Valuing Qualitative Options with Stochastic Volatility

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Abstract

We find a closed-form formula for valuing a time-switch option where its underlying asset is affected by stochastically changing market environments, and apply it to the valuation of other qualitative options such as corridor options and options in foreign exchange markets. The stochastic market environments are modeled as a Markov regime-switching process. This analytic formula provides us a rapid and accurate valuation scheme for valuing qualitative options with stochastic volatility.

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1 Introduction

It is well known that the volatility of financial securities, such as stocks and bonds, tends to change over time depending on market environments of economic, political factors and business cycle. Many researchers have studied the problem of valuing options when the volatility of underlying asset is stochastic and there have been a number of important results (e.g., Hull and White [11], Wiggins [17], Stein and Stein [15], Heston [10]). Most of such researchers have believed that stochastic volatility models can provide us some explanations for smile feature of Black-Scholes implied volatility(called as a *volatility smile*). However, there is an disadvantage of difficulty in handling when one applies stochastic volatility models to the valuation of options.

By the way, in 1995 Pechtl [14] classified the types of options into *quantitative* and *qualitative*. Quantitative option is an option of classical type and its payout is represented as a limit function of linear combinations of that of binary options. Different from this, the payout of qualitative option is determined by the number of time units for which the underlying asset process stays in a prefixed price interval. Qualitative options are exotic and path-dependent, therefore it is very difficult to get a closed-form formula for their values. Representative examples of such qualitative options are a *time-switch option* and a *corridor option*. Pechtl [14] obtained a closed-form formula for the value of a time-switch option, considering non-stochastic volatility model. Also Fusai [7] provided a semi-analytic formula for valuing corridor options where the underlying asset was evolved by constant volatility.

This paper is focused on providing a closed-form formula for valuing time-switch option under stochastic market environments and applying it to valuing other qualitative options. We model the stochastic market environments by assuming that the drift and volatility terms of the underlying asset process are governed by a Markov *regime-switching* process, which was firstly introduced by Hamilton [9].

Regime-switching models are planned to capture discrete changes in the economic events that influence financial time series. Recently several researches using these models have been fulfilled. For instance, So *et al.* [16] generalized usual stochastic volatility model to encompass Markov regime-switching properties, and Bollen *et al.* [2] investigated the ability of regime-switching models to capture the dynamics of foreign exchange rates. Fuh *et al.* [6] provided a closed-form formula for the value of a European call option and showed several interesting empirical investigations, which told us that regime-switching models can produce the three empirical phenomena, asymmetric leptokurtic features, volatility smile and *volatility clustering*. By using the moment-generating function of the concept of occupation time, Edwards [5] suggested a general way to incorporate regime-switching models in financial models. Duan *et al.* [4] established a class of GARCH option models under regime switching. Furthermore, there exist a number of researches for option pricing when the fluctuations of underlying asset price are based on a notion of *inside information* (see Naik [13], Buffington and Elliott [3], Duan *et al.* [4], Edwards [5], and Guo [8]).

On the other hand, some numerical methods for valuing options with regime-switching volatility have been developed. In particular, Bollen [1] developed a lattice method(called *pentanomial tree*) in order to value both European and American types of options where their underlying assets varied by a Markov regime-switching process. Jang and Koo [12] provided a way that one can approximate analytically the value of American puts with regime-switching volatility. Using it, they derived some interesting features of early exercise boundaries of American puts.

Most of such exiting literatures have considered only the valuation of quantitative options, not qualitative options. As far as we know, there is not any result which provides us an analytic valuation formula for qualitative options with stochastic volatility. The importance of this paper stems from this point of view.

The rest of this paper proceeds as follows. Section 2 explains an option pricing model with regime-switching volatility. Section 3 provides us a closed-form formula for valuing a time-switch option. In Section 4, we give a numerical implementation. Some applications to other qualitative options are presented in Section 5. Finally, Section 6 concludes. All proofs are in Appendix.

2 The Model

Throughout the paper, the market is assumed to be frictionless: there are no transaction costs, no taxes, no restrictions on short sales and no difference between borrowing and lending rates. Suppose that the market has one riskless asset(bond or cash account) and one risky asset(stock). Also, we consider that all activities occur on a filtered *complete* probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}$ is the *augmentation* of a filtration generated by one-dimensional Brownian motion B_t and two Poisson processes $\varphi_H(t)$, $\varphi_L(t)$ on $[0,\infty)$. Two Poisson processes are independent of each other and independent of B_t.

We assume that the underlying asset(stock) price dynamics S_t satisfies the following evolution equation:

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dB_t, \tag{2.1}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are the drift and volatility of the stock process, and both change according to a continuous-time Markov chain.

The Markov chain is independent of B_t and moves between two states. We call such states *high*("H") and *low*("L") regimes. The corresponding pairs of drifts and volatilities are denoted by (μ_H, σ_H) and (μ_L, σ_L), respectively. Define the intensity $\lambda_i (i \in \{H, L\})$ of Poisson process φ_i as the rate of leaving regime i. Namely, the random time τ_i of the leaving regime i has an exponential distribution with intensity λ_i, thus satisfies the following:

$$P(\tau_i > t) = e^{-\lambda_i t}, \quad i \in \{H, L\}.$$

Then the transition density over a time interval $[s, s+t]$, $P_{ij}(t) = P\{\sigma_{s+t} = \sigma_j | \sigma_s = \sigma_i\}$, is given by

$$P_{ii}(t) = \frac{\lambda_i}{\lambda_i + \lambda_j} e^{-(\lambda_i+\lambda_j)t} + \frac{\lambda_j}{\lambda_i + \lambda_j} = 1 - P_{ij}(t),$$

where $i, j \in \{H, L\}$, and s, $t \geq 0$. For more details, see Edwards [5].

Since the market is enable to be completed by the arguments in Guo [8], we assume that there is a *risk-neutral measure* $\widetilde{P}$. Applying Girsanov's theorem to Equation (2.1), we can change it to be

$$dS_t = rS_t dt + \sigma(t)S_t d\widetilde{B}_t, \tag{2.2}$$

where r is a riskfree interest rate and $\widetilde{B}_t$ is a standard Brownian motion under $\widetilde{P}$. By using usual calculus, the equation (2.2) is solved as follows,

$$S_t = S_0 \exp\left\{ rt - \frac{1}{2}\int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s) d\widetilde{B}_s \right\}. \tag{2.3}$$

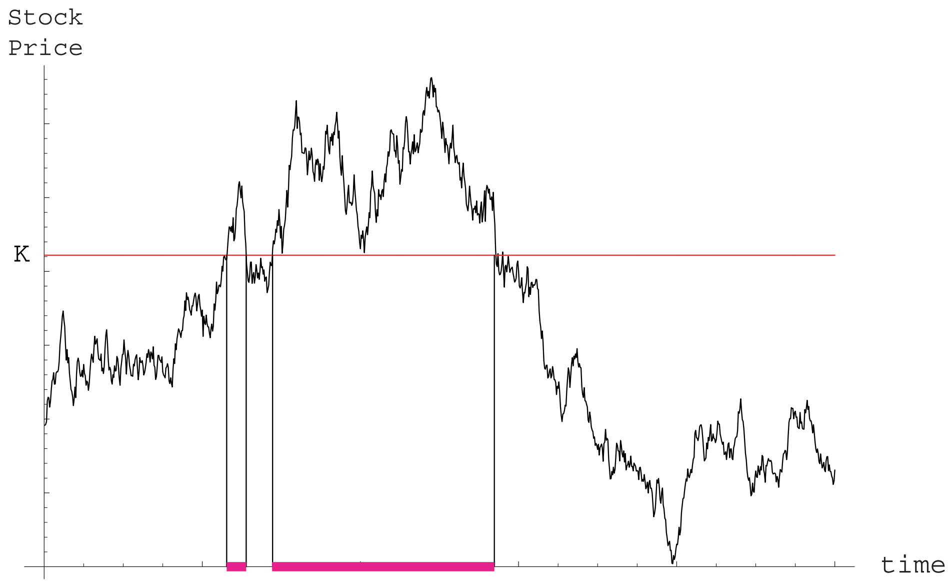

Figure 1: **Payout profile of a time-switch option**

3 The Valuation of a Time-Switch Option

As mentioned in Pechtl [14], a time-switch option has a 'qualitative' property, where writer's payment is determined by multiplying a fixed amount by the number of days during the period that the price of the underlying asset goes up above a prefixed barrier K. Figure 1 depicts its payout profile. The aim of this paper is to provide a closed-form formula, which represents the price of the time-switch option on the stock with stochastic volatility described in the previous section.

Firstly, a random variable should be defined by

$$I_t := \begin{cases} 1, & \text{if } \sigma(t) = \sigma_H, \\ 0, & \text{if } \sigma(t) = \sigma_L. \end{cases}$$

Also, T_t denotes the occupation time that the stock process is in regime H from time 0 till time t, that is,

$$T_t := \int_0^t I_s ds.$$

Following the argument in Pechtl [14], the initial value of the time-switch option can be written as

$$e^{-rT}\widetilde{E}\Big[A\int_0^T 1_{\{S_t>K\}}dt\Big], \tag{3.1}$$

where T is the maturity time of the option, A a fixed amount, K a prefixed barrier, and $1_{\{\cdot\}}$ an *indicator* function. $\widetilde{E}$ implies the expectation with respect to the probability $\widetilde{P}$.

Applying Fubini's theorem and the relationship of (2.3), (3.1) is converted into

$$
\begin{aligned}
& e^{-rT} A \int_0^T \widetilde{E}\left[1_{\{S_t > K\}}\right] dt \\
&= e^{-rT} A \int_0^T \widetilde{P}\{S_t > K\} dt \\
&= e^{-rT} A \int_0^T \widetilde{P}\left\{\frac{1}{2}\int_0^t \sigma(s)^2 ds - \int_0^t \sigma(s) d\widetilde{B}_s < \ln\frac{S_0}{K} + rt\right\} dt \\
&= e^{-rT} A \int_0^T \widetilde{P}\left\{\frac{1}{2}\left((\sigma_H^2 - \sigma_L^2)T_t + \sigma_L^2 t\right) - (\sigma_H - \sigma_L)\int_0^t I_s d\widetilde{B}_s - \sigma_L \widetilde{B}_t < \ln\frac{S_0}{K} + rt\right\} dt.
\end{aligned}
\tag{3.2}
$$

In order to calculate the probability in the last term of (3.2), we verify the properties of

$$X(t) := \int_0^t I_s d\widetilde{B}_s.$$

General properties of Ito integrals tell us that the mean of $X(t)$ is zero and the variance of $X(t)$ is $\widetilde{E}[T_t]$. If T_t is deterministic for time t, we can obtain the following lemma.:

Lemma 3.1. *Given $0 \le T_t = k \le t$, $X(t)$ is normally distributed with mean zero and variance k. Furthermore, this is true even though k is a deterministic function with respect to t.*

Proof. See Appendix A. □

In order to calculate the probability in the last term of (3.2), we need to get the correlation between $X(t)$ and $\widetilde{B}_t$.

Lemma 3.2. *Given $0 \le T_t = k \le t$, the correlation between $X(t)$ and $\widetilde{B}_t$ is $\sqrt{\frac{k}{t}}$.*

Proof. See Appendix B. □

The following Lemma is well-known, hence we use it without any proof.

Lemma 3.3. *Let Y_1 and Y_2 be standard normal variables with correlation coefficient ρ. Then for arbitrary constants a, b, c, d and k,*

$$E[e^{cY_1 + dY_2} 1_{\{aY_1 + bY_2 \ge k\}}] = e^{(c^2 + d^2 + 2\rho cd)/2} \mathcal{N}\left(\frac{ac + bd + \rho(ad + bc) - k}{\sqrt{a^2 + b^2 + 2\rho ab}}\right),$$

where $\mathcal{N}$ denotes a cumulative standard normal distribution.

In particular,

$$P\{aY_1 + bY_2 \le k\} = E[1_{\{aY_1 + bY_2 \le k\}}] = \mathcal{N}\left(\frac{k}{\sqrt{a^2 + b^2 + 2\rho ab}}\right).$$

The probability density function(PDF) of the occupation time T_t has discovered by Naik [13], Guo [8] and Fuh [6], thus we can reach our main result.

Theorem 3.4. *Let $v_i(T;K,A)(i \in \{H,L\})$ be the value of a time-switch option in regime i with maturity T, fixed multiplier A and prefixed barrier K, then it satisfies the following:*

$$\begin{aligned} v_i(T;K,A) &= e^{-rT}\widetilde{E}\left[A\int_0^T 1_{\{S_t>K\}}dt \middle| \sigma(0)=\sigma_i\right] \\ &= e^{-rT}A\int_0^T \Big\{ \int_0^t \mathcal{N}(d_1(t,u))f_i(t,u)du \\ &\qquad + \delta_L(i)\mathcal{N}(d_2(t))e^{-\lambda_L t} + \delta_H(i)\mathcal{N}(d_3(t))e^{-\lambda_H t}\Big\}dt, \end{aligned} \tag{3.3}$$

where $\mathcal{N}$ is a cumulative standard normal distribution. Here,

$$\begin{aligned} d_1(t,u) &:= \frac{\ln S_0/K + rt - \frac{1}{2}(\sigma_H^2-\sigma_L^2)u - \frac{1}{2}\sigma_L^2 t}{\sqrt{(\sigma_H-\sigma_L)^2u + \sigma_L^2 t + 2(\sigma_H-\sigma_L)\sigma_L u}}\ , \\ d_2(t) &:= d_1(t,0) = \frac{\ln S_0/K + rt - \frac{1}{2}\sigma_L^2 t}{\sigma_L\sqrt{t}}\ , \\ d_3(t) &:= d_1(t,t) = \frac{\ln S_0/K + rt - \frac{1}{2}\sigma_H^2 t}{\sigma_H\sqrt{t}}\ , \\ \delta_i(j) &:= \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j. \end{cases} \end{aligned}$$

Also $f_i(t,u)$ is a PDF of T_t where $\sigma(0)=\sigma_i$ such that

$$\begin{aligned} f_H(t,u) &:= e^{-\lambda_L(t-u)-\lambda_H u}\Bigg(\left(\frac{\lambda_H\lambda_L u}{t-u}\right)^{1/2} J_{-1}\left(2(\lambda_H\lambda_L u(t-u))^{1/2}\right) \\ &\qquad + \lambda_H J_0\left(2(\lambda_H\lambda_L u(t-u))^{1/2}\right)\Bigg), \\ f_L(t,u) &:= e^{-\lambda_l t}\delta_0(u) + e^{-\lambda_L(t-u)-\lambda_H u}\Bigg(\left(\frac{\lambda_H\lambda_L (t-u)}{u}\right)^{1/2} J_1\left(2(\lambda_H\lambda_L u(t-u))^{1/2}\right) \\ &\qquad + \lambda_L J_0\left(2(\lambda_H\lambda_L u(t-u))^{1/2}\right)\Bigg), \end{aligned}$$

where $J_a(z)$ is the modified Bessel function defined by

$$J_a(z) := \left(\frac{z}{2}\right)^a \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(a+n+1)}.$$

Proof. See Appendix C. □

The analytic formula of (3.3) contains a double integration, but it can be calculated by using a numerical approximation scheme such as *trapezoidal rule* and *Gaussian quadrature rule*. Nowadays there are many programming languages, for example `mathematica`, with a package containing functions which help us to calculate such integrals directly. In Section 4, we show numerical results obtained by a Gaussian quadrature rule.

parameters			Monte-Carlo				Analytic	
			Benchmark		2000 trials			
S	σ_H	λ_H	v_H	v_L	v_H	v_L	v_H	v_L
100	0.26	1.0	1.3299	1.0025	1.2944	0.9651	1.3281	1.0083
100	0.26	2.0	1.3071	1.0024	1.3140	0.9874	1.2927	1.0041
100	0.40	1.0	1.4904	1.0350	1.5502	1.0539	1.4923	1.0327
100	0.40	2.0	1.4531	1.0278	1.4295	0.9763	1.4620	1.0288
110	0.26	1.0	2.5365	2.7864	2.5913	2.7934	2.5367	2.7856
110	0.26	2.0	2.5653	2.7933	2.5482	2.7558	2.5770	2.7918
110	0.40	1.0	2.3689	2.7533	2.3747	2.7418	2.3633	2.7489
110	0.40	2.0	2.4099	2.7594	2.4079	2.7578	2.4154	2.7574
120	0.26	1.0	3.5943	4.1627	3.5706	4.1436	3.6014	4.1594
120	0.26	2.0	3.6835	4.1726	3.6975	4.2085	3.6898	4.1715
120	0.40	1.0	3.1633	4.0735	3.1366	4.0696	3.1601	4.0803
120	0.40	2.0	3.2690	4.0955	3.3557	4.0894	3.2746	4.0975
CPU time(sec)			-		140		8	

Table 1: **The values of time-swith options obtained by the analytic solution in Theorem 3.4 and the Monte-Carlo method by Bollen [1].** Default parameters are $A = 5$, $K = 110$, $T = 1$, $\sigma_L = 0.1313$, $r = 0.06$, and $\lambda_L = 0.5$. Initial stock prices S, high-regime volatilities σ_H and high-regime intensities λ_H are displayed in 1-3 columns. The values in columns 4-5(Benchmark) are obtained by the Monte-Carlo method with 365 time steps and 50000 trials and the values in column 6-7 are obtained by the Monte-Carlo method with 365 time steps and 2000 trials. Columns 8-9 represent the values obtained by the analytic formula. CPU time is the mean time required to compute the values for each parameter. All routines are programmed using `mathematica` language and run on a 1.2-GHz Athlon computer.

4 Implementation

As already mentioned in the previous section, the closed-from of the value of the time-switch option contains a double integration. Also it seems that the integrand is very complicated. But we found that this integration could be easily approximated by various numerical methods, such as Gaussian quadrature rules, for wide ranges of parameter spaces.

In Table 1 we presented the values of time-switch options obtained by the analytic formula in Theorem 3.4 and the Monte-Carlo method used in Bollen [1]. We use default parameters of $A = 5$, $K = 110$, $T = 1$, $\sigma_L = 0.1313$, $r = 0.06$, and $\lambda_L = 0.5$. Note that the parameters in rows $1-4$, $5-8$, $9-12$ represent the values of out-of-the-money , at-the-money, and in-the-money options, respectively. Compared with the Monte-Carlo method, our analytic formula provides us a more rapid valuation technique. The table shows that the results by the Monte-Carlo method with 2000 trials are calculated in 140 seconds on the average, while those by the analytic solution are calculated only in 8 seconds. Nevertheless, the analytic solution provides us more accurate results than the Monte-Carlo method, considering the 'Benchmark' cases as exact values of the options.

5 Applications to Other Qualitative Options

Theorem 3.4 provides us a useful tool for valuing other qualitative options and we give some examples in this section.

We call the option described in Section 3 an *up-and-in time-switch(UITS) option*, because its payout is determined by the number of days when the price of the underlying stock goes up above a fixed level. On the contrary, a *down-and-in time-switch(DITS) option* is an option which its payout is determined by the number of days when the price of the underlying stock goes down under a fixed level. The value of DITS options can be easily formulated by using Theorem 3.4. We present their values in the following way.:

Corollary 5.1. *The value $\bar{v}_i(T;K,A)$ of a DITS time-switch option in regime $i \in \{H, L\}$ with maturity T, prefixed barrier K and fixed amount multiplier A, is written by*

$$\bar{v}_i(T;K,A) = e^{-rT}\widetilde{E}\left[A\int_0^T 1_{\{S_t<K\}}dt\Big|\sigma(0)=\sigma_i\right],$$

and $v_i(T;K,A)$ and $\bar{v}_i(T;K,A)$ satisfy the relationship

$$v_i(T;K,A) + \bar{v}_i(K;T,A) = e^{-rT}AT,$$

where $v_i(T;K,A)$ is the corresponding UITS option.

On the other hand, a corridor option is an option whose payout is the product of a fixed amount A and the total time(days) such that the price of the underlying asset stays in a certain range, called a *corridor*. Pechtl [14] and Fusai [7] investigated corridor options where the underlying stock process is generated by a geometric Brownian motion with constant drift and volatility.

For the case where the underlying stock process is under the regime-switching environment described in Section 3, the value of a corridor option with maturity T, fixed amount multiplier A and corridor $[K_1, K_2]$ can be represented as

$$\begin{aligned}
&e^{-rT}\widetilde{E}\Big[A\int_0^T 1_{\{K_1<S_t<K_2\}}dt\Big|\sigma(0)=\sigma_i\Big]\\
&\quad= e^{-rT}A\left(\widetilde{E}\Big[\int_0^T 1_{\{S_t>K_1\}}dt\Big|\sigma(0)=\sigma_i\Big] - \widetilde{E}\Big[\int_0^T 1_{\{S_t>K_2\}}dt\Big|\sigma(0)=\sigma_i\Big]\right)\\
&\quad= v_i(T;K_1,A) - v_i(T;K_2,A),
\end{aligned}$$

where v_i represents the value of the UITS option described in Theorem 3.4. Therefore, we observe that the value of corridor option can be interpreted as the difference of the values of two UITS time-switch options with barriers different from each other.

The arguments described in the previous section is applicable to qualitative options in foreign exchange markets. For this case, we have to think of underlying asset price process as a foreign exchange rate X_t satisfying, under a risk-neutral measure $\widetilde{P}$,

$$dX_t = (r_d - r_f)X_t dt + \sigma(t)X_t d\widetilde{B_t}.$$

for a standard Brownian motion $\widetilde{B_t}$. Here, r_d and r_f are domestic and foreign risk-free interest rates, respectively. So we can obtain the values of this type of qualitative option by putting $(r_d - r_f)$ in the place of the risk-free interest rate r in Theorem 3.4.

6 Conclusion

We found a closed-form formula for valuing a time-switch option where the underlying asset process was affected by stochastically changing market environments, and applied it to the valuation of other qualitative options such as corridor options and options in currency markets. The stochastic market environments were modeled as a two-state regime-switching process. We showed that the analytic formula gave us a rapid and accurate valuation method for valuing qualitative options under stochastic market environments.

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Appendix

A Proof of Lemma 3.1

We must show that

$$\widetilde{E}\left[\exp\{uX(t)\}|T_t = k\right] = \exp\left\{\frac{1}{2}u^2 k\right\}, \quad \text{for all} \quad u \in \mathbb{R}. \tag{A.1}$$

First, let

$$\zeta_t(I) = \int_0^t I_s d\widetilde{B}_s - \frac{1}{2}\int_0^t I_s^2 ds.$$

If Novikov's condition of $\widetilde{E}[\exp\{\frac{1}{2}\int_0^t I_s^2 ds\}] < \infty$ is satisfied, the process $\exp\{\zeta_t(I)\}$ is a martingale. Since uI_t is a bounded simple function, the Novikov's condition is satisfied, so $\exp\{\zeta_t(uI)\}$ is a martingale for given $T_t = k$. Therefore,

$$\begin{aligned}
\widetilde{E}[\exp\{\zeta_t(uI)\}|T_t = k] &= \widetilde{E}\left[\exp\left\{\int_0^t uI_s d\widetilde{B}_s - \frac{1}{2}\int_0^t (uI_s)^2 ds\right\}|T_t = k\right] \\
&= \widetilde{E}[\exp\{\int_0^t uI_s d\widetilde{B}_s - \frac{1}{2}u^2 k\}|T_t = k] \\
&= \widetilde{E}[\exp\{uX_t - \frac{1}{2}u^2 k\}|T_t = k] = 1.
\end{aligned}$$

This implies (A.1).

B Proof of Lemma 3.2

If $T(t) = k$, the covariance of $X(t)$ and $\widetilde{B}(t)$ is

$$\begin{aligned}
\widetilde{E}[X(t)\widetilde{B}_t|T_t = k] &= \widetilde{E}[\int_0^t I_s d\widetilde{B}_s \widetilde{B}_t|T_t = k] = \widetilde{E}[\lim_{n\to\infty}\sum_{i=0}^{n} I_{t_i}(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})\widetilde{B}_{t_n}|T_t = k] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\Big(\widetilde{E}[I_{t_i}\widetilde{B}_{t_{i+1}}\widetilde{B}_{t_n}|T_t = k] - \widetilde{E}[I_{t_i}\widetilde{B}_{t_i}\widetilde{B}_{t_n}|T_t = k]\Big) \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\Big(\widetilde{E}[I_{t_i}|T_t = k]t_{i+1} - \widetilde{E}[I_{t_i}|T_t = k]t_i\Big) \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\widetilde{E}[I_{t_i}(t_{i+1} - t_i)|T_t = k] \\
&= \widetilde{E}[\lim_{n\to\infty}\sum_{i=0}^{n} I_{t_i}(t_{i+1} - t_i)|T_t = k] \\
&= \widetilde{E}[\int_0^t I_s ds|T_t = k] = k
\end{aligned}$$

Hence, the correlation coefficient ρ is calculated as

$$\rho = \frac{\widetilde{E}[X(t)\widetilde{B}_t|T_t = k]}{\sqrt{Var(X(t))Var(\widetilde{B}_t)}} = \frac{k}{\sqrt{kt}} = \sqrt{\frac{k}{t}}.$$

C Proof of Theorem 3.4

By Equation (3.2) and Theorem 3.1, the value of the time-switch option satisfies

$$\begin{aligned}
&v_i(T; K, A) \\
&= e^{-rT}\widetilde{E}[A\int_0^T 1_{\{S_t>K\}}dt\Big|\sigma(0) = \sigma_i] \\
&= e^{-rT}A\int_0^T\Bigg[\int_0^t \widetilde{P}\Big\{\frac{1}{2}(\sigma_H^2 - \sigma_L^2)u + \frac{1}{2}\sigma_L^2 t - (\sigma_H - \sigma_L)\sqrt{u}Y_1 - \sigma_L\sqrt{t}Y_2 < \ln\frac{S_0}{K} + rt\,\Bigg|\,T_t = u\Big\}f_i(t,u)du \\
&\quad + \widetilde{P}\Big\{\frac{1}{2}\sigma_L^2 t - \sigma_L\sqrt{t}Y_2 < \ln\frac{S_0}{K} + rt\,\Bigg|\,T_t = 0\Big\}\widetilde{P}\{T_t = 0\} \\
&\quad + \widetilde{P}\Big\{\frac{1}{2}(\sigma_H^2 - \sigma_L^2)t + \frac{1}{2}\sigma_L^2 t - (\sigma_H - \sigma_L)\sqrt{t}Y_1 - \sigma_L\sqrt{t}Y_2 < \ln\frac{S_0}{K} + rt\,\Bigg|\,T_t = t\Big\}\widetilde{P}\{T_t = t\}\Bigg]dt,
\end{aligned}$$

where Y_1 and Y_2 are two correlated standard normal random variables with correlation ρ. Combined Lemma 3.2 and 3.3 with the results about the PDF of T_t in Guo [8] and Fuh [6], we obtain the following.:

$$\begin{aligned}
v_i(T; K, A) &= e^{-rT}\widetilde{E}\left[A\int_0^T 1_{\{S_t>K\}}dt\Big|\sigma(0) = \sigma_i\right] \\
&= e^{-rT}A\int_0^T\Big\{\int_0^t \mathcal{N}(d_1(t,u))f_i(t,u)du + \delta_L(i)\mathcal{N}(d_2(t))e^{-\lambda_L t} + \delta_H(i)\mathcal{N}(d_3(t))e^{-\lambda_H t}\Big\}dt,
\end{aligned}$$

Here,

$$f_i(t,u) = \widetilde{P}\{T_t \in u | \sigma(0) = \sigma_i\}.$$

The proof is completed.