Optimal risk management: Different underlying exposures and various exercise price options

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Abstract

This paper analyzes an optimal risk management problem using put options when there are different underlying exposures. For simplicity, we assume that there are two different underlying assets and that each underlying asset is associated with several put options of different exercise prices. The objective of this study is to find an optimal solution to minimize the Value-at-Risk (VaR). Because it is difficult to solve the optimization problem that minimizes the exact VaR, we first solve an suboptimal problem that provides an upper bound of the VaR. It is proved that a suboptimal solution can be attained by choosing two put options, except for some extreme cases. Usually, one exercise price option for each underlying asset is chosen to solve the suboptimal problem. Sometimes, two options have to be chosen for one underlying asset (and none for the other underlying asset) to solve the suboptimal problem. In this case the exercise prices for the options have to be adjacent. In a numerical example, we compare the suboptimal solution with an (approximate) optimal solution that is obtained by taking minimum of the exact VaR varying hedge ratios of the put options about ten thousand times. The result shows that the suboptimal solution is a good approximation for the optimal solution. The suboptimal solution is also used for the sensitivity analysis of the hedging problem.

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1 Introduction

The importance to risk management by financial institution and corporations has grown exponentially in recent years. There are several reasons why they have incentives to manage their risk exposures. Smith and Stulz (1985) argued that risk management can lower earning volatility and then reduce tax. Froot, Sharfstein, and Stein (1993) linked risk management and the information asymmetry theory of Myers and Majluf (1984). If the information asymmetry between insiders and investors is large, then raising external capital is very costly. Because risk management can block the fall in value, risk management may reduce the possibility that corporations would have to raise external capital. Also, according to Morellec and Smith (2004), risk management can control the free cash flow problem.

Out of risks that the institution must manage market risks can be hedged with financial derivatives associated with market risks of special exposures such as exchange rates, interest rate, oil prices and so on. Using financial derivatives to manage the institutions' risk has grown exponentially in recent years. This is because various financial derivatives available to the institutions for hedging have been developed. In addition, the financial environment that institutions face changes continually, for example, New Basel Accord will make banks use more financial derivatives such as credit derivatives for them to obtain the credit risk migration.

In line with interest in risk management, Ahn, Boudoukh, Richardson, and Whitelaw (1999) examined optimal risk management using options. They provided optimal option strategy under the assumption that the financial institution uses put options to minimize their exposure's VaR.

This paper is related to Ahn *et al.*'s optimal risk management problem as follows: First, in contrast to single asset exposure of Ahn *et al.*, we consider that the financial institution has two asset exposures to the price risks of two underlying assets. Of course, if a financial institution can access basket options on multiple asset exposures, then a hedging problem of multiple asset exposures is essentially the same as a hedging problem using the single option. However, in practice it is not easy for institution to search for basket options corresponding to their exposures.

Second, our framework allows the institutions to access options with only finite exercise prices. In the framework of Ahn *et al.*, the institution chooses the optimal exercise price to minimize its VaR given a fixed cost. Therein, the institutions can always choose an put option with optimal strike price in their setting. Unlike Ahn *et al.*, in this paper we assume that the number of available exercise prices on put options is finite. In order to minimize their exposures' VaR, the institutions must select the optimal exercise prices among several exercise prices. Put differently, given a cost, the institution's aim is to select the combination of exercise prices on several put options to minimize the VaR.

When financial institutions manage the market risks of two asset exposures by minimizing its VaR, they have to answer the following questions:

- How many exercise prices on put options make a financial institution's risk management optimal?
- How much put options should be purchased ?

In the rest of this paper we first solve an suboptimal problem that provides an upper bound of the VaR because it is difficult to solve the optimization problem that minimizes the exact VaR. It is proved that a suboptimal solution can be attained by choosing two put options, except for some extreme cases. Usually, one exercise price option for each underlying asset is chosen to solve the suboptimal problem. Sometimes, two options have to be chosen for one underlying asset (and none for the other underlying asset) to solve the suboptimal problem. In this case the exercise prices for the options have to be adjacent. In a numerical example, we compare the suboptimal solution with an (approximate) optimal solution that is obtained by taking minimum of the exact VaR varying hedge ratios of the put options about ten thousand times. The result shows that the suboptimal solution is a good approximation for the optimal solution. The suboptimal solution is also used for the sensitivity analysis of the hedging problem.

2 The Model and a suboptimal solution

Suppose that a institution has exposures to the price risks of two assets whose prices at time t are denoted by $S^1(t)$ and $S^2(t)$, respectively. For $i = 1, 2$, the dynamics of the asset price $S^i(t)$ are governed by

$$dS^i(t) = \mu_i S^i(t)dt + \sigma_i S^i(t)dz_i(t) \tag{1}$$

where μ_i is the expected rate of return and σ_i is the volatility of each asset price, and $z_i(t)$ is a standard Brownian motion. We assume that $z_1(t)$ and $z_2(t)$ are dependent and let their correlation coefficient be ρ. The institution considers a risk management strategy using put options. However, in the market that there in no basket options with which the institution can manage the price risks of an asset pool consisting of two assets. Instead, the institution can access a put option associated with each asset to hedge the risk exposures. For $i = 1, 2$, we define $x^i_1, \cdots, x^i_{n_i}$ as exercise prices on put options whose have S^i as the underlying asset. That is, the number of put options associated with each underlying asset is $n_i,\ i = 1, 2$. Without loss of generality, we assume that $x_1^1 < x_2^1 < \cdots < x_{n_1}^1$ and $x_1^2 < x_2^2 < \cdots < x_{n_2}^2$.

Let $P_{x_k^i}^i(t) = P(S^i, x_k^i, r, \tau, \sigma_i)$ be at time t the market price of a put option where the exercise price is x_k^i, $\tau(= T - t)$ is the time to maturity, and $i = \{1, 2\}$. Then, at maturity date the payoff of a put option, $P_{x_k^i}^i(T) = \max(x_k^i - S^i(T), 0)$. As in Ahn *et al.*, we assume Our risk management framework works under the Black-Scholes economy and then

$$P_{x_k^i}^i(t) \quad = \quad x_k^i e^{-r\tau} N(d_1) - S^i(t) N(d_2) \tag{2}$$

where

$$d_1 \quad = \quad \frac{\ln(x_k^i / S^i(t)) - (r - {\sigma_i}^2/2)\tau}{{\sigma_i}^2\sqrt{\tau}}, \tag{3}$$

$$d_2 \quad = \quad \frac{\ln(x_k^i / S^i(t)) - (r + {\sigma_i}^2/2)\tau}{{\sigma_i}^2\sqrt{\tau}}, \tag{4}$$

and $N(\cdot)$ is the cumulative standard normal distribution.

While the institution can take the long positions, h_k^i in n_i put options with exercise prices x_k^i for each underlying asset, the risk management strategy is restricted to the maximum total costs C. Put differently, the total cost of the put option strategy cannot exceed a given amount: $\sum_{i=1}^2 \sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t) \quad \leq \quad C$. As in Ahn *et al.*, throughout this paper the institution's risk measure is the Value-at-Risk (VaR) defined as the loss at the α percent level of the distribution on the hedged position consisting of two underlying assets and the long positions h_k^i in each of the put options $P_{x_k^i}^i(t)$. Then, the institution's goal is to minimize its VaR of the hedged position. Let the VaR of the institution's hedged position denote $\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$.

The institution's minimization problem is given by

$$\text{Minimize} \qquad \mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \tag{5}$$

subject to

$$h_k^i \ \geq \ 0 \tag{6}$$

$$\sum_{i=1}^{2} \sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t) \ \leq \ C \tag{7}$$

To obtain mathematically tractable expression for $\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$, we introduce a mapping Ψ_α as follows: Let $\mathcal{C}$ be the collection of all random variables. Define a mapping $\Psi_\alpha : \ \mathcal{C} \to \mathbb{R}$ as

$$\Psi_\alpha(X) \quad = \quad -\sup\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \leq \alpha\}, \quad X \in \mathcal{C}.$$

Then, we obtain an expression for $\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$ as

$$\begin{aligned} &\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\ &= \ \Psi_\alpha\left(\sum_{i=1}^{2} S^i(T) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(T) - e^{r\tau}\sum_{i=1}^{2} S^i(t) - e^{r\tau}\sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t)\right). \end{aligned} \tag{8}$$

The following lemma states some properties of Ψ_α.

Lemma 1 *For X, $Y \in \mathcal{C}$,*

(i) $\Psi_\alpha(X + y) = \Psi_\alpha(X) - y$ *for* $y \in \mathbb{R}$

(ii) $\Psi_\alpha(cX) = c\Psi_\alpha(X)$ *for* $c \geq 0$

(iii) $\Psi_\alpha(X) \geq \Psi_\alpha(Y)$ *if* $X \leq Y$ *in distribution.*

Proof. (i) and (ii) are immediate from the definition of Ψ_α. Now we prove (iii). Suppose that $X \le Y$ in distribution. Then $\mathbb{P}(X \le x) \ge \mathbb{P}(Y \le x)$ for all $x \in \mathbb{R}$, which leads to

$$\sup\{x \in \mathbb{R} : \mathbb{P}(X \le x) \le \alpha\} \quad \le \quad \sup\{x \in \mathbb{R} : \mathbb{P}(Y \le x) \le \alpha\}.$$

Therefore,

$$\begin{aligned}
\Psi_\alpha(X) &= -\sup\{x \in \mathbb{R} : \mathbb{P}(X \le x) \le \alpha\} \\
&\ge -\sup\{x \in \mathbb{R} : \mathbb{P}(Y \le x) \le \alpha\} \\
&= \Psi_\alpha(Y),
\end{aligned}$$

which completes the proof. $\square$

By (i) of Lemma 1, (8) becomes

$$\begin{aligned}
&\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\
&= \Psi_\alpha\left(\sum_{i=1}^{2} S^i(T) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(T)\right) + e^{r\tau}\sum_{i=1}^{2} S^i(t) + e^{r\tau}\sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t). \qquad (9)
\end{aligned}$$

The analysis of the minimization problem for (9) subject to (6) and (7) is very complicated. The main difficulty comes from that $P_{x_k^i}^i(T)$ is a nonlinear function of $S^i(T)$:

$$P_{x_k^i}^i(T) \quad = \quad \max\{x_k^i - S^i(T), 0\}.$$

For the sake of mathematical convenience, we deal with a variant of $\mathrm{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$

$$\begin{aligned}
&\widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\
&\equiv \Psi_\alpha\left(\sum_{i=1}^{2} S^i(T) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i (x_k^i - S^i(T))\right) + e^{r\tau}\sum_{i=1}^{2} S^i(t) + e^{r\tau}\sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t), \qquad (10)
\end{aligned}$$

which is obtained by replacing $P_{x_k^i}^i(T)$ in (9) with $x_k^i - S^i(T)$. By (i) of Lemma 1, (10) becomes

$$\begin{aligned}
&\widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\
&= \Psi_\alpha\left(\sum_{i=1}^{2}\left(1 - \sum_{k=1}^{n_i} h_k^i\right) S^i(T)\right) + e^{r\tau}\sum_{i=1}^{2} S^i(t) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i \left(x_k^i - e^{r\tau} P_{x_k^i}^i(t)\right). \qquad (11)
\end{aligned}$$

We consider the minimization problem

$$\text{Minimize} \qquad \widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \tag{12}$$

subject to

$$h_k^i \ \geq \ 0 \tag{13}$$

$$\sum_{i=1}^{2} \sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t) \ \leq \ C \tag{14}$$

instead of the original minimization problem (5).

Remark.

1. If we consider a minimization problem concerning only one underlying asset, for an example, such as

$$\text{Minimize} \qquad \text{VaR of } S^1(T) + \sum_{k=1}^{n_1} h_k^1 P_{x_k^1}^1(T) \tag{15}$$

subject to

$$h_k^1 \ \geq \ 0$$

$$\sum_{k=1}^{n_1} h_k^1 P_{x_k^1}^1(t) \ \leq \ C,$$

then the problem is exactly the same as a variant

$$\text{Minimize} \qquad \text{VaR of } S^1(T) + \sum_{k=1}^{n_1} h_k^1 (x_k^1 - S^1(T))$$

subject to

$$h_k^1 \ \geq \ 0$$

$$\sum_{k=1}^{n_1} h_k^1 P_{x_k^1}^1(t) \ \leq \ C,$$

This optimization problem can be obtained by replacing $P_{x_k^1}^1(T)$ in (15) with $x_k^1 - S^1(T)$.

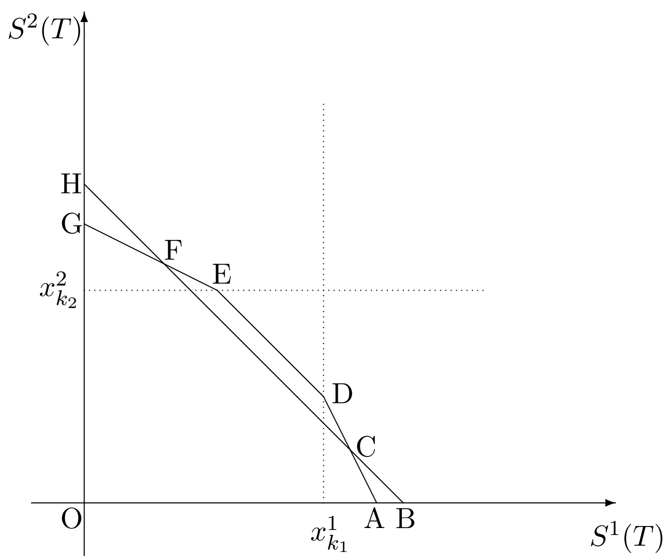

Figure 1: The regions R and $\tilde{R}$ defined as (16) and (17), when $h^1_{k_1} > 0$, $h^1_k = 0$ for $k \neq k_1$, $h^2_{k_2} > 0$ and $h^2_k = 0$ for $k \neq k_2$. The region R is the pentagon OADEG, and the region $\tilde{R}$ is the triangle OBH.

However, it can be shown that the problem (5) and its variant (12) have different solutions. Hence the problems (5) and (12) are not the same.

2. Because $P^i_{x^i_k}(T) \geq x^i_k - S^i(T)$, (iii) of Lemma 1 shows that

$$\mathrm{VaR}(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) \leq \widetilde{\mathrm{VaR}}(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}).$$

Therefore, a solution of the problem (12) provides a conservative solution for the original problem (5).

3. Define the regions R and $\tilde{R}$ by

$$
\begin{aligned}
&\left\{(S^1(T), S^2(T)) \in R\right\} \\
&= \left\{\text{Payoff}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \;\le\; -\,\text{VaR}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)\right\}, \quad (16) \\
&\left\{(S^1(T), S^2(T)) \in \tilde{R}\right\} \\
&= \left\{\widetilde{\text{Payoff}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \;\le\; -\,\widetilde{\text{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)\right\}, \quad (17)
\end{aligned}
$$

where

$$
\begin{aligned}
&\text{Payoff}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\
&= \sum_{i=1}^{2} S^i(T) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(T) - e^{r\tau}\sum_{i=1}^{2} S^i(t) - e^{r\tau}\sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t), \\
&\widetilde{\text{Payoff}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\
&= \sum_{i=1}^{2} S^i(T) + \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i (x_k^i - S^i(T)) - e^{r\tau}\sum_{i=1}^{2} S^i(t) - e^{r\tau}\sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t).
\end{aligned}
$$

Note that $\mathbb{P}\{(S^1(T), S^2(T)) \in R\} \;=\; \mathbb{P}\{(S^1(T), S^2(T)) \in \tilde{R}\} \;=\; \alpha$.

Figure 1 illustrates the regions R and $\tilde{R}$. For simplicity, we assumed that only one kind of put option is used for each underlying asset, i.e., for some k_1 and k_2,

$$h_{k_1}^1 > 0,\ h_k^1 = 0 \text{ for } k \ne k_1,\ h_{k_2}^2 > 0 \text{ and } h_k^2 = 0 \text{ for } k \ne k_2.$$

Because the probability that $(S^1(T), S^2(T))$ is in the triangles ABC or FGH, which equals the probability that $(S^1(T), S^2(T))$ is in the trapezoid CDEF, is expected to be very small, one can expect that a solution of the problem (12) is a good approximation for a solution of the original problem (5).

Now, we analyze the minimization problem (12) subject to (13) and (14). Since $\Psi_\alpha\left(\sum_{i=1}^{2}\left(1 - \sum_{k=1}^{n_i} h_k^i\right) S^i(T)\right)$ is a continuous function of h_k^i, $i = 1, 2$, $k = 1, \cdots, n_i$, the function $\widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$ which is given by (11) is also continuous in h_k^i, $i = 1, 2$, $k = 1, \cdots, n_i$. Let

$$\mathcal{H} = \left\{ (h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \in \mathbb{R}^{n_1+n_2} : h_k^i \geq 0, \sum_{i=1}^{2} \sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t) \leq C \right\}.$$

Since $\mathcal{H}$ is a compact set and $\widetilde{\mathrm{VaR}}$ is continuous, $\widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$ has a minimum on $\mathcal{H}$ and we have the following proposition:

Proposition 1 *The minimization problem (12) subject to (13) and (14) has a solution.*

Let $\mathbf{e}_k^i$, $i = 1, 2$, $k = 1, \cdots, n_i$, denote the $n_1 + n_2$-dimensional row vector with all zeros except the $(i-1)n_1 + k$th element that is one. Let $\mathcal{H}_1^*$ be a subset of $\mathcal{H}$ defined as

$$\mathcal{H}_1^* = \bigcup_{k_1=1}^{n_1} \bigcup_{k_2=1}^{n_2} \mathcal{H}_{k_1,k_2}, \tag{18}$$

where

$$\mathcal{H}_{k_1,k_2} = \left\{ h\mathbf{e}_{k_1}^1 + \frac{C - hP_{x_{k_1}^1}^1(t)}{P_{x_{k_2}^2}^2(t)} \mathbf{e}_{k_2}^2 \ : \ 0 \leq h \leq \frac{C}{P_{x_{k_1}^1}^1(t)} \right\}.$$

Define a subset $\mathcal{H}_2^*$ of $\mathcal{H}$ as

$$\mathcal{H}_2^* = \bigcup_{i=1}^{2} \bigcup_{k=1}^{n_i} \mathcal{H}_k^i, \tag{19}$$

where

$$\mathcal{H}_k^i = \begin{cases} \left\{ h\mathbf{e}_k^i + \frac{C - hP_{x_k^i}^i(t)}{P_{x_{k+1}^i}^i(t)} \mathbf{e}_{k+1}^i \ : \ 0 \leq h \leq \frac{C}{P_{x_k^i}^i(t)} \right\}, & \text{if } k < n_i, \\ \left\{ h\mathbf{e}_{n_i}^i \ : \ 0 \leq h \leq \frac{C}{P_{x_{n_i}^i}^i(t)} \right\}, & \text{if } k = n_i. \end{cases}$$

For $i = 1, 2$, let

$$K_i = \{k \in \{1, \cdots, n_i\} : P_{x_1^{\bar{i}}}^{\bar{i}}(t) < C - P_{x_k^i}^i(t) < P_{x_{n_{\bar{i}}}^{\bar{i}}}^{\bar{i}}(t)\},$$

where $\bar{i} = 3 - i$. For $k \in K_i$, define $k_{\bar{i}}^*(k) \in \{1, \cdots, n_{\bar{i}} - 1\}$ by

$$P^{\bar{i}}_{x^{\bar{i}}_{k^*_{\bar{i}}(k)}}(t) \le C - P^i_{x^i_k}(t) < P^{\bar{i}}_{x^{\bar{i}}_{k^*_{\bar{i}}(k)+1}}(t).$$

Define a subset $\mathcal{H}_3^*$ of $\mathcal{H}$ as

$$\mathcal{H}_3^* = \begin{cases} \bigcup_{i=1}^2 \{\mathbf{e}^i_k + u_k \mathbf{e}^{\bar{i}}_{k^*_{\bar{i}}(k)} + (1-u_k)\mathbf{e}^{\bar{i}}_{k^*_{\bar{i}}(k)+1} : k \in K_i\}, & \text{if } C < P^1_{x^1_{n_1}}(t) + P^2_{x^2_{n_2}}(t), \\ \{\mathbf{e}^1_{n_1} + \mathbf{e}^2_{n_2}\}, & \text{otherwise,} \end{cases} \tag{20}$$

where

$$u_k = \frac{P^i_{x^i_k}(t) + P^{\bar{i}}_{x^{\bar{i}}_{k^*_{\bar{i}}(k)+1}}(t) - C}{P^{\bar{i}}_{x^{\bar{i}}_{k^*_{\bar{i}}(k)+1}}(t) - P^{\bar{i}}_{x^{\bar{i}}_{k^*_{\bar{i}}(k)}}(t)}.$$

Theorem 1 *The minimization problem (12) subject to (13) and (14) has a solution in $\mathcal{H}^*$, i.e.,*

$$\begin{aligned} &\min\left\{ \widetilde{VaR}(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) : (h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) \in \mathcal{H} \right\} \\ = &\min\left\{ \widetilde{VaR}(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) : (h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) \in \mathcal{H}^* \right\}, \end{aligned}$$

where $\mathcal{H}^* = \mathcal{H}_1^* \cup \mathcal{H}_2^* \cup \mathcal{H}_3^*$ with (18), (19) and (20).

Note that $\mathcal{H}_1^*$ is the union of $n_1 \times n_2$ one parameter sets; $\mathcal{H}_2^*$ is the union of $n_1 + n_2$ one parameter sets; $\mathcal{H}_3^*$ is a set consisting of at most $n_1 + n_2$ points. Therefore, $\mathcal{H}^*$ is the finite union of one parameter sets adding a set of finite points. If it is guaranteed that $\widetilde{\mathrm{VaR}}$ is convex in each one parameter set, our minimization problem can be solved easily by a numerical method.

For $k_1 = 1, \cdots, n_1$ and $k_2 = 1, \cdots, n_2$, define

$$f_{k_1,k_2} : [0, \frac{C}{P^1_{x^1_{k_1}}(t)}] \to \mathbb{R}$$

as

$$f_{k_1,k_2}(h) \;=\; \widetilde{\mathrm{VaR}}\left(h\mathbf{e}^1_{k_1} + \frac{C - hP^1_{x^1_{k_1}}(t)}{P^2_{x^2_{k_2}}(t)}\mathbf{e}^2_{k_2}\right).$$

For $i = 1, 2$ and $k = 1, \cdots, n_i$, define

$$g^i_k \;:\; [0, \frac{C}{P^i_{x^i_k}(t)}] \to \mathbb{R}$$

as

$$g^i_k(h) \;=\; \begin{cases} \widetilde{\mathrm{VaR}}\left(h\mathbf{e}^i_k + \frac{C - hP^i_{x^i_k}(t)}{P^i_{x^i_{k+1}}(t)}\mathbf{e}^i_{k+1}\right) & \text{if } k < n_i, \\ \widetilde{\mathrm{VaR}}\left(h\mathbf{e}^i_k\right) & \text{if } k = n_i. \end{cases}$$

Theorem 2 below states the functions f_{k_1,k_2}, $k_1 = 1, \cdots, n_1$, $k_2 = 1, \cdots, n_2$, and g^i_k, $i = 1, 2$, $k = 1, \cdots, n_i$, are convex under a condition. To describe the condition for the convexity, we introduce some notation: Let $p : \mathbb{R}^2 \to [0, \infty)$ be the joint probability density function of $(S^1(T), S^2(T))$, i.e.,

$$p(x,y) = \begin{cases} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}xy}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[(\frac{\ln x - \mu_1}{\sigma_1})^2 - \frac{2\rho(\ln x - \mu_1)(\ln y - \mu_2)}{\sigma_1\sigma_2}(\frac{\ln y - \mu_2}{\sigma_2})^2\right]\right\} & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Partial derivatives of $p(x, y)$ are denoted by

$$\begin{aligned} p_1(x,y) &\equiv \frac{\partial}{\partial x}p(x,y) \\ &= -\frac{p(x,y)}{x}\left[1 + \frac{1}{2(1-\rho^2)}\left\{\frac{2(\ln x - \mu_1)}{\sigma_1^2} - \frac{2\rho(\ln y - \mu_2)}{\sigma_1\sigma_2}\right\}\right], \\ p_2(x,y) &\equiv \frac{\partial}{\partial y}p(x,y) \\ &= -\frac{p(x,y)}{y}\left[1 + \frac{1}{2(1-\rho^2)}\left\{\frac{2(\ln y - \mu_2)}{\sigma_2^2} - \frac{2\rho(\ln x - \mu_1)}{\sigma_1\sigma_2}\right\}\right]. \end{aligned}$$

Theorem 2 *The functions* f_{k_1,k_2}*,* $k_1 = 1, \cdots, n_1$*,* $k_2 = 1, \cdots, n_2$*, and* g_k^i*,* $i = 1, 2$*,* $k = 1, \cdots, n_i$*, are all convex if*

$$\int_{-\infty}^{\infty} u^2 [\cos\theta \; p_1(c_1(\theta) - u\sin\theta, c_2(\theta) + u\cos\theta) + \sin\theta \; p_2(c_1(\theta) - u\sin\theta, c_2(\theta) + u\cos\theta)] \, du \; > \; 0 \qquad (21)$$

for all $\theta \in [0, 2\pi)$*, where* $c_1(\theta)$ *and* $c_2(\theta)$ *are defined in Appendix B.*

Because it seems to be very difficult to check analytically whether the condition (21) holds or not, we investigate the condition (21) numerically.

3 Numerical Analysis

In this section we compare the exact VaR with our suboptimal VaR and illustrate the suboptimal VaR and optimal hedging ratios.

3.1 Exact VaR and Suboptimal VaR

To explore the difference between the exact VaR (that is, optimal solution) and our suboptimal VaR, we first calculate the optimal VaR. We calculate an (approximate) optimal solution that is obtained by taking minimum of the exact VaR varying hedge ratios of the put options about ten thousand times.

Because the calculation of the optimal VaR time consuming, we consider only two exercise prices for each underlying asset. In this subsection, baseline parameter values are as follows: the asset values at time t $S^1(t) = 100$ and $S^2(t) = 80$, the drifts of the asset values $\mu_1 = 0.1$ and $\mu_2 = 0.07$, the volatilities of the asset values $\sigma_1 = 0.4$ and $\sigma_2 = 0.2$, the correlation coefficient $\rho = 0.7$, the interest rate $r = 0.05$, the horizon $\tau = 0.5$, exercise prices for two assets $(x_1^1, x_2^1) = (95, 105)$ and $(x_1^2, x_2^2) = (75, 85)$, a hedging cost $C = 5\$$, and the VaR tail $\alpha = 5\%$.

Table 1 shows that shows that the suboptimal solution is a good approximation for the optimal solution. The greater a hedging cost is, the more accurate our suboptimal VaR is.

3.2 Optimal Hedging Ratios

This subsection shows the suboptimal VaR and optimal hedging ratios. Hereafter, the baseline parameter values are as follows: the asset values at time t $S^1(t) = 100$ and $S^2(t) = 80$, the drifts of the asset values $\mu_1 = 0.1$ and $\mu_2 = 0.07$, the volatilities of the asset values $\sigma_1 = 0.4$ and $\sigma_2 = 0.2$, the correlation coefficient $\rho = 0.7$, the interest rate $r = 0.05$, the horizon $\tau = 0.5$, exercise prices for two assets $(x_1^1, x_2^1, x_3^1) = (90, 100, 110)$ and $(x_1^2, x_2^2, x_3^2) = (70, 80, 90)$, a hedging cost $C = 5\$$, and the VaR tail $\alpha = 5\%$.

First, we analyze the effect of the various hedging costs on the suboptimal VaR and optimal hedging ratios. Table 2 shows that if an institution has enough money to hedge its exposure, then it prefers in-the-money options most. This is because using in-the-money options provides the greatest hedging benefit. For a cost of \$20, we observe that the $\widetilde{\mathrm{VaR}}$ is minimized when $h_3^1 = 1$, $h_3^2 = 0.1337$, $h_2^2 = 0.8663$ and the remaining hedge ratios are zero. If an institution's hedging cost is greater than \$20, then h_3^2 is larger than 0.1337. That is, if a hedging cost restriction were relaxed, then a institution would use more in-the-money options.

In contrast to the case that the hedging cost is \$20, for other hedging costs less than \$20 the institution cannot use the in-the-money options because the prices of the in-the-money options are much higher than those of the at-the-money or out-out-the-money options. For the lowest hedging cost, \$2, the $\widetilde{\mathrm{VaR}}$ is minimized when only one asset S^1 is hedged using a out-of-the-money option. This result may come from the fact that because S^1 is more volatile than S^2, S^1 has more influence on $\widetilde{\mathrm{VaR}}$ than S^1. Thus, when an institution is restricted to very small hedging costs, it is best policy to hold a put option on S^1.

A Proof of Theorem 1

For simplicity of notation, we introduce a function $\varphi_\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\varphi_\alpha(a,b) = \Psi_\alpha(aS^1(T) + bS^2(T)).$$

By (11), for $(h_1^1, \cdots, h_{n_1}^1; h_2^1, \cdots, h_{n_2}^2) \in \mathcal{H}$, we have

$$\begin{aligned} &\widetilde{\mathrm{VaR}}(h_1^1, \cdots, h_{n_1}^1; h_2^1, \cdots, h_{n_2}^2) \\ &= \varphi\left(1 - \sum_{k=1}^{n_1} h_k^1, 1 - \sum_{k=1}^{n_2} h_k^2\right) - \sum_{i=1}^{2}\sum_{k=1}^{n_i} h_k^i (x_k^i - e^{r\tau} P_{x_k^i}^i(t)) + e^{r\tau}\sum_{i=1}^{2} S^i(t). \end{aligned} \tag{22}$$

The following lemma is given here for later use.

Lemma 2 *If $i \in \{1,2\}$ and $0 < x < y < z$, then*

$$\frac{P_y^i(t) - P_x^i(t)}{y - x} < \frac{P_z^i(t) - P_y^i(t)}{z - y}.$$

Proof. It is well known that $P_x^i(t)$ is a strict convex function of x. From this, the result is immediate. □

Lemma 3 *If*

$$\begin{aligned} &(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \\ &\notin \bigcup_{k=1}^{n_1-1}\bigcup_{l=1}^{n_2-1} \left\{a\mathbf{e}_k^1 + b\mathbf{e}_{k+1}^1 + c\mathbf{e}_l^2 + d\mathbf{e}_{l+1}^2 \in \mathcal{H} \;:\; a \ge 0, b \ge 0, c \ge 0, d \ge 0\right\}, \end{aligned} \tag{23}$$

then $(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2)$ *is not a solution of the minimization problem (12) subject to (13) and (14).*

Proof. Suppose that (31) holds. There are $i \in \{1, 2\}$ and k_1, k_2, k_3 such that

$$1 \leq k_1 < k_2 < k_3 \leq n_i, \quad h_{k_1}^i > 0 \text{ and } h_{k_3}^i > 0.$$

We will deal with two cases separately.

Case 1:

$$\frac{h_{k_1}^i}{P_{x_{k_3}^i}^i(t) - P_{x_{k_2}^i}^i(t)} \quad \geq \quad \frac{h_{k_3}^i}{P_{x_{k_2}^i}^i(t) - P_{x_{k_1}^i}^i(t)}.$$

Let

$$\tilde{h}_k^j = \begin{cases} h_{k_1}^i - h_{k_3}^i \frac{P_{x_{k_3}^i}^i(t) - P_{x_{k_2}^i}^i(t)}{P_{x_{k_2}^i}^i(t) - P_{x_{k_1}^i}^i(t)} & \text{if } i = j, \ k = k_1 \\ h_{k_2}^i + h_{k_3}^i \frac{P_{x_{k_3}^i}^i(t) - P_{x_{k_1}^i}^i(t)}{P_{x_{k_2}^i}^i(t) - P_{x_{k_1}^i}^i(t)} & \text{if } i = j, \ k = k_2 \\ 0 & \text{if } i = j, \ k = k_3 \\ h_k^j & \text{otherwise.} \end{cases}$$

Then $\tilde{h}_k^j \geq 0$, $j = 1, 2$, $k = 1, \cdots, n_j$, and

$$\sum_{j=1}^{2} \sum_{k=1}^{n_j} \tilde{h}_k^j P_{x_k^j}^j(t) \ = \ \sum_{j=1}^{2} \sum_{k=1}^{n_j} h_k^j P_{x_k^j}^j(t) \ \leq \ C.$$

Further,

$$\sum_{k=1}^{n_j} \tilde{h}_k^j \ = \ \sum_{k=1}^{n_j} h_k^j, \quad j = 1, 2.$$

Therefore (22) leads to

$$
\begin{aligned}
&\widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_1^2,\cdots,h_{n_2}^2)-\widetilde{\mathrm{VaR}}(\tilde h_1^1,\cdots,\tilde h_{n_1}^1;\tilde h_1^2,\cdots,\tilde h_{n_2}^2)\\
=&\ (h_{k_1}^i-\tilde h_{k_1}^i)(x_{k_1}^i-e^{r\tau}P^i_{x^i_{k_1}}(t))+(h_{k_2}^i-\tilde h_{k_2}^i)(x_{k_2}^i-e^{r\tau}P^i_{x^i_{k_2}}(t))+h_{k_3}^i(x_{k_3}^i-e^{r\tau}P^i_{x^i_{k_3}}(t))\\
=&\ \frac{h_{k_3}^i}{P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t)}\Big[(P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_2}}(t))(x_{k_1}^i-e^{r\tau}P^i_{x^i_{k_1}}(t))\\
&-(P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_1}}(t))(x_{k_2}^i-e^{r\tau}P^i_{x^i_{k_2}}(t))+(P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t))(x_{k_3}^i-e^{r\tau}P^i_{x^i_{k_3}}(t))\Big]\\
=&\ \frac{h_{k_3}^i}{P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t)}\Big[(P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_2}}(t))(x_{k_1}^i-x_{k_3}^i)-(P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_1}}(t))(x_{k_2}^i-x_{k_3}^i)\Big],
\end{aligned}
$$

which is positive by Lemma 2. Therefore $(h_1^1,\cdots,h_{n_1}^1;h_1^2,\cdots,h_{n_2}^2)$ cannot be a solution of the minimization problem (12) subject to (13) and (14).

Case 2:

$$\frac{h_{k_1}^i}{P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_2}}(t)} \;<\; \frac{h_{k_3}^i}{P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t)}.$$

Let

$$
\tilde h_k^j=\begin{cases}
0 & \text{if } i=j,\ k=k_1\\
h_{k_2}^i+h_{k_1}^i\frac{P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t)}{P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_2}}(t)} & \text{if } i=j,\ k=k_2\\
h_{k_3}^i-h_{k_1}^i\frac{P^i_{x^i_{k_2}}(t)-P^i_{x^i_{k_1}}(t)}{P^i_{x^i_{k_3}}(t)-P^i_{x^i_{k_2}}(t)} & \text{if } i=j,\ k=k_3\\
h_k^j & \text{otherwise.}
\end{cases}
$$

By a similar way as in case 1, it can be shown that $\tilde h_k^j\ge 0,\ j=1,2,\ k=1,\cdots,n_j$, and $\sum_{j=1}^2\sum_{k=1}^{n_j}\tilde h_k^j P^j_{x^j_k}(t)\le C$, and that

$$\widetilde{\mathrm{VaR}}(\tilde h_1^1,\cdots,\tilde h_{n_1}^1;\tilde h_2^1,\cdots,\tilde h_{n_2}^2) \;<\; \widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2).$$

Therefore $(h_1^1,\cdots,h_{n_1}^1;h_1^2,\cdots,h_{n_2}^2)$ cannot be a solution of the minimization problem (12) subject to (13) and (14). $\square$

Recall that

$$\mathcal{H} = \{(h_1^1, \cdots, h_{n_1}^1; h_1^2, \cdots, h_{n_2}^2) \in \mathbb{R}^{n_1+n_2} : h_k^i \geq 0, \sum_{i=1}^{2} \sum_{k=1}^{n_i} h_k^i P_{x_k^i}^i(t) \leq C\}.$$

For any vector $\mathbf{h}$, let $p(\mathbf{h})$ denote the number of positive components in $\mathbf{h}$.

Lemma 4 *Let $i \in \{1, 2\}$ and $\bar{i} = 3 - i$. Suppose that $a\mathbf{e}_{k_1}^i + b\mathbf{e}_{k_2}^i + c\mathbf{e}_{k_3}^{\bar{i}} + d\mathbf{e}_{k_4}^{\bar{i}}$ minimizes (12) subject to (13) and (14), where $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $k_1 \neq k_2$, $k_3 \neq k_4$. If $acd > 0$ and $(a + b - 1)^2 + (c + d - 1)^2 > 0$, then there are a^*, c^* and d^* such that $a^*c^*d^* = 0$ and $a^*\mathbf{e}_{k_1}^i + b\mathbf{e}_{k_2}^i + c^*\mathbf{e}_{k_3}^{\bar{i}} + d^*\mathbf{e}_{k_4}^{\bar{i}}$ minimizes (12) subject to (13) and (14).*

Proof. Without loss of generality, we may assume that $i = 1$. We deal with two cases separately: '$a + b \neq 1$' and '$a + b = 1'$.

Case 1. $a + b \neq 1$.

Define a function $V : D_V \to \mathbb{R}$ as

$$V(s) = \widetilde{\mathrm{VaR}}(s\mathbf{e}_{k_1}^1 + b\mathbf{e}_{k_2}^1 + u\mathbf{e}_{k_3}^2 + v\mathbf{e}_{k_4}^2), \quad s \in D_V$$

where u and v are determined by

$$sP_{x_{k_1}^1}^1(t) + uP_{x_{k_3}^2}^2(t) + vP_{x_{k_4}^2}^2(t) = aP_{x_{k_1}^1}^1(t) + cP_{x_{k_2}^2}^2(t) + dP_{x_{k_4}^2}^2(t) \tag{24}$$

$$(1 - a - b)(1 - u - v) = (1 - s - b)(1 - c - d) \tag{25}$$

and

$$D_V = \{s \in \mathbb{R} : s \geq 0, u \geq 0,\ v \geq 0\}.$$

By (22),

$$\begin{aligned}
V(s) &= \varphi(1-s-b, 1-u-v) - s(x^1_{k_1} - e^{r\tau}P^1_{x^1_{k_1}}(t)) \\
&- b(x^1_{k_2} - e^{r\tau}P^1_{x^1_{k_2}}(t)) - u(x^2_{k_3} - e^{r\tau}P^2_{x^2_{k_3}}(t)) \\
&- v(x^2_{k_4} - e^{r\tau}P^2_{x^2_{k_4}}(t)) + e^{r\tau}\sum_{j=1}^{2} S^j(t) \qquad (26)
\end{aligned}$$

Since (25) implies

$$(1-s-b, 1-u-v) = \frac{1-s-b}{1-a-b}(1-a-b, 1-c-d),$$

we have

$$\varphi(1-s-b, 1-u-v) = \frac{1-s-b}{1-a-b}\varphi(1-a-b, 1-c-d) \quad \text{if } \frac{1-s-b}{1-a-b} \geq 0. \qquad (27)$$

Substituting (27) into (26) leads to

$$\begin{aligned}
V(s) &= \frac{1-s-b}{1-a-b}\varphi(1-a-b, 1-c-d) - s(x^1_{k_1} - e^{r\tau}P^1_{x^1_{k_1}}(t)) \\
&- b(x^1_{k_2} - e^{r\tau}P^1_{x^1_{k_2}}(t)) - u(x^2_{k_3} - e^{r\tau}P^2_{x^2_{k_3}}(t)) \\
&- v(x^2_{k_4} - e^{r\tau}P^2_{x^2_{k_4}}(t)) + e^{r\tau}\sum_{j=1}^{2} S^j(t) \quad \text{if } \frac{1-s-b}{1-a-b} \geq 0. \qquad (28)
\end{aligned}$$

Note that u and v, determined by (24) and (25), are linear functions of s on $D_V \cap \{s : \frac{1-s-b}{1-a-b} \geq 0\}$. Thus, $V(s)$ is also a linear function of s on $D_V \cap \{s : \frac{1-s-b}{1-a-b} \geq 0\}$. Since a is an interior point of $D_V \cap \{s : \frac{1-s-b}{1-a-b} \geq 0\}$ and $V(s)$ has a minimum at $s = a$, V is constant on $D_V \cap \{s : \frac{1-s-b}{1-a-b} \geq 0\}$.

Let

$$a^* = \begin{cases} \inf D_V & \text{if } \; a+b<1 \\ \sup D_V & \text{if } \; a+b>1. \end{cases}$$

Let c^* and d^* be the solution of (24) and (25) with $s = a^*$. Then $a^*c^*d^* = 0$ and $a^*\mathbf{e}^1_{k_1} + b\mathbf{e}^1_{k_2} + c^*\mathbf{e}^2_{k_3} + d^*\mathbf{e}^2_{k_4}$, which equals $a\mathbf{e}^1_{k_1} + b\mathbf{e}^1_{k_2} + c\mathbf{e}^2_{k_3} + d\mathbf{e}^2_{k_4}$, minimizes (12) subject to (13) and (14).

Case 2. $a + b = 1$.

Define a function $V_1 : D_{V_1} \to \mathbb{R}$ as

$$V_1(u) = \widetilde{\mathrm{VaR}}(a\mathbf{e}^1_{k_1} + b\mathbf{e}^1_{k_2} + u\mathbf{e}^2_{k_3} + v\mathbf{e}^2_{k_4}), \quad u \in D_{V_1}$$

where u is determined by

$$uP^2_{x_{k_3}}(t) + vP^2_{x_{k_4}}(t) = cP^2_{x_{k_3}}(t) + dP^2_{x_{k_4}}(t) \tag{29}$$

and

$$D_{V_1} = \{u \in \mathbb{R} : u \geq 0, v \geq 0\}. \tag{30}$$

Since $a + b = 1$, we have $c + d \neq 1$. By (22),

$$\begin{aligned} V_1(u) &= \varphi(0, 1-u-v) - a(x^1_{k_1} - e^{r\tau}P^1_{x^1_{k_1}}(t)) - b(x^1_{x_2} - e^{r\tau}P^2_{x_{k_2}}(t)) \\ &\quad -u(x^2_{k_3} - e^{r\tau}P^2_{x^2_{k_3}}(t)) - v(x^2_{k_4} - e^{r\tau}P^2_{x^2_{k_4}}(t)) + e^{r\tau}\sum_{j=1}^{2} S^j(t) \\ &= \frac{1-u-v}{1-c-d}\varphi(0, 1-c-d) - a(x^1_{k_1} - e^{r\tau}P^1_{x^1_{k_1}}(t)) - b(x^1_{x_2} - e^{r\tau}P^2_{x_{k_2}}(t)) \\ &\quad -u(x^2_{k_2} - e^{r\tau}P^2_{x^2_{k_2}}(t)) - v(x^2_{k_2+1} - e^{r\tau}P^2_{x^2_{k_2+1}}(t)) + e^{r\tau}\sum_{j=1}^{2} S^j(t) \quad \text{if } \frac{1-u-v}{1-c-d} \geq 0. \end{aligned}$$

Note that v, which is determined by (29), is a linear function of u. Thus, $V_1(u)$ is also a linear function of u on $D_{V_1} \cap \{u : \frac{1-u-v}{1-c-d} \geq 0\}$. Since c is an interior point of $D_{V_1} \cap \{u : \frac{1-u-v}{1-c-d} \geq 0\}$ and $V_1(u)$ has a minimum at $u = c$, V_1 is constant on $D_{V_1} \cap \{u : \frac{1-u-v}{1-c-d} \geq 0\}$.

Let $a^* = a$ and

$$c^* = \begin{cases} \inf D_{V_1} & \text{if } \ c + d < 1 \\ \sup D_{V_1} & \text{if } \ c + d > 1. \end{cases}$$

Let d^* be the solution of (29) with $u = c^*$. Then $a^*c^*d^* = 0$ and $a^*\mathbf{e}^1_{k_1} + b\mathbf{e}^1_{k_2} + c^*\mathbf{e}^2_{k_3} + d^*\mathbf{e}^2_{k_4}$, which equals $a\mathbf{e}^1_{k_1} + b\mathbf{e}^1_{k_2} + c\mathbf{e}^2_{k_3} + d\mathbf{e}^2_{k_4}$, minimizes (12) subject to (13) and (14). $\square$

Lemma 5 *If $h^i_k > 0$ for some $i \in \{1, 2\}$ and $k \in \{1, \cdots, n_i - 1\}$ and if*

$$\sum_{j=1}^{2}\sum_{l=1}^{n_j} h^j_{k_l} P^j_{x^j_{k_l}}(t) \quad < \quad C,$$

then $(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2})$ is not a solution of the minimization problem (12) subject to (13) and (14).

Proof. Suppose that $h^i_k > 0$ for some $i \in \{1, 2\}$ and $k \in \{1, \cdots, n_i - 1\}$. Choose $\epsilon \in (0, h^i_k]$ such that

$$\sum_{j=1}^{2}\sum_{l=1}^{n_j} h^j_{k_l} P^j_{x^j_{k_l}}(t) + \epsilon\left(P^i_{x^i_{n_i}}(t) - P^i_{x^i_k}(t)\right) \quad \leq \quad C.$$

Then

$$(h^1_1, \cdots, h^1_{n_1}; h^2_1, \cdots, h^2_{n_2}) + \epsilon(\mathbf{e}^i_{n_2} - \mathbf{e}^i_k) \quad \in \quad \mathcal{H}.$$

By (22),

$$\begin{aligned}
&\widetilde{\mathrm{VaR}}((h^1_1, \cdots, h^1_{n_1}; h^1_2, \cdots, h^2_{n_2}) + \epsilon(\mathbf{e}^i_{n_2} - \mathbf{e}^i_k)) \\
&= \quad \varphi\left(1 - \sum_{k=1}^{n_1} h^1_k, 1 - \sum_{k=1}^{n_2} h^2_k\right) - \sum_{i=1}^{2}\sum_{k=1}^{n_i} h^i_k(x^i_k - e^{r\tau}P^i_{x^i_k}(t)) + e^{r\tau}\sum_{i=1}^{2} S^i(t) \\
&\quad -\epsilon\left(\left(x^i_k - e^{r\tau}P^i_{x^i_k}(t)\right) - \left(x^i_{n_i} - e^{r\tau}P^i_{x^i_{n_i}}(t)\right)\right) \\
&= \quad \widetilde{\mathrm{VaR}}(h^1_1, \cdots, h^1_{n_1}; h^1_2, \cdots, h^2_{n_2}) - \epsilon\left(\left(x^i_k - e^{r\tau}P^i_{x^i_k}(t)\right) - \left(x^i_{n_i} - e^{r\tau}P^i_{x^i_{n_i}}(t)\right)\right) \\
&< \quad \widetilde{\mathrm{VaR}}(h^1_1, \cdots, h^1_{n_1}; h^1_2, \cdots, h^2_{n_2})).
\end{aligned}$$

Therefore $(h^1_1, \cdots, h^1_{n_1}; h^1_2, \cdots, h^2_{n_2})$ is not a solution of the minimization problem (12) subject to (13) and (14). $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By Lemmas 3 and 4,

$$\begin{aligned}
&\min\left\{\widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\ :\ (h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\in\mathcal{H}\right\}\\
&\quad=\ \min\left\{\widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\ :\ (h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\in\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3\right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_1 &= \bigcup_{k_1=1}^{n_1}\bigcup_{k_2=1}^{n_2}\left\{a\mathbf{e}_{k_1}^1+b\mathbf{e}_{k_2}^2\ :\ a\ge 0,b\ge 0,aP^1_{x^1_{k_1}}(t)+bP^2_{x^2_{k_2}}(t)\le C\right\},\\
\mathcal{H}_2 &= \bigcup_{i=1}^{2}\bigcup_{k=1}^{n_i-1}\left\{a\mathbf{e}_k^i+b\mathbf{e}_{k+1}^i\ :\ a\ge 0,b\ge 0,aP^i_{x^i_k}(t)+bP^i_{x^i_{k+1}}(t)\le C\right\},\\
\mathcal{H}_3 &= \bigcup_{k_1=1}^{n_1-1}\bigcup_{k_2=1}^{n_2-1}\Big\{a\mathbf{e}_{k_1}^1+b\mathbf{e}_{k_1+1}^1+c\mathbf{e}_{k_2}^2+d\mathbf{e}_{k_2+1}^2\ :\ a\ge 0,b\ge 0,c\ge 0,d\ge 0,a+b=1,\\
&\qquad c+d=1,\ aP^1_{x^1_{k_1}}(t)+bP^1_{x^1_{k_1+1}}(t)+cP^2_{x^2_{k_2}}(t)+dP^2_{x^2_{k_2+1}}(t)\le C\Big\}.
\end{aligned}$$

By Lemma 5, we have

$$\begin{aligned}
&\min\left\{\widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2):(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\in\mathcal{H}\right\}\\
&\quad=\ \min\left\{\widetilde{\mathrm{VaR}}(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2):(h_1^1,\cdots,h_{n_1}^1;h_2^1,\cdots,h_{n_2}^2)\in\mathcal{H}^*\cup\mathcal{H}_{n_1,n_2}\cup\mathcal{H}_3'\right\},
\end{aligned}\tag{31}$$

where

$$\begin{aligned}
\mathcal{H}_{n_1,n_2} &= \left\{a\mathbf{e}_{n_1}^1+b\mathbf{e}_{n_2}^2\ :\ a\ge 0,b\ge 0,aP^1_{x^1_{n_1}}(t)+bP^2_{x^2_{n_2}}(t)\le C\right\},\\
\mathcal{H}_3' &= \bigcup_{k_1=1}^{n_1-1}\bigcup_{k_2=1}^{n_2-1}\Big\{a\mathbf{e}_{k_1}^1+b\mathbf{e}_{k_1+1}^1+c\mathbf{e}_{k_2}^2+d\mathbf{e}_{k_2+1}^2: a\ge 0,b\ge 0,c\ge 0,d\ge 0,a+b=1,\\
&\qquad c+d=1,\ aP^1_{x^1_{k_1}}(t)+bP^1_{x^1_{k_1+1}}(t)+cP^2_{x^2_{k_2}}(t)+dP^2_{x^2_{k_2+1}}(t)=C\Big\}.
\end{aligned}$$

By (22), if $a+b=c+d=1$ and $a\mathbf{e}^1_{k_1}+b\mathbf{e}^1_{k_1+1}+c\mathbf{e}^2_{k_2}+d\mathbf{e}^2_{k_2+1}\in\mathcal{H}$, then

$$\begin{aligned}
&\widetilde{\mathrm{VaR}}(a\mathbf{e}^1_{k_1}+b\mathbf{e}^1_{k_1+1}+c\mathbf{e}^2_{k_2}+d\mathbf{e}^2_{k_2+1})\\
&= -a(x^1_{k_1}-e^{r\tau}P^1_{x^1_{k_1}}(t))-b(x^1_{k_1+1}-e^{r\tau}P^1_{x^1_{k_1+1}}(t)))\\
&\quad -c(x^2_{k_2}-e^{r\tau}P^2_{x^2_{k_2}}(t)))-d(x^2_{k_2}-e^{r\tau}P^2_{x^2_{k_2+1}}(t)))-e^{r\tau}\sum_{i=1}^{2}S^i(t),
\end{aligned}$$

which is a linear function of (a,b,c,d). Hence, by the theory of the linear programming, we have

$$\begin{aligned}
&\min\left\{\widetilde{\mathrm{VaR}}(h^1_1,\cdots,h^1_{n_1};h^1_2,\cdots,h^2_{n_2}):(h^1_1,\cdots,h^1_{n_1};h^1_2,\cdots,h^2_{n_2})\in\mathcal{H}'_3\right\}\\
&= \min\left\{\widetilde{\mathrm{VaR}}(h^1_1,\cdots,h^1_{n_1};h^1_2,\cdots,h^2_{n_2}):(h^1_1,\cdots,h^1_{n_1};h^1_2,\cdots,h^2_{n_2})\in\mathcal{H}^*_3\right\}\ \text{ if } \mathcal{H}'_3\neq\phi. \quad (32)
\end{aligned}$$

Now suppose that $a\mathbf{e}^1_{n_1}+b\mathbf{e}^2_{n_2}$ with $a>0$, $b>0$, $(a,b)\neq(1,1)$ and $aP^1_{x^1_{n_1}}(t)+bP^2_{x^2_{n_2}}(t)<C$ minimizes (12) subject to (13) and (14). Without loss of generality, we may assume that $a\neq 1$. Define a function $V_2:D_{V_2}\to\mathbb{R}$ as

$$V_2(s) = \widetilde{\mathrm{VaR}}\left(s\mathbf{e}^1_{n_1}+\left(1-\frac{(1-b)(1-s)}{1-a}\right)\mathbf{e}^2_{n_2}\right),\quad s\in D_{V_2},$$

where

$$D_{V_2} = \left\{s: s\geq 0, 1-\frac{(1-b)(1-s)}{1-a}\geq 0, sP^1_{x^1 n_1}(t)+\left(1-\frac{(1-b)(1-s)}{1-a}\right)P^2_{x^2 n_2}(t)\leq C\right\}.$$

By (22), we have

$$\begin{aligned}
V_2(s) &= \frac{1-s}{1-a}\varphi(1-a,a-b)-a\left(x^1_{n_1}-e^{r\tau}P^1_{x^1_{n_1}}(t)\right)\\
&\quad -b\left(x^2_{n_2}-e^{r\tau}P^2_{x^2_{n_2}}(t)\right)+e^{r\tau}\sum_{i=1}^{2}S^i(t)\qquad \text{if } \frac{1-s}{1-a}\geq 0,
\end{aligned}$$

which is a linear function of s on $D_{V_2}\cap\{s:\frac{1-s}{1-a}\geq 0\}$. Since a is an interior point of $D_{V_2}\cap\{s:\frac{1-s}{1-a}\geq 0\}$ and $V_2(s)$ has a minimum at $s=a$, V_2 is constant on $D_{V_2}\cap\{s:\frac{1-s}{1-a}\geq 0\}$.

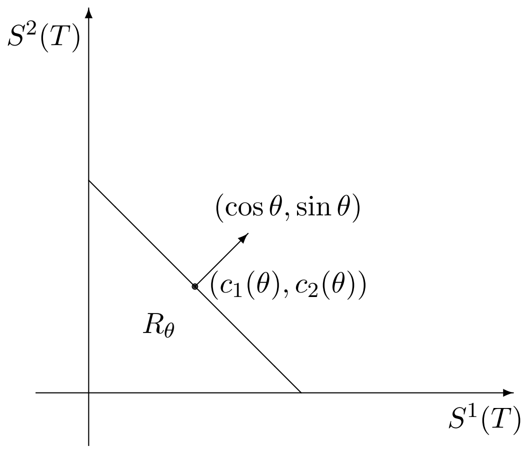

Figure 2: The region $R_\theta = \{(\cos\theta)S^1(T) + (\sin\theta)S^2(T) \leq k(\theta)\}$, and the point $(c_1(\theta), c_2(\theta)) = \mathbb{E}[(S^1(T), S^2(T))|(\cos\theta)S^1(T) + (\sin\theta)S^2(T) = k(\theta)]$

Let

$$a^* = \begin{cases} \inf D_{V_2} & \text{if } a < 1, \\ \sup D_{V_2} & \text{if } a > 1. \end{cases}$$

and $b^* = 1 - \frac{(1-b)(1-a^*)}{1-a}$. Then $a^* = 0$ or $b^* = 0$ or $a^* P^1_{x^1_{n_1}}(t) + b^* P^2_{x^2_{n_2}}(t) = C$. Thus $a^*\mathbf{e}^1_{n_1} + b^*\mathbf{e}^2_{n_2} \in \mathcal{H}^*$ and $a^*\mathbf{e}^1_{n_1} + b^*\mathbf{e}^2_{n_2}$ minimizes (12) subject to (13) and (14). This together with (31) and (32) completes the proof. $\square$

B Definition of $c_1(\cdot)$ and $c_2(\cdot)$

For $\theta \in [0, 2\pi)$, define $k(\theta)$ by

$$\mathbb{P}\{(\cos\theta)S^1(T) + (\sin\theta)S^2(T) \leq k(\theta)\} = \alpha.$$

(See Figure 2.) Let $(c_1(\theta), c_2(\theta))$, $\theta \in [0, 2\pi)$, be the conditional expectation of $(S^1(T), S^2(T))$ given $(\cos\theta)S^1(T) + (\sin\theta)S^2(T) = k(\theta)$, i.e.,

$$(c_1(\theta), c_2(\theta)) = \mathbb{E}[(S^1(T), S^2(T))|(\cos\theta)S^1(T) + (\sin\theta)S^2(T) = k(\theta)].$$

For $\theta \in [0, 2\pi)$, $(c_1(\theta), c_2(\theta))$ can be calculated by

$$
\begin{aligned}
c_1(\theta) &= \frac{\int_{-\infty}^{\infty} (k(\theta)\cos\theta - u\sin\theta)\ p(k(\theta)\cos\theta - u\sin\theta, k(\theta)\sin\theta + u\cos\theta)\ du}{\int_{-\infty}^{\infty} p(k(\theta)\cos\theta - u\sin\theta, k(\theta)\sin\theta + u\cos\theta)\ du}, \\
c_2(\theta) &= \frac{\int_{-\infty}^{\infty} (k(\theta)\sin\theta + u\cos\theta)\ p(k(\theta)\cos\theta - u\sin\theta, k(\theta)\sin\theta + u\cos\theta)\ du}{\int_{-\infty}^{\infty} p(k(\theta)\cos\theta - u\sin\theta, k(\theta)\sin\theta + u\cos\theta)\ du}.
\end{aligned}
$$

Notice that

$$
(\cos\theta)c_1(\theta) + (\sin\theta)c_2(\theta) = k(\theta).
$$

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Table 1: Exact VaR and Suboptimal VaR

C	Method	VaR	h_1^{1*}	h_2^{1*}	h_1^{2*}	h_2^{2*}
2	Exact	46.2305	0.1572	0	0.4712	0
	Suboptimal	46.5050	0.1172	0	0.6462	0
5	Exact	37.0898	0.5258	0	0.5969	0
	Suboptimal	37.2507	0.5388	0	0.5402	0
7	Exact	30.9375	0.7352	0	0.8400	0
	Suboptimal	31.3006	0.7891	0	0.6040	0
10	Exact	23.1855	1.0000	0	0.8376	0.1624
	Suboptimal	23.1855	1.0000	0	0.8376	0.1624
20	Exact	13.8452	0	1.0000	0	1.0000
	Suboptimal	13.8452	0	1.0000	0	1.0000

The exact VaR is obtained by taking minimum of the exact VaR varying hedge ratios of the put options about ten thousand times. Exact and Suboptimal denote the exact VaR and the suboptimal VaR, respectively. The parameter values are as follows: the asset values at time t, $S^1(t) = 100$ and $S^2(t) = 80$; the drifts of the asset values, $\mu_1 = 0.1$ and $\mu_2 = 0.07$; the volatilities of the asset values, $\sigma_1 = 0.4$ and $\sigma_2 = 0.2$; the correlation coefficient, $\rho = 0.7$; the interest rate, $r = 0.05$; the horizon, $\tau = 0.5$; exercise prices, $(x_1^1, x_2^1) = (95, 105)$ and $(x_1^2, x_2^2) = (75, 85)$; and the VaR tail, $\alpha = 5\%$.

Table 2: Suboptimal VaR and optimal hedge ratios

C	$\widetilde{\text{VaR}}$	h_1^{1*}	h_2^{1*}	h_3^{1*}	h_1^{2*}	h_2^{2*}	h_3^{2*}
2.0	45.6638	0.3610	0	0	0	0	0
5.0	35.2339	0.8050	0	0	0.7860	0	0
7.0	29.0226	1.0000	0	0	0.7289	0.2711	0
10.0	22.6992	0.7889	0.2111	0	0	1.0000	0
20.0	13.7260	0	0	1.0000	0	0.8663	0.1337

The parameter values are as follows: the asset values at time t, $S^1(t) = 100$ and $S^2(t) = 80$; the drifts of the asset values, $\mu_1 = 0.1$ and $\mu_2 = 0.07$; the volatilities of the asset values, $\sigma_1 = 0.4$ and $\sigma_2 = 0.2$; the correlation coefficient, $\rho = 0.7$; the interest rate, $r = 0.05$; the horizon, $\tau = 0.5$; exercise prices, $(x_1^1, x_2^1, x_3^1) = (90, 100, 110)$ and $(x_1^2, x_2^2, x_3^2) = (70, 80, 90)$; and the VaR tail, $\alpha = 5\%$.