Volatility smiles and skews under a static no arbitrage extension of the Carr-Geman-Madan valuation of options in incomplete markets

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Abstract

It is well known that the pattern of implied volatilities in foreign currency options forms a smile shape which is referred to as a volatility smile. On the other hand, the volatility skew is a general pattern of implied volatilities in equity options. In this paper, we consider the Carr-Geman-Madan valuation of options in incomplete markets on which the preference structure of the market participants are reflected. Through a simple continuous static no arbitrage extension, we examine how the smiles and skews are related.

1 Introduction

The celebrated Black-Scholes option pricing formula is derived from the complete market assumption and from the assumption that stock price process follows the geometric Brownian motion which satisfies the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \ , \tag{1.1}$$

where μ is a constant and σ a constant called volatility.

Though it is the most widely using option pricing formula until now, market data consistently reveal bias to the formula. Indeed, assume σ be unknown variable, while other variables are fixed to be constants, denotes the Black-Scholes call option price with strike K by $C(\sigma, K)$, and solve the equation

$$C(\sigma, K) = C_{ob} \ , \tag{1.2}$$

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where C_{ob} is the price obseved from the real market. Let $\sigma_I(K)$ be the solution of (1.2), the we call $\sigma_I(K)$ the implied volatility, and if the Black-Scholes is correct, then $\sigma_I(K)$ should be constant over K.

However, it is reported that $\sigma_I(K)$ shows smile for equity options before 1987 and currency options. After 1987, crash, $\sigma_I(K)$ shows skew. (see Figure 1). Rubinstein (1994) showed that the implied risk neutral distribution, after the 1987 market crash, is slightly bimodal. According to him, the bimodality coming from the lower tail ("crash-o-phobia") is quite common during the post crash period. (see Rubinstein (1994))

Many authors have tried to explain the reason of this bias. One successful, and widely accepted explanation came from the stochastic volatility model, which assume the stock price process $\{S_t\}_{t\geq 0}$ satisfies the stochastic differential equations

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t \tag{1.3}$$

$$d\sigma_t = b(\sigma_t)dt + a(\sigma_t)dZ_t \ , \tag{1.4}$$

where W_t and Z_t are Brownian motions and $\{\sigma_t\}_{t\geq 0}$ is called the volatility process, b, a are some functions for σ_t, and μ a constant. In this case, we can find the theoretical option price and if we assume the market price of options follow the theoretical price, then we can show that the implied volatility must skew.

Despite the success of the stochastic model, due to the complexity of the dependence structure between the noise of underlying and the noise of volatility process, it is hard to analyze the smile-skew relationship. Moreover, the volatility structure does not fit well.

Our aim is to provide relatively simple models and to analyze smile and skew phenomena: Indeed, we consider one period continuous Carr-Geman-Madan model (which we will refer to as CGM) in incomplete markets. Real markets are far from complete, and in incomplete markets, option price depends on the utility or the preference structure of the market participants. Our motivation is from the insight that volatility smile or skew might be a reflection of the preference structure. We extend CGM to continuous time model which satisfies the static no arbitrage conditions and includes crash-o-phobia characteristics.

In CGM model, the option price is given by the convex combination of the valuation given by test and valuation measures of the traders on which the preference structure of the participants is reflected. Indeed, CGM starts from defining an acceptable opportunity, which is drastically weakened in meaning for an arbitrage opportunity. Every reasonable person but the most risk averse would accept an opportunity with mild risks, if the gains would adequately compensate for the costs. To test whether a trading strategy is acceptable, Carr, Geman and Madan introduced a set of measures (test and valuation measures) and associated floors, non positive numbers. An investment is acceptable if and only if the expected gain under each measure exceeds its associated floor. Their main contribution is the proof of the first and second fundamental theorems which demonstrate that under suitable condition, the state pricing functions can be uniquely determined by a linear combination of the valuation measures.

Note that the CGM model is basically a one period model. The continuous time extension can be carried out easily, since European options depend only on the maturity. In the extension, we are not attempt to establish the fundamental theorems about the existence and uniqueness of the pricing density. Instead, we are more involved in the satisfaction of the minimum requirement as a option pricing machine. Since the underlying may have jumps, real market is more likely incomplete and, thus, dynamic hedging itself is costly and risky. In this case, dynamic arbitrage is had to obtain. As a result, the fact that the price satisfies the static no arbitrage conditions which Merton first derived would be a good measure for the price to be acceptable in practice.

In this paper, we do not assume a specific model for the stock price process. It may have jumps, and it may even be non-Markovian. We only assume that there are only two valuation measures Q_1, Q_2. The densities of Q_1, Q_2 are given by lognormal densities, i.e.,

$$\begin{aligned} f_{Q_1} &= \frac{\exp\left\{-\frac{1}{2\sigma_u^2 T}\left[\log(x/S_0)-(\mu_u-\sigma_u^2/2)T\right]^2\right\}}{\sqrt{2\pi T}\sigma_u x} \\ f_{Q_2} &= \frac{\exp\left\{-\frac{1}{2\sigma_d^2 T}\left[\log(x/S_0)-(\mu_d-\sigma_d^2/2)T\right]^2\right\}}{\sqrt{2\pi T}\sigma_d x}. \end{aligned} \tag{1.5}$$

The CGM model gives the European call price by

$$\begin{aligned} C(S_0,K) &= w\ BS(S_0,K,r,r-\mu_u,\sigma_u,T) \\ &+ (1-w)\ BS(S_0,K,r,r-\mu_d,\sigma_d,T), \end{aligned} \tag{1.6}$$

where $w=(e^{rT}-e^{\mu_d T})/(e^{\mu_u T}-e^{\mu_d T})$ and BS is the Black-Scholes call option formula. This model can be extended to a continuous time model by

$$\begin{aligned} C(S_0,K,t) &= w_t\ BS(S_0,K,r,r-\mu_u,\sigma_t^u,t) \\ &+ (1-w_t)\ BS(S_0,K,r,r-\mu_d(T),\sigma_d,t), \end{aligned} \tag{1.7}$$

for any $t\in(0,\bar{T}]$ and $w_t=(e^{rt}-e^{\mu_d(t)t})/(e^{\mu_u t}-e^{\mu_d(t)t})$. Note that μ_d in (1.6) becomes a function of t,i.e., $\mu_d(t)$ in (1.7). Our interest is the case that $\mu_d(t):=r-(r-m_d)t^\alpha$, with $\alpha \leq 0$ and $m_d < r$. In this case, by one test measure we expect the return becomes big negative, which reflects the crash-o-phobia phenomenon in short time. It turns out that the extended CGM price satisfies Merton's static no arbitrage conditions except one, and the remaining one condition can be almost satisfied for broad class of parameters. Here, we add 'almost' since we are not able to prove the satisfaction of the condition due to the complexity of the equation one must solve, but we can provide a strong numerical evidence.

With the pricing formula, we analytically derive a condition for the existence of a convex decreasing interval of implied volatility around the initial stock price. We also provide various shapes of smile and skew which can be obtained from the pricing formula. It turns out that for the case $\sigma_u \approx \sigma_d$, the implied volatility is almost constant, which recovers Black-Scholes. However, if $\sigma_d \gg \sigma_u$, then it shows smile and skew depends on μ_u and μ_d. Roughly, if $\mu_d \ll r < \mu_u$, then skew phenomena is obtained while smile is obtained as $\mu_d \nearrow r$. See Figure 2 and 3. This results are extended to the volatility surface. Our interpretation is that the small crash-o-phobia corresponds to smile, while big crash-o-phobia corresponds to skew. We also obtain a broad class of shapes of volatility structure which quite resemble the real market surfaces.

This paper is organized as follows. In Section 2, we introduce the CGM model of option valuation, and its continuous extension. In Section 3, we introduce the static no arbitrage conditions Merton derived, and discuss whether the continuous extension satisfies the conditions analytically and numerically. Finally, in section 4, we provide analytic and numerical evidences of smile and skew.

2 CGM valuation

2.1 CGM model

The single period lognormal model given by Carr-Geman-Madan (2001) describes an economy open for trading at date $t = 0$ and at date $t = T$. There are two assets: a non-dividend paying stock with price S_T at time T, and a bond paying one unit at time T. Let the current stock price be S_0 and bond price be e^{-rT}. There are two valuation measures Q_1, Q_2 given by lognomal distributions with mean rates of returns μ_u, μ_d and volatilities σ_u, σ_d, respectively, satisfying $\mu_d < r < \mu_u$ and $\sigma_u > \sigma_d$.

This economy is incomplete, since there is an infinite number of terminal states and there are only two assets. Carr-Geman-Madan's idea is to set up the the matrix of asset valuation test measure outcomes M in order to value a European options with maturity T. In this case, taking the appropriate expectations, M is given by:

$$M = \begin{pmatrix} S_0 e^{(\mu_u - r)T} & S_0 e^{(\mu_d - r)T} \\ e^{-rT} & e^{-rT} \end{pmatrix}. \tag{2.1}$$

Note that, for any non zero cost trading strategy, $\alpha = (\alpha_0, \alpha_1)$ which $\alpha_0 S_0 + \alpha_1 e^{-rT} = 0$ implies $\alpha_1 = -\alpha_0 S_0 e^{rT}$. Hence

$$\alpha M = \alpha_0 S_0 (e^{(\mu_u - r)T} - 1, e^{(\mu_d - r)T} - 1). \tag{2.2}$$

The fact $\mu_u > r > \mu_d$ implies, by CGM, that there is no acceptable opportunity. It turns out that the unique representative state density function is given by

$$\begin{aligned} f(x,T) &= w \frac{\exp\Big\{ -\frac{1}{2\sigma_u^2 T}\big[\log(x/S_0) - (\mu_u - \sigma_u^2/2)T\big]^2\Big\}}{\sqrt{2\pi T}\sigma_u x} \\ &+ (1-w) \frac{\exp\Big\{ -\frac{1}{2\sigma_d^2 T}\big[\log(x/S_0) - (\mu_d - \sigma_d^2/2)T\big]^2\Big\}}{\sqrt{2\pi T}\sigma_d x}, \end{aligned} \tag{2.3}$$

for $w = (e^{rT} - e^{\mu_d T})/(e^{\mu_u T} - e^{\mu_d T})$. Note that $0 < w < 1$.

Let $C(K)$ and $P(K)$ be European call option price and put option price, respectively. Then $C(K)$ and $P(K)$ are uniquely determined by (2.3). Using the Black-Scholes formula with dividend yield of $r - \mu$, we can express $C(K)$ and $P(K)$ by

$$\begin{aligned} C(K) &= wC_{BS}^u + (1-w)C_{BS}^d, \\ P(K) &= wP_{BS}^u + (1-w)P_{BS}^d, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} C_{BS}^u &:= BS^c(S_0, K, r, r-\mu_u, \sigma_u, T), \\ C_{BS}^d &:= BS^c(S_0, K, r, r-\mu_d, \sigma_d, T), \\ P_{BS}^u &:= BS^p(S_0, K, r, r-\mu_u, \sigma_u, T), \\ P_{BS}^d &:= BS^p(S_0, K, r, r-\mu_d, \sigma_d, T), \end{aligned} \tag{2.5}$$

and BS^c, BS^p are the Black-Scholes call option formula and the Black-Scholes put option formula, respectively. See Carr et al. (2001) for the details.

This single period model can be generalized to a continuous time model as explained in the next section.

2.2 Continuous extension

Consider an economy open for $0 \le T \le \bar{T}$, where $\bar{T}$ is a time horizon. For any European type contingent claim at maturity $T \le \bar{T}$, suppose the matrix of asset valuation test measure outcomes M is given by the same way of (2.1). Then we obtain the same price density which can be applied for all $T \in (0, \bar{T}]$. From (2.4), $C(T,K)$, $P(T,K)$ are given by

$$\begin{aligned} C(T,K) &= w_T\Big[S_0 e^{-(r-\mu_u(T))T}\Phi(d_+) - Ke^{-rT}\Phi(d_-)\Big] \\ &+ (1-w_T)\Big[S_0 e^{-(r-\mu_d(T))T}\Phi(\tilde{d}_+) - Ke^{-rT}\Phi(\tilde{d}_-)\Big], \\ P(T,K) &= w_T\Big[Ke^{-rT}\Phi(-d_-) - S_0 e^{-(r-\mu_u(T))T}\Phi(-d_+)\Big] \\ &+ (1-w_T)\Big[Ke^{-rT}\Phi(-\tilde{d}_-) - S_0 e^{-(r-\mu_d(T))T}\Phi(-\tilde{d}_+)\Big], \end{aligned} \tag{2.6}$$

where $w_T = (e^{rT} - e^{\mu_d(T)T})/(e^{\mu_u(T)T} - e^{\mu_d(T)T})$, $\mu_d(T) < r < \mu_u(T)$,

$$\begin{aligned} d_\pm &= \frac{\log(S_0/K) + \{\mu_u(T) \pm \sigma_u^2/2\}T}{\sigma_u\sqrt{T}}, \\ \tilde{d}_\pm &= \frac{\log(S_0/K) + \{\mu_d(T) \pm \sigma_d^2/2\}T}{\sigma_d\sqrt{T}}, \end{aligned} \tag{2.7}$$

and $\Phi(x)$ is the cdf of standard normal.

Note that we make no assumption on the stock price process, so that it may have jumps, or even it is non-Markovian. No matter how investors judge, if they select two types of lognormal valuations, the call option price turns out to be (**??**). Because of the possible jumps and non-Markovian nature, it is costly and risky to perform a dynamic hedge.

As Merton (1973) first introduced and Bergman et al. (1996), Figlewski (2002), Henderson et al. (2005), Jeon and Park (2005) used, static no arbitrage condition is a main requirement that any reasonable option pricing formula should satisfy.

We show that the pricing formula 'almost' satisfies the conditions. Here 'almost' means that it satisfies seven among the eight conditions and we can numerically justify that the remaining condition is not violated.

3 Static no arbitrage

3.1 Static no arbitrage condition

Let C and P be the prices of European call and put options, respectively, on the stock price S_t (t: current time), with strike K, maturity T, and riskless rate r. We may think of $C = C_{t,S_t}(T,K)$ and $P = P_{t,S_t}(T,K)$ as functions of maturity and strike. Merton (1973) first pointed out and Henderson et al. (2004) summarized the conditions that any option price should satisfy to avoid static arbitrages.

(i) $C_{t,S_t}(T,K)$ is a decreasing , convex function of K.

(ii) $C_{t,S_t}(T,0) = \lim_{K \searrow 0} C_{t,S_t}(T,K) = S_t$.

(iii) For $T_1 \geq T_2 \geq t$, $C_{t,S_t}(T_1, Ke^{r(T_1-t)}) \geq C_{t,S_t}(T_2, Ke^{r(T_2-t)})$.

(iv) Put-call parity: $C_{t,S_t}(T,K) - P_{t,S_t}(T,K) = S_t - Ke^{-r(T-t)}$.

(v) $\lim_{K \to \infty} C_{t,S_t}(T,K) = 0$.

(vi) For $T > 0$, $C_{t,S_t}(T, S_t e^{r(T-t)}) > 0$.

Considering the call option as a function of S_t, i.e., $C_{t,S_t}(T,K) = C_{T,K}(t,S_t)$, it should satisfy

(vii) $\lim_{S_t \searrow 0} C_{T,K}(t,S_t) = 0$.

(viii) $\lim_{S_t \nearrow \infty} \left\{ C_{T,K}(t,S_t) - (S_t - e^{-r(T-t)}K) \right\} = 0$.

(ix) $C_{T,K}(t,S_t)$ is an increasing, convex function of asset price.

Note that the conditions (vii), (viii), (ix) depend on the regularity of S_t. (see Bergman et al. (1996)) In our case, the pricing formula satisfies the conditions. (see Proposition 3.2) Now, assume the current time $t = 0$ and let $C_{0,S_0}(T,K) = C(T,K)$ be the price function of a European call option with strike price K and maturity T.

A simple proof of the condition (iii) is as follows. Assume that short selling is always possible. Suppose not the condition (iii). Then $C_1 < C_2$, where $C_1 = C(T_1, Ke^{rT_1})$ and $C_2 = C(T_2, Ke^{rT_2})$. At time $t = 0$, we buy C_1 and sell C_2. So our profit at $t = 0$ is $C_2 - C_1$. Since interest rate r is constant and we can borrow one stock, if $S_{T_2} > Ke^{rT_2}$, our profit at $t = T_2$ is $(C_2 - C_1)e^{rT_2} - (S_{T_2} - Ke^{rT_2}) + S_{T_2}$.

Else if $S_{T_2} \le Ke^{rT_2}$, our profit at $t = T_2$ is $(C_2 - C_1)e^{rT_2} + S_{T_2}$. Thus, at time $t = T_1$, if $S_{T_2} > Ke^{rT_2}$ and $S_{T_1} > Ke^{rT_1}$, our profit is $(C_2 - C_1)e^{rT_1} + Ke^{rT_1} + (S_{T_1} - Ke^{rT_1})$. Since we borrowed one stock, our net profit is $\{(C_2 - C_1)e^{rT_1} + Ke^{rT_1} + (S_{T_1} - Ke^{rT_1})\} - S_{T_1} = (C_2 - C_1)e^{rT_1} > 0$. If $S_{T_2} > Ke^{rT_2}$ and $S_{T_1} \le Ke^{rT_1}$, our net profit at $t = T_1$ is $(C_2 - C_1)e^{rT_1} + Ke^{rT_1} - S_{T_1} > 0$. Similarly, if $S_{T_2} \le Ke^{rT_2}$ and $S_{T_1} > Ke^{rT_1}$, our net profit at $t = T_1$ is $(C_2 - C_1)e^{rT_1} + (S_{T_1} - Ke^{rT_1}) > 0$. If $S_{T_2} \le Ke^{rT_2}$ and $S_{T_1} \le Ke^{rT_1}$, our net profit at $t = T_1$ is $(C_2 - C_1)e^{rT_1} > 0$. Since our net profit at $t = T_1$ is positive in all cases, so there exists a static arbitrage. Thus this completes the proof of condition (iii).

Lemma 3.1 *If $X \ge 0$, then*

$$E^{\mathbb{P}}(X) = \int_0^\infty \mathbb{P}(X > x)dx.$$

Proof. By Fubini's theorem,

$$\begin{aligned}\int_0^\infty \mathbb{P}(X > x)dx &= \int_0^\infty \int_\Omega \mathbf{1}_{\{X>x\}} d\mathbb{P}dx \\ &= \int_\Omega \int_0^\infty \mathbf{1}_{\{X>x\}} dx d\mathbb{P} \\ &= \int_\Omega \int_0^X dx d\mathbb{P} = E^{\mathbb{P}}(X),\end{aligned}$$

where $\mathbf{1}$ is a characteristic function. □

Proposition 3.2 *Let $C(T,K)$, $P(T,K)$ be given by (2.6). Then $C(T,K)$ satisfies static no arbitrage conditions : (i),(ii),(iv),(v), (vi), (vii), (viii) and (ix).*

Proof. For any $T \in (0, \bar{T}]$, let Q_1^T,Q_2^T be a probability measures and S_T a random variable such that under Q_1^T, S_T has a density f_T^1, and under Q_2^T, S_T has a density f_T^2, where

$$\begin{aligned} f_T^1(x) &:= \frac{\exp\left\{ -\frac{1}{2\sigma_u^2 T}\left[\log(x/S_0) - (\mu_u(T) - \sigma_u^2/2)T\right]^2\right\}}{\sqrt{2\pi T}\sigma_u x} \\ f_T^2(x) &:= \frac{\exp\left\{ -\frac{1}{2\sigma_d^2 T}\left[\log(x/S_0) - (\mu_d(T) - \sigma_d^2/2)T\right]^2\right\}}{\sqrt{2\pi T}\sigma_d x}. \end{aligned}$$

Then, form (2.3) and (2.4), $C(T,K)$ and $P(T,K)$ can be expressed by

$$\begin{aligned} C(T,K) &= w_T\ e^{-rT}E^{Q_1^T}\left[(S_T-K)^+\right]+(1-w_T)\ e^{-rT}E^{Q_2^T}\left[(S_T-K)^+\right], \quad (3.1)\\ P(T,K) &= w_T\ e^{-rT}E^{Q_1^T}\left[(K-S_T)^+\right]+(1-w_T)\ e^{-rT}E^{Q_2^T}\left[(K-S_T)^+\right]. \end{aligned}$$

Therefore, by Lemma 3.1,

$$\begin{aligned} C(T,K) &= w_Te^{-rT}\int_K^\infty Q_1^T(S_T>x)dx \\ &\quad +(1-w_T)e^{-rT}\int_K^\infty Q_2^T(S_T>x)dx. \end{aligned} \tag{3.2}$$

Condition (i): Differentiate both sides of (3.2) with K, we obtain

$$\partial C/\partial K = -w_Te^{-rT}Q_1^T(S_T>K)-(1-w_T)e^{-rT}Q_2^T(S_T>K)<0. \tag{3.3}$$

Similarly,

$$\partial^2 C/\partial K^2 = w_Te^{-rT}f_1^T(K)+(1-w_T)e^{-rT}f_2^T(K)>0. \tag{3.4}$$

Therefore, $C(T,K)$ is a decreasing, convex function of K.

Condition (ii): If $K\searrow 0$, then $\Phi(d_\pm)\to 1$ and $\Phi(\tilde{d}_\pm)\to 1$. Therefore, from (2.6), we have

$$\begin{aligned} \lim_{K\searrow 0} C(T,K) &= w_T\ (S_0e^{-(r-\mu_u(T))T})+(1-w_T)\ (S_0e^{-(r-\mu_d(T))T}) \\ &= S_0e^{-rT}\left[w_Te^{\mu_u(T)T}+(1-w_T)e^{\mu_d(T)T}\right] \\ &= S_0e^{-rT}e^{rT} \\ &= S_0 \end{aligned}$$

Condition (iv): put-call parity:

$$\begin{aligned} C(T,K)-P(T,K) &= w_Te^{-rT}E^{Q_1^T}\left[(S_T-K)^+-(K-S_T)^+\right] \\ &\quad +(1-w_T)e^{-rT}E^{Q_2^T}\left[(S_T-K)^+-(K-S_T)^+\right] \\ &= w_Te^{-rT}E^{Q_1^T}\left[(S_T-K)^+-(S_T-K)^-\right] \\ &\quad +(1-w_T)e^{-rT}E^{Q_2^T}\left[(S_T-K)^+-(S_T-K)^-\right] \\ &= e^{-rT}\Big(w_TE^{Q_1^T}(S_T-K)+(1-w_T)E^{Q_2^T}(S_T-K)\Big) \\ &= e^{-rT}\Big(w_TE^{Q_1^T}(S_T)+(1-w_T)E^{Q_2^T}(S_T)-K\Big) \\ &= S_0-Ke^{-rT}. \end{aligned}$$

Condition (v): We use the well known upper bound formula for normal cdf, i.e., for $y > 0$,

$$\Phi(-y) \leq \exp(-y^2/2)/\sqrt{2\pi}y. \tag{3.5}$$

Let $x = \log(K/S_0)/\sigma\sqrt{T}$, then

$$d_1 := \frac{\log(S_0/K) + (\mu_T + \sigma^2/2)T}{\sigma\sqrt{T}} = -x + a, \tag{3.6}$$

where a is some constant. For sufficient large K, $d_1 < 0$. Therefore, by (3.5),

$$\begin{aligned} 0 \leq K\Phi(d_1) &= S_0 \exp\left(\sigma\sqrt{T}x\right)\Phi(-x+a) \\ &\leq C_1 \frac{\exp(-(x-a)^2/2 + \sigma\sqrt{T}x)}{x-a} \longrightarrow 0, \end{aligned} \tag{3.7}$$

as $x \to \infty$, for some constant C_1. Thus, if $K \to \infty$, then $K\Phi(d_-)$, $K\Phi(\tilde{d}_-)$ converge to zero and $\Phi(d_+)$, $\Phi(\tilde{d}_+)$ converge to zero. Hence,

$$\lim_{K\to\infty} C(T,K) = 0.$$

Condition (vi): Since, for $T > 0$, $(S_T - S_0e^{rT})^+$ is not identically zero, from (3.2),

$$C(T, S_0e^{rT}) > 0.$$

Condition (vii): For any T and K fixed, the call price can be considered as a function of S_0, say $C_{T,K}(0,S_0) := C(T,K)$. Since $\Phi(d_\pm)$,$\Phi(\tilde{d}_\pm)$ converge to zero as $S_0 \searrow 0$, from (2.6), $C_{T,K}(0,S_0)$ converges to 0 as $S_0 \searrow 0$.

Condition (viii): From put-call parity,

$$C_{T,K}(0,S_0) - (S_0 - Ke^{-rT}) = P_{T,K}(0,S_0). \tag{3.8}$$

From (2.6),

$$\begin{aligned} P_{T,K}(0,S_0) &= w_T\Big[Ke^{-rT}\Phi(-d_-) - S_0e^{-(r-\mu_u(T))T}\Phi(-d_+)\Big] \\ &+ (1-w_T)\Big[Ke^{-rT}\Phi(-\tilde{d}_-) - S_0e^{-(r-\mu_d(T))T}\Phi(-\tilde{d}_+)\Big]. \end{aligned} \tag{3.9}$$

Let $z = \log(S_0/K)/\sigma\sqrt{T}$, then

$$d_2 := \frac{\log(S_0/K) + (\mu_T + \sigma^2/2)T}{\sigma\sqrt{T}} = z + b, \tag{3.10}$$

where b is some constant. For sufficient large S_0, $d_2 > 0$. Therefore, by (3.5),

$$\begin{aligned} 0 \le S_0\Phi(-d_2) &= K\exp\left(\sigma\sqrt{T}x\right)\Phi(-z-b) \\ &\le C_2\frac{\exp(-(z+b)^2/2+\sigma\sqrt{T}z)}{z+b} \longrightarrow 0, \end{aligned} \tag{3.11}$$

as $z \to \infty$, for some constant C_2. Thus, if $S_0 \to \infty$, then $S_0\Phi(-d_+)$, $S_0\Phi(-\tilde{d}_+)$ converge to zero and $\Phi(-d_-)$, $\Phi(-\tilde{d}_-)$ converge to zero. Hence, $P_{T,K}(0, S_0)$ converges to 0 as $S_0 \to \infty$.

Condition (xi): Using the symmetry of normal density, we can show easily that (3.1) becomes

$$\begin{aligned} C_{T,K}(0,S_0) &= w_T\Big[S_0\sigma_u\sqrt{T}e^{(\mu_u(T)-r-\sigma_u^2/2)T}\int_{-\infty}^{\gamma_u}e^{-v\sigma_u\sqrt{T}}\Phi(v)dv\Big] \\ &\quad + (1-w_T)\Big[S_0\sigma_d\sqrt{T}e^{(\mu_d(T)-r-\sigma_d^2/2)T}\int_{-\infty}^{\gamma_d}e^{-v\sigma_d\sqrt{T}}\Phi(v)dv\Big], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \gamma_u &= \big(\ \log(S_0/K)+(\mu_u(T)-\sigma_u^2/2)T\ \big)/\big(\sigma_u\sqrt{T}\ \big) \\ \gamma_d &= \big(\ \log(S_0/K)+(\mu_d(T)-\sigma_d^2/2)T\ \big)/\big(\sigma_d\sqrt{T}\ \big). \end{aligned} \tag{3.13}$$

Hence, $\dfrac{\partial C_{T,K}(0,S_0)}{\partial S_0}$ becomes

$$\begin{aligned} &w_T\ \sigma_u\sqrt{T}\Big[e^{(\mu_u(T)-r-\sigma_u^2/2)T}\int_{-\infty}^{\gamma_u}e^{-v\sigma_u\sqrt{T}}\Phi(v)dv + Ke^{-rT}\Phi(\gamma_u)/S_0\Big] \\ &+(1-w_T)\ \sigma_d\sqrt{T}\Big[e^{(\mu_d(T)-r-\sigma_d^2/2)T}\int_{-\infty}^{\gamma_d}e^{-v\sigma_d\sqrt{T}}\Phi(v)dv + Ke^{-rT}\Phi(\gamma_d)/S_0\Big]. \end{aligned}$$

Therefore,

$$\frac{\partial C_{T,K}(0,S_0)}{\partial S_0} > 0.$$

Similarly,

$$\begin{aligned} \frac{\partial^2 C_{T,K}(0,S_0)}{\partial S_0^2} &= w_T\ \sigma_u\sqrt{T}\Big[\frac{Ke^{-rT}}{S_0^2\sqrt{2\pi}}\ e^{-\gamma_u^2/2}\Big] \\ &\quad +(1-w_T)\ \sigma_d\sqrt{T}\Big[\frac{Ke^{-rT}}{S_0^2\sqrt{2\pi}}\ e^{-\gamma_d^2/2}\Big] \\ &> 0. \end{aligned} \tag{3.14}$$

In total, $C_{T,K}(0,S_0)$ is an increasing convex function of asset price.

$\square$

3.2 The validity of condition (iii)

From (2.6), we obtain

$$\begin{aligned} C(T, Ke^{rT}) &= w_T\Big[S_0 e^{-(r-\mu_u(T))T}\Phi(\delta_+) - K\Phi(\delta_-)\Big] \\ &+ (1-w_T)\Big[S_0 e^{-(r-\mu_d(T))T}\Phi(\tilde{\delta}_+) - K\Phi(\tilde{\delta}_-)\Big], \end{aligned}$$

where

$$\begin{aligned} \delta_\pm &= \frac{\log(S_0/K) + (\mu_u(T) - r \pm \sigma_u^2/2)T}{\sigma_u\sqrt{T}}, \\ \tilde{\delta}_\pm &= \frac{\log(S_0/K) + (\mu_d(T) - r \pm \sigma_d^2/2)T}{\sigma_d\sqrt{T}}. \end{aligned}$$

Therefore, by a simple calculation, we have

$$\begin{aligned} \frac{\partial C(T, Ke^{rT})}{\partial T} &= \frac{\partial w_T}{\partial T}\Big[S_0 e^{-(r-\mu_u(T))T}\Phi(\delta_+) - K\Phi(\delta_-)\Big] \\ &+ w_T\Big[S_0 e^{-(r-\mu_u(T))T}\big(-r + \frac{\partial \mu_u(T)}{\partial T}T + \mu_u(T)\big)\Phi(\delta_+) \\ &+ S_0 e^{-(r-\mu_u(T))T}\frac{1}{\sqrt{2\pi}}\, e^{-\delta_+^2/2}\, \frac{\partial \delta_+}{\partial T} - \frac{K}{\sqrt{2\pi}}\, e^{-\delta_-^2/2}\, \frac{\partial \delta_-}{\partial T}\Big] \\ &- \frac{\partial w_T}{\partial T}\Big[S_0 e^{-(r-\mu_d(T))T}\Phi(\tilde{\delta}_+) - K\Phi(\tilde{\delta}_-)\Big] \\ &+ (1-w_T)\Big[S_0 e^{-(r-\mu_d(T))T}\big(-r + \frac{\partial \mu_d(T)}{\partial T}T + \mu_d(T)\big)\Phi(\tilde{\delta}_+) \\ &+ S_0 e^{-(r-\mu_d(T))T}\frac{1}{\sqrt{2\pi}}\, e^{-\tilde{\delta}_+^2/2}\, \frac{\partial \tilde{\delta}_+}{\partial T} - \frac{K}{\sqrt{2\pi}}\, e^{-\tilde{\delta}_-^2/2}\, \frac{\partial \tilde{\delta}_-}{\partial T}\Big]. \end{aligned}$$

Assume $\mu_u(T) = m_u > 0$ and $\mu_d(T) = r - (r - m_d)T^\alpha$ with $\alpha \leq 0$.

$$
\begin{aligned}
\frac{\partial C(T, Ke^{rT})}{\partial T} &= \frac{\partial w_T}{\partial T}\Big[\ S_0 e^{-(r-m_u)T}\Phi(\delta_+) - K\Phi(\delta_-)\ \Big] \\
&+\ w_T\Big[\ S_0 e^{-(r-m_u)T}(-r+m_u)\Phi(\delta_+) \\
&+\ \frac{S_0}{2\sqrt{2\pi}\sigma_u T^{3/2}}\ \exp\big(-(r-m_u)T - \delta_+^2/2\big)
\end{aligned}
$$

Assume that ranges for variables of $\partial C/\partial T$ are as follows:

- $0.06 \leq m_u \leq 1$
- $-1 \leq m_d \leq 0.04$
- $0.15 \leq \sigma_u \leq 0.25$
- $0.34 \leq \sigma_d \leq 0.5$
- $0.8 \leq K/S_0 \leq 1.2$
- $0.05 \leq T \leq 1$
- $r = 0.05$

The mesh size for ranges are assumed to be 0.05. Then, we can calculate the value of $\partial C/\partial T$ for every mesh point. As a result, we note that the minimum value of $\partial C/\partial T$ is positive. Therefore, Condition(iii) can be almost satisfied.

4 Smile and Skew

4.1 Analytic result for smile and skew

One natural question is, for no arbitrage pricing functions, when there exist the smile or skew phenomena. A necessary condition for skew or smile with minimum volatility at $K > S_0e^{rT}$ is that there exists an interval

$$L_0 = (S_0e^{rT} - h_1, S_0e^{rT} + h_2),\ (h_1, h_2 > 0)$$

such that the corresponding implied volatility $I(K)$ is decreasing and convex on L_0.

Suppose the time to maturity T is fixed, and let $C(K) := C(T,K)$ be a given price function of a European call option with strike price K which is twice differentiable. Let $C_{BS}(K,\sigma)$ be the Black-Scholes option pricing function, where σ is the volatility of the model. The implied volatility $I(K)$ is, then, the solution of the equation

$$C(K) = C_{BS}(K, I(K)). \tag{4.1}$$

Let $I_0 := I(S_0e^{rT})$. Then we have a necessary condition as follows.

Theorem 4.1 *Suppose $C'(S_0e^{rT}) \leq -\frac{1}{2}e^{-rT}$ and*

$$C''(S_0e^{rT}) \geq \frac{e^{-2rT-I_0^2T/8}}{S_0I_0\sqrt{2\pi T}},$$

then there exist $h_1, h_2 > 0$ such that $I(K)$ is convex and decreasing on L_0, where $L_0 = (S_0e^{rT} - h_1, S_0e^{rT} + h_2)$

Proof. Since $C(K) = C_{BS}(K, I(K))$, differentiate both sides with respect to K, then

$$C'(K) = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma}\frac{dI}{dK} \tag{4.2}$$

i.e.

$$\frac{dI}{dK} = \frac{C'(K) - \frac{\partial C_{BS}}{\partial K}}{\frac{\partial C_{BS}}{\partial \sigma}}. \tag{4.3}$$

By a simple calculation we can show

$$\frac{dI}{dK} = \frac{e^{-rT}\Phi(d_2) + C'(K)}{\frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-d_1^2/2}}, \tag{4.4}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + \left(r + \frac{1}{2}I(K)^2\right)T}{I(K)\sqrt{T}} \\ d_2 &= \frac{\log(S_0/K) + \left(r - \frac{1}{2}I(K)^2\right)T}{I(K)\sqrt{T}} = d_1 - I(K)\sqrt{T}. \end{aligned}$$

Replacing K by S_0e^{rT} in the numerator we can show

$$\begin{aligned} \frac{dI}{dK}\Big|_{K=S_0e^{rT}} &= c\Big(e^{-rT}\Phi(-I_0\sqrt{T}/2) + C'(S_0e^{rT})\Big) \\ &\leq c\Big(e^{-rT}\Phi(-I_0\sqrt{T}/2) - e^{-rT}/2\Big) \ < \ 0, \end{aligned}$$

where c is a positive constant. Hence there is a decreasing interval around S_0e^{rT}.

Now, differentiate (4.2) once more, we get

$$C''(K) = \frac{\partial^2 C_{BS}}{\partial K^2} + 2\frac{\partial^2 C_{BS}}{\partial\sigma\partial K}\frac{dI}{dK} + \frac{\partial^2 C_{BS}}{\partial\sigma^2}\Big(\frac{dI}{dK}\Big)^2 + \frac{\partial C_{BS}}{\partial\sigma}\frac{d^2 I}{dK^2}. \tag{4.5}$$

By a direct calculation, we can get

$$\begin{aligned}\frac{d^2 I}{dK^2}\Big|_{K=S_0e^{rT}} &= \frac{\sqrt{2\pi}}{S_0\sqrt{T}}\ e^{I_0^2T/8}\Big\{C''(S_0e^{rT}) - \frac{e^{-2rT-I_0^2T/8}}{I_0S_0\sqrt{2\pi T}} \\ &\quad -\frac{3e^{-rT}}{2S_0}\Big[e^{-rT}\Phi(-I_0\sqrt{T}/2) + C'(S_0e^{rT})\Big] \\ &\quad +\frac{I_0\sqrt{2\pi T}}{4S_0}\ e^{I_0^2T/8}\Big[e^{-rT}\Phi(-I_0\sqrt{T}/2) + C'(S_0e^{rT})\Big]^2\Big\}.\end{aligned}$$

Thus we obtain

$$\frac{d^2 I}{dK^2}\Big|_{K=S_0e^{rT}} > 0,$$

since

$$C''(S_0e^{rT}) \geq \frac{e^{-2rT-I_0^2T/8}}{S_0I_0\sqrt{2\pi T}}\ ,$$

and

$$e^{-rT}\Phi(-I_0\sqrt{T}/2) + C'(S_0e^{rT}) \leq e^{-rT}\Phi(-I_0\sqrt{T}/2) - e^{-rT}/2 < 0.$$

□

Suppose a European call option price function $C(T,K)$ is given by (2.6). Replacing K by S_0e^{rT} in (4.1) , we get a equation as follows.

$$C_{BS}(S_0e^{rT}, I_0) = C(T, S_0e^{rT}). \tag{4.6}$$

From (4.6),

$$\begin{aligned}S_0\Phi(I_0\sqrt{T}/2) - S_0\Phi(-I_0\sqrt{T}/2) \ &= \ w_T\big[S_0e^{-(r-\mu_u(T))T}\Phi(\alpha_+) - S_0\Phi(\alpha_-)\big] \\ &+ \ (1-w_T)\big[S_0e^{-(r-\mu_T^d)T}\Phi(\tilde\alpha_+) - S_0\Phi(\tilde\alpha_-)\big],\end{aligned} \tag{4.7}$$

where

$$\alpha_\pm = \big((\mu_u(T) - r)/\sigma \pm \sigma/2\big)\sqrt{T}, \quad \tilde\alpha_\pm = \big((\mu_T^d - r)/\sigma \pm \sigma/2\big)\sqrt{T}.$$

Using the symmetry of normal cdf, from (4.7), we obtain

$$\begin{aligned} I_0 &= \frac{2}{\sqrt{T}}\Phi^{-1}\Big\{\frac{1}{2}+\frac{1}{2}w_T\big[e^{-(r-\mu_u(T))T}\Phi(\alpha_+)-\Phi(\alpha_-)\big] \\ &\quad +\frac{1}{2}(1-w_T)\big[e^{-(r-\mu_T^d)T}\Phi(\tilde{\alpha}_+)-\Phi(\tilde{\alpha}_-)\big]\Big\}. \end{aligned} \tag{4.8}$$

Theorem 4.2 *For fixed $T > 0$, let C(K):=C(T, K) be given by (2.6). Choose $\mu_u(T), \mu_T^d$ with*

$$\frac{e^{rT}-e^{\mu_T^d T}}{e^{\mu_u(T)T}-e^{\mu_T^d T}}\Phi(\alpha_-)+\frac{e^{\mu_u(T)T}-e^{rT}}{e^{\mu_u(T)T}-e^{\mu_T^d T}}\Phi(\tilde{\alpha}_-)\geq 1/2. \tag{4.9}$$

Let I_0 be given by (4.8). Suppose

$$C''(S_0e^{rT})\geq \frac{e^{-2rT-I_0^2T/8}}{S_0I_0\sqrt{2\pi T}},$$

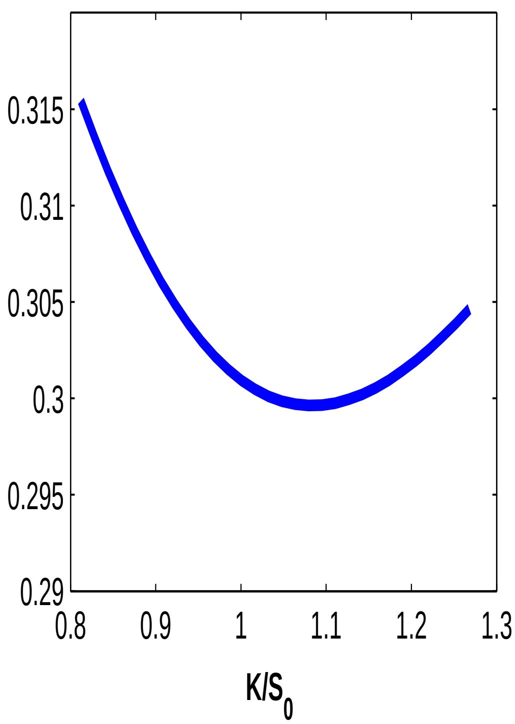
then there exist $h_1, h_2 > 0$ such that $I(K)$ is convex and decreasing on L_0, where $L_0 = (S_0e^{rT} - h_1, S_0e^{rT} + h_2)$.

Proof. From (3.3)and (4.9),

$$\begin{aligned} \frac{\partial C}{\partial K}\Big|_{K=S_0e^{rT}} &= -w_Te^{-rT}\Phi(\alpha_-)-(1-w_T)e^{-rT}\Phi(\tilde{\alpha}_-) \\ &\leq -e^{-rT}/2. \end{aligned}$$

Thus, by assumption and by Proposition 4.1, this completes the proof. □

4.2 Numerical illustration



0.315
0.31
0.305
0.3
0.295
0.29
0.8
0.9
1
1.1
1.2
1.3
K/S0
0.23
0.225
0.22
0.215
0.21
0.8
0.9
1
1.1
1.2
1.3
K/S0

Figure 1:

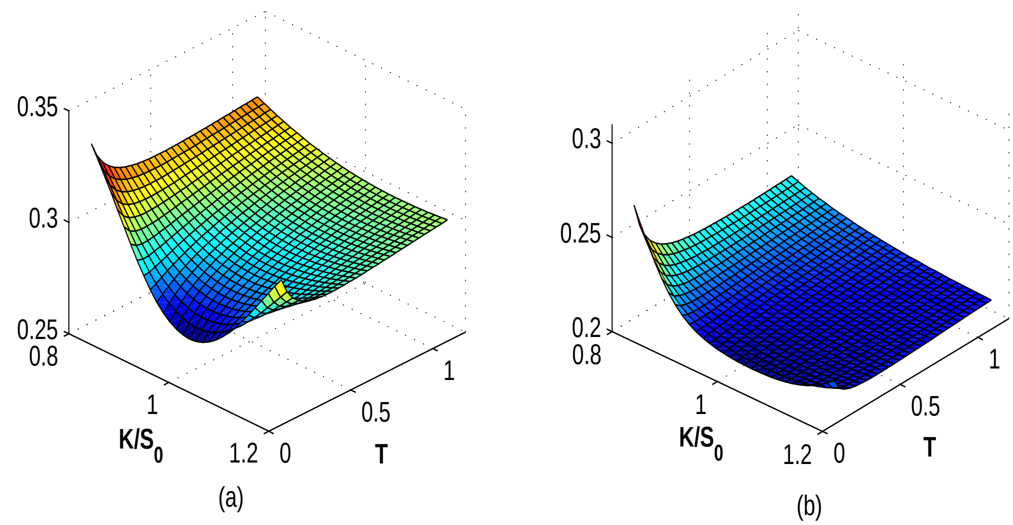


Figure 2:

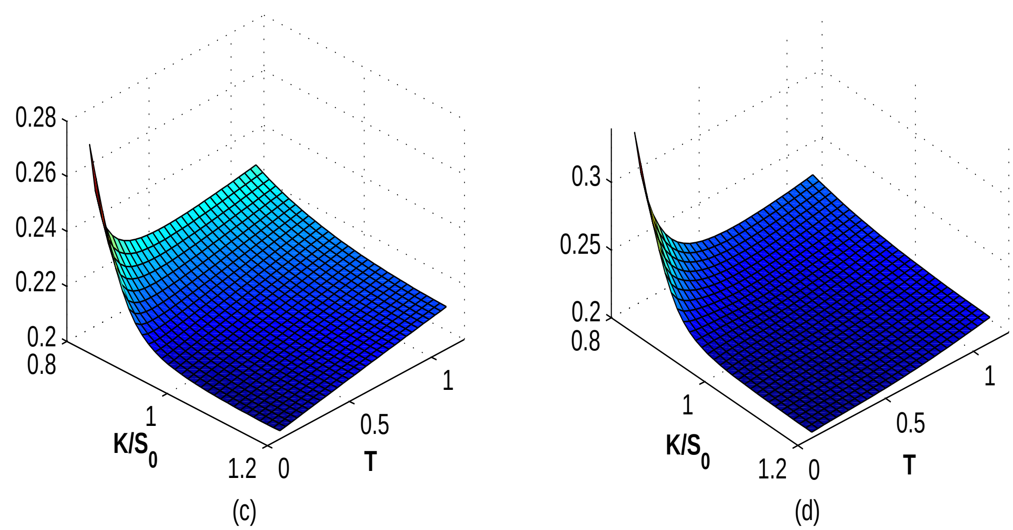


Figure 3:

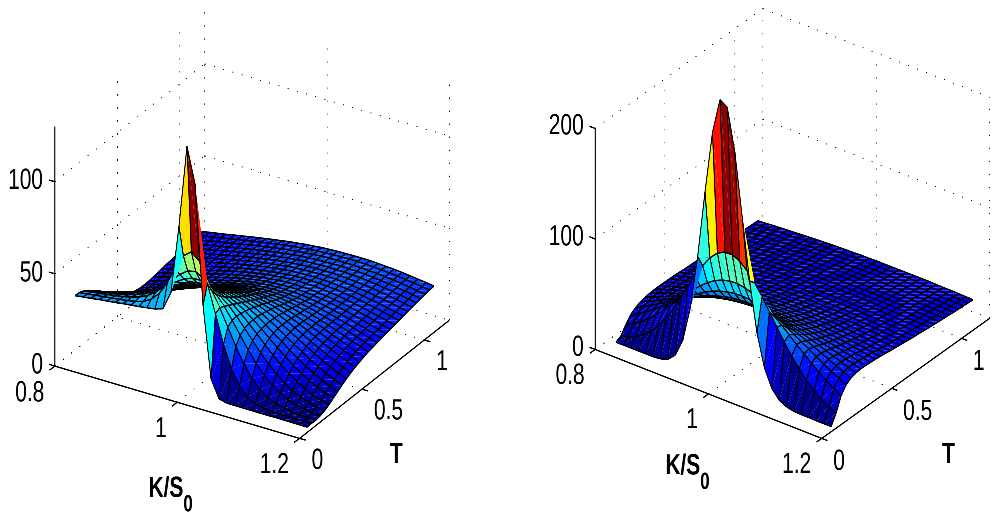


Figure 4:

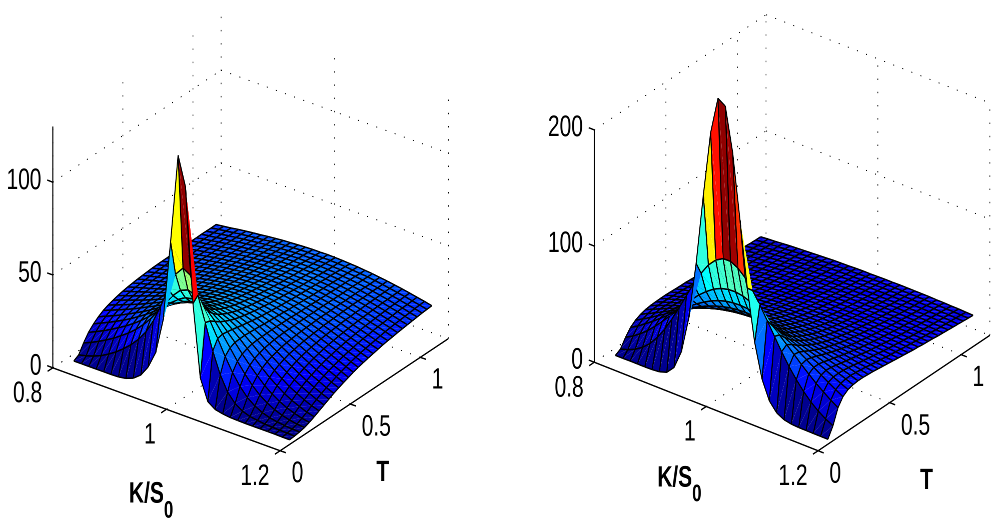


Figure 5:

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