Dynamic Arbitrage Pricing with Return-Related Market Frictions.

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Abstract

The paper presents the fundamental theorem of asset pricing (FTAP) in multi-period asset markets with return-related market frictions such as capital income taxes and transaction costs. Specifically, we show the triple equivalence between the no arbitrage condition, the existence of pricing rules, and the viability of asset prices. The net returns of assets are assumed to be a concave function of portfolios. This assumption subsumes progressive income taxes and convex transaction costs as a special case. Remarkably, the pricing rules are determined by the lowest and highest net returns and thus, characterized as simply as in the case with proportional tax schemes and transaction costs. The result is in sharp contrast to equilibrium pricing models where prices depend on hard-to-observe quantities such as nonlinear marginal net returns.

KEYWORDS: The fundamental theorem of asset pricing, arbitrage, viable prices, income taxes, transaction costs.

JEL Classification: G12, D52, C62, G11.

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I. Introduction

Capital income taxes and transaction costs affect the net returns of assets which are paid to investors. Rational investors will choose optimal portfolios at which the discounted expected value of marginal net returns with respect to a risk-neutral probability is equal to the equilibrium asset prices. Since capital income taxes and transaction costs are an important ingredient of the marginal net returns, they have impact on the asset prices in equilibrium. When tax rates are independent of income levels, marginal tax rates are constant. In this case, their effect on net returns is easily incorporated into calculating market value of asset returns. If tax schedules are nonlinear so that marginal tax rates hinge on the level of capital income, it is hard to measure them and thus, calculate the effect of income taxes on asset prices without observing capital income tax schedules and portfolio choices. For example, as shown in Ross (1987), the pricing kernel with progressive taxation depends upon the marginal effects of capital income taxes and thus, may not be observable.

The paper establishes the fundamental theorem of asset pricing (FTAP) in multi-period, finite-state asset markets with return-related market frictions such as capital income taxes and transaction costs. Specifically, we show the triple equivalence between (i) the no arbitrage condition, (ii) the existence of pricing rules, and (iii) the viability of asset prices.[1] The net returns of assets are assumed to be a concave function of portfolios. Thus, market frictions under examination subsume all the types of captial income taxes and transaction costs as far as they allow the net return functions to be concave. In particular, the current framework of market frictions encompasses progressive income taxes and convex transaction costs which lead to nonlinear net return schedules. Remarkably, the pricing rules are determined by the lowest and highest net returns and thus, characterized as simply as in the case with proportional tax schemes and transaction costs. The result is in sharp contrast to equilibrium pricing models where prices depend on hard-to-observe quantities such as nonlinear marginal net returns.

The approach of the paper to return-related market frictions depends on the superlinear functions determined by the lowest and highest marginal rate of nonlinear net return

[1] Informally speaking, asset prices are viable if optimal portfolio choices are available to investors.

functions.[2] Notably, intermediate marginal net returns do not affect the set of pricing rules which fulfill the no arbitrage condition. The idea of the paper is closely related to Dammon and Green (1987) which characterize the no tax arbitrage condition with the highest and lowest marginal tax rates under progressive tax regimes. Dammon and Green (1987) demonstrate that intermediate marginal tax rates need not be taken into account in formulating the arbitrage pricing rules with progressive taxation.

The FTAP of the paper is differentiated from the results of the literature in several respects. First, the equivalence between the no arbitrage condition and viability implies that the notion of arbitrage enables us to capture exactly the interval of equilibrium asset prices in a general setting. As shown Won (2006), well-known no-arbitrage conditions of the literature fail to pass the viability test. Second, no matter how complex the net return functions are, the pricing rules are characterized as simply as in the case with superlinear net return functions and therefore, determined independently of the marginal effect of market frictions. Third, the paper provides a unified view of different notions of arbitrage adopted in the literature which examines asset pricing implications of proportional or progressive tax regimes, and convex transaction costs in multi-period, finite-state asset markets. Fourth, investors' portfolio choices are not necessarily restricted to the set of self-financing strategies because they are allowed to consume in intermediate dates between the initial and final dates. Harrison and Kreps (1979) show that since intertemporal income transfers are free in the frictionless world, the approach to the two-date consumption model can be extended to the finite-period model where consumptions are available at all dates. Their arguments do not hold any more in the case with return-related market frictions which make intertemporal income transfers costly and affect ultimately the tradeoff between consumptions at different dates.

There exists a large body of the literature on arbitrage pricing with proportional transaction costs. Garman and Ohlson (1981), Boyle and Vorst (1992), Jouini and Kallal (1995), Kabanov (1999), Kabanov and Stricker (2001), Delbaen, Kabanov and Valkeila (200), Zhang, Xu and Deng (2002), Schachermayer (2004) among others examine the effect of propor-

[2] A function is *superlinear* if it is concave and positively homogeneous. Positively homogeneous functions are linear in the nonnegative direction of any vectors. For example, the net return functions with proportional taxes or transaction costs are superlinear.

tional transaction costs on asset pricing. The notions of arbitrage adopted in this literature are not applicable to market frictions which lead to nonlinear net returns such as progressive tax schedules. Dermody and Prisman (1993) attempt to adapt the notion of arbitrage with proportional transaction costs to the case with convex transaction costs. As illustrated in Won (2006), however, the notion of arbitrage of Dermody and Prisman (1993) fails to pass the viability test and may underestimate the interval of the viable pricing rules. Won (2006) verifies the fundamental theorem of asset pricing with convex transaction costs. Prisman (1986) and Ross (1987) introduce the notion of 'local arbitrage' to study the impact of nonlinear tax schedules on pricing rules. As illustrated in Dammon and Green (1987), however, the equivalence of the absence of local arbitrage and viability of pricing rule fails because local arbitrages exist in equilibrium in general.[3]

The paper is organized as follows. In Section II, multi-period, finite-state markets with return-related market frictions are described. The unifying notion of arbitrage is defined in terms of the superlinear function determined by the lowest and highest net returns, and the main consequences of the paper is presented in Section III. Concluding remarks are given in Section IV.

II. The Model

Asset markets are assumed to persist over finite time periods, $t = 0, 1, \ldots, T$. Let $\Omega = \{1, 2, \ldots, S\}$ denote a finite partition of states of nature. The revelation of information is described by a collection of partitions of Ω, $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}$, where $\mathcal{F}_t$ is finer than $\mathcal{F}_{t-1}$ (i.e. $\sigma \in \mathcal{F}_t$ and $\sigma' \in \mathcal{F}_{t-1}$ imply that $\sigma \subset \sigma'$ or $\sigma \cap \sigma' = \emptyset$) for all $t = 1, \ldots, T$.[4] We assume that $\mathcal{F}_0 = \{\Omega\}$. The information available at time $t = 0, 1, \ldots, T$ is described by the set $\sigma \in \mathcal{F}_t$ of the states of nature. We set $D = \bigcup_{t=0}^{T} \mathcal{F}_t$ and $D_{-T} = \bigcup_{t=0}^{T-1} \mathcal{F}_t$. An element in D is called a node or an event and D is called an event tree. In particular, σ_t in D denotes an event in $\mathcal{F}_t$. For each $\sigma_t \in \mathcal{F}_t$, let σ_t^- denote the event which immediately precedes σ_t, σ_t^+ the set of events which immediately succeed σ_t, and D_{σ_t} the set of events

[3]Local arbitrage is defined with respect to some reference portfolio. Thus there may exist a local arbitrage in general at portfolio choices which differ from the reference portfolio.

[4]For more details on the stochastic economy, see Magill and Shafer (1991) or Magill and Quinzii (1996).

which consist of σ_t and all the events succeeding σ_t. The set D_{σ_t} is a subtree at σ_t. For some positive integer n, let $\mathcal{L}(D_{-T}, \mathbb{R}^n)$ denote the collection of all $\mathbb{R}^n$-valued functions on D_{-T}. For brevity, $\mathcal{L}^n$ will be used instead of $\mathcal{L}(D_{-T}, \mathbb{R}^n)$. Let $\#D$ and $\#D_{-T}$ denote the number of elements in D and $\#D_{-T}$, respectively. Then $\mathcal{L}^n$ is the Euclidean space of dimension $(\#D_{-T}) \times n$. Let L denote the set of all real-valued functions defined on D. We set $L_+ = \{x \in L : x(\sigma) \geq 0, \sigma \in D\}$ and $L_{++} = \{x \in L : x(\sigma) > 0, \sigma \in D\}$.

We assume that a single good is available in each state of time $t = 0, \ldots, T$.[5] There are J long-lived assets issued at time 0 and traded in each state of time $t = 0, \ldots, T-1$. Allowing for some notational abuse, we also denote the set of assets by J. A price process of asset j is a function $q_j : D_{-T} \to \mathbb{R}$ and a trading strategy is a function $\theta : D_{-T} \to \mathbb{R}^J$. Thus, $q = (q_1, \ldots, q_J)$ and θ are a point in $\mathcal{L}^J$. More specifically, $q^j(\sigma)$ and $\theta^j(\sigma)$ denote a price and a position of asset j, and $q(\sigma) \in \mathbb{R}^J$ and $\theta(\sigma) \in \mathbb{R}^J$ denote prices and positions of J assets at the node $\sigma \in D$. For a price-event pair (q, σ) in $\mathcal{L}^J \times D$, let $R(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ denote the net return schedule which is derived from deducting transaction costs from the gross return. Specifically, if a trading strategy $\theta \in \mathcal{L}^J$ is chosen at the price q, the net return $R(\theta, q; \sigma)$ will be delivered to the investor in the event σ. For a price $q \in \mathcal{L}^J$, let $R(\cdot, q)$ denote the function which assigns each $\sigma \in D$ to $R(\cdot, q; \sigma)$. Thus, for a trading strategy $\theta \in \mathcal{L}^J$, $R(\theta, q)$ is a $\#D$-dimensional net return vector.

We impose the following conditions on the net return functions.

Assumption 1: For each $q \in \mathcal{L}^J$ and $\sigma \in D$, $R(0, q; \sigma) = 0$ and $R(\cdot, q; \sigma)$ is concave and continuous.

Assumption 2: For a price $q \in \mathcal{L}^J$, let $\theta \neq 0$ be a point in $\mathcal{L}^J$ with $\lim_{\lambda\to\infty} R(\lambda\theta, q)/\lambda = 0$. Then there exists a nonzero $\gamma \in \mathcal{L}^J$ such that

$$\lim_{\lambda\to\infty} \frac{R(\lambda\gamma, q)}{\lambda} = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{R(\lambda(\theta+\gamma), q)}{\lambda} > 0.^6$$

The condition $R(0, q; \sigma) = 0$ of Assumption 1 means that no portfolio holding pays noth-

[5]It is straightforward to incorporate the case of multiple goods into the current setting.

[6]The inequality $>$ indicates a vector inequality such that for two vectors x, y in a Euclidean space, $x > y$ means that each component of x is larger than that of y and $x \neq y$.

ing. The concavity of $R(\cdot, q; \sigma)$ is suitable to characterizing the effect of progressive tax schedules and proportional transaction cost functions on net returns. To check Assumption 2, we introduce some notation. For each $q \in \mathcal{L}^J$ and $\sigma \in D$, we define the function $V(\cdot, q; \sigma) : \mathcal{L}^J \to \mathbb{R}$ such that for all $\theta \in \mathcal{L}^J$,

$$V(\theta, q; \sigma) = \lim_{\lambda \to \infty} \frac{R(\lambda\theta, q; \sigma)}{\lambda}. \tag{1}$$

For each $q \in \mathcal{L}^J$, let $V(\cdot, q)$ denote a collection of functions $V(\cdot, q; \sigma)$'s for all $\sigma \in D$.[7] By Lemma 3.1 stated in the next section, we have $V(\theta' + \gamma', q) \geq V(\theta', q) + V(\gamma', q) \geq 0$ for any θ' and γ' with $V(\theta', q) = 0$ and $V(\gamma', q) = 0$. The condition of Assumption 2 goes little further by imposing that for any θ with $V(\theta, q) = 0$, there exists γ with $V(\gamma, q) = 0$ such that $V(\theta + \gamma, q) > 0$.

For each $q \in \mathcal{L}^J$, we set $\mathcal{C}(q) = \{\theta \in \mathcal{L}^J : V(\theta, q) = V(-\theta, q) = 0\}$. Since $R(\cdot, q)$ is concave, it follows that for all $\theta \in \mathcal{L}^J$, $v \in \mathcal{C}(q)$, and $\lambda \in \mathbb{R}$, $R(\theta + \lambda\gamma) = R(\theta, q)$.[8] In particular, $R(\lambda\gamma) = 0$ for all $\gamma \in \mathcal{C}(q)$ and $\lambda \in \mathbb{R}$. Suppose that $\mathcal{C}(q)$ is not trivial, i.e., $\mathcal{C}(q) \neq \{0\}$. Then $\mathcal{L}^J$ is nontrivially decomposed as a direct sum of $\mathcal{C}(q)$ and its orthogonal complement $\mathcal{C}(q)^\perp$. Portfolios in $\mathcal{C}(q)$ do not affect the opportunity set of income transfers. In this case, nothing will be changed when we work with $\mathcal{C}(q)^\perp$ and the restriction of $R(\cdot, q)$ to $\mathcal{C}(q)^\perp$. Thus, from now on, we will assume that $\mathcal{C}(q) = \{0\}$ to avoid unnecessary complications.

To examine the relationship between arbitrage-free prices and viability of asset prices, we introduce an agent who has the endowment of consumptions $e \in L_+$ and preferences represented by a utility function $u : L_+ \to \mathbb{R}$.[9] We make the following assumption.

Assumption 3: The function u is continuous and strictly increasing.[10]

For a price $q \in \mathcal{L}^J$, the agent chooses $(x^*, \theta^*) \in L_+ \times \mathcal{L}^J$ which solves the optimization

[7] Recall that $V(v, q)$ has the same dimension as the differential of $R(v, q)$.

[8] The set $\mathcal{C}(q)$ is called the constancy space of $R(\cdot, q)$. For details, see Rockafellar (1970).

[9] It is implicitly assumed that a single consumption good is available in each state of the economy.

[10] The function u is strictly increasing if $u(x) > u(x')$ for any x, x' in L_+ which satisfy $x(\sigma) \geq x'(\sigma)$ for all $\sigma \in D$ and $x \neq x'$.

problem:

$$\max_{(x,\theta)} u(x)$$

subject to the budget set

$$\mathcal{B}(q) = \{(x,\theta) \in L_+ \times \mathcal{L}^J : x - e \leq R(\theta, q)\}.$$

The demand correspondence $\xi(q)$ is the set of optimal choices in $L_+ \times \mathcal{L}^J$ which solve the above optimization problem. Viability of asset prices are defined as follows.

Definition 2.1. An asset price $q \in \mathcal{L}^J$ is *viable* if $\xi(q) \neq \emptyset$.

We analyze the tax arbitrage model of Dammon and Green (1987) in the current framework.

Example 1: Let R denote the pre-tax return matrix and $\overline{R}$ the taxable portion of R. Let θ and q be a point in $\mathcal{L}^J$. For each $\sigma \in D$,

$$R(\theta, q; \sigma) = \begin{cases} -q \cdot \theta, & \sigma = \sigma_0 \\ R_\sigma \cdot \theta - T(\overline{R}_\sigma \cdot \theta), & \sigma \neq \sigma_0 \end{cases}$$

where R_σ and $\overline{R}_\sigma$ are the returns and the taxable portion of returns in state σ, respectively, and T is the tax schedule. It is assumed in Dammon and Green (1987) that

(DG1) T is a nondecreasing, convex, and continuous function of taxable income with $T(0) = 0$ and there exists $c > 0$ such that T is differentiable at all y's with $|y| > c$.

Since T is continuous and convex, for each $q \in \mathcal{L}^J$ and $\sigma \in D$, $R(\cdot, q; \sigma)$ is continuous and concave. Thus, it satisfies Assumption 1. On the other hand, by L'Hôpital's rule, for all $\sigma \neq \sigma_0$, we have[11]

$$V(\theta, q; \sigma) = \lim_{\lambda \to \infty} \frac{R(\lambda\theta, q; \sigma)}{\lambda} = \begin{cases} (R_\sigma - \overline{t}\overline{R}_\sigma) \cdot \theta, & \text{if } \overline{R}_\sigma \cdot \theta \geq 0 \\ (R_\sigma - \underline{t}\overline{R}_\sigma) \cdot \theta, & \text{if } \overline{R}_\sigma \cdot \theta < 0 \end{cases},$$

[11] For details, see Dammon and Green (1987).

We set $\bar{t} = \lim_{y \to +\infty} dT(y)/dy$ and $\underline{t} = \lim_{y \to -\infty} dT(y)/dy$. The rate $\bar{t}$ is the highest marginal tax rate and $\underline{t}$ the lowest marginal tax rate. Note that $\underline{t} = \bar{t}$ if T represents the linear tax schedule. If T is linear, $\theta \in \mathcal{C}(q)$ if only if $V(\theta, q) = 0$.

Now we check that the tax arbitrage model satisfies Assumption 2. If T is linear, Assumption 2 holds trivially because $\mathcal{C}(q) = \{0\}$. Thus, we will assume that T is not linear, i.e., $\underline{t} < \bar{t}$. Notice that $\theta \in \mathcal{C}(q)$ if and only if $V(\theta, q) = 0$, and $\overline{R} \cdot \theta = 0$. For each $\theta' \in \mathcal{L}^J$, we define the sets $D^+(\theta') = \{\sigma \in D \backslash \{\sigma_0\} : \overline{R}_\sigma \theta' > 0\}$ and $D^-(\theta') = \{\sigma \in D \backslash \{\sigma_0\} : \overline{R}_\sigma \theta' < 0\}$. Let θ be a nonzero point in $\mathcal{L}^J$ with $V(\theta, q) = 0$. Since $\mathcal{C}(q) = \{0\}$, we have $\theta \notin \mathcal{C}(q)$ and therefore, $\overline{R} \cdot \theta \neq 0$. Then either $D^+(\theta) \neq \emptyset$ or $D^-(\theta) \neq \emptyset$. If $D^+(\theta) \neq \emptyset$, we choose $\gamma \in \mathcal{L}^J$ such that $V(\gamma, q) = 0$, $\overline{R} \cdot \gamma \neq 0$, and $D^+(\theta) \cap D^-(\gamma) \neq \emptyset$. Let $\sigma \in D^+(\theta) \cap D^-(\gamma)$. Then we have $\overline{R}_\sigma(\theta + \alpha\gamma) > 0$ for sufficiently small $\alpha > 0$. It follows that

$$\begin{aligned} V(\theta + \alpha\gamma, q; \sigma) &= (R_\sigma - \bar{t}\overline{R}_\sigma) \cdot (\theta + \alpha\gamma) \\ &= \alpha(R_\sigma - \bar{t}\overline{R}_\sigma) \cdot \gamma \\ &> \alpha(R_\sigma - \underline{t}\overline{R}_\sigma) \cdot \gamma = 0 \end{aligned}$$

Similarly, if $D^-(\theta) \neq \emptyset$, we choose $\gamma \in \mathcal{L}^J$ such that $V(\gamma, q) = 0$, $\overline{R} \cdot \gamma \neq 0$, and $D^-(\theta) \cap D^+(\gamma) \neq \emptyset$. Let $\sigma \in D^-(\theta) \cap D^+(\gamma)$. Then we have $\overline{R}_\sigma(\theta + \alpha\gamma) < 0$ for sufficiently small $\alpha > 0$. It follows that

$$\begin{aligned} V(\theta + \alpha\gamma, q; \sigma) &= (R_\sigma - \underline{t}\overline{R}_\sigma) \cdot (\theta + \alpha\gamma) \\ &= \alpha(R_\sigma - \underline{t}\overline{R}_\sigma) \cdot \gamma \\ &> \alpha(R_\sigma - \bar{t}\overline{R}_\sigma) \cdot \gamma = 0 \end{aligned}$$

In short, there exists γ such that $V(\gamma, q) = 0$ and $V(\theta + \alpha\gamma, q) > 0$ for some $\alpha \in (0, 1)$. Thus, Assumption 2 is fulfilled in the tax arbitrage model.

III. Main Results

We provide a notion of arbitrage which is appropriate to studying the effect of return-related market frictions on asset pricing.

Definition 3.1: An asset price $q \in \mathcal{L}^J$ admits *no arbitrage opportunities* if there is no $\theta \in \mathcal{L}^J$ which satisfies $V(\theta, q) > 0$.[12]

Definition 3.1 is an extension of the notions of arbitrage appearing in the literature. Suppose that for all θ, q in $\mathcal{L}^J$, there exists $R(q)$ such that $R(\theta, q) = R(q) \cdot \theta$. Then it holds trivially that $V(\theta, q) = R(q) \cdot \theta$. Frictionless markets belong to this class of net return functions. Another spacial case is that for all $q \in \mathcal{L}^J$, $R(\cdot, q)$ is superlinear and thus, satisfies $R(\theta, q) = V(\theta, q)$. Examples are asset markets which are subject to proportional taxes or transaction costs.

Before going to the main results of the paper, we provide a useful characterization of the function V.

Lemma 3.1: For a pair $(q, \sigma) \in \mathcal{L}^J \times D$, the following hold.

i) For all $\theta \in \mathcal{L}^J$ and $\lambda \geq 0$, $V(\lambda\theta, q; \sigma) = \lambda V(\theta, q; \sigma)$.

ii) $V(\cdot, q; \sigma)$ is a concave function.

iii) For all θ, γ in $\mathcal{L}^J$, $V(\theta + \gamma, q; \sigma) \geq V(\theta, q; \sigma) + V(\gamma, q; \sigma)$.

iv) For a sequence $\{\theta^n\}$ in $\mathcal{L}^J$ with $\|\theta^n\| \to \infty$ such that $\theta^n/\|\theta^n\|$ converges to a point $\hat{\theta}$, we have

$$V(\hat{\theta}, q) \geq \liminf_{n\to\infty} \frac{R(\theta^n, q)}{\|\theta^n\|}.$$

PROOF : i) This result is immediate from the definition of V.

ii) Let α be a number in $[0, 1]$. It follows from (1) that for points θ, θ' in $\mathcal{L}^J$,

$$\begin{aligned}
V(\alpha\theta + (1-\alpha)\theta', q; \sigma) &= \lim_{\lambda\to\infty} \tfrac{R(\lambda(\alpha\theta+(1-\alpha)\theta'), q;\sigma)}{\lambda} \\
&\geq \lim_{\lambda\to\infty} \tfrac{\alpha R(\lambda\theta, q;\sigma)+(1-\alpha)R(\lambda\theta', q;\sigma)}{\lambda} \quad \text{(by concavity of } R(\cdot, q; \sigma)) \\
&= \alpha \lim_{\lambda\to\infty} \tfrac{R(\lambda\theta, q;\sigma)}{\lambda} + (1-\alpha) \lim_{\lambda\to\infty} \tfrac{R(\lambda\theta', q;\sigma)}{\lambda} \\
&= \alpha V(\theta, q; \sigma) + (1-\alpha) V(\theta', q; \sigma)\cdot
\end{aligned}$$

[12] Let v and v' be vectors in a Euclidean space. Then $v \geq v'$ implies v is greater than or equal to v' in a component-wise manner; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that each component of v is greater than the counterpart of v'.

This implies that $V(\cdot, q; \sigma)$ is a concave function.

iii) By i) and ii), we obtain

$$\begin{aligned} V(\theta + \gamma) &= V(2(\theta/2 + \gamma/2)) \\ &= 2V(\theta/2 + \gamma/2) \\ &\geq V(\theta) + V(\gamma). \end{aligned}$$

iv) For each $(q, \sigma) \in \mathcal{L}^J \times D$, we define the set

$$G(q, \sigma) = \{(\theta, \mu) \in \mathcal{L}^J \times \mathbb{R} : R(\theta, q; \sigma) \geq \mu\},$$

and denote the recession cone of $G(q, \sigma)$ by $G_\infty(q, \sigma)$.[13] Since $R(\cdot, q; \sigma)$ is continuous and concave, $G(q, \sigma)$ and $G_\infty(q, \sigma)$ are closed and convex. Clearly, $(\theta^n, R(\theta^n, q; \sigma)) \in G(q, \sigma)$ for each n.

By the concavity of $R(\cdot, q; \sigma)$ along with $R(0, q; \sigma) = 0$, we have $R(\theta^n, q; \sigma))/\|\theta^n\| \leq R(\theta^n/\|\theta^n\|, q; \sigma))$ for sufficiently large n. Since $R(\theta^n/\|\theta^n\|, q; \sigma)) \to R(\hat{\theta}, q; \sigma)) < \infty$. $\{R(\theta^n, q; \sigma))/\|\theta^n\|\}$ is bounded. Thus, $\{R(\theta^n, q; \sigma))/\|\theta^n\|\}$ has a subsequence convergent to $\mu(q, \sigma) \in \mathbb{R}$. (For notational simplicity, we will keep the index n to denote the convergent subsequence.) Then $\lim_{n\to\infty} R(\theta^n, q; \sigma))/\|\theta^n\| = \mu(q, \sigma)$. Since $(\hat{\theta}, \mu(q, \sigma)) \in G_\infty(q, \sigma)$, we have $(\lambda\hat{\theta}, \lambda\mu(q, \sigma)) \in G_\infty(q, \sigma)$ for all $\lambda > 0$. Recalling that $R(0, q; \sigma) = 0$, we have $(\lambda\hat{\theta}, \lambda\mu(q, \sigma)) \in G(q, \sigma)$ or $R(\lambda\hat{\theta}, q; \sigma)) \geq \lambda\mu(q, \sigma)$. It follows that

$$\lim_{\lambda\to\infty} \frac{R(\lambda\hat{\theta}, q; \sigma))}{\lambda} \geq \mu(q, \sigma) = \lim_{n\to\infty} \frac{R(\theta^n, q; \sigma)}{\|\theta^n\|} \geq \liminf_{n\to\infty} \frac{R(\theta^n, q)}{\|\theta^n\|}.$$

□

In particular, i) and iii) of Lemma 3.1 show that $V(\cdot, q; \sigma)$ is superlinear for all $q \in \mathcal{L}^J$ and $\sigma \in D$.

Let Λ denote the set of no arbitrage prices. Then we see that

$$\Lambda = \{q \in \mathcal{L}^J : V(\theta, q) \not> 0 \text{ for all } \theta \in \mathcal{L}^J\},$$

[13] A vector $v \in \mathcal{L}^J$ is called the direction of recession of the convex set $Z \in \mathcal{L}^J$ if for some $z \in Z$ and all $\lambda \geq 0$, $z + \lambda v \in Z$. The recession cone of Z is the set of all the directions of recession for Z. For details, see Rockafellar (1970).

where $\not>$ denotes the negation of the vector inequality $>$. Most literature on asset valuation by arbitrage focuses on verifying the equivalence between the no arbitrage conditions and the existence of pricing functionals. If the notions of arbitrage do not pass viability test, however, they fail to exactly characterize asset pricing in equilibrium. We shows that the no arbitrage condition of Definition 3.1 is equivalent to viability. Thus the no arbitrage condition of Definition 3.1 provides a coherent conceptual framework for studying asset pricing, portfolio choice problem, or equilibrium in markets with return-related market frictions such as taxes and transaction costs.

Theorem 3.1: Under Assumptions 1-3, $q \in \Lambda$ if and only if $\xi(q) \neq \emptyset$.

PROOF : ($\Leftarrow$) For a price $q \in \mathcal{L}^J$, suppose that $\xi(q) \neq \emptyset$. Then there exists a point $(x, \theta) \in \xi(q)$. Suppose that there exists a nonzero $v \in \mathcal{L}^J$ such that $V(v, q) > 0$. It follows from the concavity of $R(\cdot, q)$ that for all $\lambda > 1$,

$$\begin{aligned} R(\theta + v, q) &= R\left[\frac{1}{\lambda}(\theta + \lambda v, q) + \left(1 - \frac{1}{\lambda}\right)\theta\right] \\ &\geq \frac{1}{\lambda}R(\theta + \lambda v, q) + \left(1 - \frac{1}{\lambda}\right)R(\theta, q) \end{aligned}$$

By passing to the limit, it follows from iv) of Lemma 3.1 that

$$\begin{aligned} R(\theta + v, q) &\geq \lim_{\lambda \to \infty} \frac{1}{\lambda}R(\theta + \lambda v, q) + \lim_{\lambda \to \infty}\left(1 - \frac{1}{\lambda}\right)R(\theta, q) \\ &= V(v, q) + R(\theta, q) \\ &> R(\theta, q) \end{aligned}$$

By the strict monotonicity of u, this result implies that there exists $x' \in L_+$ such that $u(x') > u(x)$ and $(x, \theta + \lambda v) \in B(q)$, which contradicts the optimality of (x, θ) in $B(q)$.

($\Rightarrow$) Let q be a price in Λ. We set

$$X(q) = \{x \in L_+ : (x, \theta) \in \mathcal{B}(q) \quad \text{for some} \quad \theta \in \mathcal{L}^J\}.$$

First we claim that each $X(q)$ is closed and bounded, i.e., compact. To show that $X(q)$ is closed, we choose $\{x^n\}$ in $X(q)$ which converges to a point $x \in L_+$. For each n, we can choose $\theta^n \in \mathcal{L}^J$ such that $(x^n, \theta^n) \in \mathcal{B}(q)$. We claim that $\{\theta^n\}$ is bounded. Otherwise, let

$\theta^n/\|\theta^n\| \to \hat{\theta}$. Since $\{x^n\}$ is bounded, we have $\lim_{n\to\infty} R(\theta^n, q)/\|\theta^n\| \geq 0$. By v) of Lemma 3.1, we have $V(\hat{\theta}, q) \geq \lim_{n\to\infty} R(\theta^n, q)/\|\theta^n\| \geq 0$. Since $q \in \Lambda$, we must have $V(\hat{\theta}, q) = 0$. Recalling that $\hat{\theta} \neq 0$, by Assumption 2, there exists $\eta \in \mathcal{L}^J$ such that $V(\hat{\theta} + \eta) > 0$, which contradicts the fact that $q \in \Lambda$. Thus, $\{\theta^n\}$ is bounded. Since $\{(x^n, \theta^n)\}$ is bounded, it has a subsequence convergent to a point $(x, \theta) \in L_+ \times \mathcal{L}^J$. This implies that $(x, \theta) \in \mathcal{B}(q)$ and therefore, $X(q)$ is closed in L_+.

Now we show that $X(q)$ is bounded. Suppose that it is unbounded. Then there exists $\{x^n\}$ in $X(q)$ such that $\|x^n\| \to \infty$. Let $\{\theta^n\}$ be a sequence in $\mathcal{L}^J$ such that $(x^n, \theta^n) \in \mathcal{B}(q)$ for each n, i.e., $x^n - e \leq R(\theta^n, q)$. Noting that $\|x^n\| \to \infty$, we have $\|\theta^n\| \to \infty$. Since $\{\theta^n/\|\theta^n\|\}$ is bounded, it has a subsequence which converges to a nonzero point $\hat{\theta} \in \mathcal{L}^J$. By the concavity of $R(\cdot, q; \sigma)$ along with $R(0, q; \sigma) = 0$, it follows from iv) of Lemma 3.1 that for each n,

$$0 \leq \lim_{n\to\infty} \left(\frac{x^n}{\|\theta^n\|} - \frac{e}{\|\theta^n\|} \right) \leq \lim_{n\to\infty} \frac{R(\theta^n, q)}{\|\theta^n\|} \leq V(\hat{\theta}, q).$$

The same argument made above leads to a contradiction. Thus, $X(q)$ is bounded. Since $X(q)$ is closed, we conclude that $X(q)$ is compact.

Since u is continuous and $X(q)$ is compact, there exists $x \in L_+$ which satisfies $u(x) \geq u(z)$ for all $z \in X(q)$ and therefore, $\theta \in \mathcal{L}^J$ such that $(x, \theta) \in \xi(q)$. □

The following shows the equivalence between the no arbitrage condition and the existence of pricing rules.

Theorem 3.2: Under Assumption 1, the following two statements are equivalent.

i) $q \in \Lambda$.

ii) There exists $\pi \gg 0$ such that $\pi \cdot V(\theta, q) \leq 0$ for all $\theta \in \mathcal{L}^J$.

PROOF : ii) ⇒ i): Suppose that $q \notin \Lambda$. Then there exists $\theta \in \mathcal{L}^J$ such that $V(\theta, q) > 0$. Since $\lambda \gg 0$, this implies that $\lambda \cdot V(\theta, q) > 0$, which leads to a contradiction.

i) ⇒ ii): Suppose that $q \in \Lambda$. We define the set

$$Z(q) = \{y \in L : y \leq V(\theta, q), \theta \in \mathcal{L}^J\}.$$

By i) and ii) of Lemma 3.1, $Z(q)$ is a closed, convex cone.

Let Δ denote the set $\{y \in L_+ : \sum_{\sigma \in D} y(\sigma) = 1\}$. Clearly, Δ is compact and convex. Then $q \in \Lambda$ is equivalent to the condition that $Z(q) \cap (L_+ \setminus \{0\}) = \emptyset$ or $Z(q) \cap \Delta = \emptyset$. Since $Z(q)$ is a closed, convex cone, by the separating hyperplane theorem there exists a nonzero $\pi \in L$ such that

$$\sup_{\theta \in \mathcal{L}^J} \pi \cdot V(\theta, q) < \inf_{y \in \Delta} \pi \cdot y.$$

In particular, we see that

$$0 = \pi \cdot V(0, q) \leq \sup_{\theta \in \mathcal{L}^J} \pi \cdot V(\theta, q) < \inf_{y \in \Delta} \pi \cdot y.$$

Thus we have $\inf_{y \in \Delta} \pi \cdot y > 0$, which implies that $\pi \in L_{++}$. Let $\theta \in \mathcal{L}^J$. Then for each $\lambda > 0$, we have $\pi \cdot V(\lambda\theta, q) < \inf_{y \in \Delta} \pi \cdot y$. Recalling that $V(\lambda\theta, q) = \lambda V(\theta, q)$ for all $\lambda > 0$, we obtain $\pi \cdot V(\theta, q) < \inf_{y \in \Delta}(\pi \cdot y)/\lambda$ for all $\lambda > 0$, and therefore, $\pi \cdot V(\theta, q) \leq 0$. □

Theorem 3.1 and 3.2 leads to a full extension of the FTAP of Harrison and Kreps (1979) and Dybvig and Ross (1989) to asset markets with return-related market frictions.

Theorem 3.3: Under Assumptions 1-3, the following statements are equivalent.

(i) $q \in \Lambda$.

(ii) There exists $\pi \in L_{++}$ such that $\pi \cdot V(\theta, q) \leq 0$ for all $\theta \in \mathcal{L}^J$.

(iii) $\xi(q) \neq \emptyset$.

IV. Conclusion

The fundamental theorem of asset pricing is presented in the presence of return-related market frictions such as progressive income taxes. Specifically, Theorem 3.3 states the triple equivalence among the no arbitrage condition, the existence of pricing rules, and the viability of asset prices. The result relies on no specific functional form and thus, can find applications to a broad range of return-related market frictions. In particular, no matter how complex the structure of return-related market frictions looks like, the viable pricing rules are determined by the lowest and highest marginal net returns, and thus, characterized as simply as in the case with the superlinear net functions.

Theorem 3.3 can be applied to real-world examples of the net return function. In this case, one immediate issue is to find lowest and highest marginal net returns to get the corresponding superlinear function $V(\cdot, \cdot)$. Another interesting problem is to extend the results of Theorem 3.3 to asset markets where the net return functions are not concave or certain form of restrictions are imposed on individual portfolio choices.

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