Optimal Investment, Consumption and Retirement Decision with Disutility and Liquidity Constraints

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Abstract

In this paper we consider general consumption, portfolio and retirement optimization problems in which a working investor has liquidity constraints. Closed-form solutions are obtained for the utility maximization problems and numerical procedures are given for the general utility function under the liquidity constraints. The numerical results for a special utility function, for example, the constant relative risk aversion(CRRA) utility function, suggest that the restriction to borrow future income makes the investor retire in a lower critical wealth level than in the case of no liquidity constraints.

Keywords : Liquidity constraints, general utility function, consumption, portfolio selection, retirement, disutility, labor income

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1 Introduction

The optimal consumption and portfolio choice problems have been developed in various strands and under the rational constraints. There were many literature about the optimal

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consumption and portfolio selection problem in continuous time after Merton [14, 15]. (See [4], [9] and [10] etc.) Nowadays the voluntary retirement is considered importantly and the financial market might enable this voluntary retirement. (See [5] and [6].) This problem can be mathematically formulated under the framework of Karatzas and Wang [12].

In this paper, we consider general optimization problems in which an infinitely-lived working investor who has liquidity constraints can choose her retirement time. During the period the investor works, she receives a labor income and has disutility which indicates a degree of repugnance about labor. The liquidity constraints mean that the investor should hold nonnegative wealth in the financial market. In other words, she cannot borrow her money from her future labor income. So the liquidity constraints restrict the investment and the consumption of the investor. Since the labor income is nontradable, in general, this liquidity constraints are more realistic. For computational convenience, only two assets are considered, which are a risky asset and a riskless asset. This assumption makes the value function and the optimal policies be expressed explicitly though the utility function is general. So we also give the numerical procedures for the general utility function under the liquidity constraints.

There are some results on similar issues. Choi and Shim [2] solved a general optimal consumption and portfolio problem with labor income, disutility and the voluntary retirement, but they didn't consider the liquidity constraints. (We will consider this model as the *benchmark model*.) Farhi and Panageas [6] considered not only the voluntary retirement but also the liquidity constraints but they did not think about the case of general utility. The voluntary retirement was also considered in [2], [3] and [5].

The numerical results suggest that the liquidity constraints which prevent borrowing against future income make the investor retire in a lower critical wealth level than in the case of no liquidity constraints. We show the results with the constant relative risk aversion(CRRA) utility function in Section 5. In that section, it can be also shown that there is a jump in portfolio at the retirement time wealth level.

The structure of the paper is as follows. In Section 2, the financial market setup is discussed. In Section 3, we consider the utility maximization problem with the liquidity constraints. The optimal policies of our problem are given in Section 4. In Section 5, CRRA utility functions are applied to the solutions derived in Section 4 and there are some numerical results. The concluding remarks are given in Section 6.

2 The Financial Market Setup

We assume that there are two assets which are one riskless asset with constant interest rate $r > 0$ and one risky asset S_t satisfying the following stochastic differential equation(SDE) $dS_t/S_t = \mu dt + \sigma dB_t,\ S_0 > 0$, where μ and σ are positive constants and B_t is a standard Brownian motion on suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}\}_{t=0}^{\infty}$ is the augmentation of the filtration generated by B_t and $\mathbb{P}$ is the probability measure on $\mathcal{F}$. (See Section 1.7 of Karatzas and Shreve [11] for the detailed mathematical construction in the infinite horizon setting.)

We define the market-price-of-risk, the discount process, the exponential martingale and the state-price-density process, respectively, as

$$\theta \triangleq \frac{\mu - r}{\sigma}, \quad \zeta_t \triangleq \exp\{-rt\}, \quad Z_t \triangleq \exp\left\{-\theta B_t - \frac{1}{2}\theta^2 t\right\} \quad \text{and} \quad H_t \triangleq \zeta_t Z_t.$$

We also define an equivalent martingale measure as $\widetilde{\mathbb{P}}(A) \triangleq \mathbb{E}[Z_T \mathbf{1}_A]$, for any given fixed $T > 0$ and any $A \in \mathcal{F}_T$. Then the Girsanov theorem gives that the new process $\widetilde{B}_t \triangleq B_t + \theta t$, for $0 \le t \le T$, is a standard Brownian motion under the new measure $\widetilde{\mathbb{P}}$.

Let c_t be an $\mathcal{F}_t$-progressively measurable consumption process $c_t : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\int_0^\infty c_s ds < \infty$, almost surely(a.s.) and π_t be an $\mathcal{F}_t$-progressively measurable portfolio process $\pi_t : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ with $\int_0^\infty \pi_s^2 ds < \infty$, a.s.

Let τ be an $\mathcal{F}$-stopping time considered as the retirement time and X_t be the wealth process with the initial endowment $X_0 = x \ge 0$. Then for the given consumption process c_t and portfolio process π_t, X_t evolves

$$dX_t = \left[rX_t + \pi_t(\mu - r) - c_t + \epsilon \mathbf{1}_{\{0 \le t < \tau\}}\right] dt + \pi_t \sigma dB_t \tag{2.1}$$

$$= \left[rX_t - c_t + \epsilon \mathbf{1}_{\{0 \le t < \tau\}}\right] dt + \pi_t \sigma d\widetilde{B}_t, \tag{2.2}$$

where ϵ is a constant labor income before retirement time τ. Then the pair (c, π) is called admissible if $X_t \ge 0$ for all $0 \le t \le \tau$. Since the investor cannot borrow future income the wealth level should always be nonnegative, that is,

$$X_t \ge 0, \quad \text{for all } t \ge 0. \tag{2.3}$$

This assumption (2.3) is called the liquidity constraint(or the borrowing constraint).

By multiplying the discount process ζ_t on both sides to the wealth process (2.2) and integrating from 0 to t for $0 \le t \le \tau$, we have

$$\zeta_t X_t + \int_0^t \zeta_s c_s ds = x + \int_0^t \zeta_s \epsilon ds + \int_0^t \zeta_s \pi_s \sigma d\widetilde{B}_s. \tag{2.4}$$

For an admissible pair (c,π) before the retirement time τ, the third term on the right-hand side of (2.4) is a continuous local martingale bounded below and hence a supermartingale under the new measure $\widetilde{\mathbb{P}}$ by Fatou's lemma. Then the optional sampling theorem gives

$$\widetilde{\mathbb{E}}\left[\zeta_\tau X_\tau + \int_0^\tau \zeta_s c_s ds - \int_0^\tau \zeta_s \epsilon ds\right] \le x, \quad \text{for all } \tau \in \mathcal{S},$$

where $\mathcal{S}$ denotes the set of all $\mathcal{F}$-stopping time τ's and consequently the Bayes rule gives

$$\mathbb{E}\left[H_\tau X_\tau + \int_0^\tau H_s c_s ds - \int_0^\tau H_s \epsilon ds\right] \le x, \quad \text{for all } \tau \in \mathcal{S}. \tag{2.5}$$

To guarantee the existence of a optimal portfolio process, the wealth process X_t should have the form of

$$\begin{aligned} X_t &= \frac{1}{Z_t}\widetilde{\mathbb{E}}\left[\zeta_\tau X_\tau + \int_t^\tau \zeta_s (c_s - \epsilon) ds \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{1}{H_t}\mathbb{E}\left[H_\tau X_\tau + \int_t^\tau H_s (c_s - \epsilon) ds \,\middle|\, \mathcal{F}_t\right], \quad \text{for all } 0 \le t \le \tau. \end{aligned}$$

(See the proof of Lemma 6.3 of Karatzas and Wang [12].) So the liquidity constraint (2.3) implies

$$\mathbb{E}\left[\int_t^\tau H_s c_s ds + H_\tau X_\tau - \int_t^\tau H_s \epsilon ds \,\middle|\, \mathcal{F}_t\right] \ge 0, \quad \text{for all } 0 \le t \le \tau. \tag{2.6}$$

(See He and Pagès [7] and Farhi and Panageas [6].)

3 The Optimization Problem

We now define a general utility function.

Definition 3.1. *A function $u:(0,\infty) \to \mathbb{R}$ is a utility function if it is strictly increasing, strictly concave, continuously differentiable and satisfying*

$$u'(0+) \triangleq \lim_{c \downarrow 0} u'(c) = \infty, \quad u'(\infty) \triangleq \lim_{c \uparrow \infty} u'(c) = 0.$$

With an admissible plan (c,π) and an initial endowment x, the immortal investor's expected discounted utility is given by

$$J(x;c,\pi,\tau) \triangleq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{u(c_t) - l\mathbf{1}_{\{0 \le t < \tau\}}\right\} dt\right], \tag{3.1}$$

where $\beta > 0$ is a subjective discount factor which contains moral hazard rate and $l > 0$ is a constant disutility which comes from labor. Let $\mathcal{A}(x)$ be the admissible class of the triple (c,π,τ) which makes (3.1) well-defined. i.e., $\mathbb{E}\left[\int_0^\infty e^{-\beta t}(u(c_t) - l\mathbf{1}_{\{0 \le t < \tau\}})^- dt\right] < \infty$, where $u^- \triangleq \max(-u, 0)$.

Problem For the given initial endowment x and the utility function $u(\cdot)$ in Definition 3.1, the main problem is to find the value function defined by

$$\begin{aligned} V(x) &= \sup_{(c,\pi,\tau)\in\mathcal{A}} J(x;c,\pi,\tau) \\ &= \sup_{(c,\pi,\tau)\in\mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\{u(c_t)-l\}dt + e^{-\beta\tau}U(X_\tau)\right], \end{aligned} \tag{3.2}$$

subject to the liquidity constraint (2.6) and the budget constraint (2.5).

Lemma 3.1. *The value function $U(\cdot)$ in the equation (3.2) is given by*

$$\begin{aligned} U(x) &= \frac{2(\lambda^{**})^{n_+}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^{**}}\frac{zI_1(z)-u(I_1(z))}{z^{n_++1}}dz \\ &\quad - \frac{2(\lambda^{**})^{n_-}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^{**}}\frac{zI_1(z)-u(I_1(z))}{z^{n_-+1}}dz + (\lambda^{**})x, \end{aligned} \tag{3.3}$$

where $\hat{y} > 0$ is an arbitrary constant, $I_1(\cdot)$ is the inverse function of $u'(\cdot)$ and λ^{**} is determined by the following algebraic equation

$$\begin{aligned} &-\frac{2n_+(\lambda^{**})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^{**}}\frac{zI_1(z)-u(I_1(z))}{z^{n_++1}}dz \\ &\qquad + \frac{2n_-(\lambda^{**})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^{**}}\frac{zI_1(z)-u(I_1(z))}{z^{n_-+1}}dz = x. \end{aligned} \tag{3.4}$$

Here $n_+ > 1$ and $n_- < 0$ are two roots of the following quadratic equation

$$\frac{1}{2}\theta^2 n^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n - \beta = 0. \tag{3.5}$$

Proof. The procedure of proof of this lemma is similar to the results of Shin, Lim, and Choi [16]. □

Remark 3.1. *If the utility function is CRRA type, then the value function $U(\cdot)$ after retirement is a classical Merton's solution. (See Section 5.)*

Remark 3.2. *The real number $\hat{y}$ in Lemma 3.1 is chosen as an arbitrary positive constant, but we will consider $\hat{y}$ as the boundary value which makes zero wealth level for later computation. (See Variational Inequality 1 in Appendix B.)*

In order to find the value function (3.2), for a fixed stopping time $\tau \in \mathcal{S}$, we will consider the reduced utility maximization problem for which only consumption and portfolio pair $(c,\pi) \in \Pi_\tau(x)$ are considered, where $\Pi_\tau(x)$ is the set of pair (c,π) satisfying $(c,\pi,\tau) \in \mathcal{A}(x)$ under the framework of Karatzas and Wang [12]. (The whole process is given in Appendix A and B.)

4 The Optimal Values

Now we will present the value function and the optimal policies based on Appendix A and B.

Theorem 4.1. *Let the critical wealth level*

$$\begin{aligned}\bar{x} &= I_2(\bar{y}) \\ &= -\frac{2n_+\bar{y}^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\bar{y}}\frac{zI_1(z)-u(I_1(z))}{z^{n_++1}}dz+\frac{2n_-\bar{y}^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\bar{y}}\frac{zI_1(z)-u(I_1(z))}{z^{n_-+1}}dz,\end{aligned}$$

then the value function $V(x)$ is given by

$$V(x)=\begin{cases} C_1(\lambda^*)^{n_+}+C_2(\lambda^*)^{n_-}+(\lambda^*)x \\ \quad +\frac{2(\lambda^*)^{n_+}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^*}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\ \quad -\frac{2(\lambda^*)^{n_-}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^*}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz, & \text{if } 0\le x<\bar{x}, \\ U(x) & , \text{if } x\ge\bar{x} \end{cases}$$

where λ^* is the solution of the following algebraic equation

$$\begin{aligned}-n_+C_1(\lambda^*)^{n_+-1}-n_-C_2(\lambda^*)^{n_--1}-\frac{2n_+(\lambda^*)^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^*}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\ +\frac{2n_-(\lambda^*)^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{\lambda^*}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz=x. \qquad (4.1)\end{aligned}$$

Since $\tilde{V}(\cdot)$ given in Appendix A is strictly convex, $\tilde{V}'(\cdot)$ is a strictly increasing function. Therefore one-to-one correspondence between $\lambda^*\in(\bar{y},\hat{y})$ and $x\in(0,\bar{x})$ is induced. The next lemma is useful to find the optimal portfolio.

Proposition 4.1. *Let $y_t^{\lambda^*}$ and $y_t^{\lambda^{**}}$ be solutions of the SDE (B.1) with initial values $y_0=\lambda^*$ and $y_0=\lambda^{**}$ respectively, then*

$$\begin{aligned}X^*(t)=&-n_+C_1(y_t^{\lambda^*})^{n_+-1}-n_-C_2(y_t^{\lambda^*})^{n_--1} \\ &-\frac{2n_+(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\ &+\frac{2n_-(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}}\frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \qquad (4.2)\end{aligned}$$

and

$$\begin{aligned}X^{**}(t)=&-\frac{2n_+(y_t^{\lambda^{**}})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^{**}}}\frac{zI_1(z)-u(I_1(z))}{z^{n_++1}}dz \\ &+\frac{2n_-(y_t^{\lambda^{**}})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^{**}}}\frac{zI_1(z)-u(I_1(z))}{z^{n_-+1}}dz \qquad (4.3)\end{aligned}$$

are the optimal wealth processes for $0\le t<\tau$ and $t\ge\tau$, respectively.

Proof. Substituting $y_t^{\lambda^*}$ for λ^* into the equation (4.1) in Theorem 4.1, then the one-to-one corresponding property leads the equation (4.2). Similarly substituting $y_t^{\lambda^{**}}$ for λ^{**} into the equation (3.4) in Lemma 3.1, the equation (4.3) is induced. □

The optimal portfolio can be obtained by comparing the coefficients of the SDE of the optimal wealth processes (4.2) and (4.3) in Proposition 4.1 with those of the wealth process (2.1). The next theorem gives the results.

Theorem 4.2. *The optimal consumption, portfolio and retirement time* (c^*, π^*, τ^*) *are given, respectively, by*

$$c_t^* = \begin{cases} I_1(y_t^{\lambda^*}), & \text{if } \ 0 \le X_t < \bar{x} \\ I_1(y_t^{\lambda^{**}}), & \text{if } \ X_t \ge \bar{x}, \end{cases}$$

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\Big\{n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\ \quad + \frac{2}{\theta^2}\frac{l + y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon) - u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\ \quad + \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\ \quad - \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz\Big\}, & \text{if } \ 0 \le X_t < \bar{x} \\ \frac{2}{\sigma\theta}\Big\{\frac{y_t^{\lambda^{**}} I_1(y_t^{\lambda^{**}}) - u(I_1(y_t^{\lambda^{**}}))}{y_t^{\lambda^{**}}} \\ \quad + \frac{n_+(n_+-1)(y_t^{\lambda^{**}})^{n_+-1}}{n_+-n_-}\int_{\hat{y}}^{y_t^{\lambda^{**}}} \frac{zI_1(z)-u(I_1(z))}{z^{n_++1}}dz \\ \quad - \frac{n_-(n_--1)(y_t^{\lambda^{**}})^{n_--1}}{n_+-n_-}\int_{\hat{y}}^{y_t^{\lambda^{**}}} \frac{zI_1(z)-u(I_1(z))}{z^{n_-+1}}dz\Big\}, & \text{if } \ X_t \ge \bar{x}, \end{cases}$$

and

$$\tau^* = \inf\{t > 0 \mid X^*(t) \ge \bar{x}\}.$$

Proof. See Appendix C. □

5 Optimal Policies Under a CRRA Utility Class (Numerical Approaches)

In this section, the methods developed in the previous sections are applied to a CRRA utility class. A CRRA utility function is defined by

$$u(c) \triangleq \begin{cases} \frac{1}{1-\gamma}c^{1-\gamma}, & \text{if } \ \gamma > 0 \text{ and } \gamma \ne 1, \\ \log c, & \text{if } \ \gamma = 1, \end{cases}$$

where γ is an investor's coefficient of relative risk aversion. Let K be the Merton's constant which is defined by

$$K \triangleq r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0.$$

Then, for $\gamma > 0$ and $\gamma \neq 1$, from Lemma 3.1, $U(\cdot)$ is given by

$$U(x) = \frac{1}{K^\gamma}\frac{1}{1-\gamma}x^{1-\gamma}.$$

The value function $U(\cdot)$ is exactly same as that of the classical infinite Merton's problem. The dual value functions (A.3) and (A.4) become

$$\tilde{u}(y) = \frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}$$

and

$$\tilde{U}(y) = \frac{\gamma}{K(1-\gamma)}y^{-\frac{1-\gamma}{\gamma}},$$

respectively. Therefore from Proposition B.1, $v(y)$ is the function of the form

$$v(y) = \begin{cases} c_1 y^{n_+} + c_2 y^{n_-} + \frac{\gamma}{K(1-\gamma)}y^{-\frac{1-\gamma}{\gamma}} + \frac{\epsilon}{r}y - \frac{l}{\beta}, & \text{if } \ \bar{y} < y \leq \hat{y}, \\ \frac{\gamma}{K(1-\gamma)}y^{-\frac{1-\gamma}{\gamma}}, & \text{if } \ 0 < y \leq \bar{y}. \end{cases}$$

provided that the inequality (B.4) holds. (Here it is easily shown that $c_1 y^{n_+} + c_2 y^{n_-} + \frac{\epsilon}{r}y - \frac{l}{\beta} > 0$, for $\bar{y} < y \leq \hat{y}$, using the procedure of the proof of Proposition B.1.) Then the coefficients c_1, c_2, $\hat{y}$ and the free boundary value $\bar{y}$ are determined from the following lemma.

Lemma 5.1. *Assume that $\bar{y} = \xi\hat{y}$, for some $\xi \in (0,1)$, then we obtain the following algebraic equation with respect to ξ*

$$\frac{K\frac{l}{\beta}n_+n_-\{(n_+-1)(n_-\xi-\xi^{n_-})-(n_--1)(n_+\xi-\xi^{n_+})\}}{n_-(n_+-1)\left(\frac{1}{\gamma}+n_--1\right)\xi^{n_++1}-n_+(n_--1)\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-+1}-\frac{1}{\gamma}(n_+-n_-)\xi^{n_++n_-}}$$
$$-\left[\frac{\frac{rl}{\beta\epsilon}n_+n_-\left\{\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-}-\left(\frac{1}{\gamma}+n_--1\right)\xi^{n_+}\right\}}{n_+(n_--1)\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-+1}-n_-(n_+-1)\left(\frac{1}{\gamma}+n_--1\right)\xi^{n_++1}+\frac{1}{\gamma}(n_+-n_-)\xi^{n_++n_-}}\right]^{-\frac{1-\gamma}{\gamma}} = 0. \tag{5.1}$$

If we can determine the value of ξ from the equation (5.1), then

$$\hat{y} = \frac{\frac{rl}{\beta\epsilon}n_+n_-\left\{\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-}-\left(\frac{1}{\gamma}+n_--1\right)\xi^{n_+}\right\}}{n_+(n_--1)\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-+1}-n_-(n_+-1)\left(\frac{1}{\gamma}+n_--1\right)\xi^{n_++1}+\frac{1}{\gamma}(n_+-n_-)\xi^{n_++n_-}},$$

$$c_1 = \frac{\frac{\epsilon}{r}(n_--1)\hat{y} - \frac{1}{K}\left(\frac{1}{\gamma}+n_--1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}}}{n_+(n_+-n_-)\hat{y}^{n_+}},$$

and

$$c_2 = \frac{-\frac{\epsilon}{r}(n_+-1)\hat{y} + \frac{1}{K}\left(\frac{1}{\gamma}+n_+-1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}}}{n_-(n_+-n_-)\hat{y}^{n_-}}.$$

Proof. The equations (B.7), (B.8), (B.11) and (B.12) in the proof of Proposition B.1 become respectively

$$c_1 = \frac{-\frac{l}{\beta}n_- - \frac{\epsilon}{r}(1-n_-)\bar{y}}{(n_+ - n_-)\bar{y}^{n_+}}, \tag{5.2}$$

$$c_2 = \frac{\frac{l}{\beta}n_+ - \frac{\epsilon}{r}(n_+ - 1)\bar{y}}{(n_+ - n_-)\bar{y}^{n_-}}, \tag{5.3}$$

$$c_1 = \frac{\frac{\epsilon}{r}(n_- - 1)\hat{y} - \frac{1}{K}\left(\frac{1}{\gamma} + n_- - 1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}}}{n_+(n_+ - n_-)\hat{y}^{n_+}} \tag{5.4}$$

and

$$c_2 = \frac{-\frac{\epsilon}{r}(n_+ - 1)\hat{y} + \frac{1}{K}\left(\frac{1}{\gamma} + n_+ - 1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}}}{n_-(n_+ - n_-)\hat{y}^{n_-}}. \tag{5.5}$$

Assume that $\bar{y} = \xi\hat{y}$, for some $\xi \in (0,1)$, then from the equations (5.2) and (5.4) and from the equations (5.3) and (5.5) we derive the following equations, respectively,

$$\frac{\epsilon}{r}(n_- - 1)(n_+\xi - \xi^{n_+})\hat{y} - \frac{l}{\beta}n_+n_- = -\frac{\xi^{n_+}}{K}\left(\frac{1}{\gamma} + n_- - 1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}} \tag{5.6}$$

and

$$\frac{\epsilon}{r}(n_+ - 1)(n_-\xi - \xi^{n_-})\hat{y} - \frac{l}{\beta}n_+n_- = -\frac{\xi^{n_-}}{K}\left(\frac{1}{\gamma} + n_+ - 1\right)\hat{y}^{-\frac{1-\gamma}{\gamma}}. \tag{5.7}$$

From the equations (5.6) and (5.7) we obtain the algebraic equation (5.1) with respect to ξ provided that

$$\hat{y} = \frac{\frac{rl}{\beta\epsilon}n_+n_-\left\{\left(\frac{1}{\gamma} + n_+ - 1\right)\xi^{n_-} - \left(\frac{1}{\gamma} + n_- - 1\right)\xi^{n_+}\right\}}{n_+(n_- - 1)\left(\frac{1}{\gamma} + n_+ - 1\right)\xi^{n_-+1} - n_-(n_+ - 1)\left(\frac{1}{\gamma} + n_- - 1\right)\xi^{n_++1} + \frac{1}{\gamma}(n_+ - n_-)\xi^{n_++n_-}}.$$

So if we can determine the value of ξ from the equation (5.1), then we can also confirm $\hat{y}$, $\bar{y}$, c_1 and c_2 successively. □

Lemma 5.2. *Left-hand side of the equation (5.1) are denoted by $G(\xi)$ then $G(\cdot) = 0$ has a unique solution in an interval* $(0,1)$.

Proof. Set $f(\xi) = \frac{K\frac{l}{\beta}n_+n_-\{(n_+-1)(n_-\xi - \xi^{n_-}) - (n_- - 1)(n_+\xi - \xi^{n_+})\}}{n_-(n_+-1)\left(\frac{1}{\gamma}+n_- - 1\right)\xi^{n_++1} - n_+(n_- - 1)\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-+1} - \frac{1}{\gamma}(n_+ - n_-)\xi^{n_++n_-}}$ and $g(\xi) = \frac{\frac{rl}{\beta\epsilon}n_+n_-\left\{\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-} - \left(\frac{1}{\gamma}+n_- - 1\right)\xi^{n_+}\right\}}{n_+(n_- - 1)\left(\frac{1}{\gamma}+n_+-1\right)\xi^{n_-+1} - n_-(n_+-1)\left(\frac{1}{\gamma}+n_- - 1\right)\xi^{n_++1} + \frac{1}{\gamma}(n_+ - n_-)\xi^{n_++n_-}}$, then

$$G(\xi) = f(\xi) - (g(\xi))^{-\frac{1-\gamma}{\gamma}}.$$

It is easily shown that $f(1) = 0$ and $g(1) > 0$. So $G(1) < 0$. Also it is easily seen that $\lim_{\xi\to 0+} g(\xi) = +\infty$. Hence the limit of $G(\xi)$ as $\xi \to 0+$ can be represented by

$$\lim_{\xi\to 0+} G(\xi) = \lim_{\xi\to 0+} g(\xi)\left(f(\xi)/g(\xi) - g(\xi)^{-\frac{1}{\gamma}}\right) = +\infty,$$

since $0 < \lim_{\xi \to 0+} \left(f(\xi)/g(\xi) - g(\xi)^{-\frac{1}{\gamma}} \right) = \frac{\epsilon K(n_+ - 1)}{r\left(\frac{1}{\gamma} + n_+ - 1\right)} < +\infty$. The continuity of $G(\xi)$ guarantees the existence of $\xi \in (0,1)$.

The uniqueness of ξ can be shown using the inequality (3) in the Variational Inequality 1. Let $\hat{\xi}(\neq \xi)$ be an another solution of $G(\cdot)$, then the free boundary value $\bar{y}'$ is also different from $\bar{y}$. Without loss of generality, assume that $\bar{y} > \bar{y}'$, then there exists $\check{y} \in (\bar{y}', \bar{y})$. Since $\phi(t,y) > e^{-\beta t}\tilde{U}(y)$ for $\bar{y}' < y < \bar{y}$, this contradicts to $\phi(t,y) = e^{-\beta t}\tilde{U}(y)$ for $0 < y \leq \bar{y}$. □

The value function $V(\cdot)$ in Theorem 4.1 becomes

$$V(x) = \begin{cases} c_1(\lambda^*)^{n_+} + c_2(\lambda^*)^{n_-} + \frac{\gamma}{K(1-\gamma)}(\lambda^*)^{-\frac{1-\gamma}{\gamma}} + \left(x + \frac{\epsilon}{r}\right)(\lambda^*) - \frac{l}{\beta} & \text{, if } 0 \leq x < \bar{x} \\ \frac{1}{K^\gamma}\frac{1}{1-\gamma}x^{1-\gamma} & \text{, if } x \geq \bar{x} \end{cases}$$

where λ^* is determined from the following algebraic equation

$$-n_+c_1(\lambda^*)^{n_+-1} - n_-c_2(\lambda^*)^{n_--1} + \frac{1}{K}(\lambda^*)^{-\frac{1}{\gamma}} - \frac{\epsilon}{r} = x, \text{ for } 0 \leq x < \bar{x}$$

and the critical wealth level is given by $\bar{x} = \frac{1}{K}\bar{y}^{-\frac{1}{\gamma}}$.

From Theorem 4.2, the optimal consumption, portfolio and retirement time are arranged, respectively, by

$$c_t^* = \begin{cases} (y_t^{\lambda^*})^{-\frac{1}{\gamma}}, & \text{if } 0 \leq X_t < \bar{x} \\ KX_t, & \text{if } X_t \geq \bar{x}, \end{cases}$$

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\left\{n_+(n_+-1)c_1(y_t^{\lambda^*})^{n_+-1} \right. \\ \qquad \left. + n_-(n_--1)c_2(y_t^{\lambda^*})^{n_--1} + \frac{1}{K\gamma}(y_t^{\lambda^*})^{-\frac{1}{\gamma}}\right\}, & \text{if } 0 \leq X_t < \bar{x} \\ \frac{\theta}{\sigma\gamma}X_t, & \text{if } X_t \geq \bar{x} \end{cases}$$

and

$$\tau^* = \inf\left\{t > 0 \mid X^*(t) \geq \bar{x}\right\},$$

where the optimal wealth process before retirement is given by

$$X^*(t) = -n_+c_1(y_t^{\lambda^*})^{n_+-1} - n_-c_2(y_t^{\lambda^*})^{n_--1} + \frac{1}{K}(y_t^{\lambda^*})^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

The figure 1 and 2 show the graphical results for the CRRA utility function with $\gamma = 2$. As seen in the figure 1 and 2, we have a lower critical wealth level in our model with the liquidity constraints than in the benchmark model. The figure 1 represents the optimal portfolio of risky assets. There are big jump at the critical wealth level and the amount of investment is very high quite near the retirement wealth level but after

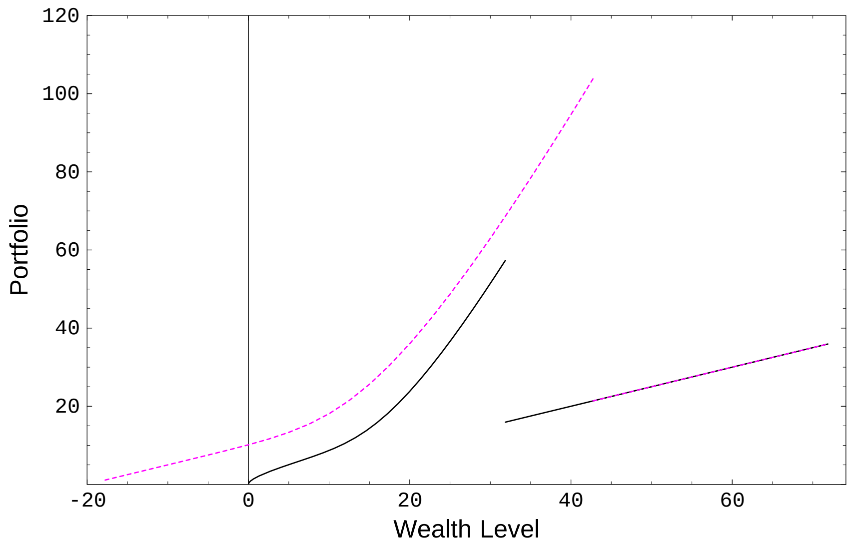

Figure 1: *The optimal portfolio for the CRRA utility function with* $\gamma = 2$ ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \epsilon = 0.2$ and $l = 0.5$). *The solid line shows our model and the dotted line shows the benchmark model.*

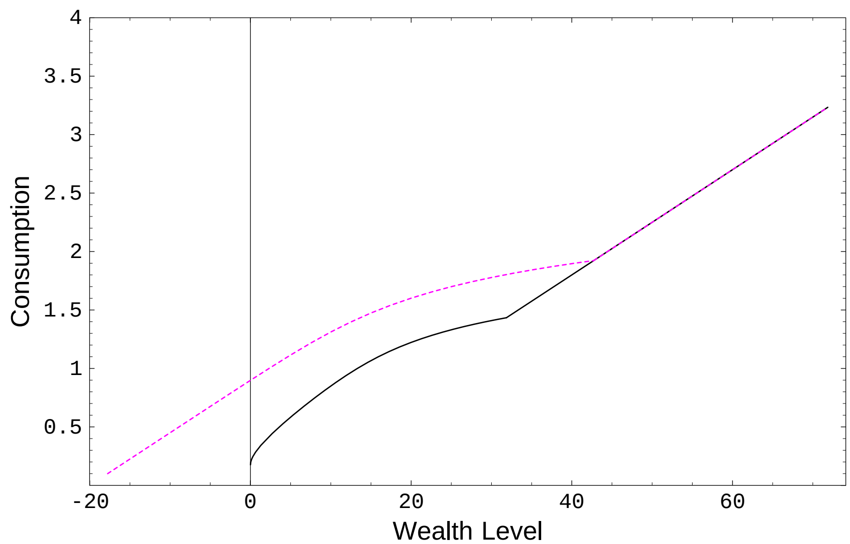

Figure 2: *The optimal consumption for the CRRA utility function with* $\gamma = 2$ ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \epsilon = 0.2$ and $l = 0.5$). *The solid line shows our model and the dotted line shows the benchmark model.*

retirement the increasing rate follows the classical Merton's solution. This means that near the retirement wealth level, the investor invests more money in the risky asset to retire as soon as possible. The figure 1 shows the investor in our model invests less in the risky asset than in the benchmark model before retirement. The figure 2 represents the optimal consumption process. Unlike the portfolio process, the consumption process is continuous at the retirement wealth level but the increasing rate of consumption is different before and after retirement. Near the critical wealth level, the increasing rate of consumption is almost zero and this is because the investor gathers her money so as to retire as soon as possible. In our model, the investor consumes less than in the benchmark model before retirement. This results show that the restriction to borrow future income induces the reduced financial activity.

Now we consider the case of $\gamma = 1$, i.e., the utility function is given by log-type such as $u(c) = \log c$. The procedure is very similar to the case of $\gamma > 0$ and $\gamma \neq 1$. So the results are shown briefly.

From Lemma 3.1, $U(\cdot)$ is given by $U(x) = \frac{1}{\beta^2}\left(\beta \log \beta x - \beta + r + \frac{1}{2}\theta^2\right)$, and the dual value functions (A.3), (A.4) and $v(y)$ in Proposition B.1 are given, respectively, by $\tilde{u}(y) = \log \frac{1}{y} - 1$, $\tilde{U}(y) = \frac{1}{\beta^2}\left(-\beta \log y - 2\beta + r + \frac{1}{2}\theta^2\right)$ and

$$v(y) = \begin{cases} c_1 y^{n_+} + c_2 y^{n_-} + \frac{1}{\beta^2}\left(-\beta \log y - 2\beta + r + \frac{1}{2}\theta^2\right) + \frac{\epsilon}{r} y - \frac{l}{\beta}, & \text{if } \bar{y} < y \leq \hat{y}, \\ \frac{1}{\beta^2}\left(-\beta \log y - 2\beta + r + \frac{1}{2}\theta^2\right), & \text{if } 0 < y \leq \bar{y} \end{cases}$$

provided that the inequality (B.4) holds. The coefficients c_1, c_2, $\hat{y}$ and the free boundary value $\bar{y}$ are also determined from the following lemma which is very similar to Lemma 5.1 and 5.2.

Lemma 5.3. *Assume that $\bar{y} = \xi \hat{y}$, for some $\xi \in (0, 1)$, then we obtain the following algebraic equation with respect to ξ*

$$n_-(n_+ - 1)(n_-\xi - \xi^{n_-})\left(l n_+ - \xi^{n_+}\right) - n_+(n_- - 1)(n_+\xi - \xi^{n_+})\left(l n_- - \xi^{n_-}\right) = 0. \quad (5.8)$$

If we can determine the value of ξ from the equation (5.8), then $\hat{y} = \frac{\frac{l}{\beta} n_+ n_- - \frac{n_-}{\beta}\xi^{n_+}}{\frac{\epsilon}{r}(n_- - 1)(n_+\xi - \xi^{n_+})}$, $c_1 = \frac{\frac{\epsilon}{r}(n_- - 1)\hat{y} - \frac{n_-}{\beta}}{n_+(n_+ - n_-)\hat{y}^{n_+}}$ and $c_2 = \frac{-\frac{\epsilon}{r}(n_+ - 1)\hat{y} + \frac{n_+}{\beta}}{n_-(n_+ - n_-)\hat{y}^{n_-}}$. Furthermore, left-hand side of the equation (5.8) is denoted by $F(\xi)$ then $F(\cdot) = 0$ has a unique solution in an interval $(0, 1)$.

The value function $V(\cdot)$ in Theorem 4.1 becomes

$$V(x)=\begin{cases} c_1(\lambda^*)^{n_+}+c_2(\lambda^*)^{n_-}+\frac{1}{\beta^2}\left(-\beta\log\lambda^*-2\beta+r+\frac{1}{2}\theta^2\right) & \\ \quad +\left(x+\frac{\epsilon}{r}\right)\lambda^*-\frac{l}{\beta} & ,\text{if } \ 0\le x<\bar{x} \\ \frac{1}{\beta^2}\left(\beta\log\beta x-\beta+r+\frac{1}{2}\theta^2\right) & ,\text{if } \ x\ge\bar{x} \end{cases}$$

where λ^* is determined from the following algebraic equation

$$-n_+c_1(\lambda^*)^{n_+-1}-n_-c_2(\lambda^*)^{n_--1}+\frac{1}{\beta\lambda^*}-\frac{\epsilon}{r}=x, \ \text{for} \ \ 0\le x<\bar{x}$$

and the critical wealth level is given by $\bar{x}=\frac{1}{\beta\bar{y}}$.

From Theorem 4.2, the optimal consumption, portfolio and retirement time are given, respectively, by

$$c_t^*=\begin{cases} 1/y_t^{\lambda^*}, & \text{if } \ 0\le X_t<\bar{x} \\ \beta X_t, & \text{if } \ X_t\ge\bar{x}, \end{cases}$$

$$\pi_t^*=\begin{cases} \frac{\theta}{\sigma}\left\{n_+(n_+-1)c_1(y_t^{\lambda^*})^{n_+-1}\right. & \\ \qquad \left.+n_-(n_--1)c_2(y_t^{\lambda^*})^{n_--1}+\frac{1}{\beta y_t^{\lambda^*}}\right\}, & \text{if } \ 0\le X_t<\bar{x} \\ \frac{\theta}{\sigma}X_t, & \text{if } \ X_t\ge\bar{x} \end{cases}$$

and

$$\tau^*=\inf\left\{t>0 \mid X^*(t)\ge\bar{x}\right\},$$

where the optimal wealth process before retirement is given by

$$X^*(t)=-n_+c_1(y_t^{\lambda^*})^{n_+-1}-n_-c_2(y_t^{\lambda^*})^{n_--1}+\frac{1}{\beta y_t^{\lambda^*}}-\frac{\epsilon}{r}.$$

Remark 5.1. *For the benchmark model, refer to Section 5 of Choi and Shim [2].*

There are some graphical results for the log utility function. The changes of optimal values are similar to the power type utility function. The figure 3 and 4 show the optimal portfolio and the optimal consumption rate respectively.

6 Concluding Remarks

In this paper, we investigate the general optimization problem in which an immortal working investor who has liquidity constraints can choose the retirement time. During the period the investor works, she receives a constant labor income and has disutility which comes from working. Although the utility function is general we obtain the closed-form solutions for the utility maximization problem by the martingale approaches. The

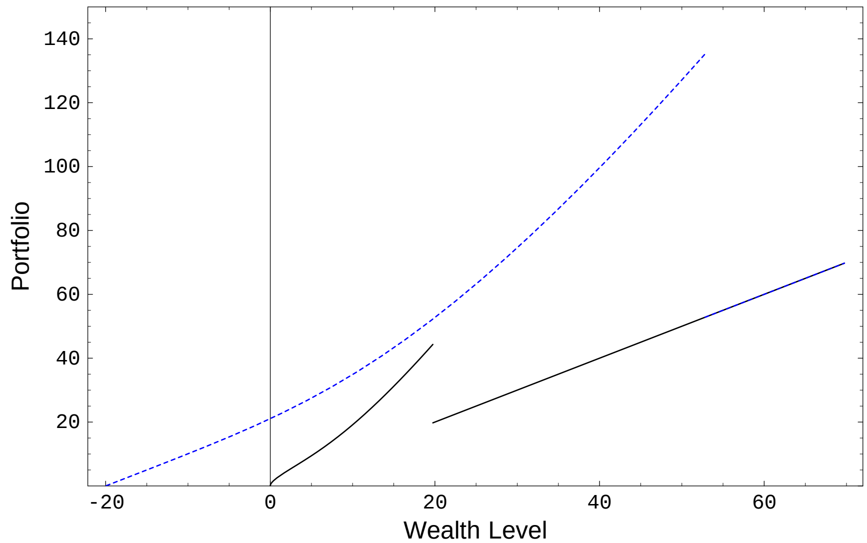

Figure 3: *The optimal portfolio for the log utility function* ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \epsilon = 0.2$ and $l = 0.5$). *The solid line shows our model and the dotted line shows the benchmark model.*

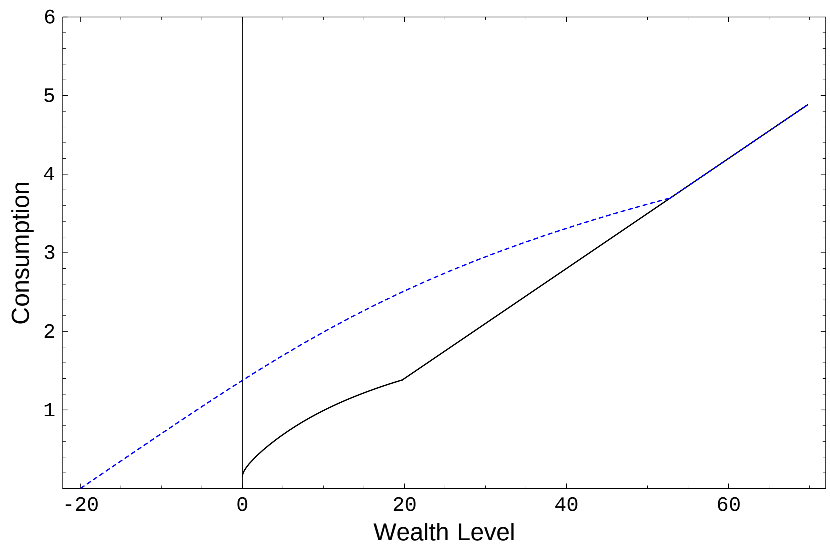

Figure 4: *The optimal consumption for the log utility function* ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \epsilon = 0.2$ and $l = 0.5$). *The solid line shows our model and the dotted line shows the benchmark model.*

retirement time is the optimal stopping time obtained by solving a variational inequality which is a free boundary value problem. For the special case of a CRRA utility class, in both the optimal portfolio and the optimal consumption, the liquidity constraints reduce not only the investment and the consumption but also the critical wealth level. In other words, the restriction to borrow future income induces the reduced financial activity.

A Duality Approaches

For any fixed $\tau \in \mathcal{S}$, we consider the following utility maximization problem

$$V_\tau(x) \triangleq \sup_{(c,\pi)\in\Pi_\tau(x)} J(x;c,\pi,\tau). \tag{A.1}$$

The solution of the problem (A.1) can be derived by the following ways. For any $(c,\pi,\tau) \in \mathcal{A}(x)$ and any real number $\lambda > 0$, we obtain the following inequality about a positive process D_t which is a non-increasing process with $D_0 = 1$. (See Extended Appendix of Farhi and Panageas [6].)

$$\begin{aligned}
J(x;c,\pi,\tau) = \mathbb{E}&\left[\int_0^\tau e^{-\beta t}\{u(c_t) - l - \lambda D_t e^{\beta t} H_t c_t\}dt\right.\\
&\left.+\, e^{-\beta\tau}\{U(X_\tau) - \lambda D_\tau e^{\beta\tau} H_\tau X_\tau\}\right] + \lambda\mathbb{E}\left[\int_0^\tau D_t H_t c_t dt + D_\tau H_\tau X_\tau\right]\\
\le \mathbb{E}&\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda D_t e^{\beta t} H_t)dt + e^{-\beta\tau}\tilde{U}(\lambda D_\tau e^{\beta\tau} H_\tau) - \int_0^\tau e^{-\beta t} l dt\right]\\
&+\lambda\mathbb{E}\left[\int_0^\tau D_t H_t c_t dt + D_\tau H_\tau X_\tau\right],
\end{aligned} \tag{A.2}$$

where the dual utility functions $\tilde{u}(\cdot)$ and $\tilde{U}(\cdot)$ are defined by

$$\tilde{u}(y) = \max_{c>0}\{u(c) - cy\} = u(I_1(y)) - yI_1(y), \tag{A.3}$$

and

$$\tilde{U}(y) = \max_{x>0}\{U(x) - xy\} = U(I_2(y)) - yI_2(y), \tag{A.4}$$

respectively. Here $I_1(\cdot)$ is defined in Lemma 3.1 and $I_2(\cdot)$ is the inverse function of $U'(\cdot)$. Then $\tilde{u}(\cdot)$ and $\tilde{U}(\cdot)$ are strictly decreasing and strictly convex.

Since we assume $D_0 = 1$, by the integration by parts and the constraints (2.5) and

(2.6), the second term of the right hand side of the inequality (A.2) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\tau D_t H_t c_t dt + D_\tau H_\tau X_\tau\right] \\
&= \mathbb{E}\left[\int_0^\tau D_t H_t (c_t - \epsilon) dt + D_\tau H_\tau X_\tau + \int_0^\tau D_t H_t \epsilon dt\right] \\
&= \mathbb{E}\left[\int_0^\tau D_t H_t \epsilon dt + \int_0^\tau H_t c_t dt - \int_0^\tau H_t \epsilon dt + H_\tau X_\tau\right] \\
&\quad + \mathbb{E}\left[\int_0^\tau \mathbb{E}\left[\int_t^\tau H_s c_s ds + H_\tau X_\tau - \int_t^\tau H_s \epsilon ds \middle| \mathcal{F}_t\right] dD_t\right] \\
&\le \mathbb{E}\left[\int_0^\tau D_t H_t \epsilon dt\right] + x,
\end{aligned}$$

provided that dD_t is not zero, that is, D_t is not a constant. Therefore the inequality (A.2) becomes

$$\begin{aligned}
J(x; c, \pi, \tau) &\le \mathbb{E}\left[\int_0^\tau e^{-\beta t} \tilde{u}(\lambda D_t e^{\beta t} H_t) dt + e^{-\beta\tau} \tilde{U}(\lambda D_\tau e^{\beta\tau} H_\tau) - \int_0^\tau e^{-\beta t} l dt\right] \\
&\quad + \mathbb{E}\left[\int_0^\tau \lambda D_t H_t \epsilon dt\right] + \lambda x \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{\tilde{u}(\lambda D_t e^{\beta t} H_t) dt + \lambda D_t e^{\beta t} H_t \epsilon - l\right\} dt\right. \\
&\quad \left. + e^{-\beta\tau} \tilde{U}(\lambda D_\tau e^{\beta\tau} H_\tau)\right] + \lambda x.
\end{aligned}$$

So for any fixed $\tau \in \mathcal{S}$, $V_\tau(x) \le \inf_{\{\lambda>0, D_t>0\}}[\tilde{J}(\lambda, D_t; \tau) + \lambda x]$, where

$$\tilde{J}(\lambda, D_t; \tau) = \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{\tilde{u}(\lambda D_t e^{\beta t} H_t) + \lambda D_t e^{\beta t} H_t \epsilon - l\right\} dt + e^{-\beta\tau} \tilde{U}(\lambda D_\tau e^{\beta\tau} H_\tau)\right],$$

and this inequality holds as equality if

$$c_t = I_1(\lambda D_t e^{\beta t} H_t), \quad X_\tau = I_2(\lambda D_\tau e^{\beta\tau} H_\tau), \text{ for all } 0 \le t \le \tau, \tag{A.5}$$

$$\mathbb{E}\left[\int_0^\tau H_t c_t dt + H_\tau X_\tau - \int_0^\tau H_t \epsilon dt\right] = x, \tag{A.6}$$

and

$$\mathbb{E}\left[\int_t^\tau H_s c_s ds + H_\tau X_\tau - \int_t^\tau H_s \epsilon ds \middle| \mathcal{F}_t\right] = 0. \tag{A.7}$$

Hence with the conditions (A.5), (A.6) and (A.7), $V_\tau(x)$ is obtained by $V_\tau(x) = \inf_{\{\lambda>0, D_t>0\}}[\tilde{J}(\lambda, D_t; \tau) + \lambda x]$. Therefore the value function $V(\cdot)$ is given by

$$V(x) = \sup_{\tau\in\mathcal{S}} V_\tau(x) = \sup_{\tau\in\mathcal{S}} \inf_{\{\lambda>0, D_t>0\}} \left[\tilde{J}(\lambda, D_t; \tau) + \lambda x\right].$$

It is not easy, however, to get an explicit solution by this method and also it does not guarantee the existence of the solution. Karatzas and Wang [12], joined with the results of He and Pagès [7] resolved the problem by interchanging $\sup_{\tau\in\mathcal{S}}$ and $\inf_{\{\lambda>0, D_t>0\}}$.

We define

$$\tilde{V}(\lambda) \triangleq \sup_{\tau \in \mathcal{S}} \inf_{D_t > 0} \tilde{J}(\lambda, D_t; \tau) = \inf_{D_t > 0} \sup_{\tau \in \mathcal{S}} \tilde{J}(\lambda, D_t; \tau), \tag{A.8}$$

then the following proposition gives the value function $V(\cdot)$.

Proposition A.1. *If $\tilde{V}(\lambda)$ exists and is differentiable for $\lambda > 0$, then*

$$V(x) = \inf_{\lambda > 0} \left[\tilde{V}(\lambda) + \lambda x \right]$$

holds for every $x \in (0, \infty)$.

Proof. See Theorem 8.5 and Corollary 8.7 of Karatzas and Wang [12]. □

Remark A.1. *For any fixed $\tau \in \mathcal{S}$, $\tilde{V}(\cdot)$ is strictly convex and strictly decreasing. (See Section 5 of He and Pagès [7] and Lemma 8.1 of Karatzas and Wang [12].)*

B Solutions to the Problem

In order to find $\tilde{V}(\lambda)$, two problems are considered at a time. One is a minimization problem with the control D_t and the other is an optimal stopping problem. We get a Bellman equation which can be solved explicitly and a variational inequality which can be induced by the dual functions of the optimal stopping problem. In the next section, the function $\tilde{V}(\lambda)$ is obtained. In addition, the value function and the optimal policies are also derived explicitly.

In order to obtain $\tilde{V}(\lambda)$ we define

$$\phi(t, y) = \sup_{\tau > t} \inf_{D_t > 0} \mathbb{E}^{y_t = y} \left[\int_t^{\tau} e^{-\beta s} \{ \tilde{u}(y_s) + \epsilon y_s - l \} ds + e^{-\beta \tau} \tilde{U}(y_\tau) \right],$$

where $y_t = \lambda D_t e^{\beta t} H_t,\ y_0 = \lambda > 0$. So the following SDE is induced by Itô's formula

$$\frac{dy_t}{y_t} = \frac{dD_t}{D_t} + (\beta - r) dt - \theta dB_t. \tag{B.1}$$

Now we assume that D_t solves a differential equation of the form $dD_t = -\psi(t) D_t dt$ for some $\psi(t) \geq 0$.

Then we can get the following Bellman equation

$$\min \left\{ \mathcal{L}\phi(t, y) + e^{-\beta t} \{ \tilde{u}(y) + \epsilon y - l \}, -\frac{\partial \phi}{\partial y} \right\} = 0, \tag{B.2}$$

where the differential operator is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r) y \frac{\partial}{\partial y} + \frac{1}{2} \theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

(For more detailed reference, see Section 5 of He and Pagès [7]).

Now let D_t^* be the optimal solution of the Bellman equation (B.2), then the optimal stopping time problem can be derived by the following modified variational inequality.

Variational Inequality 1. *Find the free boundary $\bar{y}$, $\hat{y}$ which makes zero wealth level and a function $\tilde{\phi}(\cdot,\cdot) \in C^1((0,\infty)\times\mathbb{R}^+) \cap C^2((0,\infty)\times(\mathbb{R}^+ \setminus \{\bar{y}\}))$ satisfying*

(1) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}(y) + \epsilon y - l\} = 0,\ \bar{y} < y \le \hat{y}$

(2) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}(y) + \epsilon y - l\} \le 0,\ 0 < y \le \bar{y}$

(3) $\tilde{\phi}(t,y) > e^{-\beta t}\tilde{U}(y),\ y > \bar{y}$

(4) $\tilde{\phi}(t,y) = e^{-\beta t}\tilde{U}(y),\ 0 < y \le \bar{y},$

(5) $\frac{\partial\tilde{\phi}}{\partial y}(t,y) \le 0,\ 0 < y \le \hat{y}$

(6) $\frac{\partial\tilde{\phi}}{\partial y}(t,y) = 0,\ y \ge \hat{y}$

for all $t > 0$, with boundary conditions

$$\frac{\partial\tilde{\phi}}{\partial y}(t,\hat{y}) = 0 \quad \text{and} \quad \frac{\partial^2\tilde{\phi}}{\partial y^2}(t,\hat{y}) = 0. \tag{B.3}$$

This Variational Inequality 1 is a free boundary value problem which solution is the solution to the optimal stopping problem (A.8).

Remark B.1. *If we substitute (3.3) in Lemma 3.1 into (A.4), we obtain $\tilde{U}(\cdot)$ as follows:*

$$\begin{aligned}\tilde{U}(y) = {} & \frac{2y^{n_+}}{\theta^2(n_+ - n_-)}\int_{\hat{y}}^{y}\frac{zI_1(z) - u(I_1(z))}{z^{n_+ + 1}}dz \\ & - \frac{2y^{n_-}}{\theta^2(n_+ - n_-)}\int_{\hat{y}}^{y}\frac{zI_1(z) - u(I_1(z))}{z^{n_- + 1}}dz.\end{aligned}$$

Remark B.2. *In the benchmark model, the conditions (B.3) are not considered. The conditions (B.3) are considered under the nonnegative wealth constraints. (See the proof of Lemma 4 in Appendix of Dybvig and Liu [5] and Subsection 6.2 of Choi, Shim, and Shin [3].)*

Theorem B.1. *(Variational inequality for optimal stopping) If $\tilde{\phi}(t,y)$ satisfies the Variational Inequality 1, then*

$$\tilde{\phi}(t,y) = \phi(t,y) = \sup_{\tau > t}\inf_{D_t > 0}\mathbb{E}^{y_t = y}\left[\int_t^\tau e^{-\beta s}\{\tilde{u}(y_s) + \epsilon y_s - l\}ds + e^{-\beta\tau}\tilde{U}(y_\tau)\right]$$

and

$$\tau^* = \inf\{s > t \mid y_s \le \bar{y}\} < \infty, \quad a.s.$$

is the optimal stopping time.

Proof. See Theorem 10.4.1 of Øksendal [13]. □

Next remark and proposition give the value function $V(\cdot)$.

Remark B.3. *If $\phi(t,y)$ is a solution to the Variational Inequality 1, then $\tilde{V}(\lambda) = \phi(0,\lambda)$.*

Proposition B.1. *Let*

$$v(y) = \begin{cases} C_1 y^{n_+} + C_2 y^{n_-} \\ \quad + \frac{2y^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_+ + 1}} dz \\ \quad - \frac{2y^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_- + 1}} dz, & \text{if } \bar{y} < y \le \hat{y}, \\ \frac{2y^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{zI_1(z) - u(I_1(z))}{z^{n_+ + 1}} dz \\ \quad - \frac{2y^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{zI_1(z) - u(I_1(z))}{z^{n_- + 1}} dz, & \text{if } 0 < y \le \bar{y}, \end{cases}$$

then $\tilde{\phi}(t,y) = e^{-\beta t} v(y)$ is a solution to Variational Inequality 1 provided that

$$\bar{y} \le \frac{l}{\epsilon} \le \hat{y}, \tag{B.4}$$

where the coefficients C_1, C_2, $\hat{y}$ and the free boundary value $\bar{y}$ are determined numerically. (See the proof.)

Proof. We consider the partial differential equation(PDE) (1) of the Variational Inequality 1,

$$\frac{\partial \phi}{\partial t} + (\beta - r) y \frac{\partial \phi}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial \phi^2}{\partial y^2} + e^{-\beta t}\left(u(I_1(y)) - yI_1(y) + \epsilon y - l\right) = 0, \text{ for } \bar{y} < y \le \hat{y}. \tag{B.5}$$

If we guess the trial solution of the form $\phi(t,y) = e^{-\beta t} v(y)$, then the PDE (B.5) is rewritten as an ordinary differential equation(ODE) for $v(y)$, that is,

$$\frac{1}{2}\theta^2 y^2 v''(y) + (\beta - r) y v'(y) - \beta v(y) + u(I_1(y)) - yI_1(y) + \epsilon y - l = 0. \tag{B.6}$$

Setting $v(y) = y^n$, then the homogenous equation of ODE (B.6) can be reduced to the quadratic equation (3.5) with roots $n_- < 0$ and $n_+ > 1$. So the homogenous ODE (B.6) has a general solution of the form $v_h(y) = C_1 y^{n_+} + C_2 y^{n_-}$, for some constants C_1 and C_2. Using the variation of parameters, the particular solution can be obtained easily. By the superposition principle, the solution of the ODE (B.6) is given by

$$v(y) = C_1 y^{n_+} + C_2 y^{n_-} + \frac{2y^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_+ + 1}} dz \\ - \frac{2y^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_- + 1}} dz.$$

Now in order to determine the coefficients C_1, C_2, $\hat{y}$ which makes zero wealth level and the free boundary value $\bar{y}$, we use the fact that $v(\cdot)$ has the smooth fit condition at $y = \bar{y}$, i.e. $v(\cdot)$ is continuous and has C^1-condition at $y = \bar{y}$. Thus we obtain C_1 and C_2 as follows:

$$C_1 = -\frac{\frac{l}{\beta}n_- - \frac{\epsilon}{r}(n_- - 1)\bar{y}}{(n_+ - n_-)\bar{y}^{n_+}} + \frac{\frac{l}{\beta}n_- - \frac{\epsilon}{r}(n_- - 1)\hat{y}}{(n_+ - n_-)\hat{y}^{n_+}}, \tag{B.7}$$

and

$$C_2 = \frac{\frac{l}{\beta}n_+ - \frac{\epsilon}{r}(n_+ - 1)\bar{y}}{(n_+ - n_-)\bar{y}^{n_-}} - \frac{\frac{l}{\beta}n_+ - \frac{\epsilon}{r}(n_+ - 1)\hat{y}}{(n_+ - n_-)\hat{y}^{n_-}}, \tag{B.8}$$

respectively. In order to find $\hat{y}$, we use the boundary conditions (B.3), which are rewritten as $v'(\hat{y}) = 0$ and $v''(\hat{y}) = 0$, i.e.,

$$v'(\hat{y}) = n_+ C_1 \hat{y}^{n_+ - 1} + n_- C_2 \hat{y}^{n_- - 1} = 0 \tag{B.9}$$

and

$$v''(\hat{y}) = n_+(n_+ - 1)C_1\hat{y}^{n_+ - 2} + n_-(n_- - 1)C_2\hat{y}^{n_- - 2} + \frac{2\left[l + \hat{y}(I_1(\hat{y}) - \epsilon) - u(I_1(\hat{y}))\right]}{\theta^2\hat{y}^2} = 0. \tag{B.10}$$

With the previous equations (B.9) and (B.10), we also obtain C_1 and C_2 as follows:

$$C_1 = -\frac{\frac{2}{\theta^2}\left[l + \hat{y}(I_1(\hat{y}) - \epsilon) - u(I_1(\hat{y}))\right]}{n_+(n_+ - n_-)\hat{y}^{n_+}}, \tag{B.11}$$

and

$$C_2 = \frac{\frac{2}{\theta^2}\left[l + \hat{y}(I_1(\hat{y}) - \epsilon) - u(I_1(\hat{y}))\right]}{n_-(n_+ - n_-)\hat{y}^{n_-}}, \tag{B.12}$$

respectively. In order to get the boundary value $\hat{y}$, assume that $\bar{y} = \xi\hat{y}$, for some $\xi \in (0, 1)$, then the equations (B.7) and (B.11) imply the following equation

$$\left[\frac{\epsilon}{r}(n_- - 1)(\xi^{n_+} - \xi) + \frac{2\epsilon}{\theta^2}\xi\right]\hat{y} - \frac{2l}{\theta^2} = \frac{2}{\theta^2}\left[\hat{y}I_1(\hat{y}) - u(I_1(\hat{y}))\right]\xi^{n_+}. \tag{B.13}$$

Similarly, the equations (B.8) and (B.12) imply the following equation

$$\left[\frac{\epsilon}{r}(n_+ - 1)(\xi^{n_-} - \xi) + \frac{2\epsilon}{\theta^2}\xi\right]\hat{y} - \frac{2l}{\theta^2} = \frac{2}{\theta^2}\left[\hat{y}I_1(\hat{y}) - u(I_1(\hat{y}))\right]\xi^{n_-}. \tag{B.14}$$

From the equations (B.13) and (B.14) we get the following equation

$$\hat{y} = \frac{\frac{2l}{\theta^2}(\xi^{n_+} - \xi^{n_-})}{\frac{\epsilon}{r}\left\{(n_+ - n_-)\xi^{n_+ + n_-} - (n_+ - 1)\xi^{n_+ + 1} + (n_- - 1)\xi^{n_- + 1}\right\} + \frac{2\epsilon}{\theta^2}(\xi^{n_+ + 1} - \xi^{n_- + 1})}. \tag{B.15}$$

So if we substitute (B.15) into (B.13) or (B.14), and if we can get the equation about ξ, then we compute the value of ξ provided that it exists uniquely. (Here we can't show the existence and the uniqueness of ξ but we will show those for the special case of CRRA utility class in Section 5.) Therefore $\hat{y}$, $\bar{y}$, C_1 and C_2 are also determined successively.

It is easily shown that the inequality (2) of the Variational Inequality 1 holds by the condition (B.4). The inequality (5) of the Variational Inequality 1 also holds since $v(\cdot)$ is a decreasing function. (See Remark A.1 and B.3.)

For the inequality (3) of the Variational Inequality 1, we consider two functions

$$f_1(y) = C_1 y^{n_+} + C_2 y^{n_-} + \frac{2y^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_+ + 1}} dz$$
$$- \frac{2y^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{l + z(I_1(z) - \epsilon) - u(I_1(z))}{z^{n_- + 1}} dz$$

and

$$f_2(y) = \frac{2y^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{zI_1(z) - u(I_1(z))}{z^{n_+ + 1}} dz - \frac{2y^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{y} \frac{zI_1(z) - u(I_1(z))}{z^{n_- + 1}} dz.$$

Then it is easily seen that $\mathcal{L}\{e^{-\beta t} f_1(y)\} + e^{-\beta t}(\tilde{u}(y) + \epsilon y - l) = 0$ and $\mathcal{L}\{e^{-\beta t} f_2(y)\} + e^{-\beta t}\tilde{u}(y) = 0$. So if we define $f(y) \triangleq f_1(y) - f_2(y)$, for $\bar{y} < y < \hat{y}$, then we see that $\mathcal{L}\{e^{-\beta t} f(y)\} = e^{-\beta t}(l - \epsilon y)$, i.e.

$$\frac{1}{2}\theta^2 y^2 f''(y) + (\beta - r) y f'(y) - \beta f(y) = l - \epsilon y, \tag{B.16}$$

with $f(\bar{y}) = 0$ and $f'(\bar{y}) = 0$, by construction and definition of $\bar{y}$ and $f'(\hat{y}) = 0$, by the equation (B.9). So if we differentiate the ODE (B.16) with respect to y, we obtain

$$\frac{1}{2}\theta^2 y^2 f'''(y) + (\theta^2 + \beta - r) y f''(y) - r f'(y) = -\epsilon. \tag{B.17}$$

Suppose that $f'(\underline{y})$ attains a non-positive minimum in $\underline{y} \in (\bar{y}, \hat{y})$, then we directly obtain that $f'''(\underline{y}) > 0$ and $f''(\underline{y}) = 0$. However the left hand side of the ODE (B.17) is positive but the right hand side of (B.17) is negative. It is a contradiction. So $f'(\cdot)$ cannot attain any non-positive minimum in an interval $(\bar{y}, \hat{y})$. Since $f'(\bar{y}) = f'(\hat{y}) = 0$, $f'(y) > 0$, for $\bar{y} < y < \hat{y}$, $f(\cdot)$ is increasing for $\bar{y} < y < \hat{y}$, and consequently $f(\cdot) > 0$, for $\bar{y} < y < \hat{y}$. □

Remark B.4. *If $\bar{y}$ is the free boundary value in Proposition B.1 , then the optimal stopping time is given by*

$$\tau_y = \inf\{s > t \mid y_s \leq \bar{y}\} < \infty, \quad a.s.$$

C Proof of Theorem 4.2

The optimal retirement time is the same expression in Remark B.4 because there is one-to-one correspondence between X_t and $y_t^{\lambda^*}$ and $\bar{x}$ corresponds to $\bar{y}$.

The optimal consumption process is obtained while the problem is remodeled by the duality approaches. The condition (A.5) shows the optimal consumption process before retirement. While proving Lemma 3.1, the similar condition to (A.5) appears. Also the optimal consumption process after retirement is easily determined.

In order to seek the optimal portfolio process the quadratic equation (3.5) with two roots n_+ and n_- is required. The optimal portfolio comes from the optimal wealth process as mentioned before. In Proposition 4.1, the optimal wealth process for $0 \leq X_t < \bar{x}$ is given. So by applying Itô's formula to the optimal wealth process (4.2), the differential form is given by

$$
\begin{aligned}
dX^*(t) = &\Big[-n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-2} - n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--2} \\
&- \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-2}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&+ \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--2}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&- \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^2}\Big](dy_t^{\lambda^*}) \\
&+\frac{1}{2}\Big[-n_+(n_+-1)(n_+-2)C_1(y_t^{\lambda^*})^{n_+-3} - n_-(n_--1)(n_--2)C_2(y_t^{\lambda^*})^{n_--3} \\
&- \frac{2n_+(n_+-1)(n_+-2)(y_t^{\lambda^*})^{n_+-3}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&+ \frac{2n_-(n_--1)(n_--2)(y_t^{\lambda^*})^{n_--3}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&- \frac{2(n_++n_--3)}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{(y_t^{\lambda^*})^3} - \frac{2}{\theta^2}\frac{I_1(y_t^{\lambda^*})-\epsilon}{(y_t^{\lambda^*})^2}\Big](dy_t^{\lambda^*})^2 \\
= &\Big[-n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} - n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&- \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&+ \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&- \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Big]\{(\beta-r)dt-\theta dB_t\} \\
&+\frac{1}{2}\Big[-n_+(n_+-1)(n_+-2)C_1(y_t^{\lambda^*})^{n_+-1} - n_-(n_--1)(n_--2)C_2(y_t^{\lambda^*})^{n_--1} \\
&- \frac{2n_+(n_+-1)(n_+-2)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&+ \frac{2n_-(n_--1)(n_--2)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)} \int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&- \frac{2(n_++n_--3)}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}} \\
&- \frac{2}{\theta^2}I_1(y_t^{\lambda^*}) + \frac{2}{\theta^2}\epsilon\Big]\theta^2 dt
\end{aligned}
$$

$$
\begin{aligned}
&= r\Bigg[-n_+C_1(y_t^{\lambda^*})^{n_+-1} - n_-C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad - \frac{2n_+(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad + \frac{2n_-(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz\Bigg]dt \\
&+ \theta^2\Bigg[n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad + \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad - \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon))-u(I_1(z))}{z^{n_-+1}}dz \\
&\quad + \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Bigg]dt - I_1(y_t^{\lambda^*})dt + \epsilon dt \\
&- n_+C_1\left[\frac{1}{2}\theta^2n_+^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n_+ - \beta\right](y_t^{\lambda^*})^{n_+-1}dt \\
&- n_-C_2\left[\frac{1}{2}\theta^2n_-^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n_- - \beta\right](y_t^{\lambda^*})^{n_--1}dt \\
&- \frac{2n_+(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\left[\frac{1}{2}\theta^2n_+^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n_+ - \beta\right]\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dzdt \\
&+ \frac{2n_-(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\left[\frac{1}{2}\theta^2n_-^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n_- - \beta\right]\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dzdt \\
&- \left(n_+ + n_- - 1 + \frac{2(\beta - r)}{\theta^2}\right)\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}dt \\
&+ \theta\Bigg[n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad + \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad - \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&\quad + \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Bigg]dB_t
\end{aligned}
$$

$$
\begin{aligned}
&= r\Big[-n_+C_1(y_t^{\lambda^*})^{n_+-1} - n_-C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad - \frac{2n_+(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad + \frac{2n_-(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz\Big]\,dt \\
&+\theta^2\Big[n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad + \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad - \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&\quad + \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Big]\,dt \\
&- I_1(y_t^{\lambda^*})dt + \epsilon dt + \theta\Big[n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&\quad + \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&\quad - \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&\quad + \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Big]\,dB_t.
\end{aligned}
$$

Now if we compare this differential form with the wealth dynamics (2.1), then the optimal portfolio is obtained by

$$
\begin{aligned}
\pi_t^* = \frac{\theta}{\sigma}\Big[&n_+(n_+-1)C_1(y_t^{\lambda^*})^{n_+-1} + n_-(n_--1)C_2(y_t^{\lambda^*})^{n_--1} \\
&+ \frac{2n_+(n_+-1)(y_t^{\lambda^*})^{n_+-1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_++1}}dz \\
&- \frac{2n_-(n_--1)(y_t^{\lambda^*})^{n_--1}}{\theta^2(n_+-n_-)}\int_{\hat{y}}^{y_t^{\lambda^*}} \frac{l+z(I_1(z)-\epsilon)-u(I_1(z))}{z^{n_-+1}}dz \\
&+ \frac{2}{\theta^2}\frac{l+y_t^{\lambda^*}(I_1(y_t^{\lambda^*})-\epsilon)-u(I_1(y_t^{\lambda^*}))}{y_t^{\lambda^*}}\Big].
\end{aligned}
$$

In addition, the optimal consumption process is also confirmed by similar comparing

$$c_t^* = I_1(y_t^{\lambda^*}).$$

This value is exactly same with the value derived by the duality approaches. For $X_t \geq \bar{x}$, the optimal wealth process (4.3) is given in Proposition 4.1. So a similar method is also applied.

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