frac{C_0(x)}{r} + \frac{1}{r}(U'(C_0(x)))^{\lambda_+}\int_0^{C_0(x)}\frac{d\theta}{(U'(\theta))^{\lambda_+}}\} - \frac{\lambda_-\epsilon}{r}.$$

By differentiation it is easily checked that $-\lambda_-\{x - \frac{c}{r} + \frac{1}{r}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}\}$ is a decreasing function of c. Since $C(x;\hat{B}) < C_0(x)$, we have

$$\frac{V'(x)}{-V''(x)} > \frac{V_0'(x)}{-V_0''(x)}$$

for $-\frac{\epsilon}{r} < x < z^*$. Inequality (4.7) is proved similarly.

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