Optimal Portfolio, Consumption-Leisure and Retirement Choice Problem with CES Utility*

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Abstract

We study optimal portfolio, consumption-leisure and retirement choice of an infinitely-lived economic agent whose instantaneous preference is characterized by a constant elasticity of substitution(CES) function of consumption and leisure. We integrate in one model the optimal consumption-leisure-work choice, optimal portfolio selection, and the optimal stopping problem in which the agent chooses her retirement time. The economic agent derives utility from both consumption and leisure, and is able to adjust her supply of labor flexibly above a certain minimum work-hour, and also has a retirement option. We solve the problem analytically by considering a variational inequality arising from the dual functions of the optimal stopping problem. The optimal retirement time is characterized as the first time when her wealth exceeds a certain critical level. We provide the critical wealth level for retirement and characterize the optimal consumption-leisure and portfolio policies before and after retirement in closed forms. We also derive properties of the optimal policies. In particular, we show that consumption in general jumps around retirement.

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1

1 Introduction

We study optimal portfolio, consumption-leisure and retirement choice of an infinitely-lived economic agent whose instantaneous preference is characterized by a constant elasticity of substitution(CES) function of consumption and leisure. The problem is formulated mathematically as a mixture of portfolio and consumption-leisure choice and an optimal stopping time problem. In the consumption-leisure and portfolio choice problem the agent chooses optimally consumption, leisure and portfolio (c, l, π) and in the optimal stopping problem she chooses the time of retirement τ.

The economic agent in this paper derives utility from both consumption and leisure, and is able to adjust her supply of labor flexibly above a certain minimum work-hour, and also has a retirement option. Namely, the agent chooses her hours of work every day and eventually a retirement time considering the trade-off between the utility effect of leisure and the wealth effect of labor. Before retirement the agent receives income at a rate proportional to her hours of work. We assume that she has to work minimum hours in order to keep her employment, but is free to choose her hours of work above this minimum level. Retirement is an irreversible decision that allows the agent to enjoy leisure full-time at the cost of foregone wage income.

We solve the problem analytically by considering a variational inequality arising from the dual functions of the optimal stopping problem. A general property of the solution is that the optimal retirement time is the first time when her wealth exceeds a certain critical level. We provide the critical wealth level for retirement and characterize the optimal consumption-leisure and portfolio policies before and after retirement in closed forms.

We can derive a few results concerning the behavior of the optimal policies. We first compare the optimal strategies in our model with those in a different model where the agent does not have a retirement option and is always forced to work at least for minimum hours (for instance, under slavery) and show that the agent consumes less and invests more

2

in the risky asset in our model where she has a retirement option than she does when there is no such option (see Figure 1 and 2). Intuitively, with a retirement option at hand the agent tries to accumulate her wealth fast enough to reach the critical level. After attaining the wealth level, she is able to enjoy leisure full-time; this incentive lets her save more and take more risk. Consistent with this intuitive explanation, it can be shown that typically as the agent's wealth approaches the critical level she tends to invest in the risky asset more aggressively. We also show that generally there is a consumption jump at the critical wealth level; the hours of leisure jumps around retirement and consumption jumps as well. Some numerical examples are provided. And we also consider the liquidity constraints. To model the liquidity constraints for the future income arises another free boundary in a variational inequality problem, which eventually requires massive algebraic calculation, especially in the case of CES utility, but it is still tractable in a reasonable condition.

Bodie, Merton, and Samuelson (1992) have studied an optimal consumption and investment problem of an economic agent who has flexibility in labor supply and shown that flexibility in labor supply tends to increase the agent's risk taking in market securities. Bodie et al. (2004) have studied a similar problem in the context of retirement planning, i.e., there is a fixed time of retirement and the agent chooses consumption and investment in preparation for a scheduled retirement. However, they have not solved for the agent's optimal choice of retirement time, as we have done in this paper.

Karatzas and Wang (2000) first studied a discretionary stopping problem by using the martingale method. They have introduced the family of stopping time problems to reduce the problems into easy forms. Choi and Koo (2005) have studied the effect of a preference change around a discretionary stopping time. Jeanblanc, Lakner, and Kadam (2004) have solved a problem of an agent under obligation to pay a debt at a fixed rate who can declare bankruptcy by using the dynamic programming method. Choi and Shim (2006) have studied a problem in which a wage earner can choose consumption/investment policies, and the time to retire considering a trade-off between income and disutility from labor by using the dynamic programming method. But they have not considered the consumption-leisure choice problem as we have done in this paper.

He and Pagès (1993) have studied the optimal consumption and portfolio problem in which the agent is subject to the liquidity constraints. (See also Dybvig and Liu (2005).)

Farhi and Panageas (2007) have independently studied a model similar to ours. They

have considered the optimal consumption and portfolio problem with retirement time. They have solved this problem with Cobb-Douglas utility and a binomial choice of leisure ($l_1 = 1$ or $\bar{l}$ when retire) with the liquidity constraints and the finite horizon problem with retirement. The main difference between our paper and theirs consists of two aspects. First, in this paper we employ a general CES utility to model choice between consumption and leisure, whereas they have confined their attention to the Cobb-Douglas utility function which is a special case of the CES function. Second, we allow a continuum of choice between labor and leisure, while they have considered a binomial choice, to work or to retire. Whether labor supply is flexible or not is largely an empirical question. However, consideration of flexible labor supply has contributed greatly to the understanding of macroeconomic labor supply, consumption and asset prices (see e.g. Mankiw, Rotemberg, and Summers (1985), Eichenbaum, Hansen, and Singleton (1988), Kennan (1988), Basak (1999)). Furthermore, the general CES specification allows the elasticity of substitution between labor and leisure to take any positive real numbers, and therefore, becomes an essential feature when labor supply flexibility is introduced in the model (see Basak (1999)).

The rest of this paper proceeds as follows. Section 2 provides the financial market model and Section 3 describes the optimization problem. In Section 4 we introduce the results of the duality approach. Section 5 provides a value function by solving a free boundary value problem and characterizes the optimal policies for an agent with general CES utility. Some numerical examples are also provided. In Section 6 we consider the case in which the agent is subject to the liquidity constraints. Section 7 concludes. All detailed proofs in this paper are given in Appendix.

2 The Financial Market Model

2.1 The Economy

We consider a continuous-time financial market with an infinite-time horizon. Assume that there are $N + 1$ assets. One asset is a riskless asset with a constant interest rate $r > 0$ and the others are N risky assets(or stocks) whose price processes are governed by the stochastic differential equation(SDE) with constant parameters, the column vector $b = (b_j)$ of mean rates of return and a volatility matrix $\sigma = (\sigma_{jk})$ $j, k = 1, \cdots, N$
$dS_j(t) = S_j(t) \left[b_j dt + \sum_{k=1}^{N} \sigma_{jk} dB_k(t) \right]$, where $B(t) = (B_1(t), B_2(t), \cdots, B_N(t))^\top$ is an

4

N-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the superscript $\top$ denotes the transpose of a matrix or a vector. $\{\mathcal{F}_t\}_{t=0}^{\infty}$ is the augmentation under $\mathbb{P}$ of the natural filtration generated by $B(t)$. We assume that the matrix σ is nonsingular.

We now define the market price of risk, the discount process, the exponential martingale, and the state-price-density process (or pricing kernel), respectively, by $\theta \triangleq \sigma^{-1}(b - r\mathbf{1}_N)$, $\zeta(t) \triangleq \exp\{-rt\}$, $Z_0(t) \triangleq \exp\left\{-\theta^{\top}B(t) - \frac{1}{2}||\theta||^2 t\right\}$, and $H(t) \triangleq \zeta(t)Z_0(t) = \exp\left\{-\left(r + \frac{1}{2}||\theta||^2\right)t - \theta^{\top}B(t)\right\}$, where $\mathbf{1}_N = (1, \cdots, 1)^{\top}$ denotes the column vector of N ones. For each given fixed $T > 0$, we define the equivalent martingale measure $\tilde{\mathbb{P}}(A) \triangleq \mathbb{E}[Z_0(T)\mathbf{1}_A]$, for $A \in \mathcal{F}_T$. By Girsanov's Theorem, we obtain the new process $\tilde{B}(t) = B(t) + \theta t$, $0 \le t \le T$, which is a standard Brownian motion under $\tilde{\mathbb{P}}$.

Let X_t be the wealth of an agent at time t and let $\pi_j(t)$ be the amount invested in the risky asset $S_j(t)$ at time t, then $X_t - \sum_{j=1}^{N} \pi_j(t)$ can be invested in the riskless asset. $\pi_t \triangleq (\pi_1(t), \pi_2(t), \cdots, \pi_N(t))^{\top}$ is $\mathcal{F}_t$-measurable adapted such that $\int_0^t ||\pi_s||^2 ds < \infty$, for all $t \ge 0$, almost surely(a.s.). Let $c_t \ge 0$ be the agent's consumption rate at time t; it is progressively measurable with respect to $\mathcal{F}_t$ and nonnegative a.s..

A novel feature of our model is that the agent chooses between labor and leisure. The sum of rates of labor and leisure is assumed to be a constant $\bar{L}$. Therefore, if we let $l_t \ge 0$ be the rate of leisure at time t, which is progressively measurable with respect to $\mathcal{F}_t$, then $\bar{L} - l_t$ is the rate of work at time t, so that the (effective) total hours of work between t and s is given by $\int_t^s (\bar{L} - l_u)du$.

We assume that the wage rate is a constant w. Therefore, if the agent works at the rate $\bar{L} - l_t$, then she receives labor income at the rate $w(\bar{L} - l_t)$ at time t. We also assume that c_t satisfies $\int_0^t c_s ds < \infty$, for all $t \ge 0$, a.s..

The agent chooses her retirement time τ which is assumed to be an $\mathcal{F}_t$-stopping time. We assume that the agent should work at least at the rate $\bar{L} - L > 0$ in order to keep her employment before retirement. That is, $L > 0$ is the maximum rate of leisure the agent can choose before retirement. Furthermore, we assume that retirement is a once-and-for-all decision, i.e., the agent cannot come back to work after retirement.[1] The assumptions

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[1]The assumption that there is a positive minimum for the rate of labor is necessary in order to guarantee the existence of a finite retirement time. If we allow the agent can voluntarily choose a zero labor rate before retirement, then she will never retire. Choosing a zero labor rate is a better option than retirement, because the agent has an option to work alive by choosing the former. An obvious shortcoming of the assumption is

can be summarized as

$$(2.1) \qquad 0 \leq l_t \leq L < \bar{L}, \quad \text{for} \ \ 0 \leq t < \tau \quad \text{and} \quad l_t = \bar{L}, \quad \text{for} \ \ t \geq \tau.$$

In particular, the agent cannot enjoy leisure full-time before retirement.

A consumption-leisure-portfolio plan of the agent is a triple (c, l, π). Let $X_t = X_t^{(c,l,\pi,x)}$ be the agent's wealth process corresponding to a given consumption-leisure-portfolio plan (c, l, π) with initial wealth $X_0 = x$. Therefore the agent's wealth process X_t evolves according as

$$(2.2) \qquad dX_t = [rX_t + \pi_t^\top (b - r\mathbf{1}_N) - c_t + w(\bar{L} - l_t)]dt + \pi_t^\top \sigma dB(t).$$

The consumption-leisure-portfolio plan (c, l, π) is called admissible until the stopping time τ if $X_\tau^{(c,l,\pi,x)} \geq 0$ and $X_t^{(c,l,\pi,x)} > -\frac{w\bar{L}}{r}$ for $0 \leq t < \tau$.[2] After retirement the agent faces the liquidity constraints $X_t \geq 0$ for all $t \geq \tau$ a.s..[3]

Under $\tilde{\mathbb{P}}$ the wealth process (2.2) is rewritten as

$$(2.3) \qquad dX_t = [rX_t - c_t + w(\bar{L} - l_t)]dt + \pi_t^\top \sigma d\tilde{B}(t).$$

By Itô's formula, we have

$$(2.4) \qquad \zeta(t)\left(X_t + \frac{w\bar{L}}{r}\right) + \int_0^t \zeta(s)\left(c_s + wl_s\right)ds = x + \frac{w\bar{L}}{r} + \int_0^t \zeta(s)\pi_s^\top \sigma d\tilde{B}(s).$$

For an admissible plan (c, l, π) until a stopping time τ, the third term on right-hand side of equation (2.4) is a continuous $\mathbb{P}$-local martingale bounded below and therefore a supermartingale by Fatou's Lemma. Thus the optional sampling theorem implies

$$(2.5) \qquad \mathbb{E}\left[\int_0^\tau H(t)\left(c_t + wl_t\right)dt + H(\tau)\left(X_\tau + \frac{w\bar{L}}{r}\right)\right] \leq x + \frac{w\bar{L}}{r},$$

for all $\tau \in \mathcal{S}$ where $\mathcal{S}$ denotes the set of all $\mathcal{F}$-stopping time τ's.

2.2 The Utility Function

The agent has the following von Neumann-Morgenstern utility function:

$$\mathcal{U} \triangleq \int_0^\infty e^{-\beta t} u(c_t, l_t) dt$$

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that we cannot deal with involuntary unemployment in our model.

[2]We consider $\frac{w\bar{L}}{r}$ as the present value of the future income of the agent. So she is able to consume and invest as long as her wealth level does not fall below $-\frac{w\bar{L}}{r}$.

[3]We will consider the case in which the agent has the liquidity constraints for all $t \geq 0$ in Section 6.

where $\beta > 0$ is the subjective discount rate and a CES(Constant Elasticity of Substitution) utility function is defined by

$$(2.6) \qquad u(c,l) \triangleq \frac{\{\alpha c^\rho + (1-\alpha)l^\rho\}^{\frac{1-\gamma}{\rho}}}{1-\gamma}, \ \rho < 1, \ \rho \neq 0, \ 0 < \alpha < 1, \ \gamma > 0 \text{ and } \gamma \neq 1.$$

Stiglitz (1969) studied behavior toward risk with multiple commodities and defined coefficient of relative (or absolute) risk aversion as that of the indirect utility function of income when the relative prices of the commodities are fixed. In (2.6) γ is the Stiglitz coefficient of relative risk aversion when we think of consumption and leisure as two different commodities, $1/(1-\rho)$ is the elasticity of substitution between consumption and leisure and α is a parameter that measures the share of consumption's contribution to agent's period utility.

We introduce a convex dual function $\tilde{u}(\cdot)$ of a concave function $u(\cdot,\cdot)$ by $\tilde{u}(y) \triangleq \sup_{\{c \geq 0, \ L \geq l \geq 0\}}[u(c,l) - (c+wl)y]$. Let $I_{1,c}(\cdot)$ be the inverse function of $\frac{\partial u}{\partial c}(\cdot, L)$, then we obtain

$$\tilde{u}(y) = A_1 y^{-\frac{1-\gamma}{\gamma}} \mathbf{1}_{\{y \geq \tilde{y}\}} + [u(I_{1,c}(y), L) - \{I_{1,c}(y) + wL\}y]\mathbf{1}_{\{0 < y \leq \tilde{y}\}},$$

where

$$A_1 \triangleq \frac{\gamma}{1-\gamma}\alpha^{\frac{1-\gamma}{\rho\gamma}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{(1-\rho)(1-\gamma)}{\rho\gamma}}$$

and

$$\tilde{y} \triangleq \alpha^{\frac{1-\gamma}{\rho}}\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{\gamma}{1-\rho}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho}}L^{-\gamma}.$$

It can be easily shown that $\tilde{u}(\cdot)$ is strictly decreasing and strictly convex.

3 The Optimization Problem

The following assumption is a standard assumption to make the optimization problem well-defined and holds throughout the paper without further comments.

Assumption 3.1.
$$K \triangleq r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}||\theta||^2 > 0.$$

Now the agent's problem is to maximize her expected utility:

$$J(x; c, l, \pi, \tau) \triangleq \mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t, l_t)dt\right]$$

subject to the budget constraint (2.5) and the leisure constraint (2.1). We rewrite $J(x; c, l, \pi, \tau)$ as

$$(3.1) \qquad J(x; c, l, \pi, \tau) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} u(c_t, l_t)dt + \int_\tau^\infty e^{-\beta t} u(c_t, \bar{L})dt\right]$$
$$= \mathbb{E}\left[\int_0^\tau e^{-\beta t} u(c_t, l_t)dt + e^{-\beta \tau} U(X_\tau^{(c,l,\pi,x)})\right].$$

Then the value function is defined by

$$V(x) \triangleq \sup_{(c,l,\pi,\tau)\in\mathcal{A}(x)} J(x; c, l, \pi, \tau),$$

where $\mathcal{A}(x)$ is the set of all admissible plans (c, l, π, τ) such that

$$\mathbb{E}\left[\int_0^\tau e^{-\beta t} u^-(c_t, l_t)dt + e^{-\beta \tau} U^-(X_\tau^{(c,l,\pi,x)})\right] < \infty,$$

where $u^- \triangleq \max(-u, 0)$.

Remark 3.1. *We see that the optimal policies after retirement are obtained using the solution of the Merton problem. (See a similar result in Proposition 2.4 of Jeanblanc, Lakner, and Kadam (2004).) That is, for any $\tau \in \mathcal{S}$ and $\mathcal{F}_\tau$-measurable random variable η there exists a consumption-portfolio plan $\{(\hat{c}_t, \hat{\pi}_t) : t \geq \tau\}$ such that*

$$\mathbb{E}\left[e^{-\beta \tau} U(\eta)\mathbf{1}_{\{\tau<\infty\}}\right] = \mathbb{E}\left[\int_\tau^\infty e^{-\beta t} u(\hat{c}_t, \bar{L})dt\right],$$

where the wealth dynamics is governed by

$$dX_t = [rX_t + \hat{\pi}_t^\top(b - r\mathbf{1}_N) - \hat{c}_t]dt + \hat{\pi}_t^\top \sigma dB(t) \ , t \geq \tau, \ X_\tau = \eta.$$

We can also develop $U(\cdot)$ according to the framework of Karatzas et al. (1986).

For notational simplicity, we let $u_{\bar{L}}(c) \triangleq u(c, \bar{L})$.

Lemma 3.1. *$U(\cdot)$ in (3.1) is given by*

$$U(x) = J_0(C(x, 0)),$$

where $C(\cdot, 0)$ is the inverse function of $X_0(\cdot)$,

$$X_0(c) = \frac{c}{r} - \frac{2}{||\theta||^2(m_+ - m_-)}\left[\frac{(u'_{\bar{L}}(c))^{m_+}}{m_+}\int_0^c \frac{dz}{(u'_{\bar{L}}(z))^{m_+}} + \frac{(u'_{\bar{L}}(c))^{m_-}}{m_-}\int_c^\infty \frac{dz}{(u'_{\bar{L}}(z))^{m_-}}\right],$$

$$\frac{1}{2}||\theta||^2 m^2 - \left(r - \beta - \frac{1}{2}||\theta||^2\right)m - r = 0,$$

where are two real roots, $m_- < -1$ and $m_+ > 0$, and

$$J_0(c) = \frac{u_{\bar{L}}(c)}{\beta} - \frac{2}{||\theta||^2(n_+ - n_-)}\left[\frac{(u'_{\bar{L}}(c))^{n_+}}{n_+}\int_0^c \frac{dz}{(u'_{\bar{L}}(z))^{m_+}} + \frac{(u'_{\bar{L}}(c))^{n_-}}{n_-}\int_c^\infty \frac{dz}{(u'_{\bar{L}}(z))^{m_-}}\right]$$

8

with $n_+ = m_+ + 1$ and $n_- = m_- + 1$. *Moreover the optimal portfolio and consumption are*

$$\hat{\pi}_t = -\sigma^{-1}\theta \frac{U'(X_t)}{U''(X_t)} \quad and \quad \hat{c}_t = c_{**},$$

where c_{**} is the solution to the following algebraic equation

$$\alpha(c_{**})^{\rho-1} \left\{ \alpha(c_{**})^\rho + (1-\alpha)\bar{L}^\rho \right\}^{\frac{1-\gamma-\rho}{\rho}} = U'(X_t).$$

Proof. In the case of $\rho < 0$ and $\gamma > 1$, we have $u_{\bar{L}}(0) = -\infty$. The proof is done by Theorem 11.4 of Karatzas et al (1986). In the other case, we have $u_{\bar{L}}(0) < \infty$. Then, we apply Theorem 10.1 of Karatzas et al (1986). □

Now we introduce a convex dual function $\tilde{U}(\cdot)$ of a concave function $U(\cdot)$ by $\tilde{U}(y) \triangleq \sup_{\{x \geq 0\}} \left[U(x) - \left(x + \frac{w\bar{L}}{r} \right) y \right]$. Let $I_2(\cdot)$ be the inverse function of $U'(\cdot)$, then we obtain

$$\tilde{U}(y) \triangleq U(I_2(y)) - \left\{ I_2(y) + \frac{w\bar{L}}{r} \right\} y.$$

The dual function $\tilde{U}(\cdot)$ is also strictly decreasing and strictly convex.

4   The Martingale Method

We now proceed to solve the optimization problem using a martingale approach. For any fixed stopping time $\tau \in \mathcal{S}$, $\Pi_\tau(x)$ is denoted by the set of consumption-leisure-portfolio plan (c, l, π) for which $(c, l, \pi, \tau) \in \mathcal{A}(x)$. Define the following utility maximization problem

$$V_\tau(x) \triangleq \sup_{(c,l,\pi)\in\Pi_\tau(x)} J(x; c, l, \pi, \tau).$$

For a Lagrange multiplier $\lambda > 0$, we define a dual value function

(4.1)
$$\tilde{V}(\lambda; \tau) \triangleq \sup_{(c,l,\pi)\in\Pi_\tau(x)} \left(J(x; c, l, \pi, \tau) - \lambda \mathbb{E} \left[\int_0^\tau H(t)(c_t + wl_t)dt + H(\tau) \left(X_\tau^{(c,l,\pi,x)} + \frac{w\bar{L}}{r} \right) \right] \right).$$

Then

$$\tilde{V}(\lambda; \tau) = \mathbb{E} \left[\int_0^\tau e^{-\beta t} \tilde{u}(\lambda e^{\beta t} H(t))dt + e^{-\beta \tau} \tilde{U}(\lambda e^{\beta \tau} H(\tau)) \right].$$

9

For a fixed τ we can determine optimal policies $(c_\lambda, l_\lambda, \pi_\lambda)$ to problem (4.1). We consider the first order conditions, for $0 \le t < \tau$

(4.2)
$$\frac{\partial u}{\partial c}(c_\lambda(t), l_\lambda(t)) = \lambda e^{\beta t} H(t) \quad \text{and} \quad \frac{\partial u}{\partial l}(c_\lambda(t), l_\lambda(t)) = w\lambda e^{\beta t} H(t), \text{ on } \{0 \le l \le L\},$$

(4.3) $\quad \frac{\partial u}{\partial c}(c_\lambda(t), L) = \lambda e^{\beta t} H(t) \quad \text{and} \quad l_\lambda(t) = L, \text{ on } \{l \ge L\}.$

By the duality argument, We also have

(4.4)
$$c_\lambda(t) + wl_\lambda(t) = -\tilde{u}'(\lambda e^{\beta t} H(t)), \ 0 \le t < \tau.$$

The optimal replicating portfolio π_λ, satisfying the wealth process

(4.5)
$$X_\tau^{(c_\lambda, l_\lambda, \pi_\lambda, x)} = -\frac{w\bar{L}}{r} - \tilde{U}'(\lambda e^{\beta \tau} H(\tau)),$$

exists by the following lemma, that is, there exists the optimal portfolio π_λ corresponding to $(c_\lambda, l_\lambda, X_\tau)$ in (4.2), (4.3), (4.4), and (4.5).

Lemma 4.1. *For any $\tau \in \mathcal{S}$, any $\mathcal{F}_\tau$-measurable random variable B with $\mathbb{P}[B > 0] = 1$, any progressively measurable process $c_t \ge 0$ and $l_t \ge 0$ that satisfy*

$$\mathbb{E}\left[\int_0^\tau H(t)(c_t + wl_t)dt + H(\tau)\left(B + \frac{w\bar{L}}{r}\right)\right] = x + \frac{w\bar{L}}{r},$$

there exists portfolio $\pi(\cdot)$ such that, a.s.

$$X_t^{(c,l,\pi,x)} > -\frac{w\bar{L}}{r}, \ 0 \le t < \tau \quad \text{and} \quad X_\tau^{(c,l,\pi,x)} = B.$$

Proof. This lemma is similar to Lemma 6.3 of Karatzas and Wang (2000). $\qquad\square$

Now for a fixed τ, let $(c_\lambda, l_\lambda, \pi_\lambda)$ be the policies that provide the supremum value in (4.1). We have from the definition of $V_\tau(x)$ and $\tilde{V}(\lambda; \tau)$

$$V(x) = \sup_{\tau \in \mathcal{S}} V_\tau(x) = \sup_{\tau \in \mathcal{S}} \inf_{\lambda > 0}\left[\tilde{V}(\lambda; \tau) + \lambda x + \frac{\lambda w\bar{L}}{r}\right].$$

For a given Lagrange multiplier λ, we define

$$\tilde{V}(\lambda) \triangleq \sup_{\tau \in \mathcal{S}} \tilde{V}(\lambda; \tau).$$

Then the following proposition provides the value function $V(\cdot)$ which is an analogue to Karatzas and Wang (2000).

10

Proposition 4.1. *If $\tilde{V}(\lambda)$ exists and is differentiable for $\lambda > 0$, then*

$$(4.6) \qquad V(x) = \inf_{\lambda > 0} \left[\tilde{V}(\lambda) + \lambda x + \frac{\lambda w \bar{L}}{r} \right]$$

for any $x \in (-\frac{w\bar{L}}{r}, \infty)$. In particular, the optimal policies are represented by (4.2), (4.3), (4.4) and (4.5) with λ attaining the minimum value at (4.6).

Proof. See Section 8, especially Theorem 8.5 and consequently Corollary 8.7 of Karatzas and Wang (2000). □

5   A Solution to the Free Boundary Value Problem

We define $y_t^\lambda \triangleq \lambda e^{\beta t} H(t)$. Then $\tilde{V}(\lambda; \tau)$ is rewritten as $\tilde{V}(\lambda; \tau) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} \tilde{u}(y_t^\lambda) dt + e^{-\beta \tau} \tilde{U}(y_\tau^\lambda) \right]$. Let $y_t = \lambda \exp\{(\beta - r - \frac{1}{2}||\theta||^2)t - \theta^\top B(t)\}$. By Itô's formula, we have

$$(5.1) \qquad dy_t = y_t\{(\beta - r)dt - \theta^\top dB(t)\}.$$

It can be easily seen that y_t^λ is a unique strong solution to (5.1) with initial value $y_0^\lambda = \lambda$. We consider the following optimal stopping problem

$$(5.2) \qquad \phi(t, y) = \sup_{\tau > t} \mathbb{E}^{y_t = y} \left[\int_t^\tau e^{-\beta s} \tilde{u}(y_s) ds + e^{-\beta \tau} \tilde{U}(y_\tau) \right],$$

where we denote $\mathbb{E}^{y_t = y} = \mathbb{E}^y$ for simplicity. We consider the differential operator $\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r)y\frac{\partial}{\partial y} + \frac{1}{2}||\theta||^2 y^2 \frac{\partial^2}{\partial y^2}$ acting on a mapping $\psi : (0, \infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. A solution to the following free boundary value problem is a solution to the optimal stopping problem (5.2).

Variational Inequality 1. *Find a free boundary $\bar{y} > 0$ and a function $\tilde{\phi}(\cdot, \cdot) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times (\mathbb{R}^+ \setminus \{\bar{y}\}))$ satisfying*

(1) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\tilde{u}(y) = 0$, $\bar{y} < y$

(2) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\tilde{u}(y) \leq 0$, $0 < y \leq \bar{y}$

(3) $\tilde{\phi}(t, y) \geq e^{-\beta t}\tilde{U}(y)$, $\bar{y} \leq y$

(4) $\tilde{\phi}(t, y) = e^{-\beta t}\tilde{U}(y)$, $0 < y \leq \bar{y}$

for all $t > 0$.

We consider the quadratic equation

$$(5.3) \qquad \frac{1}{2}||\theta||^2 n^2 + \left(\beta - r - \frac{1}{2}||\theta||^2\right) n - \beta = 0$$

with two roots $n_+ > 1$ and $n_- < 0$, for the next proposition. The next proposition provides a solution to Variational Inequality 1.

11

Proposition 5.1. *We consider the function*

$$
v(y) = \begin{cases}
C_2 y^{n_-} + \dfrac{A_1}{K} y^{-\frac{1-\gamma}{\gamma}} & \text{if } y \geq \tilde{y} \\[2mm]
D_1 y^{n_+} + D_2 y^{n_-} - \dfrac{wL}{r} y + \dfrac{2 y^{n_+}}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{y} \dfrac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz & \\[3mm]
\quad - \dfrac{2 y^{n_-}}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{y} \dfrac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz & \text{if } \bar{y} < y \leq \tilde{y} \\[3mm]
U(I_2(y)) - \left(I_2(y) + \dfrac{w\bar{L}}{r} \right) y & \text{if } 0 < y \leq \bar{y},
\end{cases}
$$

where $\bar{y}$, C_2, D_1, and D_2 are determined by the following algebraic equations

$$
D_1 = -\frac{1}{n_+ - n_-} \left(\frac{1-\gamma}{\gamma} + n_- \right) \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma} - n_+} + \frac{1 - n_-}{n_+ - n_-} \frac{wL}{r} \tilde{y}^{1 - n_+},
$$

$$
D_1 \bar{y}^{n_+} + \frac{2 \bar{y}^{n_+}}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{\bar{y}} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz
$$
$$
+ \frac{n_-}{n_+ - n_-} U(I_2(\bar{y})) + \frac{1 - n_-}{n_+ - n_-} \left(I_2(\bar{y}) + \frac{w(\bar{L} - L)}{r} \right) \bar{y} = 0,
$$

from which $\bar{y}$ is derived,

$$
D_2 = \frac{n_+}{n_+ - n_-} \bar{y}^{-n_-} U(I_2(\bar{y})) - \frac{n_+ - 1}{n_+ - n_-} \left(I_2(\bar{y}) + \frac{w(\bar{L} - L)}{r} \right) \bar{y}^{1 - n_-}
$$
$$
+ \frac{2}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{\bar{y}} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz,
$$

and

$$
C_2 = -\frac{1}{n_+ - n_-} \left(\frac{1 - \gamma}{\gamma} + n_+ \right) \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma} - n_-} - \frac{n_+ - 1}{n_+ - n_-} \frac{wL}{r} \tilde{y}^{1 - n_-} + D_2.
$$

Then $\tilde{\phi}(t, y) = e^{-\beta t} v(y)$ is a solution to Variational Inequality 1 provided that

$$
(5.4) \quad -\beta U(I_2(y)) + r y I_2(y) - \frac{1}{2} ||\theta||^2 y^2 I_2'(y) + u(I_{1,c}(y), L) - y I_{1,c}(y) + w(\bar{L} - L) y \leq 0,
$$

for $0 < y \leq \bar{y}$,

$$
D_1 y^{n_+} + D_2 y^{n_-} + \frac{w(\bar{L} - L)}{r} y + \frac{2 y^{n_+}}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{y} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz
$$
$$
(5.5) \quad - \frac{2 y^{n_-}}{||\theta||^2 (n_+ - n_-)} \int_{\tilde{y}}^{y} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz - U(I_2(y)) + y I_2(y) \geq 0,
$$

for $\bar{y} < y \leq \tilde{y}$, and

$$
(5.6) \quad C_2 y^{n_-} + \frac{A_1}{K} y^{-\frac{1-\gamma}{\gamma}} - U(I_2(y)) + y I_2(y) + \frac{w\bar{L}}{r} y \geq 0,
$$

for $y \geq \tilde{y}$.

Proof. See Appendix A. $\qquad\square$

12

Theorem 5.1. *If the pair $(\bar{y}, \tilde{\phi}(t, y))$ is a solution to Variational Inequality 1, then $\tilde{\phi}(t, y)$ coincides with $\phi(t, y)$ of (5.2) and an optimal stopping time is given by*

$$\tau_y = \inf \left\{ s > t \mid y_s \leq \bar{y} \right\} < \infty, \ a.s.$$

Proof. We directly obtain this result from Theorem 10.4.1 of Øksendal (1998). ∎

By the results of Theorem 5.1 and Proposition 4.1, we obtain the value function $V(x)$. Since $\tilde{V}(\lambda)$ is obtained from $\phi(t, y)$ at $t = 0$, $y = \lambda$ and consequently $\tilde{V}(\lambda) = v(\lambda)$. The optimal stopping time corresponding to λ is characterized as $\tau_\lambda = \inf \left\{ t > 0 \mid y_t^\lambda \leq \bar{y} \right\}$. So we obtain the value function $V(x)$ using the following theorem.

Theorem 5.2. *Let*

$$(5.7) \qquad \tilde{x} = -n_- C_2 \tilde{y}^{n_- - 1} + \frac{1 - \gamma}{\gamma} \frac{A_1}{K} \tilde{y}^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r} \quad and \quad \bar{x} = I_2(\bar{y}).$$

Then, the value function is

$$V(x) = \begin{cases} C_2(\lambda_{**})^{n_-} + \frac{A_1}{K}(\lambda_{**})^{-\frac{1-\gamma}{\gamma}} + (\lambda_{**})x + \frac{w\bar{L}}{r}(\lambda_{**}) & if \ -\frac{w\bar{L}}{r} < x \leq \tilde{x} \\ \\ D_1(\lambda^{**})^{n_+} + D_2(\lambda^{**})^{n_-} + \frac{w(\bar{L}-L)}{r}(\lambda^{**}) + (\lambda^{**})x \\ \quad + \frac{2(\lambda^{**})^{n_+}}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\lambda^{**}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz \\ \quad - \frac{2(\lambda^{**})^{n_-}}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\lambda^{**}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz & if \ \tilde{x} \leq x < \bar{x} \\ \\ U(x) & if \ x \geq \bar{x}, \end{cases}$$

where λ_{**} and λ^{**} are determined from the following algebraic equations

$$(5.8) \qquad -n_- C_2(\lambda_{**})^{n_- - 1} + \frac{1 - \gamma}{\gamma} \frac{A_1}{K}(\lambda_{**})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r} = x, \ for \ -\frac{w\bar{L}}{r} < x \leq \tilde{x}$$

and

$$- n_+ D_1(\lambda^{**})^{n_+ - 1} - n_- D_2(\lambda^{**})^{n_- - 1} - \frac{2n_+(\lambda^{**})^{n_+ - 1}}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\lambda^{**}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz$$

$$(5.9)$$

$$+ \frac{2n_-(\lambda^{**})^{n_- - 1}}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\lambda^{**}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz - \frac{w(\bar{L} - L)}{r} = x, \ for \ \tilde{x} \leq x < \bar{x}.$$

Remark 5.1. *It is easily seen the one-to-one correspondences between $\lambda_{**} \in (\tilde{y}, \infty)$ and $x \in (-\frac{w\bar{L}}{r}, \tilde{x})$ at (5.8) and $\lambda^{**} \in (\bar{y}, \tilde{y})$ and $x \in (\tilde{x}, \bar{x})$ at (5.9) using decreasing property of (5.8) and (5.9) with respect to λ_{**} and λ^{**} respectively.*

13

The optimal stopping times τ^* are determined by means of (5.9) such that

$$(5.10) \qquad \tau^* = \tau_{\lambda^{**}} = \inf\left\{ t > 0 \mid y_t^{\lambda^{**}} \leq \bar{y} \right\}.$$

We now assume that there is 1 risky asset to represent the optimal solution without notational complexity. Let $y_t^{\lambda_{**}}$ and $y_t^{\lambda^{**}}$ be solutions of SDE (5.1) with an initial value $y_0 = \lambda_{**}$ and $y_0 = \lambda^{**}$, respectively. In order to find an optimal portfolio we consider the optimal wealth process. In order to obtain the optimal wealth process we substitute $y_t^{\lambda^{**}}$ for λ^{**} into (5.9). Then

$$X^{**}(t) = -n_+ D_1(y_t^{\lambda^{**}})^{n_+-1} - n_- D_2(y_t^{\lambda^{**}})^{n_--1} - \frac{w(\bar{L}-L)}{r}$$

$$- \frac{2n_+(y_t^{\lambda^{**}})^{n_+-1}}{\theta^2(n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^{**}}} \frac{zI_{1,c}(z) - u(I_{1,c}(z),L)}{z^{n_++1}} dz$$

$$(5.11) \qquad + \frac{2n_-(y_t^{\lambda^{**}})^{n_--1}}{\theta^2(n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^{**}}} \frac{zI_{1,c}(z) - u(I_{1,c}(z),L)}{z^{n_-+1}} dz.$$

Similarly we substitute $y_t^{\lambda_{**}}$ for λ_{**} into (5.8). Then

$$(5.12) \qquad X_{**}(t) = -n_- C_2(y_t^{\lambda_{**}})^{n_--1} + \frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r}.$$

Theorem 5.3. *The optimal policies are given by* $(c^*, l^*, \pi^*, \tau^*)$ *such that*

$$c_t^* = \begin{cases} \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho\gamma}}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} & \text{if } -\frac{w\bar{L}}{r} < X_t \leq \tilde{x} \\ I_{1,c}(y_t^{\lambda^{**}}) & \text{if } \tilde{x} \leq X_t < \bar{x} \\ c_{**} & \text{if } X_t \geq \bar{x} \end{cases},$$

$$l_t^* = \begin{cases} \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho\gamma}}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} & \text{if } -\frac{w\bar{L}}{r} < X_t \leq \tilde{x} \\ L & \text{if } \tilde{x} \leq X_t < \bar{x} \\ \bar{L} & \text{if } X_t \geq \bar{x} \end{cases},$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\left\{n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_--1} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}}\right\} & \text{if } -\frac{w\bar{L}}{r} < X_t \leq \tilde{x} \\ \frac{\theta}{\sigma}\left\{n_+(n_+ - 1)D_1(y_t^{\lambda^{**}})^{n_+-1} + n_-(n_- - 1)D_2(y_t^{\lambda^{**}})^{n_--1}\right. & \\ \quad + \frac{2n_+(n_+-1)(y_t^{\lambda^{**}})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\tilde{y}}^{y_t^{\lambda^{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_++1}}dz & \\ \quad - \frac{2n_-(n_--1)(y_t^{\lambda^{**}})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\tilde{y}}^{y_t^{\lambda^{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_-+1}}dz & \\ \left.\quad + \frac{2}{\theta^2}\frac{y_t^{\lambda^{**}}I_{1,c}(y_t^{\lambda^{**}})-u(I_{1,c}(y_t^{\lambda^{**}}),L)}{y_t^{\lambda^{**}}}\right\} & \text{if } \tilde{x} \leq X_t < \bar{x} \\ -\frac{\theta}{\sigma}\frac{U'(X_t)}{U''(X_t)} & \text{if } X_t \geq \bar{x} \end{cases}$$

with

$$\tau^* = \inf\{t > 0 \mid X^{**}(t) \geq \bar{x}\}.$$

14

In this case c_{**} is a solution to the following algebraic equation

$$\alpha(c_{**})^{\rho-1}\left\{\alpha(c_{**})^{\rho}+(1-\alpha)L^{\rho}\right\}^{\frac{1-\gamma-\rho}{\rho}}=U'(X_t).$$

Proof. See Appendix B. $\qquad\square$

Now we will compare the optimal policies in our model with those when the agent does not have a retirement option, i.e., τ is forced to be infinite. This is an optimization problem of an agent who chooses $0 \le l_t \le L$. In the case of no retirement option, the optimal consumption and portfolio are given as follows:

$$c_{t,N}^{*} = \begin{cases} \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(1+w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho\gamma}}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}} & \text{if } -\frac{w\bar{L}}{r} < X_t \le \tilde{x}_N \\ I_{1,c}(y_{t,N}^{\lambda_{**}}) & \text{if } X_t \ge \tilde{x}_N \end{cases}$$

and

$$\pi_{t,N}^{*} = \begin{cases} \frac{\theta}{\sigma}\left\{n_{-}(n_{-}-1)(C_2-D_2)(y_{t,N}^{\lambda_{**}})^{n_{-}-1}+\frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}}\right\} & \text{if } -\frac{w\bar{L}}{r} < X_t \le \tilde{x}_N \\ \frac{\theta}{\sigma}\left\{n_{+}(n_{+}-1)D_1(y_t^{\lambda^*})^{n_{+}-1} \right. & \\ \quad +\frac{2n_{+}(n_{+}-1)(y_{t,N}^{\lambda_{**}})^{n_{+}-1}}{\theta^2(n_{+}-n_{-})}\int_{\tilde{y}}^{y_{t,N}^{\lambda_{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_{+}+1}}dz & \\ \quad -\frac{2n_{-}(n_{-}-1)(y_{t,N}^{\lambda_{**}})^{n_{-}-1}}{\theta^2(n_{+}-n_{-})}\int_{\tilde{y}}^{y_{t,N}^{\lambda_{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_{-}+1}}dz & \\ \quad \left.+\frac{2}{\theta^2}\frac{y_{t,N}^{\lambda_{**}}I_{1,c}(y_{t,N}^{\lambda_{**}})-u(I_{1,c}(y_{t,N}^{\lambda_{**}}),L)}{y_{t,N}^{\lambda_{**}}}\right\} & \text{if } X_t \ge \tilde{x}_N \end{cases},$$

where the wealth level corresponding to $\tilde{y}$ is given by

$$(5.13) \qquad \tilde{x}_N = -n_{-}(C_2-D_2)\tilde{y}^{n_{-}-1}+\frac{1-\gamma}{\gamma}\frac{A_1}{K}\tilde{y}^{-\frac{1}{\gamma}}-\frac{w\bar{L}}{r}$$

and the optimal wealth processes are given by

$$X_N^{**}(t) = -n_{+}D_1(y_{t,N}^{\lambda_{**}})^{n_{+}-1}-\frac{w(\bar{L}-L)}{r}-\frac{2n_{+}(y_{t,N}^{\lambda_{**}})^{n_{+}-1}}{\theta^2(n_{+}-n_{-})}\int_{\tilde{y}}^{y_{t,N}^{\lambda_{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_{+}+1}}dz$$

$$(5.14)$$

$$+\frac{2n_{-}(y_{t,N}^{\lambda_{**}})^{n_{-}-1}}{\theta^2(n_{+}-n_{-})}\int_{\tilde{y}}^{y_{t,N}^{\lambda_{**}}}\frac{zI_{1,c}(z)-u(I_{1,c}(z),L)}{z^{n_{-}+1}}dz$$

and

$$(5.15) \qquad X_{**,N}(t) = -n_{-}(C_2-D_2)(y_{t,N}^{\lambda_{**}})^{n_{-}-1}+\frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}}-\frac{w\bar{L}}{r}.$$

Remark 5.2. *It can be easily seen that $\tilde{x} > \tilde{x}_N$ from comparing (5.7) with (5.13) provided that $D_2 > 0$. (Here we assume that $D_2 > 0$.)*

Proposition 5.2. *Suppose that the agent has a CES period utility function, then the agent in our model consumes less and invest more before retirement than she does when there is no retirement option. In other words,*

$$c_{t,N}^* > c_t^* \qquad for \ -\frac{w\bar{L}}{r} < X_t < \bar{x}$$

and

$$\pi_{t,N}^* < \pi_t^* \qquad for \ -\frac{w\bar{L}}{r} < X_t < \bar{x}.$$

Proof. See Appendix C. □

Figures 1 and 2 provide illustrations of results concerning the optimal policies in Proposition 5.2. The agent consumes less and takes risk more before the wealth of the agent reaches the critical level, because she has an incentive to reach the level and retire fast enough. This tendency becomes more apparent as the wealth level becomes closer to the critical level. A notable feature of the consumption behavior is its discontinuity at the critical wealth level. This is because the leisure rate is discontinuous at the level.

Several numerical examples are given. Figure 3 shows that the critical wealth level is an increasing function of the maximum leisure rate L (with $\bar{L}$ fixed). It is clear that if the difference between the maximum rate of leisure during employment and the rate of leisure after retirement is smaller, then the agent will choose to retire with higher wealth. Figure 4 illustrates that as the wage rate w becomes higher, the critical wealth level becomes higher (it exhibits an approximately linear relationship); the higher the wage rate, the higher the agent's income while she works, so she tends to retire at a higher wealth level. Figure 5 shows that the critical wealth level is an increasing function of α, which is a weight for consumption in the period utility function. With a higher α, consumption contributes more to the agent's utility, therefore, the agent tends to retire at a higher wealth level.
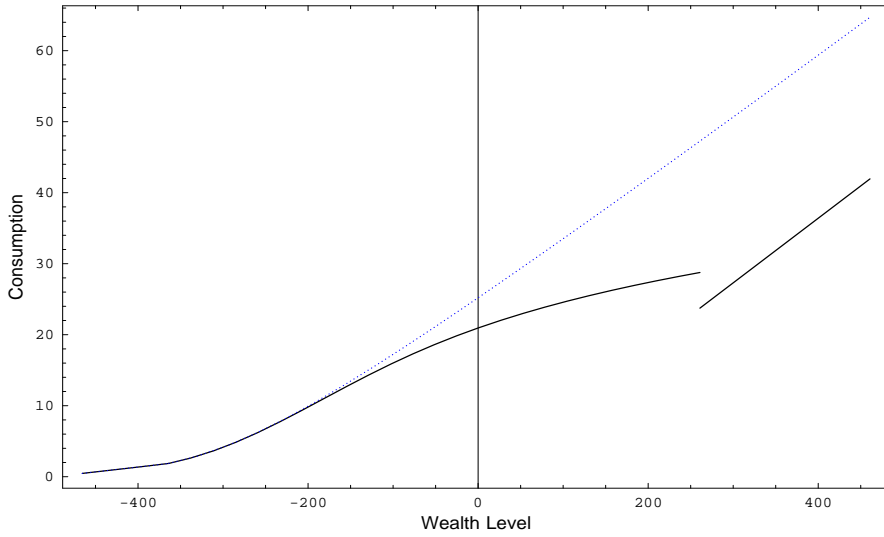
Figure 1: Optimal consumption ($\beta = 0.1$, $r = 0.02$, $b = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $\alpha = 0.2$, $\bar{L} = 1$, $L = 0.75$, $w = 10$ and $\rho \to 0$). Dotted line gives optimal consumption for the case where the agent does not have a retirement option, i.e., τ is forced to be infinite, and the solid line gives optimal consumption for the case in our model where retirement time τ is chosen optimality.
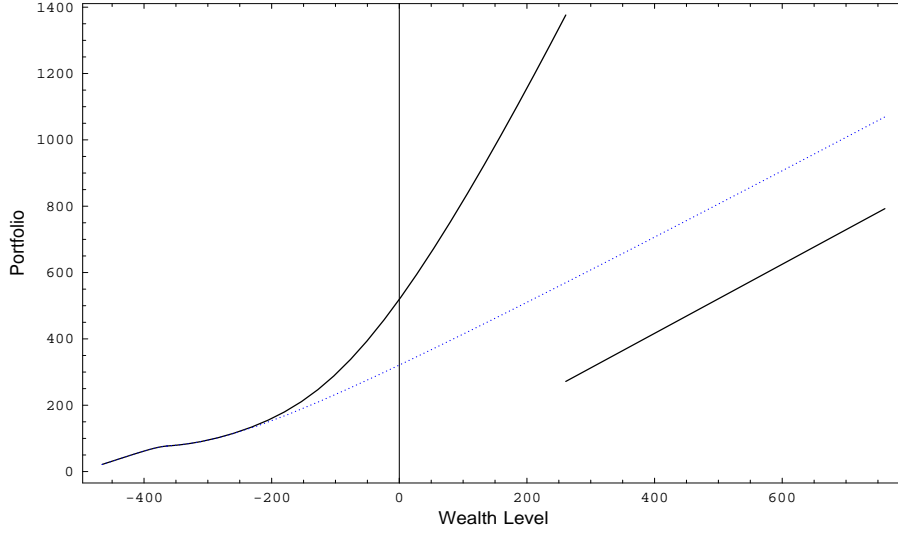
17

Figure 2: Optimal investment in the risky asset ($\beta = 0.1$, $r = 0.02$, $b = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $\alpha = 0.2$, $\bar{L} = 1$, $L = 0.75$, $w = 10$ and $\rho \to 0$). Dotted line gives optimal investment in the risky asset for the case where the agent does not have a retirement option, i.e., τ is forced to be infinite, and the solid line gives optimal investment in the risky asset for the case in our model where retirement time τ is chosen optimality.
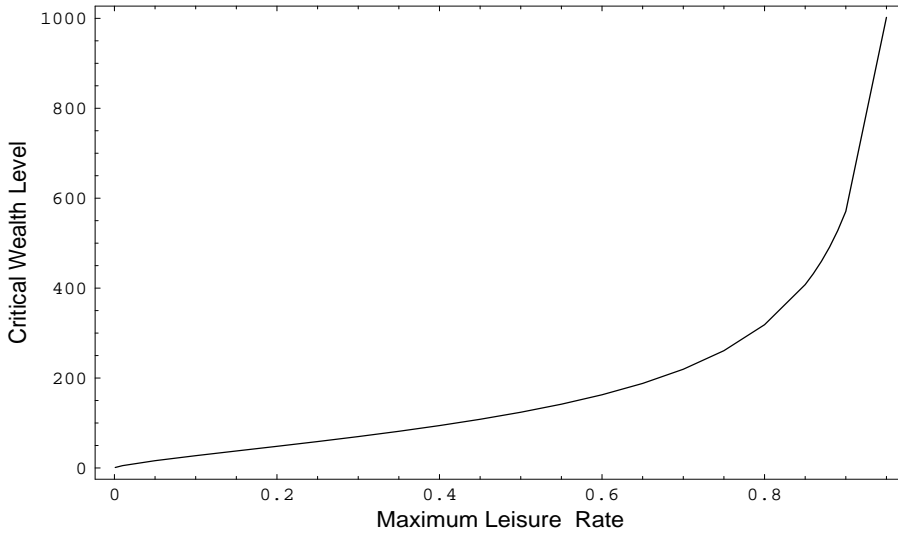

Figure 3: The critical wealth level $\bar{x}$ as a function of the maximum leisure rate L ($\beta = 0.1$, $r = 0.02$, $b = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $\alpha = 0.2$, $\bar{L} = 1$, $w = 10$ and $\rho \to 0$).

18

Figure 4: The critical wealth level $\bar{x}$ as a function of the wage rate w ($\beta = 0.1$, $r = 0.02$, $b = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $\alpha = 0.2$, $\bar{L} = 1$, $L = 0.75$ and $\rho \to 0$).
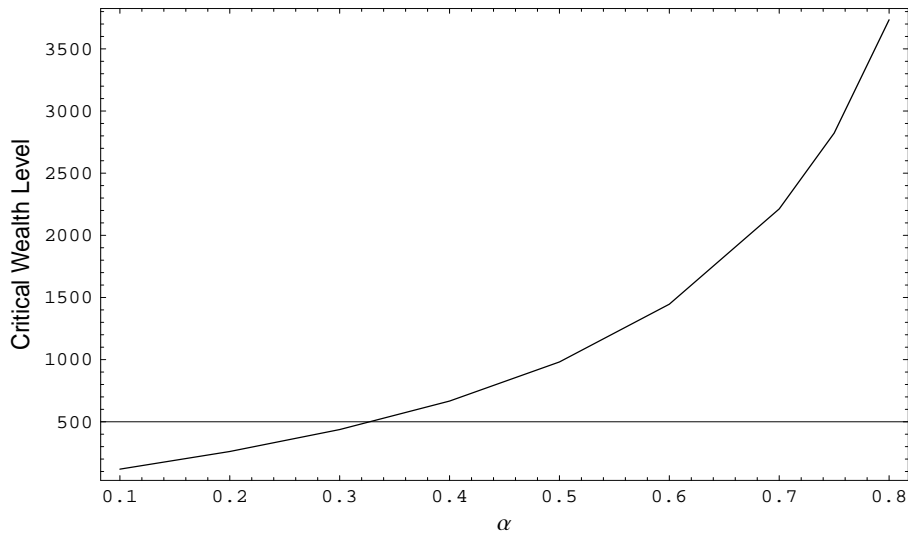

Figure 5: The critical wealth level $\bar{x}$ as a function of α ($\beta = 0.1$, $r = 0.02$, $b = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $\bar{L} = 1$, $L = 0.75$, $w = 10$ and $\rho \to 0$).

6 Liquidity Constraints

6.1 Introduction

In this section we will consider the optimization problem in which the agent faces liquidity constraints, $X_t \geq 0$ for all $0 \leq t \leq \tau$.[4] From (2.3) we obtain

$$\zeta(\tau)X_\tau - \zeta(t)X_t = \int_t^\tau \zeta(s)\left(-c_s + w(\bar{L} - l_s)\right)ds + \int_t^\tau \zeta(s)\pi_s^\top \sigma d\tilde{B}(s).$$

Consequently, we have the following budget constraint (cf. (2.5))

$$\mathbb{E}\left[\int_0^\tau H(t)\left((c_t + wl_t) - w\bar{L}\right)dt + H(\tau)X_\tau\right] \leq x$$

and the liquidity constraint, $X_t \geq 0$ for all $0 \leq t \leq \tau$, takes the form

$$(6.1) \qquad 0 \leq \mathbb{E}_t\left[\int_t^\tau \frac{H(s)}{H(t)}\left((c_s + wl_s) - w\bar{L}\right)ds + \frac{H(\tau)}{H(t)}X_\tau\right], \quad \text{for all } 0 \leq t \leq \tau.$$

(See He and Pagès (1993) and Farhi and Panageas (2007).) Similar to He and Pagès (1993) and Farhi and Panageas (2007), we consider $D_t > 0$ which is non-increasing with $D_0 = 1$.[5]

---

[4]Since the character of the problem under liquidity constraints is now well-understood, here we sketch derivation of a solution. For a detailed explanation the reader is referred to He and Pagès (1993), Dybvig and Liu (2005) and Farhi and Panageas (2007).

[5]D_t has an intuitive interpretation as the integral of shadow prices of the liquidity constraints. See He and Pagès (1993).

For any real number $\lambda > 0$, we have

$$
\begin{aligned}
J(x;c,l,\pi,\tau) &= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{u(c_t,l_t) - \lambda D_t e^{\beta t}H(t)\left(c_t + wl_t\right)\right\}dt + e^{-\beta\tau}\left\{U(X_\tau) - \lambda D_\tau e^{\beta\tau}H(\tau)X_\tau\right\}\right] \\
&\quad + \lambda\mathbb{E}\left[\int_0^\tau D_t H(t)\left(c_t + wl_t\right)dt + D_\tau H(\tau)X_\tau\right] \\
&\leq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda D_t e^{\beta t}H(t))dt + e^{-\beta\tau}\tilde{U}_1(\lambda D_\tau e^{\beta\tau}H(\tau))\right] \\
&\quad + \lambda\mathbb{E}\left[\int_0^\tau D_t H(t)\left(c_t + wl_t\right)dt + D_\tau H(\tau)X_\tau\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda D_t e^{\beta t}H(t))dt + e^{-\beta\tau}\tilde{U}_1(\lambda D_\tau e^{\beta\tau}H(\tau))\right] \\
&\quad + \lambda\mathbb{E}\left[\int_0^\tau D_t H(t)\left(c_t + wl_t\right)dt - \int_0^\tau w\bar{L}D_t H(t)dt + D_\tau H(\tau)X_\tau + \int_0^\tau w\bar{L}D_t H(t)dt\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda D_t e^{\beta t}H(t))dt + e^{-\beta\tau}\tilde{U}_1(\lambda D_\tau e^{\beta\tau}H(\tau))\right] \\
&\quad + \lambda\mathbb{E}\left[\int_0^\tau w\bar{L}D_t H(t)dt + H(\tau)X_\tau + \int_0^\tau H(t)\left(c_t + wl_t\right)dt - \int_0^\tau w\bar{L}H(t)dt\right] \\
&\quad + \lambda\mathbb{E}\left[\int_0^\tau H(t)\mathbb{E}_t\left[\frac{H(\tau)}{H(t)}X_\tau + \int_t^\tau \frac{H(s)}{H(t)}\left(c_s + wl_s\right)ds - \int_t^\tau w\bar{L}\frac{H(s)}{H(t)}ds\right]dD_t\right] \\
&\leq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda D_t e^{\beta t}H(t))dt + e^{-\beta\tau}\tilde{U}_1(\lambda D_\tau e^{\beta\tau}H(\tau))\right] + \lambda\mathbb{E}\left[\int_0^\tau w\bar{L}D_t H(t)dt\right] + \lambda x \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(\tilde{u}(\lambda D_t e^{\beta t}H(t)) + w\bar{L}\lambda D_t e^{\beta t}H(t)\right)dt + e^{-\beta\tau}\tilde{U}_1(\lambda D_\tau e^{\beta\tau}H(\tau))\right] + \lambda x \\
&\triangleq \tilde{V}(\lambda, D_t;\tau) + \lambda x,
\end{aligned}
$$

where

$$
\tilde{U}_1(y) \triangleq \sup_{\{x\geq 0\}}\left[U(x) - xy\right] = U(I_2(y)) - yI_2(y).
$$

The second inequality is obtained from constraint (6.1) and $dD_t \leq 0$. In the last equality, we have defined $\tilde{V}(\lambda, D_t;\tau)$. For a fixed $\tau \in \mathcal{S}$, $V_\tau(x) \leq \inf_{\{\lambda>0,\ D_t>0\}}[\tilde{V}(\lambda, D_t;\tau) + \lambda x]$ and the equality holds if for $0 \leq t < \tau$

$$
\frac{\partial u}{\partial c}(c_t, l_t) = \lambda D_t e^{\beta t}H(t) \quad\text{and}\quad \frac{\partial u}{\partial l}(c_t, l_t) = w\lambda D_t e^{\beta t}H(t),\ \text{on } \{0 \leq l_t \leq L\},
$$

$$
\frac{\partial u}{\partial c}(c_t, L) = \lambda D_t e^{\beta t}H(t) \quad\text{and}\quad l_t = L,\ \text{on } \{l_t \geq L\},
$$

$$
X_\tau = I_2(\lambda D_\tau e^{\beta\tau}H(\tau)),
$$

$$
\mathbb{E}\left[\int_0^\tau H(t)\left((c_t + wl_t) - w\bar{L}\right)dt + H(\tau)X_\tau\right] = x
$$

and

$$
\mathbb{E}_t\left[\int_t^\tau \frac{H(s)}{H(t)}\left((c_s + wl_s) - w\bar{L}\right)ds + \frac{H(\tau)}{H(t)}X_\tau\right] = 0
$$

21

for all $0 \leq t \leq \tau$ where dD_t is non-zero, i.e., $D.$ is not a constant over a neighborhood of t. So we obtain $V_\tau(x) = \inf_{\{\lambda>0,\ D_t>0\}}[\tilde{V}(\lambda, D_t; \tau) + \lambda x]$. Therefore the value function $V(\cdot)$ is obtained by

$$V(x) = \sup_{\tau \in \mathcal{S}} V_\tau(x) = \sup_{\tau \in \mathcal{S}} \inf_{\{\lambda>0,\ D_t>0\}} [\tilde{V}(\lambda, D_t; \tau) + \lambda x].$$

Similar to Karatzas and Wang (2000) we redefine

$$\tilde{V}(\lambda) \triangleq \sup_{\tau \in \mathcal{S}} \inf_{D_t>0} \tilde{V}(\lambda, D_t; \tau) = \inf_{D_t>0} \sup_{\tau \in \mathcal{S}} \tilde{V}(\lambda, D_t; \tau)$$

and we obtain the value function

$$V(x) = \inf_{\lambda>0} \left[\tilde{V}(\lambda) + \lambda x \right].$$

(cf. Proposition 4.1)

6.2   The Optimization Problem

For simplicity we again assume that there is only one risky asset, i.e., $N = 1$. In order to obtain $\tilde{V}(\lambda)$ we define

$$\phi(t, z) = \sup_{\tau>t} \inf_{D_t>0} \mathbb{E}^{z_t=z} \left[\int_t^\tau e^{-\beta s} \{\tilde{u}(z_s) + w\bar{L}z_s\}ds + e^{-\beta\tau}\tilde{U}_1(z_\tau) \right],$$

where $z_t = \lambda D_t e^{\beta t} H_t$, $z_0 = \lambda > 0$. Itô's formula implies $\frac{dz_t}{z_t} = \frac{dD_t}{D_t} + (\beta - r)dt - \theta dB_t$.

Then we obtain the following Bellman equation

(6.2)
$$\min \left\{ \mathcal{L}\phi(t, z) + e^{-\beta t}\{\tilde{u}(z) + w\bar{L}z\}, -\frac{\partial\phi}{\partial z} \right\} = 0,$$

where the differential operator is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r)z\frac{\partial}{\partial z} + \frac{1}{2}\theta^2 z^2 \frac{\partial^2}{\partial z^2}.$$

(For a more detailed derivation of a similar equation, see Section 5 of He and Pagès (1993)).

Now let D_t^* be the optimal solution of the Bellman equation (6.2), then the optimal stopping time problem can be derived by the following changed variational inequality.

Variational Inequality 2. *Find a free boundary $\bar{z} > 0$, $\hat{z}$ which makes zero wealth level and a function $\tilde{\phi}(\cdot, \cdot) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times (\mathbb{R}^+ \setminus \{\bar{z}\}))$ satisfying*

(1) $\frac{\partial\tilde{\phi}}{\partial z}(t, z) = 0$, $z \geq \hat{z}$

(2) $\frac{\partial\tilde{\phi}}{\partial z}(t, z) \leq 0$, $0 < z \leq \hat{z}$

22

(3) $\mathcal{L}\tilde{\phi} + e^{-\beta t}(\tilde{u}(z) + w\bar{L}z) = 0$, $\bar{z} < z \le \hat{z}$

(4) $\mathcal{L}\tilde{\phi} + e^{-\beta t}(\tilde{u}(z) + w\bar{L}z) \le 0$, $0 < z \le \bar{z}$

(5) $\tilde{\phi}(t,z) \ge e^{-\beta t}\tilde{U}_1(z)$, $z > \bar{z}$

(6) $\tilde{\phi}(t,z) = e^{-\beta t}\tilde{U}_1(z)$, $0 < z \le \bar{z}$,

for all $t > 0$, with the boundary conditions

$$(6.3) \qquad \frac{\partial \tilde{\phi}}{\partial z}(t,\hat{z}) = 0 \quad and \quad \frac{\partial^2 \tilde{\phi}}{\partial z^2}(t,\hat{z}) = 0.^6$$

Now we need to divide the problem into two cases: one is the case where $0 < \bar{z} < \tilde{y} < \hat{z}$ and the other is the case where $0 < \bar{z} < \hat{z}(< \tilde{y})$.[7] The following proposition gives the optimal policies for each case.

Proposition 6.1. 1. (The Case Where $0 < \bar{z} < \tilde{y} < \hat{z}$) The optimal policies are given by $(c^*, l^*, \pi^*, \tau^*)$ such that

$$c_t^* = \begin{cases} \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho\gamma}}(y_t^{\lambda*})^{-\frac{1}{\gamma}} & if \ 0 \le X_t \le \tilde{x}_1 \\ I_{1,c}(y_t^{\lambda*}) & if \ \tilde{x}_1 \le X_t < \bar{x}_1 \\ c_{**} & if \ X_t \ge \bar{x}_1 \end{cases},$$

$$l_t^* = \begin{cases} \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\right)^{\frac{1-\rho-\gamma}{\rho\gamma}}(y_t^{\lambda*})^{-\frac{1}{\gamma}} & if \ 0 \le X_t \le \tilde{x}_1 \\ L & if \ \tilde{x}_1 \le X_t < \bar{x}_1 \\ \bar{L} & if \ X_t \ge \bar{x}_1 \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\left\{n_+(n_+ - 1)c_1(y_t^{\lambda*})^{n_++1} + n_-(n_- - 1)c_2(y_t^{\lambda*})^{n_--1} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda*})^{-\frac{1}{\gamma}}\right\} & if \ 0 \le X_t \le \tilde{x}_1 \\ \frac{\theta}{\sigma}\left\{n_+(n_+ - 1)d_1(y_t^{\lambda*})^{n_++1} + n_-(n_- - 1)d_2(y_t^{\lambda*})^{n_--1}\right. \\ \quad + \frac{2n_+(n_+-1)(y_t^{\lambda*})^{n_++1}}{\theta^2(n_+-n_-)}\int_{\tilde{y}}^{y_t^{\lambda*}}\frac{zI_{1,c}(z) - u(I_{1,c}(z),L)}{z^{n_++1}}dz \\ \quad - \frac{2n_-(n_--1)(y_t^{\lambda*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\tilde{y}}^{y_t^{\lambda*}}\frac{zI_{1,c}(z) - u(I_{1,c}(z),L)}{z^{n_-+1}}dz \\ \quad \left. + \frac{2}{\theta^2}\frac{y_t^{\lambda*}I_{1,c}(y_t^{\lambda*}) - u(I_{1,c}(y_t^{\lambda*}),L)}{y_t^{\lambda*}}\right\} & if \ \tilde{x}_1 \le X_t < \bar{x}_1 \\ -\frac{\theta}{\sigma}\frac{U'(X_t)}{U''(X_t)} & if \ X_t \ge \bar{x}_1 \end{cases}$$

with

$$\tau^* = \inf\{t > 0 \mid X^*(t) \ge \bar{x}_1\}.$$

---

[6]Equation (1) of Variational Inequality 2 implies that $\frac{\partial^2 \tilde{\phi}}{\partial z^2}(t,z) = 0$ for all $z > \hat{z}$. By the smooth-pasting, i.e., C^2-condition of $\tilde{\phi}(t,\cdot)$ we obtain $\frac{\partial \tilde{\phi}}{\partial z}(t,\hat{z}) = 0$ and $\frac{\partial^2 \tilde{\phi}}{\partial z^2}(t,\hat{z}) = 0$ at $z = \hat{z}$.

[7]This case converges to that of Farhi and Panageas (2007) as ρ approaches 0.

In this case the optimal wealth processes are given by

$$X_*(t) = -n_+ c_1 (y_t^{\lambda_*})^{n_+ - 1} - n_- c_2 (y_t^{\lambda_*})^{n_- - 1} + \frac{1-\gamma}{\gamma} \frac{A_1}{K} (y_t^{\lambda_*})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r}$$

and

$$X^*(t) = -n_+ d_1 (y_t^{\lambda^*})^{n_+ - 1} - n_- d_2 (y_t^{\lambda^*})^{n_- - 1} - \frac{w(\bar{L} - L)}{r}$$

$$- \frac{2n_+ (y_t^{\lambda^*})^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz$$

$$+ \frac{2n_- (y_t^{\lambda^*})^{n_- - 1}}{\theta^2 (n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^*}} \frac{z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz.$$

2. (The Case Where $0 < \bar{z} < \hat{z}(< \tilde{y})$) The optimal policies are given by $(c^*, l^*, \pi^*, \tau^*)$

such that

$$c_t^* = \begin{cases} I_{1,c}(y_t^{\lambda^*}) & \text{if } 0 \le X_t < \bar{x}_1 \\ c_{**} & \text{if } X_t \ge \bar{x}_1 \end{cases},$$

$$l_t^* = \begin{cases} L & \text{if } 0 \le X_t < \bar{x}_1 \\ \bar{L} & \text{if } X_t \ge \bar{x}_1 \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma} \left\{ n_+(n_+ - 1) d_1 (y_t^{\lambda^*})^{n_+ - 1} + n_-(n_- - 1) d_2 (y_t^{\lambda^*})^{n_- - 1} \right. \\ \quad + \frac{2n_+(n_+ - 1)(y_t^{\lambda^*})^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{z}}^{y_t^{\lambda^*}} \frac{-w(\bar{L}-L)z + z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz \\ \quad - \frac{2n_-(n_- - 1)(y_t^{\lambda^*})^{n_- - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{z}}^{y_t^{\lambda^*}} \frac{-w(\bar{L}-L)z + z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz \\ \quad \left. + \frac{2}{\theta^2} \frac{-w(\bar{L}-L) y_t^{\lambda^*} + y_t^{\lambda^*} I_{1,c}(y_t^{\lambda^*}) - u(I_{1,c}(y_t^{\lambda^*}), L)}{y_t^{\lambda^*}} \right\} & \text{if } 0 \le X_t < \bar{x}_1 \\ -\frac{\theta}{\sigma} \frac{U'(X_t)}{U''(X_t)} & \text{if } X_t \ge \bar{x}_1 \end{cases}$$

with

$$\tau^* = \inf \{ t > 0 \mid X^*(t) \ge \bar{x}_1 \}.$$

In this case the optimal wealth process is given by

$$X^*(t) = -n_+ d_1 (y_t^{\lambda^*})^{n_+ - 1} - n_- d_2 (y_t^{\lambda^*})^{n_- - 1}$$

$$- \frac{2n_+ (y_t^{\lambda^*})^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{z}}^{y_t^{\lambda^*}} \frac{-w(\bar{L} - L)z + z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz$$

$$+ \frac{2n_- (y_t^{\lambda^*})^{n_- - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{z}}^{y_t^{\lambda^*}} \frac{-w(\bar{L} - L)z + z I_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz.$$

(Other coefficients are given in Appendix D.)

7 Conclusion

We have solved an optimal portfolio and consumption-leisure and retirement choice problem of an agent who has a period utility function with a constant elasticity of substitution(CES) between consumption and leisure and a constant Stiglitz relative risk aversion coefficient. We have obtained a closed form solution by solving a free boundary value problem by using a martingale approach. We have provided the optimal policies in closed forms and characterized properties of the optimal policies.

The optimal retirement time is characterized by the first hitting time of a threshold wealth level. Generally consumption jumps around retirement due to discontinuity in leisure rates. We have also considered the case with liquidity constraints, i.e., the case where the agent cannot borrow against her future labor income.

A Proof of Proposition 5.1

We consider the PDE (1) of Variational Inequality 1

$$(A.1) \qquad \mathcal{L}\phi + e^{-\beta t}\tilde{u}(y) = 0, \quad y > \bar{y},$$

with a boundary condition $\phi(t, \bar{y}) = e^{-\beta t}\tilde{U}(\bar{y})$. First we consider the equation (A.1) for $y \geq \tilde{y}$. That is,

$$\frac{\partial \phi}{\partial t} + (\beta - r)y\frac{\partial \phi}{\partial y} + \frac{1}{2}||\theta||^2 y^2 \frac{\partial^2 \phi}{\partial y^2} + e^{-\beta t}A_1 y^{-\frac{1-\gamma}{\gamma}} = 0, \quad y \geq \tilde{y}.$$

Guessing a solution form $\phi(t, y) = e^{-\beta t}v(y)$, then we derive

$$v(y) = C_2 y^{n_-} + \frac{A_1}{K}y^{-\frac{1-\gamma}{\gamma}}, \quad y \geq \tilde{y},$$

by solving the second order ordinary differential equation(ODE), where

$$\frac{1}{2}||\theta||^2 n^2 + \left(\beta - r - \frac{1}{2}||\theta||^2\right)n - \beta = 0$$

with two roots $n_+ > 1$ and $n_- < 0$. Similarly, for $\bar{y} < y \leq \tilde{y}$, we also derive

$$v(y) = D_1 y^{n_+} + D_2 y^{n_-} - \frac{wL}{r}y + \frac{2y^{n_+}}{||\theta||^2(n_+ - n_-)}\int_{\tilde{y}}^y \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}}dz$$

$$- \frac{2y^{n_-}}{||\theta||^2(n_+ - n_-)}\int_{\tilde{y}}^y \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}}dz.$$

From the principle of the C^1-condition at $y = \tilde{y}$ and $y = \bar{y}$, we derive the coefficients C_2, D_1, D_2 and the free boundary $\bar{y}$ as follows:

$$D_1 = -\frac{1}{n_+ - n_-}\left(\frac{1-\gamma}{\gamma} + n_-\right)\frac{A_1}{K}\tilde{y}^{-\frac{1-\gamma}{\gamma}-n_+} + \frac{1-n_-}{n_+ - n_-}\frac{wL}{r}\tilde{y}^{1-n_+},$$

$$D_1 \bar{y}^{n_+} + \frac{2\bar{y}^{n_+}}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\bar{y}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}} dz$$

$$+ \frac{n_-}{n_+ - n_-} U(I_2(\bar{y})) + \frac{1 - n_-}{n_+ - n_-} \left(I_2(\bar{y}) + \frac{w(\bar{L} - L)}{r} \right) \bar{y} = 0,$$

from which $\bar{y}$ is derived,

$$D_2 = \frac{n_+}{n_+ - n_-} \bar{y}^{-n_-} U(I_2(\bar{y})) - \frac{n_+ - 1}{n_+ - n_-} \left(I_2(\bar{y}) + \frac{w(\bar{L} - L)}{r} \right) \bar{y}^{1 - n_-}$$

$$+ \frac{2}{||\theta||^2(n_+ - n_-)} \int_{\tilde{y}}^{\bar{y}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}} dz,$$

and

$$C_2 = -\frac{1}{n_+ - n_-} \left(\frac{1 - \gamma}{\gamma} + n_+ \right) \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma} - n_-} - \frac{n_+ - 1}{n_+ - n_-} \frac{wL}{r} \tilde{y}^{1 - n_-} + D_2.$$

Now we will show that $v''(\cdot)$ is continuous at $y = \tilde{y}$. It is sufficient to show that

$$n_+(n_+ - 1)D_1\tilde{y}^{n_+} + n_-(n_- - 1)(D_2 - C_2)\tilde{y}^{n_-} - \frac{1 - \gamma}{\gamma^2} \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma}}$$

$$+ \frac{2}{||\theta||^2} \{ -u(I_{1,c}(\tilde{y}), L) + I_{1,c}(\tilde{y})\tilde{y} \} = 0,$$

with C_2, D_1 and D_2 obtained from the previous processes. Now we show the statement as follows:

$$\text{(LHS)} = -\frac{1}{n_+ - n_-} \left\{ n_+(n_+ - 1) \left(\frac{1 - \gamma}{\gamma} + n_- \right) - n_-(n_- - 1) \left(\frac{1 - \gamma}{\gamma} + n_+ \right) \right\} \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma}}$$

$$- (n_+ - 1)(n_- - 1)\frac{wL}{r}\tilde{y} - \frac{1 - \gamma}{\gamma^2}\frac{A_1}{K}\tilde{y}^{-\frac{1-\gamma}{\gamma}} + \frac{2}{||\theta||^2} \{ -u(I_{1,c}(\tilde{y}), L) + I_{1,c}(\tilde{y})\tilde{y} \}$$

$$= -\left\{ (n_+ + n_- - 1)\frac{1 - \gamma}{\gamma} + n_+ n_- + \frac{1 - \gamma}{\gamma^2} \right\} \frac{A_1}{K} \tilde{y}^{-\frac{1-\gamma}{\gamma}}$$

$$+ \frac{2}{||\theta||^2} wL\tilde{y} - \frac{2}{||\theta||^2} \left(A_1\tilde{y}^{-\frac{1-\gamma}{\gamma}} + wL\tilde{y} \right) = 0$$

since

$$-\frac{2}{||\theta||^2}K = (n_+ + n_- - 1)\frac{1 - \gamma}{\gamma} + n_+ n_- + \frac{1 - \gamma}{\gamma^2}$$

and

$$A_1\tilde{y}^{-\frac{1-\gamma}{\gamma}} + wL\tilde{y} = u(I_{1,c}(\tilde{y}), L) - I_{1,c}(\tilde{y})\tilde{y}$$

by definition of $\tilde{u}(\cdot)$.

The inequality of (2) of Variational Inequality 1 is equivalent to (5.4) and the inequality of (3) of Variational Inequality 1 is equivalent to (5.5) and (5.6).

B   Proof of Theorem 5.3

Note that the optimal stopping time τ^* is a rewriting of (5.10) by using the optimal wealth process $X^{**}(t)$ of (5.11). Also we can easily see the optimal consumption and leisure from Lemma 3.1 and (4.2), (4.3), (4.4), and (4.5). So it is enough to show that the given consumption, leisure and portfolio processes [8] generate the optimal wealth processes $X_{**}(t)$ and $X^{**}(t)$ of (5.12) and (5.11), respectively.

For $-\frac{w\bar{L}}{r} < X_t \leq \tilde{x}$, applying Itô's formula to (5.12), then we have

$$
\begin{aligned}
dX_{**}(t) &= \left[-n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 2} - \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1+\gamma}{\gamma}} \right] dy_t^{\lambda_{**}} \\
&\quad + \frac{1}{2}\left[-n_-(n_- - 1)(n_- - 2)C_2(y_t^{\lambda_{**}})^{n_- - 3} + \frac{1-\gamma^2}{\gamma^3}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1+2\gamma}{\gamma}} \right] (dy_t^{\lambda_{**}})^2 \\
&= \left[-n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 1} - \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right][(\beta - r)dt - \theta dB(t)] \\
&\quad + \frac{1}{2}\left[-n_-(n_- - 1)(n_- - 2)C_2(y_t^{\lambda_{**}})^{n_- - 1} + \frac{1-\gamma^2}{\gamma^3}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right]\theta^2 dt \\
&= r\left[-n_- C_2(y_t^{\lambda_{**}})^{n_- - 1} + \frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r} \right] dt \\
&\quad + \theta^2\left[n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 1} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right] dt \\
&\quad - \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}} \right)^{\frac{1-\rho-\gamma}{\rho\gamma}} (y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} dt \\
&\quad + w\left[\bar{L} - \alpha^{\frac{1-\gamma}{\rho\gamma}}\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}}\left(1 + w\left(\frac{\alpha w}{1-\alpha}\right)^{\frac{1}{\rho-1}} \right)^{\frac{1-\rho-\gamma}{\rho\gamma}} (y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right] dt \\
&\quad - \frac{1-\gamma}{\gamma}\left[\frac{\beta - r}{\gamma} + r + \frac{\gamma - 1}{2\gamma^2}\theta^2 - K \right]\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} dt \\
&\quad - n_- C_2\left[\frac{1}{2}\theta^2 n_-^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n_- - \beta \right](y_t^{\lambda_{**}})^{n_- - 1} dt \\
&\quad + \theta\left[n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 1} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right] dB(t).
\end{aligned}
$$

Here we can see that the fifth and the sixth terms of the right hand side of the last equality are equal to zero by definitions of K and n's.(See equation (5.3).) For the last term, if we correspond

$$
\pi_t^* \sigma = \theta\left[n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 1} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} \right],
$$

---

[8]For $0 \leq t < \tau$, the optimal portfolio is unknown yet. We will compare the coefficients of the wealth dynamics with those of the SDEs obtained from applying Itô's formula to (5.11) and (5.12). Then we can determine the optimal portfolio for each case.

then we obtain

$$dX_{**}(t) = [rX_{**}(t) + \pi_t^*(b - r) - c_t^* + w(\bar{L} - l_t^*)]dt + \pi_t^*\sigma dB(t).$$

So the optimal wealth is induced from the strategies (c^*, l^*, π^*) for $-\frac{w\bar{L}}{r} < X_t \leq \tilde{x}$.

Similarly, for $\tilde{x} \leq X_t < \bar{x}$, we also obtain the optimal portfolio

$$
\begin{aligned}
\pi_t^*\sigma = \theta &\left[n_+(n_+ - 1)D_1(y_t^{\lambda^{**}})^{n_+-1} + n_-(n_- - 1)D_2(y_t^{\lambda^{**}})^{n_--1} \right.\\
&+ \frac{2n_+(n_+ - 1)(y_t^{\lambda^{**}})^{n_+-1}}{\theta^2(n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^{**}}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_++1}}dz \\
&- \frac{2n_-(n_- - 1)(y_t^{\lambda^{**}})^{n_--1}}{\theta^2(n_+ - n_-)} \int_{\tilde{y}}^{y_t^{\lambda^{**}}} \frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_-+1}}dz \\
&\left.+ \frac{2}{\theta^2} \frac{y_t^{\lambda^{**}}I_{1,c}(y_t^{\lambda^{**}}) - u(I_{1,c}(y_t^{\lambda^{**}}), L)}{y_t^{\lambda^{**}}} \right].
\end{aligned}
$$

Thus we obtain

$$dX^{**}(t) = [rX^{**}(t) + \pi_t^*(b - r) - c_t^* + w(\bar{L} - l_t^*)]dt + \pi_t^*\sigma dB(t).$$

So the optimal wealth is induced from the strategies (c^*, l^*, π^*) for $\tilde{x} \leq X_t < \bar{x}$.

C   Proof of Proposition 5.2

We see that $X^{**}(t)$ and $X_{**}(t)$ in (5.11) and (5.12) are decreasing with respect to $y_t^{\lambda^{**}}$ and $y_t^{\lambda_{**}}$ respectively. (we can see from Remark 5.1.) For fixed X_t, so we can see that

$$y_t^{\lambda^{**}} > y_{t,N}^{\lambda^{**}} \quad \text{and} \quad y_t^{\lambda_{**}} > y_{t,N}^{\lambda_{**}}$$

by comparing (5.11) with (5.14) and (5.12) with (5.15), and by the assumption $D_2 > 0$ in Remark 5.2. Thus we obtain from comparing c_t^* with $c_{t,N}^*$

$$c_{t,N}^* > c_t^* \qquad \text{for } -\frac{w\bar{L}}{r} < X_t < \bar{x}.^9$$

Next, we compare π_t^* with $\pi_{t,N}^*$. For $-\frac{w\bar{L}}{r} < X_t \leq \tilde{x}$ and fixed, we obtain the following equality from (5.12) and (5.15),

$$
\begin{aligned}
X_t &= -n_-C_2(y_t^{\lambda_{**}})^{n_--1} + \frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r} \\
&= -n_-(C_2 - D_2)(y_{t,N}^{\lambda_{**}})^{n_--1} + \frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r}.
\end{aligned}
$$

---

^{9}In fact, we need to divide the interval $-\frac{w\bar{L}}{r} < X_t < \bar{x}$ into three cases: $-\frac{w\bar{L}}{r} < X_t < \tilde{x}_N$, $\tilde{x}_N < X_t < \tilde{x}$ and $\tilde{x} < X_t < \bar{x}$. The first and the third cases are obvious. The second case is also trivial since $\tilde{y}(=\tilde{y}_N)$ correspond to $\tilde{x}$ and $\tilde{x}_N$, respectively and the optimal consumption process is increasing with respect to the wealth process.

We multiply both sides of the above equality by $n_- - 1$, then we obtain

$$- n_-(n_- - 1)C_2(y_t^{\lambda_{**}})^{n_- - 1} + (n_- - 1)\frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}}$$

$$= -n_-(n_- - 1)(C_2 - D_2)(y_{t,N}^{\lambda_{**}})^{n_- - 1} + (n_- - 1)\frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}}$$

and we rewrite the equality as

$$- \frac{\sigma}{\theta}\pi_t^* + (n_- - 1)\frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_t^{\lambda_{**}})^{-\frac{1}{\gamma}}$$

$$= -\frac{\sigma}{\theta}\pi_{t,N}^* + (n_- - 1)\frac{1-\gamma}{\gamma}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}} + \frac{1-\gamma}{\gamma^2}\frac{A_1}{K}(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}}.$$

Sequentially,

$$\frac{\sigma}{\theta}\pi_{t,N}^* = \frac{\sigma}{\theta}\pi_t^* + \left(n_- - 1 + \frac{1}{\gamma}\right)\frac{1-\gamma}{\gamma}\frac{A_1}{K}\left\{(y_{t,N}^{\lambda_{**}})^{-\frac{1}{\gamma}} - (y_t^{\lambda_{**}})^{-\frac{1}{\gamma}}\right\}.$$

If we substitute $\frac{\gamma-1}{\gamma}$ for n into (5.3), then

$$\frac{1}{2}\theta^2\left(\frac{\gamma-1}{\gamma}\right)^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)\left(\frac{\gamma-1}{\gamma}\right) - \beta = -K < 0.$$

So we obtain $n_- - 1 + \frac{1}{\gamma} < 0$. Since $y_t^{\lambda_{**}} > y_{t,N}^{\lambda_{**}}$, we obtain

$$\pi_{t,N}^* < \pi_t^* \qquad \text{for} \quad -\frac{w\bar{L}}{r} < X_t \le \tilde{x}_N.$$

Similarly we obtain

$$\pi_{t,N}^* < \pi_t^* \qquad \text{for } \tilde{x}_N \le X_t < \bar{x}.$$

D   Coefficients in Proposition 6.1

D.1   The Case Where $0 < \bar{z} < \tilde{y} < \hat{z}$ in Proposition 6.1

If we can determine the values of $\bar{z}$ and $\hat{z}$ from the following two equations

$$\frac{1}{n_-}\left(n_+ + \frac{1-\gamma}{\gamma}\right)\frac{1-\gamma}{\gamma}\frac{A_1}{K}\hat{z}^{-\left(n_- + \frac{1-\gamma}{\gamma}\right)} - \frac{n_+ - 1}{n_-}\frac{w\bar{L}}{r}\hat{z}^{1-n_-}$$

$$= -\frac{w(\bar{L} - L)}{r}(n_+ - 1)\bar{z}^{1-n_-} + \frac{2}{\theta^2}\int_{\tilde{y}}^{\bar{z}}\frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}}dz$$

$$+ n_+\bar{z}^{-n_-}U(I_2(\bar{z})) - (n_+ - 1)\bar{z}^{1-n_-}I_2(\bar{z})$$

$$- \left(n_+ + \frac{1-\gamma}{\gamma}\right)\frac{A_1}{K}\tilde{y}^{-\left(n_- + \frac{1-\gamma}{\gamma}\right)} - \frac{wL}{r}(n_+ - 1)\tilde{y}^{1-n_-}$$

29

and

$$-\frac{1}{n_+}\left(n_- + \frac{1-\gamma}{\gamma}\right)\frac{1-\gamma}{\gamma}\frac{A_1}{K}\hat{z}^{-\left(n_+ + \frac{1-\gamma}{\gamma}\right)} + \frac{n_- - 1}{n_+}\frac{w\bar{L}}{r}\hat{z}^{1-n_+}$$

$$= \frac{w(\bar{L} - L)}{r}(n_- - 1)\bar{z}^{1-n_+} - \frac{2}{\theta^2}\int_{\tilde{y}}^{\bar{z}}\frac{zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}}dz$$

$$- n_-\bar{z}^{-n_+}U(I_2(\bar{z})) + (n_- - 1)\bar{z}^{1-n_+}I_2(\bar{z})$$

$$+ \left(n_- + \frac{1-\gamma}{\gamma}\right)\frac{A_1}{K}\tilde{y}^{-\left(n_+ + \frac{1-\gamma}{\gamma}\right)} + \frac{wL}{r}(n_- - 1)\tilde{y}^{1-n_+},$$

then we have

$$c_1 = -\frac{1}{n_+(n_+ - n_-)}\left(n_- + \frac{1-\gamma}{\gamma}\right)\frac{1-\gamma}{\gamma}\frac{A_1}{K}\hat{z}^{-\left(n_+ + \frac{1-\gamma}{\gamma}\right)} + \frac{n_- - 1}{n_+(n_+ - n_-)}\frac{w\bar{L}}{r}\hat{z}^{1-n_+}$$

and

$$c_2 = \frac{1}{n_-(n_+ - n_-)}\left(n_+ + \frac{1-\gamma}{\gamma}\right)\frac{1-\gamma}{\gamma}\frac{A_1}{K}\hat{z}^{-\left(n_- + \frac{1-\gamma}{\gamma}\right)} - \frac{n_+ - 1}{n_-(n_+ - n_-)}\frac{w\bar{L}}{r}\hat{z}^{1-n_-}.$$

Moreover we can also obtain

$$d_1 = c_1 - \frac{1}{n_+ - n_-}\left(n_- + \frac{1-\gamma}{\gamma}\right)\frac{A_1}{K}\tilde{y}^{-\left(n_+ + \frac{1-\gamma}{\gamma}\right)} - \frac{n_- - 1}{n_+ - n_-}\frac{wL}{r}\tilde{y}^{1-n_+}$$

and

$$d_2 = c_2 + \frac{1}{n_+ - n_-}\left(n_+ + \frac{1-\gamma}{\gamma}\right)\frac{A_1}{K}\tilde{y}^{-\left(n_- + \frac{1-\gamma}{\gamma}\right)} + \frac{n_+ - 1}{n_+ - n_-}\frac{wL}{r}\tilde{y}^{1-n_-}.$$

Critical wealth levels are given by $\bar{x}_1 = I_2(\bar{z})$ and $\tilde{x}_1 = -n_+ c_1 \tilde{y}^{n_+ - 1} - n_- c_2 \tilde{y}^{n_- - 1} + \frac{1-\gamma}{\gamma}\frac{A_1}{K}\tilde{y}^{-\frac{1}{\gamma}} - \frac{w\bar{L}}{r}$.

D.2 The Case Where $0 < \bar{z} < \hat{z}(< \tilde{y})$ in Proposition 6.1

If we can determine the values of $\bar{z}$ and $\hat{z}$ from the following two equations

$$\frac{1}{n_-}\frac{2}{\theta^2}\left(-w(\bar{L} - L)\hat{z} + \hat{z}I_{1,c}(\hat{z}) - u(I_{1,c}(\hat{z}), L)\right)\hat{z}^{-n_-}$$

$$= \frac{2}{\theta^2}\int_{\hat{z}}^{\bar{z}}\frac{-w(\bar{L} - L)z + zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_- + 1}}dz$$

$$+ n_+\bar{z}^{-n_-}U(I_2(\bar{z})) - (n_+ - 1)\bar{z}^{1-n_-}I_2(\bar{z})$$

and

$$-\frac{1}{n_+}\frac{2}{\theta^2}\left(-w(\bar{L} - L)\hat{z} + \hat{z}I_{1,c}(\hat{z}) - u(I_{1,c}(\hat{z}), L)\right)\hat{z}^{-n_+}$$

$$= -\frac{2}{\theta^2}\int_{\hat{z}}^{\bar{z}}\frac{-w(\bar{L} - L)z + zI_{1,c}(z) - u(I_{1,c}(z), L)}{z^{n_+ + 1}}dz$$

$$- n_-\bar{z}^{-n_+}U(I_2(\bar{z})) + (n_- - 1)\bar{z}^{1-n_+}I_2(\bar{z}),$$

30

then we have

$$d_1 = -\frac{1}{n_+(n_+ - n_-)}\frac{2}{\theta^2}\left(-w(\bar{L} - L)\hat{z} + \hat{z}I_{1,c}(\hat{z}) - u(I_{1,c}(\hat{z}), L)\right)\hat{z}^{-n_+}$$

and

$$d_2 = \frac{1}{n_-(n_+ - n_-)}\frac{2}{\theta^2}\left(-w(\bar{L} - L)\hat{z} + \hat{z}I_{1,c}(\hat{z}) - u(I_{1,c}(\hat{z}), L)\right)\hat{z}^{-n_-}.$$

Critical wealth level is given by $\bar{x}_1 = I_2(\bar{z})$.

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