Default Risk of Life Annuity and the Annuity Puzzle*

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ABSTRACT

In this paper, we consider an optimal consumption and investment problem for a partially annuitized individual in the presence of the default risk of insurance company. We verified the relationships between individual's optimal consumption and investment behaviors and the default risk of insurance company.

Numerical results tell us that the existence of the default risk can give a significant effect on the optimal behaviors of such individual. Furthermore, using the numerical results we found that the individual who invests large portion of her wealth in annuities has a strong demand for selling or refunding her annuities when facing the default risk of insurance company, therefore we suggest an opinion for the annuity puzzle that individual's recognition of the default risk of insurance company may interrupt the growth of annuity markets.

1 Introduction

As individuals in retirement live much longer than before, the optimal investment problem for such retirees becomes more important issue. In fact, in 1970 the expected life of a 65 year old male living in the Unites States was 13.8 years but in 2006 it is 17.2 years,[1] and the tendency of longer longevity is parallelled across almost all OECD countries. Thus how individuals in retirement invest in various financial assets, e.g. bonds, stocks and annuities, becomes a substantial question for such individuals and the financial consulting companies as well as insurance companies.

Since the seminal paper of Yaari (1965), lots of researchers have studied the optimal investment problem in the presence of life annuity market. Yaari (1965) found that for a individual (or a consumer) who has Von Neumann-Morgenstern expected utility, an uncertain lifetime and no bequest motive it is optimal to annuitize all of their savings. He assumed that the annuity market is *actuarially fair*, that is, the interest rate on actuarial notes is fairly determined by considering the mortality risk of the individual and the interest rate on regular notes. His results were extended by Davidoff et al. (2005), and they showed that the sufficient conditions for the full annuitization need not impose the Von Neumann-Morgenstern expected utility on the individual, nor need annuities be actuarially fair.

The results of Yaari (1965) and Davidoff et al. (2005) were obtained under the strong assumptions that individual's bequest motives did not exist and there was no other market except annuity and bond markets. However, in reality, some (may be the whole) parts of people, e.g. persons having children, seem to have bequest motives and there exist lots of financial markets where retirees trade actively. Related to the bequest motives, Friedman and Warshawsky (1990) and others have insisted on the point that it can reduce the demand for annuitization. In addition, there are some papers, e.g. Milevsky and Young (2007), solving the optimal investment problem by assuming the existence of the markets which are neither

[1] This data is contained in the 'Cohort Life Expectancy' table in *The 2007 Annual Report of The Board of Trustees of the Federal Old-Age and Survivors Insurance and Federal Disability Insurance Trust Funds*.

bond market nor annuity market.

This paper is devoted to solving an optimal consumption and investment problem for a partially annuitized retiree. An optimal consumption and investment problem for non-annuitized investors was firstly studied by Merton (1969, 1971), and he investigated individual's optimal consumption and investment behaviors taking part in bond and stock markets. Different from him, we will investigate the behaviors of a partially annuitized retiree. And we also assume that such individual have bequest motives and an uncertain lifetime.

On the other hand, history says that the insurance companies was used to being defaulted when some economic shocks were arrived. For examples, in the United States Executive Life Insurance Co., First Capital Life Insurance Co., Monarch Life Insurance Co. and more defaulted after 1990. A detailed list of defaulted insurance companies can be found in Chen and Suchanecki (2007). For this reason, some literatures such as Grosen and Jørgensen (2002), Bernard et al. (2005) and Chen and Suchanecki (2007) considered the default risk of insurance companies in order to value the liabilities of such companies.

This fact gives us the motivation of modelling the possibility of the default of insurance company which sells annuities to annuity buyers. Thus in this paper we assume that annuities can be defaulted, that is, we consider the case where an individual takes the default risk of insurance company.

The existence of both a partially annuitized individual and a defaultable insurance company makes optimal consumption and investment problem more interesting. The first contribution of this paper is to verify relationships between individual's optimal behaviors and the expected lifetime of insurance company. We believe that our study is the first one that investigates the effect of the default risk of insurance companies on the optimal economic behaviors of a retired individual.[2] Numerical results tell us that the expected lifetime of insurance company can give a significant effect on the optimal behaviors of the retirees holding

[2]There are some papers, for example Babbel and Merill (2007), which studied optimal annuitization problem for individuals preparing retirement. We employ a model, a little simple but easy to implement than that, in order to obtain the further results such as implications for the annuity puzzle.

life annuities.

By using numerical results, we also obtain implications with respect to the well-known *annuity puzzle*. For this, we employ two concepts measuring quantitatively the impact of default risk of an annuity on its policyholders: implicit value of annuity and default risk premium. The implicit value of annuity is considered as the policyholder's subjective reservation price of the annuity when she is offered to buy or sell a small amount of the annuity, and the default risk premium is the additional expected rate of return from the annuity in order for her to be induced to accept the insurer's default risk. We find that the implicit value of an annuity for a retiree heavily investing her wealth in it can be lower than its actuarially fair price, and the default risk premium of an annuity can be relatively very big compared with the original interest rates of the annuity. In the sequel, we find some numerical evidences that individuals' recognition of default risks of insurance companies may hamper the growth of annuity markets. This is the main contribution of our paper.

This remainder of this paper is organized as follows. We set up our model, which contain insurance company facing default risk, and clarify the optimal consumption and investment problem in Section 2. We explore two stationary cases of our model in Section 3 and define two important concepts: implicit value and default risk premium. We provide numerical implications in Section 5 and Section 6 concludes.

2 The Basic Model

We study a retired individual's consumption and investment problem. The financial market consists of a risk-free asset (bank account or bond), a risky asset (stock), and an annuity.

The individual receives income from the annuity which she accumulated with an insurance company before retirement.[3] We assume that the annuity is a fixed life annuity with no

[3]In this paper we consider only annuities issued by private insurance companies. However, all the results except those regarding default risk premium are valid also for the case where the annuity is from a public source such as the Social Security Administration.

guarantee period,[4] thus she receives income at a constant rate from the insurance company until death unless the company defaults. We let the fixed rate be \$$\varepsilon$ per unit time (a year). For basic information about life annuities we refer to Milevsky and Young (2007).

The individual is assumed to have a bequest motive and want to maximize her utility by investing both risk-free and risky assets and consuming out of her income and wealth.[5] Denote the consumption rate process at time t by c_t. Her lifetime is modeled as a random time τ and the corresponding *hazard rate* (or *intensity*)[6] is denoted as λ_t. The insurance company selling the annuity to the individual is assumed to be exposed to default risk, which is expressed by another random time τ_D with hazard rate ϕ_t. We also assume that, if the insurance company defaults at time t, the individual obtains I_t amount of lump-sum income as a recovery. In other words, we assume that the policyholder always can liquidate her annuities even though she suffers a loss from the default, or we consider I_t as the net worth of the recovery of the annuity at the default time.[7]

The price process P_t of the risk-free asset evolves according to the equation

$$dP_t = rP_t dt,$$

where r is a risk-free interest rate. The price S_t of the risky asset follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where μ and σ are positive constant and B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t\}_{t \geqslant 0}$ be the augmentation of the natural filtration generated by pairly independent random variables $\tau,\ \tau_D,\ I_t$ and $B_t (t \geqslant 0)$.

[4] We affirm that the same results as those in the paper are also derived for the case where there is a variable life annuity in the market.

[5] We assume the individual is not restricted by short-selling constraints. In other words, she can borrow money from the bank at the risk-free interest rate and take a short position of the stock without any cost.

[6] For more details of hazard rate, we refer to Bielecki and Jeanblanc (2007).

[7] The in-depth explanation of I_t in real world will be exhibited in Section 3.

If we define

$$\begin{cases} F_t = P\{\tau \le t \mid \mathcal{F}_t\}, \\ G_t = P\{\tau_D \le t \mid \mathcal{F}_t\}, \end{cases}$$

and assume that they are absolutely continuous, then we can rewrite them as

$$\begin{cases} F_t = 1 - exp\Big(- \int_0^t \lambda_s ds\Big) \\ G_t = 1 - exp\Big(- \int_0^t \phi_s ds\Big), \end{cases}$$

by definition of the hazard rate (see Bielecki and Jeanblanc (2007)).

The individual's problem is to maximize her lifetime happiness by choosing consumption and investment: her happiness is measured by a utility function $u(\cdot)$ and bequest function $v(\cdot)$. That is, her problem can be stated as finding the following value function:

$$V(w,t) = \sup_{\{c,\pi\}\in\mathcal{A}(w)} E\left[\int_t^\tau (1-\alpha)e^{-\rho(s-t)}u(c_s)ds + \alpha e^{-\rho(\tau-t)}v(W_\tau) \;\Big|\; \mathcal{F}_t\right], \tag{1}$$

where individual's wealth process W is subject to

$$dW_s = [rW_s + \pi_s(\mu - r) - c_s + \varepsilon]\, ds + \sigma\pi_s dB_s, \;\; W_t = w > 0, \quad \text{for} \;\; t \le s < \tau \wedge \tau_D.$$

Here, ρ is the individual's constant subjective discount rate, α is the constant weight representing the relative importance of the individual's motive for bequest, and π_s is the amount invested in the risky assets at time s. As in Merton (1969, 1971), we let $\mathcal{A}(\cdot)$ be a usual set of *admissible* controls making the value of $V(\cdot,\cdot)$ finite.

According to the dynamic programming principle (see Merton 1971 and Moore and Young 2006) we obtain the following Hamilton-Jacobi-Bellman(HJB) equation:

$$\begin{aligned} \frac{\partial V}{\partial t} + \sup_c \Big[(1-\alpha)u(c) - c\frac{\partial V}{\partial w}\Big] + \sup_\pi \Big[\pi(\mu - r)\frac{\partial V}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 V}{\partial w^2}\Big] + (rw+\varepsilon)\frac{\partial V}{\partial w} \\ + \lambda_t\Big[\alpha v(w) - V\Big] + \phi_t\Big[E[J(w+I_t,t)|\mathcal{F}_t] - V\Big] = \rho V \end{aligned} \tag{2}$$

for $t < \tau$, with a boundary condition

$$\lim_{s\to\infty} E\Big[\exp\Big(- \int_t^s (\rho + \lambda_u)du\Big) \; V(W_s, s) \;\Big|\; \mathcal{F}_t\Big] = 0.$$

Here, $J(\cdot,\cdot)$ represents the value function for the situation where the insurance company has already defaulted, and hence, the individual's wealth evolves by

$$dW_s = [rW_s + \pi_s(\mu - r) - c_s]ds + \sigma\pi_s dB_s, \quad \text{for} \;\; t \le s \le \tau,$$

and satisfies the HJB equation:

$$\begin{aligned} \frac{\partial J}{\partial t} + \sup_c \Big[(1-\alpha)u(c) - c\frac{\partial J}{\partial w}\Big] + \sup_\pi \Big[\pi(\mu - r)\frac{\partial J}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 J}{\partial w^2}\Big] \\ + rw\frac{\partial J}{\partial w} + \lambda_t\Big[\alpha v(w) - J\Big] = \rho J, \end{aligned} \tag{3}$$

subject to a boundary condition

$$\lim_{s\to\infty} E\Big[\exp\Big(-\int_t^s(\rho + \lambda_u)du\Big)\; J(W_s, s) \;\Big|\; \mathcal{F}_t\Big] = 0.$$

As a result, the individual's optimal policy can be derived one obtains solutions to the two HJB equations (3) and (2) in turn.

In general, the HJB equations are hard to solve analytically or numerically under the circumstance where the intensities of λ_t and ϕ_t are time-varying and there is stochastically changing recovery I_t. Therefore, to make our analysis easier, we construct a simple stationary model which can reflect most of all important properties of the basic model.

3 A Stationary Model

In this section we introduce a stationary model by firstly assuming λ_t and ϕ_t to be non-negative constants. Let

$$\lambda_t = \lambda, \;\; \text{and} \;\; \phi_t = \phi,$$

then both the individual's lifetime τ and the default time τ_D of the insurance company are exponentially distributed with intensities λ and ϕ, respectively. Therefore, the expected lifetimes of the individual and the insurance company are equal to $\frac{1}{\lambda}$ and $\frac{1}{\phi}$, respectively.

We also assume that the individual has a constant relative risk aversion (CRRA) type utility with relative risk aversion (RRA) coefficient γ. In other words, we let

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

We also assume that $v(\cdot) = u(\cdot)$.[8]

Under the set-up, the HJB equation (3) becomes

$$\sup_c \left[(1-\alpha)\frac{c^{1-\gamma}}{1-\gamma} - c\frac{\partial J}{\partial w}\right] + \sup_\pi \left[\pi(\mu - r)\frac{\partial J}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 J}{\partial w^2}\right] + rw\frac{\partial J}{\partial w} + \lambda\left[\alpha\frac{w^{1-\gamma}}{1-\gamma} - J\right] = \rho J. \quad (4)$$

Fortunately, this equation can be solved analytically by a method similar to that in Merton (1971) and the following theorem provides the solution.

Theorem 3.1 *Under the stationary model, the equation (4) has the following solution:*

$$J(W_t, t) = K\frac{W_t^{1-\gamma}}{1-\gamma}, \quad (5)$$

where K satisfies

$$\frac{\gamma}{1-\gamma}(1-\alpha)^{\frac{1}{\gamma}}K^{\frac{\gamma-1}{\gamma}} + \left[r + \frac{(\mu - r)^2}{2\gamma\sigma^2} - \frac{\lambda+\rho}{1-\gamma}\right]K + \frac{\alpha\lambda}{1-\gamma} = 0.$$

We next turn to equation (2). Due to the stochastically changing lump-sum recovery, this kind of equation cannot be solved. Only with constant I_t we can get a numerical solution, but one should be careful when importing such constant recovery model because the loss of the policyholder can occur from various sources.

In practice, policyholders of a fixed life annuity are protected in part or in full against a substantial loss throughout rehabilitation or liquidation of the insurance company. In the United States, when an insurance company in a state goes under, the insurance commissioner of the state typically places the company in rehabilitation by an order of the state court. If the company's financial status are so severe that it cannot be rehabbed, then the insurance commissioner liquidates it.

[8] Alternatively we may assume that $v(w) = A\frac{w^{1-\gamma}}{1-\gamma}$ for some positive A. For instance, if the individual cares about the utility of her posterity in all generations and all her posterity share the same relative risk aversion, then v can be arranged to be equal to the value function of an infinitely lived individual, which typically takes the above form. However, having α as a parameter to control the individual's weight for the bequest motive, additional parameter A is redundant.

Amid rehabilitation, it is possible to occur a substantial loss for the policyholders in some ways. Firstly, the policyholders may have only limited access to their annuity accounts, namely, they may lose consumption and investment opportunities due to such limitation. For example, in 2002 the North Carolina commissioner allowed policyholders of the annuities, which is sold by London Pacific Life & Annuity, to take out only 10 - 20% of the value of their accounts, and in the early 1990's policyholders of the annuities sold by Mutual Benefit Life Insurance were restricted to access their accounts for eight years. Secondly, during rehabilitation the insurance commissioner may lower an annuity's interest payments below the rate quoted by the insurance company. This really happened in the early 1990's: the commissioner cut the annuity payments, which should be paid by Executive Life Insurance, at least initially by about 30%.

On the other hand, in the United States if an insurance company is in liquidation the policyholders of the company's annuities immediately enter into the coverage of the guaranty association in each state.[9] However, the coverage is generally limited to $100,000 in cash value for defaulted annuities, the policyholders are vulnerable to substantial losses. Moreover, some guaranty associations have the right to lower the interest rate, which be paid by the liquidated insurance company, if they consider the rate to be unreasonably high. Thus in general policyholders of an annuity have a potential fear against the liquidation of the insurer.

Consequently, the recovery I_t should be interpreted as the liquidated price of the annuity at annuity's default time after being considered all possible losses mentioned above.

The stationary model in this paper is the case where I_t is constantly zero. However, as stated previously, the recovery can hardly become zero in practice. We resolve this inconsistency by considering the default intensity ϕ of the insurance company as the intensity reflecting all possible losses of the policyholder. Therefore, ϕ under this model should be less than or equal to the default intensity of the insurance company, since non-zero recovery

[9]These guaranty associations voluntarily established the National Organization of Life and Health Insurance Guaranty Associations (NOLHGA), and one can find the full story of the coverage mechanism in the NOLHGA website: http://www.nolhga.com.

mitigates policyholder's potential loss-phobia occurring from the default of insurance company. During 1976-2002, the average frequency of *financially impaired companies* (FICs) in the U.S. annuity industry is 1.02%.[10] Thus, under this model, it seems to reasonable to take a restriction for ϕ such as

$$0 < \phi \leq 0.0102.$$

Under this set-up, given an analytic expression (5) we can rewrite (2) as

$$\begin{aligned}\sup_c \Big[(1-\alpha)\frac{c^{1-\gamma}}{1-\gamma} \quad & -c\frac{\partial V}{\partial w}\Big] + \sup_\pi \Big[\pi(\mu - r)\frac{\partial V}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 V}{\partial w^2}\Big] \\ & +(rw+\varepsilon)\frac{\partial V}{\partial w} + \lambda\Big[\alpha\frac{w^{1-\gamma}}{1-\gamma} - V\Big] + \phi\Big[K\frac{w^{1-\gamma}}{1-\gamma} - V\Big] = \rho V.\end{aligned} \tag{6}$$

The first-order conditions of (6) yield the optimal consumption strategy

$$c^* = \Big(\frac{1}{1-\alpha}\cdot\frac{\partial V}{\partial w}\Big)^{-1/\gamma}$$

and the optimal investment strategy

$$\pi^* = -\frac{(\mu - r)\cdot(\partial V/\partial w)}{\sigma^2\cdot(\partial^2 V/\partial w^2)}.$$

The existence of the first-derivative term, $(rw+\varepsilon)V_w$, prohibits us from solving the equation analytically. Therefore, throughout the paper we use the Markov Chain Approximation Method (MCAM) of Kushner (1990) in order to approximate the value function $V(w)$ and the controls c and π. We explain the details of the MCAM in Appendix.[11]

4 Definition of Terminologies

For further analysis, we define the individual's subjective measures of the price of the annuity and the default risk of the insurance company.

[10]The estimation is in Exhibit 34 of the special report of A.M. Best Company entitled *Best's Insolvency Study: Life/Health U.S. Insurers 1976-2002*, which is published in December 2004. According to the report, a FIC is defined as of the first official action containing rehabilitation and liquidation taken by the insurance department in its state of domicile, whereby the insurer can no longer conduct normal insurance operations. Thus we consider the FIC frequency as a proxy of the maximum value of the intensity for losses.

[11]Unfortunately, the MCAM can be used only for the case where the coefficient of risk aversion γ is in the interval (0,1). We could not find any numerical scheme to solve the problem for other γ's.

Definition 4.1 *Denote the value function with respect to initial wealth w, fixed rate ε of income from the annuity, and default intensity ϕ of insurance company as $V(w,\varepsilon,\phi)$. Then the* implicit value *of the annuity (with respect to w, ε and ϕ) is defined by*

$$\frac{\partial V(w,\varepsilon,\phi)}{\partial \varepsilon} \Big/ \frac{\partial V(w,\varepsilon,\phi)}{\partial w}.$$

And the default risk premium *(with respect to w, ε and ϕ) is defined by p satisfying*

$$V(w,\varepsilon+p,\phi) = V(w,\varepsilon,0).$$

The implicit value of the annuity is the marginal rate of substitution between annuity and financial wealth, and can be regarded as the individual's subjective reservation price of the annuity when she is offered to buy or sell a small amount of the annuity. Koo (1998) has considered a similar implicit value of human capital.

The default risk premium is the additional expected rate of return from the annuity in order for her to be induced to accept the insurance company's default risk.

5 Numerical Implications

5.1 Benchmark Parameters

Now we choose carefully benchmark parameters for further numerical works.

Firstly, we take the loss intensity parameter $\phi = 0.3\%$, that is, the expected time when the individual loses all of her investment amount in annuities is 333 years. [12]

We set the other market parameters to be, for the both models, $r = 2\%$, $\mu = 7\%$, $\sigma = 20\%$, and individual's subjective parameters to be $\gamma = 0.8$, $\rho = 2\%$, $\lambda = 5\%$. We think of the

[12] The parameters are chosen conservatively, since the policyholder of an annuity usually are protected by some guaranty program. In fact, according to Exhibit 39 of the Moody's Investor Service report (2001), for the period 1981-2000 the average one-year default rate of the all corporate bonds is about 1.44% (note that it is bigger than the FIC frequency) and, according to Exhibit 20 of the same report, for the period 1981-2000 the estimated default recovery rate of the senior secured corporate bond is only about 50%. Therefore the loss intensity of the annuity in the paper is chosen very conservatively compared with that of the senior secured corporate bond.

risk-free rate r as the one which is inflation-adjusted, so we set it to be only 2%. Note that we set the individual's expected lifetime to be 20 years.

We fix the annuity rate ε paid by the insurance company to be \$1, and we will consider several types of individual with initial wealth different each other.

Now, for further analysis, we clarify the concept of the 'actuarially fair' price of an annuity. The actuarially fair price is determined by both individual's mortality rate and expected losses from the default of insurer: using our notation, the actuarially fair price of an annuity which gives a fixed rate of return ε should be defined as

$$
\begin{aligned}
&E\Big[\int_0^{\tau_D\wedge\tau}\varepsilon e^{-rs}ds\Big]\\
&=\lambda\phi\int_0^\infty\int_0^\infty\Big(\int_0^{x\wedge y}\varepsilon e^{-rs}ds\Big)e^{-\lambda x}e^{-\phi y}dxdy,\\
&=\lambda\phi\int_0^\infty\int_0^y\Big(\int_0^x\varepsilon e^{-rs}ds\Big)e^{-\lambda x}e^{-\phi y}dxdy+\lambda\phi\int_0^\infty\int_y^\infty\Big(\int_0^y\varepsilon e^{-rs}ds\Big)e^{-\lambda x}e^{-\phi y}dxdy\\
&=\frac{\varepsilon}{r+\lambda+\phi}.
\end{aligned}
$$

Note that individual's annuity investment portion against her total wealth at initial time can be calculated as

$$
\frac{\text{(actuarially fair price of annuities)}}{\text{(initial wealth }\ w)+\text{(actuarially fair price of annuities)}}=\frac{\varepsilon}{w(r+\lambda+\phi)+\varepsilon}. \tag{7}
$$

Therefore, we can induce individual's initial investment amount in annuities from her initial wealth level.

5.2 Optimal consumption and investment

Figure 1 exhibits graphs showing the optimal consumption and investment of individuals with positive bequest motives.

Figure 1(a) shows the optimal consumption-to-wealth ratios (OCWRs) c^*/w for individuals with three different initial wealth levels.[13] For the individuals with low initial wealth

[13] Note that, from (7), we can calculate individual's annuity investment portion at initial time for each wealth level. In fact, together with the benchmark parameters it is 30% where $w = 32$, 60% where $w = 9.1$, and 90% where $w = 1.5$.

levels, the OCWRs decrease as loss intensity ϕ increases. That is, an individual with partially annuitized wealth reduces her consumption when facing the higher default risk of the annuity issuer: She recognizes that her expected income from the annuity becomes smaller as the default risk increases. As a matter of fact, for the case where $w = 1.5$ the OCWR of the model changes from 75% to 55% as ϕ changes from 0 to 0.02. On the contrary, for the case where $w = 9.1$ or $w = 32$ the OCWRs do not seem to change much under the same circumstances. This fact can be interpreted as follows: an investor having a relatively bigger annuity position against her initial wealth changes her OCWR by a larger proportion whenever the credit event of the insurance company occurs.

Figure 1(b) shows the optimal investment-to-wealth ratios (OIWRs) π^*/w for individuals with different initial wealth levels. The OIWRs decrease for all the cases as loss intensity ϕ increases. This implies that an investor with partially annuitized wealth optimally reduces her investment in the risky asset whenever the default risk of the insurance company increases. This is because an increase in the default risk of insurance company increases the risk in her annuity income and reduces her tolerance for risk in financial assets.[14] However, the absolute values of the slopes of each graph in Figure 1(b) become bigger as individual's initial wealth decreases.

[Insert Figure 1 here.]

In Figure 2, we show the OCWRs and OIWRs for individuals having no bequest motive. The graphs in Figure 2 show properties similar to those shown in Figure 1.

[Insert Figure 2 here.]

5.3 Actuarially fair price, implicit value, and the annuity puzzle

In his seminal work Yaari (1996) showed that full annuitization is optimal for retirees if the market is complete and actuarially fair. In spite of his result, there is a consensus among

[14] See Kimball (1993) for this decrease in risk tolerance with respect to an increase in the background risk.

economists that private annuity markets across the world are quite thin, and few retirees buy life annuities. This apparent contradiction between theory and evidence is called the *annuity puzzle*.

Economists have shown interest in the annuity puzzle and tried to give explanations from a variety of perspectives: the adverse selection problem facing insurance companies (Mitchell et al. 1999), the longer longevity of married couples (Brown and Poterba 2000), the bequest motives of annuity buyers (Brown 2001, Johnson et al. 2004), and so on. We refer to Section 2 of Gentry and Rothschild (2006) and Babbel and Merill (2007) as sources providing a brief history about this line of research. In spite of such trials, most economists still believe that the annuity puzzle cannot be resolved by any one factor. In this paper, we propose another explanation based on the default possibility of insurance companies.

The actuarially fair price should be equal to the *market price* of the annuity if the market for the annuity is competitive and there are no frictions (e.g. taxes and transaction costs) and investors are risk neutral. The real financial market does not satisfy these assumptions and there is an attempt to measure the difference between the market price of an annuity and the actuarially fair price. For example, the money's worth framework by Mitchell et al. (1999) measures how much actuarially fair value a premium of \$1 has. They have estimated that a premium of \$1 generates an annuity whose actuarially fair value equal to \$0.85 by using the mortality table for annuity buyers. Therefore, the market price is higher than the actuarially fair price according to them.

Figure 3 shows the implicit values of an annuity (defined in Section 4), its actuarially fair price and its market price[15] as a function of the loss intensity ϕ.

In Figure 3(a) and 3(b), the implicit value of an annuity for the individual having initial wealth $w = 1$ or 2 or 3 is smaller than its market price for almost all ϕ's. And we also observe

[15]By using the result of Mitchell et al. (1999), we set the market price of the annuity to be

$$\text{(actuarially fair price of annuity)}/0.85.$$

that the difference, the implicit value minus the market price, decreases as ϕ increases, and this fact implies that, as the default probability of insurance company increases, the individual with partially annuitized wealth has bigger demand for selling or refunding some parts of her annuities. Note that, from (7), the individual having initial wealth $w = 3$ is considered as the one who is investing about $79\% - 83\%$ of her wealth in annuities. Therefore Figure 3(a) also tells us that such demand is bigger for an individual investing a bigger portion of her wealth in annuities. In Figure 3(b) we observe essentially the same results as above for an individual with no bequest motive. We can observe that these results are also true even though we consider the actuarially fair price as a benchmark price instead of the market price.

[Insert Figure 3 here.]

Figure 4 shows the implicit value, the actuarially fair price, and the market price of an annuity as a function of initial wealth w.

We observe that an individual having a weight of $\alpha = 0.5$ for bequest motive and initial wealth less than 11 has a demand for selling or refunding part of her annuities (Figure 4(a)), and so does an individual having no bequest motive and initial wealth less than 7.5 (Figure 4(b)). Since the individual with $w = 11$ ($w = 7.5$) is the one who 55% (65%, resp.) of her wealth invests in annuities, the results can be interpreted that an individual who has already invested a sufficiently big portion of her wealth in annuities and has a strong demand for selling or refunding annuities when facing a higher default risk of her annuity providers, even though she does not have any bequest motives. Therefore, it might be possible to say that individuals' recognition of the default possibility of annuity providers can hamper the growth of annuity markets.

[Insert Figure 4 here.]

Figure 5 displays the increasing and convex property of the implicit value of an annuity when risk aversion γ changes. As you can see in Figure 5(a), the implicit value of the annuity for the individual investing 60% of her wealth in annuity (i.e., $w = 9.1$) is less than its market price, and that for the individual investing 73% of her wealth in annuity (i.e., $w = 5.1$) is even less than its actuarially fair price, for all γ. The slope of the graph become smaller as w decreases, that is, The implicit values of the annuity across individuals who investing the same big portion of their wealth in annuity and having different risk aversions are not very different.[16]

[Insert Figure 5 here.]

5.4 The Default risk premium

In Figure 6 $\sim$ 10, we exhibit the relationships between the default risk premium p and parameters ϕ, μ, σ, γ, and α.

[Insert Figure 6 - 10 here.]

We observe that default risk premium p is larger than 5% for almost all cases, even though we employ a little conservative parameters for measuring the default risk of annuity. Furthermore, for some cases such as the case of Figure 7, default risk premium are over than 10% for economically acceptable volatility $\mu = 4\%$. This implies that, in general, the individual requires to receive more than 5% extra return from her annuity position to be induced to bear the default risk of insurance company. This implies that the presence of the default risk of insurance companies can have a significant impact on the optimal behaviors of annuity investors. Also this fact supports our opinion toward the annuity puzzle in the following way: if default risk premium is significantly big, annuity policyholders are apt to sell and refund their annuity.

We also observe the following facts from the figures:

[16]The result may provide us with an prospect toward the annuity puzzle for the case where $\gamma \geq 1$: an individual who investing sufficiently big portion of her wealth in annuities has a strong demand for selling or refunding her annuities when she faces a substantial loss from her annuity position.

1. Naturally, default risk premium p increases as loss intensity ϕ increases. (Figure 6)

2. Default risk premium p decreases as the expected rate μ of return on the risky asset increases. This is also because an increase in the expected rate of return on the risky asset decreases the individual's cash demand. (Figure 7)

3. Default risk premium p increases as the volatility, σ, of the risky asset increases. This is because an increase in the volatility of the risky asset makes the investment opportunity facing the individual worse. (Figure 8)

4. Default risk premium p increases as the coefficient γ representing the individual's RRA increases. This is because an individual with bigger risk aversion tends to exhibit more aversion to the default risk of the insurance company. (Figure 9)

5. Default risk premium p decreases as the weight α for bequest motive increases. This is because an individual who derives more happiness by consuming her assets than by bequeathing fortunes to her posterity has more cash demand. (Figure 10)

6 Conclusion

In this paper we have studied a retiree's optimal consumption and investment behavior when the retiree has partially annuitized wealth. We have assumed that the insurance company, the annuity provider, is subject to default risk. We have casted the problem as a continuous-time consumption and portfolio selection problem and derived the Hamilton-Jacobi-Bellman equation for the problem.

By solving the equation numerically, we have shown that the existence of the default risk of the insurance company can have a significant effect on the optimal behavior of a retiree. We have also found that an individual who has already invested a sufficiently large portion of her wealth in annuities has a strong demand for selling or refunding her annuities when facing a heightened default risk of insurance company. This fact suggests a new potential explanation for the annuity puzzle: individuals' recognition of default risks of insurance companies may hamper the growth of annuity markets.

Appendix

The details of the Markov chain approximation method.

By the dynamic programming principle, we can rewrite (1) as the following:

$$
\begin{aligned}
&e^{-\rho t}V(W_t,t) = \sup_{\{c,\pi\}} E\left[\int_t^{t+h}(1-\alpha)e^{-\rho s}u(c_s)ds + e^{-\rho(t+h)}V(W_{t+h},t+h)|\mathcal{F}_t\right] \\
&= \sup_{\{c,\pi\}}\Big\{e^{-\int_t^{t+h}\phi_s ds}e^{-\int_t^{t+h}\lambda_s ds}E\left[\int_t^{t+h}(1-\alpha)e^{-\rho s}u(c_s)ds + e^{-\rho(t+h)}V(W_{t+h},t+h)|\mathcal{F}_t\right] \\
&\quad + e^{-\int_t^{t+h}\phi_s ds}(1-e^{-\int_t^{t+h}\lambda_s ds})E\left[\int_t^{t+h}(1-\alpha)e^{-\rho s}u(c_s)ds + \alpha e^{-\rho(t+h)}u(W_{t+h})|\mathcal{F}_t\right] \\
&\quad + (1-e^{-\int_t^{t+h}\phi_s ds})e^{-\int_t^{t+h}\lambda_s ds}E\left[\int_t^{t+h}(1-\alpha)e^{-\rho s}u(c_s)ds + e^{-\rho(t+h)}J(W_{t+h},t+h)|\mathcal{F}_t\right] \\
&\quad + (1-e^{-\int_t^{t+h}\phi_s ds})(1-e^{-\int_t^{t+h}\lambda_s ds})E\left[\int_t^{t+h}(1-\alpha)e^{-\rho s}u(c_s)ds + \alpha e^{-\rho(t+h)}u(W_{t+h})|\mathcal{F}_t\right]\Big\},
\end{aligned}
$$

and $W_t = w$.

If $\lambda_t = \lambda$ and $\phi_t = \phi$ (i.e., in the stationary model), this relationship is approximately converted into

$$
\begin{cases}
e^{-\rho t}V(w_i) \approx \sup_{\{c_i,\pi_i\}}\Big\{(1-\alpha)e^{-\rho t}u(c_i)\Delta t + \alpha\lambda\Delta t e^{-\rho t}u(w_i) + \phi\Delta t e^{-\lambda\Delta t}e^{-\rho(t+\Delta t)}J(w_i) \\
\qquad + e^{-\lambda\Delta t}(1-\phi\Delta t)e^{-\rho(t+\Delta t)}E[V(w)|\mathcal{F}_t]\Big\}, \\
W_t = w_i,
\end{cases}
\tag{8}
$$

where $\{w_i\}_{i=0}^n$ is the set of discretized wealths and Δt is a small time interval. Then we know that for a given $W_t = w_i$

$$E[V(w)|\mathcal{F}_t] = \sum_{j=0}^n p_{ij}^{c,\pi}V(w_j),$$

where $p_{ij}^{c,\pi}$ denotes the control-dependent transition probability from w_i to w_j. We take

$$\{w_i\}_{i=0}^n = \{ih \mid 0 \leqslant i \leqslant n\}$$

and the transition probabilities

$$\begin{cases} p_{i,i+1}^{c,\pi} &= \left\{\frac{1}{2}\sigma^2\pi^2 + h[(\mu-r)\pi + r(ih) + \varepsilon]\right\}/Q \ , \\ p_{i,i-1}^{c,\pi} &= \left\{\frac{1}{2}\sigma^2\pi^2 + hc\right\}/Q \ , \\ p_{i,i}^{c,\pi} &= 1 - p_{i,i+1}^{c,\pi} - p_{i,i-1}^{c,\pi} = 1 - \left\{\sigma^2\pi^2 + h[c + (\mu-r)\pi + rih + \varepsilon]\right\}/Q \end{cases}$$

for $1 \leqslant i \leqslant n-1$, where

$$Q = \max_{0\leqslant c,\pi \leqslant Knh,\ 0\leqslant i\leqslant n} p_{i,i+1}^{c,\pi} + p_{i,i-1}^{c,\pi} = \sigma^2(Knh)^2 + nh^2[(\mu-r)K + r + K] + h\varepsilon$$

and K is an appropriate bound on the controls. Also we let the other transition probabilities be

$$\begin{cases} p_{0,0}^{c,\pi} = 1, \\ p_{n,n-1}^{c,\pi} = \left\{\frac{1}{2}\sigma^2\pi^2 + hc\right\}/Q, \\ p_{n,n}^{c,\pi} = 1 - p_{n,n-1}^{c,\pi}, \end{cases}$$

As a result, we can obtain the following Markov chain approximation formula from (8):

$$\begin{cases} V_i \approx \sup_{\{c_i,\pi_i\}}\{(1-\alpha)u(c_i)\Delta t + \alpha\lambda\Delta t u(w_i) & +\phi\Delta t e^{-\lambda\Delta t}e^{-\rho\Delta t}J_i \\ & +e^{-\lambda\Delta t}(1-\phi\Delta t)e^{-\rho\Delta t}\sum_{j=0}^{n} p_{ij}^{c,\pi}V_j\}, \\ V_i = V(w_i), & \\ J_i = J(w_i). & \end{cases}$$

Thus we can calculate approximate value function V_i and approximate optimal controls c_i and π_i by using the arguments in Fitzpatrick and Fleming (1991).

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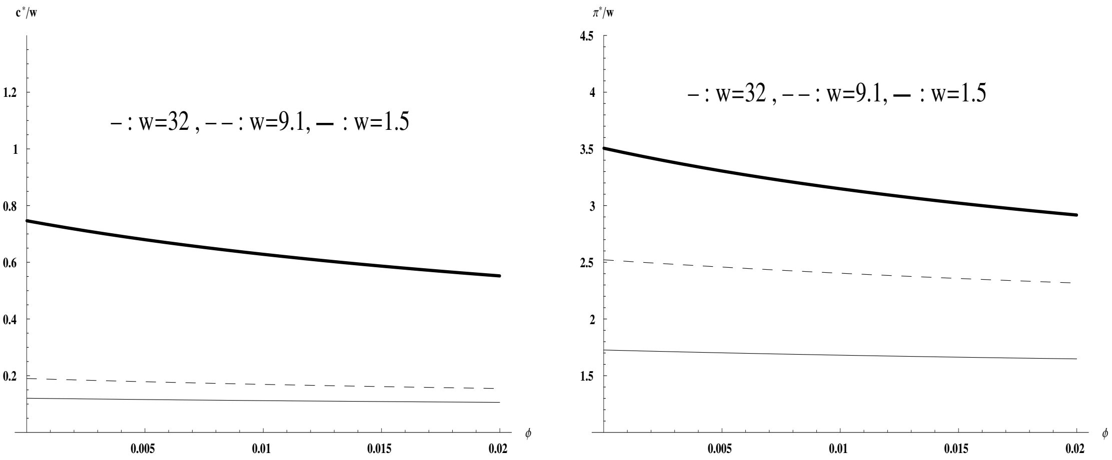
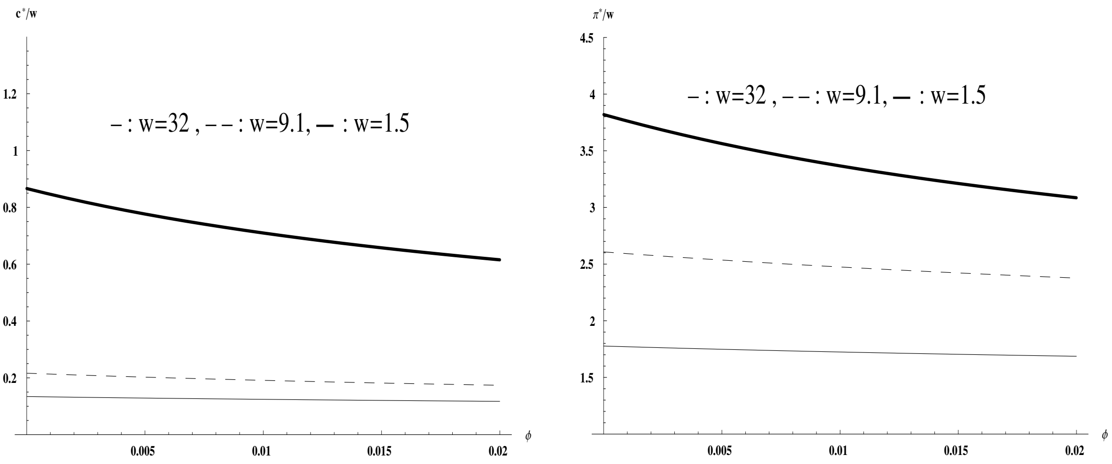


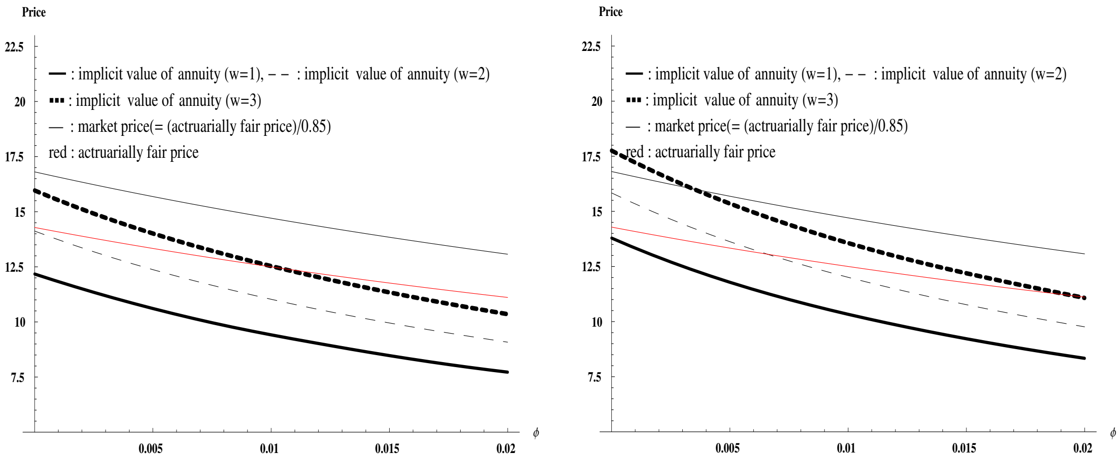
(a) OCWR (b) OIWR

Figure 1: The relationship between default intensity and optimal consumption and investment in the presence of individual's bequest motives. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \alpha = 0.5.$



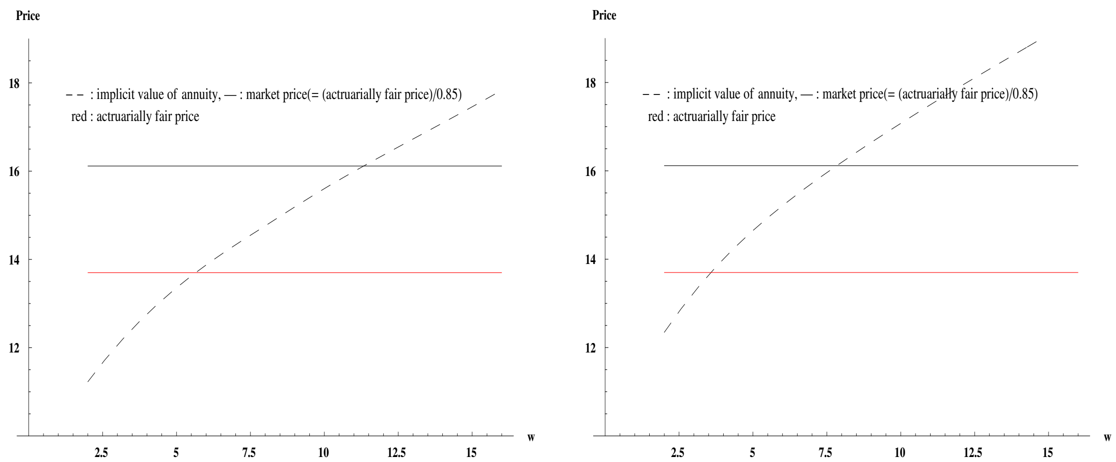
(a) OCWR (b) OIWR

Figure 2: The relationship between default intensity and optimal consumption and investment in the absence of individual's bequest motives. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \alpha = 0.$



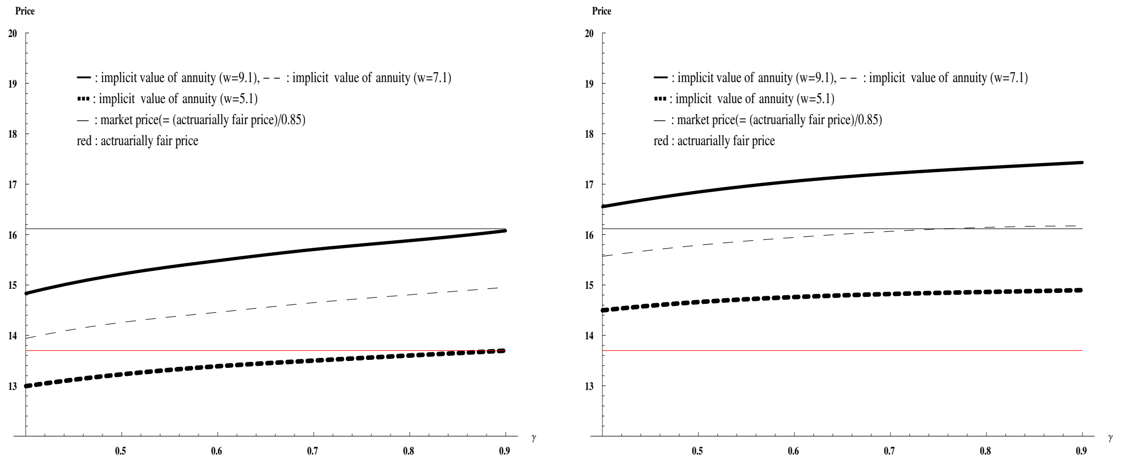
(a) $\alpha = 0.5$ (b) $\alpha = 0$

Figure 3: The implicit values and market price of an annuity as a function of ϕ. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1$.



(a) $\alpha = 0.5$ (b) $\alpha = 0$

Figure 4: The implicit value and market price of an annuity as a function of w. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \phi = 0.003$.



(a) $\alpha = 0.5$ (b) $\alpha = 0$

Figure 5: The implicit value and market price of an annuity as a function of γ. The parameters are $\lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \phi = 0.003,\ w = 9.1$.

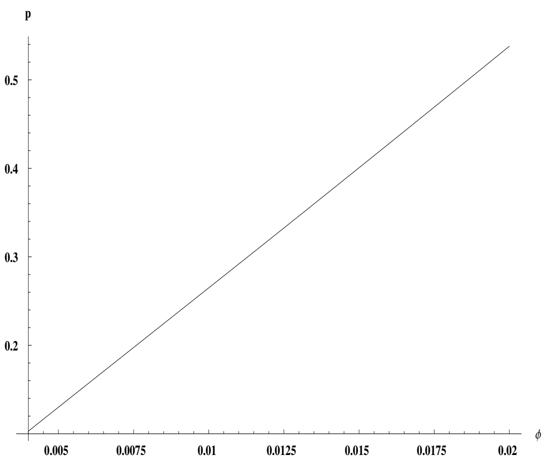


Figure 6: The relationship between default intensity ϕ and default risk premium p. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \alpha = 0.5,\ w = 9.1$.

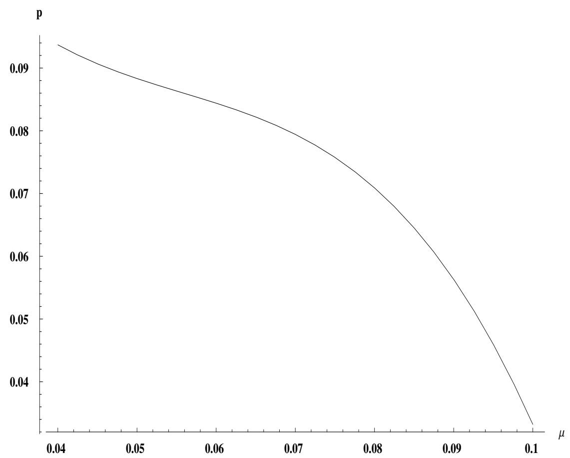


Figure 7: The relationship between expected rate of return μ of the risky asset and default risk premium p. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \sigma = 0.2,\ \varepsilon = 1,\ \alpha = 0.5,\ w = 9.1,\ \phi = 0.003$.

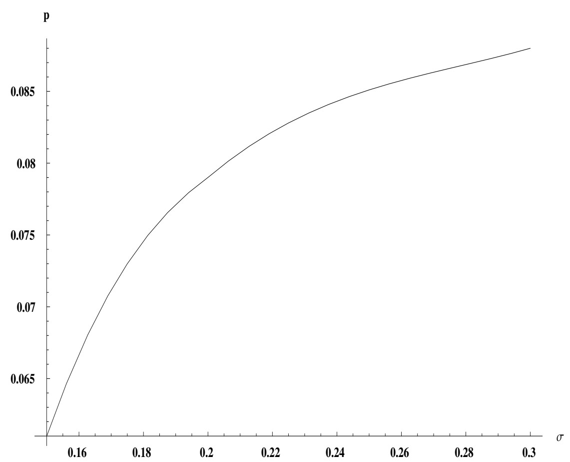


Figure 8: The relationship between volatility σ of the risky asset and default risk premium p. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \varepsilon = 1,\ \alpha = 0.5,\ w = 9.1,\ \phi = 0.003$.

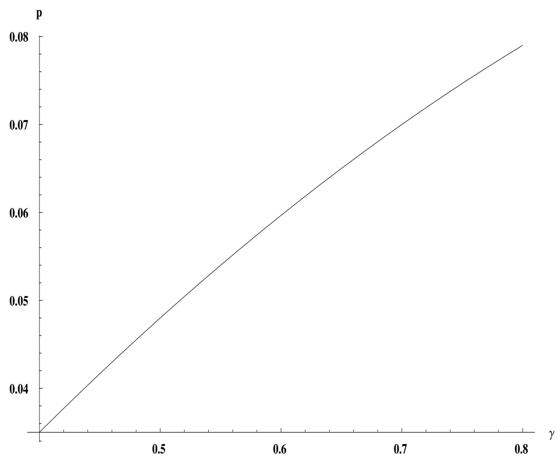


Figure 9: The relationship between the coefficient γ of an individual's relative risk aversion and default risk premium p. The parameters are $\lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ \alpha = 0.5,\ w = 9.1,\ \phi = 0.003$.

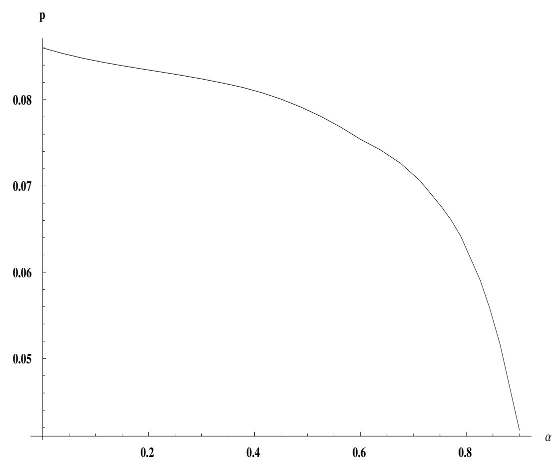


Figure 10: The relationship between the weight α for an individual's bequest motives and default risk premium p. The parameters are $\gamma = 0.8,\ \lambda = 0.05,\ r = 0.02,\ \rho = 0.02,\ \mu = 0.07,\ \sigma = 0.2,\ \varepsilon = 1,\ w = 9.1,\ \phi = 0.003$.