Investment under Ambiguity and Regime-Switching Environment

U Jin Choi[*] Kwangmoon Kim[†] Minsuk Kwak[‡]

May 5, 2009

Abstract

We consider all or nothing investment problem with a finite time horizon when the investment opportunity set is changing stochastically over time, especially under Markovian regime-switching environment, and a decision maker faces ambiguity of parameters governing profit flow dynamics of the investment. We apply α-Maxmin Expected Utility(α-MEU) preferences to reflect the ambiguity seeking attitude of decision maker and provide semi-explicit formulas for the expected value of investment and the critical present value of the profit flow. Numerical results show that the critical present value of the profit flow depends on the business cycle and the tendency of ambiguity seeking is mitigated in case of project whose profit flow is dependent on regime-switching environment.

[*]Department of Mathematical Science, KAIST, 373-1, Guseong-dong Yuseong-gu, Daejeon, Korea. Tel: +82-42-350-2726, E-mail: ujinchoi@kaist.ac.kr

[†]Department of Mathematical Science, KAIST, 373-1, Guseong-dong Yuseong-gu, Daejeon, Korea. Tel: +82-42-350-2766, Fax: +82-42-350-2766, E-mail: KwangmoonKim@gmail.com

[‡]Department of Mathematical Science, KAIST, 373-1, Guseong-dong Yuseong-gu, Daejeon, Korea. Tel: +82-42-350-2766, Fax: +82-42-350-2766, E-mail: MinsukKwak@gmail.com

JEL classification : D81, G11

Keywords: Investment, Markovian regime-switching, the business cycle, Knightian uncertainty, Ambiguity, α-Maxmin Expected Utility

1 Introduction

We consider a risk-neutral entrepreneur of the firm who wants to start innovative business or project with initial sunk cost. In order to decide whether to invest or not at the present time, she evaluates the investment. In this paper, we assume that she faces ambiguity and considers the business cycle.

We assume that a decision maker faces ambiguity about future profit flow. Recently, Knightian uncertainty or ambiguity is used to explain economical and behavioral issues. It is to consider problems with environment on a set of probability measures instead of single probability measure. Maxmin framework established by Gilboa and Schmeidler (1989), which is developed to intertemporal framework by Chen and Epstein (2002), ignores positive effect of ambiguity. In this framework, decision makers only look at the worst scenario, i.e., the attitude toward ambiguity is extremely negative. However, all decision makers are not completely pessimistic. Hence, it is natural that there are existing literature reflecting various attitudes of decision makers such as smooth ambiguity, α-Maxmin Expected Utility(α-MEU) preferences.

The α-MEU preferences introduced by Ghiradato et al. (2004); Olszewski (2007) involve ambiguity loving(seeking) attitude of decision maker. It is a convex combination of two extreme cases, i.e., the worst case and the best case. Hence, the framework using α-MEU preferences makes it available to examine the effect of attitude toward ambiguity. In the context of investment problem, Heath and Tversky (1991) points out entrepreneurs tend to look at the best scenario when they face ambiguity since they have overconfidence. Schröder (2007) studies an all or nothing problem and the irreversible investment problem extending Nishimura and Ozaki (2007) by using α-MEU preferences. He finds that if a decision maker has a small fraction of optimism, he is eager to invest significantly because of the presence of ambiguity. It is well harmonized with behavioral economics and psychology literature. But Nishimura and Ozaki (2007); Schröder (2007) don't consider the business cycle and only deal with infinite time horizon even though there are many projects which have a given fixed maturity time or life-time in real-life economy. In Driffill *et al.* (2003), the authors deal with entry and exit problem under regime-switching environment. But they don't consider ambiguity.

Taking regime-switching environment, first introduced by Hamilton (1989, 1990), into account financial and behavioral economics is meaningful and important. For example, Jang et al. (2007) highlights that the effect of transaction costs under regime-switching environment is much bigger than that of costs without regime switching. And Fuh *et al.* (2003) explains empirical investigation like volatility smile, volatility clustering etc. using hidden Markov regime-switching model.

If the profit flow of the investment depends on the business cycle, a decision maker has to concern it. When contraction begins, the collapse or decreasing of price of product and bankruptcy of related firms can occur. So the profit flow of project in contraction period is different from that in expansion period. When decision maker who evaluates project is not myopic, it will be more penetrating to consider the business cycle.

So we consider an all or nothing investment problem in finite time horizon under ambiguity, when the investment opportunity set is changing stochastically over time, especially under Markovian regime-switching. We assume that the entrepreneur knows whether the business cycle is an expansion or a contraction at the present time. In real-life, this is natural because there are many reports of economic experts such as economists, analysts about current economic situation and the business cycle, e.g. coincident composite index and preceding index.

We derive semi-explicit formulas of the expected value of investment and the critical present value of the profit flow. In fact, the framework we use in this paper is a generalization of (Nishimura and Ozaki, 2007; Schröder, 2007). For computation, we use moment generating function of occupation time which is suggested by Edwards (2005). And we present a decision maker, who faces ambiguity and considers regime-switching environment, does not exactly invest like Schröder (2007). The result of this paper shows that, when the investment period is short, it is important to know whether it is a recession or an expansion. And the tendency of ambiguity seeking is mitigated when the decision maker considers the business cycle.

This paper is organized as follows. In Section 2, we describe an all or nothing investment problem under ambiguous environment and the business cycle. Then we define a decision maker's problem in Section 3. Section 4 presents numerical examples and comparative statics. Section 6 concludes.

2 Profit flow under ambiguous and regime-switching environment

2.1 Regime-switching environment

We assume that there are two regimes: regime E, C which means "expansion", "contraction", respectively. When a market-independent Poisson processes $N(t)$ with intensity λ_i, $i \in \{E, C\}$ jumps, the state change from regime i to regime $j \neq i$ occurs.

We consider a probability space $(\Omega, \mathcal{F}, P)$, where the filtration $\{\mathcal{F}_t\}$ is generated by a standard Brownian motion $B(t)$ and independent Poisson process $N(t)$, i.e. $\mathcal{F}_t = \sigma\{B(s), N(s) \mid 0 \leq s \leq t\}$. P represents a single original probability measure from which a family of absolutely continuous so-called real-world probability measures are generated. Each stochastic process in this paper is adapted.

The profit flow under regime-switching environment and probability measure P is defined by

$$\frac{d\pi(t)}{\pi(t)} = \mu(t)dt + \sigma(t)dB(t), \tag{1}$$

where

$$\mu(t) = \begin{cases} \mu_E, & \text{expansion,} \\ \mu_C, & \text{contraction,} \end{cases} \qquad \sigma(t) = \begin{cases} \sigma_E, & \text{expansion,} \\ \sigma_C, & \text{contraction,} \end{cases} \tag{2}$$

and the present profit flow $\pi(0) \in \mathbb{R}$. We assume that $\pi(0)$ is observable. A decision maker can compute the difference between revenue and expenditure if the project is operated at the present time.

2.2 Ambiguous Environment

We assume that the entrepreneur faces ambiguity on parameters governing the profit flow dynamics. (Nishimura and Ozaki, 2007; Schröder, 2007) solve investment problem in which the project is in ambiguous environment. Nishimura and Ozaki (2007) study irreversible investment problem with Maxmin expected utility theory is proposed by Gilboa and Schmeidler (1989) and extended, recently, to continuous time model by many attempts including (Hansen and Sargent, 2001; Hansen *et al.*, 2006; Chen and Epstein, 2002). Pathak (2002) has a good survey by dividing literature such as the recursive multiple priors framework and the robust control model and discuss what is linked well to Gilboa-Schmeidler[1]. Schröder (2007) generalize Nishimura and Ozaki (2007) using α-MEU preferences. The results in these papers serve as a benchmark for later analysis of effect of regime-switching environment.

The set of density generators are defined by

$$(\theta_t) \in \Theta \subset K = [-k,\, k], \tag{3}$$

where the degree of ambiguity k is nonrandom(κ-ignorance) and is a perceived degree of

[1]Hansen *et al.* (2006) present two robust control problems. One of them referred as constraint robust control problem is related naturally to Gilboa-Schmeidler.

ambiguity. Hence $\mathcal{P} = \{Q^\theta | \theta \in \Theta\}$ where Q^θ is made out of original probability measure P is rectangular. It satisfies dynamic consistency. (For details, refer Chen and Epstein (2002); Nishimura and Ozaki (2007)). Since the investor should decide at the present time, we naturally assume there is no learning.

In regime $i \in \{E, C\}$, the standard Brownian motion under Q^θ is, by Girsanov's theorem(see e.g. Øksendal (2003)), defined as

$$B(t)^\theta = B(t) + \int_0^t \theta(s) ds. \tag{4}$$

Hence, the dynamics of profit flow with respect to Q^θ is given by

$$\frac{d\pi(t)}{\pi(t)} = (\mu(t) - \sigma(t)\theta(t))dt + \sigma(t)dB(t)^\theta, \tag{5}$$

subject to (2), (3).

By the Itô formula, the profit flow of project with ambiguity under regime-switching environment is

$$\pi(t) = \pi(0) \exp\left[\int_0^t \mu(s) - \sigma(s)\theta(s) - \frac{1}{2}\sigma^2(s)\ ds + \int_0^t \sigma(s)\ dB(s)^\theta\right], \tag{6}$$

subject to (2), (3).

3 A decision maker's problem

In this paper, we use α-MEU preferences framework. The expected value of stochastic function $f(x)$ is a convex combination of the best scenario term with weight α and the worst scenario term with weight $1 - \alpha$. Here, $\alpha \in [0, 1]$ is a degree of optimism reflecting personal character of decision maker. Hence, the framework using α-MEU preferences make it available to examine the effect of attitude toward ambiguity. The following equation represent so-called the α-expectation.

$$E^\alpha[f(x)] = \alpha \sup_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta}[f(x) \mid \sigma(t) = \sigma_i] + (1 - \alpha) \inf_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta}[f(x) \mid \sigma(t) = \sigma_i], \quad i = E,\ C, \tag{7}$$

where $E_t^{Q^\theta}[\cdot|\sigma(t)=\sigma_i]$ is the expectation with respect to Q^θ conditional on $\mathcal{F}(t)$ given that current regime i is known. For simplicity, we assume that the firm's decision maker knows whether the market is contraction or expansion at time t.

If we assume that ρ is the firm's subjective discount rate and T is the exit time of the project, the α-expected value, at time 0, of the investment with the profit stream under ambiguity and regime-switching environment using α-MEU preferences is given by,

$$V_i(\pi(0)|\ \alpha) = E_0^\alpha\left[\int_0^T e^{-\rho s}\ \pi(s)\ ds\right] \tag{8}$$

$$= \alpha \sup_{Q^\theta\in\mathcal{P}} E_0^{Q^\theta}\left[\int_0^T e^{-\rho s}\ \pi(s)\ ds\middle|\ \sigma(0)=\sigma_i\right] + (1-\alpha)\inf_{Q^\theta\in\mathcal{P}} E_0^{Q^\theta}\left[\int_0^T e^{-\rho s}\ \pi(s) ds\middle|\ \sigma(0)=\sigma_i\right],\quad i=E,\ C, \tag{9}$$

Hence, the firm's entrepreneur evaluates

$$F_i(\pi(0)|\alpha) = \max\{\ V_i(\pi(0))\ |\ \alpha) - I,\ 0\},\quad i=E,\ C, \tag{10}$$

the value of option to invest, where I is the initial sunk cost of the project. If F_i is positive, the entrepreneur will invest the project. If F_i is less than 0, the project must be abandoned.

We adopt the following assumption in order to ensure problem make sense:

Assumption 1. *For a nonnegative real constant k,*

$$k < (\rho-\mu_i)/\sigma_i,\quad i=E,\ C. \tag{11}$$

Perceived degree of ambiguity k restricted by (11) is large enough to describe the market. If we calculate inequality (11) using parameter in Section 5, we have $k < 0.3117$ for an economic expansion and $k < 0.2310$ for an economic contraction. Note that in this paper, k does not change depending on regime-switching.

4 Investment decision under ambiguity and the business cycle

4.1 Evaluating the project

We borrow the method of dealing regime-switching from Jang and Roh (2007). For computation, we use moment generating function of occupation time which is suggested by Edwards (2005). For calculation of the expected value of the investment $V_i(\pi(t) \mid \alpha)$, we define a stochastic process I_t as

$$I(t) \triangleq \begin{cases} 1, & \text{expansion} \\ 0, & \text{contraction.} \end{cases} \tag{12}$$

Define the occupation time that the profit flow is in high regime from time 0 to time t as

$$\zeta(s) \triangleq \int_0^s I(t)dt. \tag{13}$$

Theorem 1. *When the output of the project is given by Equation (6) subject to (2), (3) the following equations is satisfied.*

$$\sup_{Q^\theta \in \mathcal{P}} E^{Q^\theta} \left[\int_0^T e^{-\rho s} \; \pi(s)ds \middle| \sigma(0) = \sigma_i \right]$$
$$= \pi(0) \int_0^T \left[\exp\left((-\rho + \mu_C + k\sigma_C) \, s \right) \int_0^s \exp\left((\mu_E - \mu_C + k(\sigma_E - \sigma_L)) \, u \right) f_i(s,u)du \right] ds, \tag{14}$$

$$\inf_{Q^\theta \in \mathcal{P}} E^{Q^\theta} \left[\int_0^T e^{-\rho s} \; \pi(s)ds \middle| \sigma(0) = \sigma_i \right]$$
$$= \pi(0) \int_0^T \left[\exp\left((-\rho + \mu_C - k\sigma_C) \, s \right) \int_0^s \exp\left((\mu_E - \mu_C - k(\sigma_E - \sigma_L)) \, u \right) f_i(s,u)du \right] ds, \tag{15}$$

where $f_i(s,u)$ *is a PDF of* $\zeta(s)$ *where* $\sigma(0)=\sigma_i$ *such that*

$$f_H(s,u) \triangleq e^{-\lambda_H s}\delta_0(s-u) + e^{-\lambda_L(s-u)-\lambda_H u}\Bigg(\left(\frac{\lambda_H\lambda_L u}{s-u}\right)^{1/2} I_1\Big(2(\lambda_H\lambda_L u(s-u))^{1/2}\Big) + \lambda_H I_0\Big(2(\lambda_H\lambda_L u(s-u))^{1/2}\Big)\Bigg), \quad (16)$$

$$f_L(s,u) \triangleq e^{-\lambda_L s}\delta_0(s-u) + e^{-\lambda_L(s-u)-\lambda_H u}\Bigg(\left(\frac{\lambda_H\lambda_L (s-u)}{u}\right)^{1/2} I_1\Big(2(\lambda_H\lambda_L u(s-u))^{1/2}\Big) + \lambda_L I_0\Big(2(\lambda_H\lambda_L u(s-u))^{1/2}\Big)\Bigg), \quad (17)$$

where $I_a(z)$ *is the modified Bessel function defined by*

$$I_a(z) \triangleq \left(\frac{z}{2}\right)^a \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(a+n+1)}.$$

Proof. *See Appendix A.*

We get the following results.

Corollary 1. *The* α*-expected value of the project in regime* $i \in \{E,\ C\}$*, for a decision maker who has a degree of optimism* $\alpha \in [0,1]$ *is given by*

$$V_i(\pi(0)|\alpha) = \pi(0)\Bigg\{\alpha\int_0^T\left[\exp\left((-\rho+\mu_C+k\sigma_C)\,s\right)\int_0^s \exp\left((\mu_E+k\sigma_E-\mu_C-k\sigma_L)\,u\right)f_i(s,u)du\right]ds + (1-\alpha)\int_0^T\left[\exp\left((-\rho+\mu_C-k\sigma_C)\,s\right)\int_0^s \exp\left((\mu_E-k\sigma_E-\mu_C+k\sigma_L)\,u\right)f_i(s,u)du\right]ds\Bigg\} \quad (18)$$

where f_i *are same as Equation (16), (17), respectively.*

Proof. *It is straightforward.*

For convenience, We denote ϕ_i^α as

$$\phi_i^\alpha = \alpha\int_0^T\left[\exp\left((-\rho+\mu_C+k\sigma_C)\,s\right)\int_0^s \exp\left((\mu_E-\mu_C+k(\sigma_E-\sigma_L))\,u\right)f_i(s,u)du\right]ds + (1-\alpha)\int_0^T\left[\exp\left((-\rho+\mu_C-k\sigma_C)\,s\right)\int_0^s \exp\left((\mu_E-\mu_C-k(\sigma_E-\sigma_L))\,u\right)f_i(s,u)du\right]ds. \quad (19)$$

Then we can rewrite $V_i(\pi(0)|\alpha)$ as

$$V_i(\pi(0)|\alpha) = \pi(0)\phi_i^{\alpha}. \tag{20}$$

When a decision maker has a complete ambiguity aversion, she only thinks about the worst possible scenario.

Corollary 2. *(Perfectly pessimistic) The α-expected value of the project in regime $i \in \{E,\ C\}$ for a decision maker who has a degree of optimism $\alpha = 0$is given by*

$$V_i(\pi(0)|\alpha = 0) = \pi(0)\int_0^T \left[\exp\left(\left(-\rho + \mu_C - k\sigma_C\right)s\right)\int_0^s \exp\left(\left(\mu_E - \mu_C - k(\sigma_E - \sigma_C\right)u\right)f_i(s,u)du\right]ds, \tag{21}$$

where f_i are same as Equation (16), (17), respectively.

Proof. *It is straightforward.*

When a decision maker's belief consists of only one probability measure, the perceived degree of ambiguity equals to 0, i.e. $k = 0$. The dynamics of profit flow is given by Equation (1), (2). The value of the investment is easily obtained by inserting $k = 0$ to Equation (18) as follows:

$$V_i(\pi(0)) = E_0^P\left[\int_0^T e^{-\rho s}\ \pi(s)ds \middle| \sigma(0) = \sigma_i\right]. \tag{22}$$

Here, $E_0^P[|\sigma(0) = \sigma_i]$ is the expectation with respect to reference probability measure P conditional on $\mathcal{F}(0)$ given that current regime i is known. In this setting, the investment opportunity set is stochastically is changing over time without ambiguity.

Corollary 3. *(Without ambiguity) When $k = 0$ i.e. the dynamics of profit flow is given by Equation (1), (2), the expected value of project $V_i(\pi(0))$ in regime $i \in \{E,\ C\}$ is represented as*

$$V_i(\pi(0)) = \pi(0)\int_0^T \left[\exp\left(\left(-\rho + \mu_C\right)s\right)\int_0^s \exp\left(\left(\mu_E - \mu_C\right)u\right)f_i(s,u)du\right]ds, \tag{23}$$

where f_i are same as Equation (16), (17), respectively.

Proof. *It is straightforward.*

Similarly, if we denote ϕ_i as

$$\phi_i = \int_0^T \left[\exp\left((-\rho + \mu_C)\, s \right) \int_0^s \exp\left((\mu_E - \mu_C)\, u \right) f_i(s, u) du \right] ds, \tag{24}$$

then $V_i(\pi(0)) = \pi(0)\, \phi_i$.

When decision makers don't adopt a regime-switching environment but faces ambiguity about the profit flow, we denote the expected mean return for one-state model as μ, and the volatility as σ in a reference probability measure P.

Corollary 4. *(One-state Model) When the dynamics of profit flow is given by Equation (5) subject to (3), the expected value of project $V^T(\pi(0)|\alpha)$ is represented as*

$$V^T(\pi(0)|\alpha) = \pi(0) \left(\frac{\alpha(1 - \exp(-(\rho - k\sigma - \mu)T))}{\rho - k\sigma - \mu} + \frac{(1-\alpha)(1 - \exp(-(\rho + k\sigma - \mu)T))}{\rho + k\sigma - \mu} \right) \tag{25}$$

Proof. *By Simple calculation and using definition of f_i, we can prove.*

If we denote ϕ^α as

$$\phi^\alpha = \left(\frac{\alpha(1 - \exp(-(\rho - k\sigma - \mu)T))}{\rho - k\sigma - \mu} + \frac{(1-\alpha)(1 - \exp(-(\rho + k\sigma - \mu)T))}{\rho + k\sigma - \mu} \right) \tag{26}$$

then $V(\pi(0)) = \pi(0)\phi^\alpha$.

Corollary 4 is a finite time horizon version of Schröder (2007). Take the investment period goes to infinity, i.e. $T \to \infty$, then we get, by Assumption 1, the result of Schröder (2007) as follows:

$$V^\infty(\pi(0)|\alpha) = \pi(0) \left(\frac{\alpha}{\rho - k\sigma - \mu} + \frac{(1-\alpha)}{\rho + k\sigma - \mu} \right). \tag{27}$$

4.2 The critical present level of the profit flow

We present the critical present value of the profit flow π_i^*, $i \in \{E,\ C\}$ of the investment. It is defined as what makes the value of option to invest $F_i(\pi(t)|\alpha)$ to be 0. If observable profit flow $\pi_i(t)$ is bigger than π_i^*, decision makers are willing to invest. Otherwise she gives up investing. Note that the critical present value of the profit flow is regime-dependent except for one-state model in Corollary 4.

1. Under ambiguity and regime-switching environment, the critical present value of the profit flow is
$$\pi_i^* = \frac{I}{\phi_i^\alpha}, \qquad i \in \{E,\ C\},\ \alpha \in [0,1]. \tag{28}$$

2. Under ambiguous and regime-switching environment, the critical present value of the profit flow of complete ambiguity aversion($\alpha = 0$) is
$$\pi_i^* = \frac{I}{\phi_i^{\alpha=0}}, \qquad i \in \{E,\ C\}. \tag{29}$$

3. Under regime-switching environment, the critical present value of the profit flow is
$$\pi_i^* = \frac{I}{\phi_i}, \qquad i \in \{E,\ C\}. \tag{30}$$

4. Under ambiguity and one-state model, the critical present value of the profit flow is
$$\pi^* = \frac{I}{\phi^\alpha}, \qquad \alpha \in [0,\ 1]. \tag{31}$$

5 Comparative Statics

The parameters are calibrated for the growth rate of the profit flow for individual firm in Driffill *et al.* (2003) in which they are modified from the parameters of the growth rate of GDP in Hamilton (1989) using some transformation for an annual intertemporal model. The parameters are following : $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3$. For one-state model, the expected mean return $\mu = 0.049548513$, the volatility $\sigma = 0.2$ are used. We fix $\rho = 0.1,\ \pi(0) = 1,\ I = 10$ like as Schröder (2007).

In Table 1, 2, we present the expected values of investment and the critical present value of the investment with varying the attitude toward ambiguity α, the perceived level of ambiguity k, the investment period T, respectively. For each case, the upper value corresponds when initial state is an economical expansion and the lower value corresponds when initial state is an economical recession.

5.1 a change in time horizon

Since we assume that profit flow of the investment is geometric Brownian motion modified ambiguity and regime-switching, it is always nonnegative. Hence the investment which has longer period implies bigger the expected value of investment and smaller the critical present value of the profit. These are well shown by the Table 1, 2.

$1/\lambda_E = 1/0.293 \cong 3.413($ $1/\lambda_C = 1/3.413 \cong 0.293)$ is the expected waiting time to jump from expansion(recession) to recession(expansion). If the project has a short life, it is very important to distinguish whether it is expansion or recession.[2] Longer investment period deduces the gap from the initial state. From a computational point of view, we can regard $T = 450$ as infinity.

5.2 a change(an increase) in ambiguity

The expected value of the investment $V_i(\pi(0) = 1|\alpha)$ as a function of perceived level of ambiguity k is shown in Figure 1, 3. The investment value under the contraction regime is bigger than that under the expansion regime. It is well harmonized with the well known fact in context of real option theory e.g. Dixit and Pindyck (1994) that an increase of volatility of the real option increases the value of real option. Only when the investor is completely pessimistic, $\alpha = 0$, increasing ambiguity decreases the value of investment. Our

[2]We don't need to know how long the state have been stayed because of memorylessness of the exponential distribution.

analysis suggests that when the observable present profit flow is same, it is preferred to invest the investment under contraction regime. It is well displayed the function of critical level of profit flow in Figure 2 and 4.

We display our model with one-state model. In case of $T = 450$, for $\alpha = 0.5, \text{or } 1$, the slope of the expected value of the investment of the one-state model is much bigger than that of our model when the perceived level of ambiguity k is around 0.12. This shows that when the investor concern about the business cycle, the positive effect of the ambiguity is mitigated. We can check that in Figure 4. When $k = 0$, i.e. there is no ambiguity of the profit flow, the critical present value of profit flow π^* of one-state model is located between that of when initial state is an economic expansion and that of when initial state is an economic contraction. This relationship is broken when k is around 0.1 for each $\alpha = 0,\ 0.5$. In case of $k = 0.2$ and $\alpha = 0$ or 0.5, furthermore, the critical present value of one-state model is half of that of our model. The tendency of ambiguity seeking in Shröder model is much decreased.

When the investment period of the project T equals 5, Figure 1 and 2 show that if the investment period is short, the effect of ambiguity seeking is more mitigated and the critical present value of the profit flow depends on the business cycle. Since the risk of regime-switching(regime risk) is reflected, the critical present value of the profit flow π_i^* is bigger than that of one-state model even though the volatility of contraction regime is bigger than that of one-state model. It shows that when the investment period is short, considering the business cycle is very important to make investment decision.

5.3 a change(an increase) in optimism

From Equation (18), we get

$$\begin{aligned}
&\frac{\partial V_i(\pi(0)|\alpha)}{\partial \alpha} \\
&= \pi(0) \left\{ \int_0^T \left[\exp\left((-\rho + \mu_C + k\sigma_C)\, s \right) \int_0^s \exp\left((\mu_E - \mu_C + k(\sigma_E - \sigma_C))\, u \right) f_i(s,u) du \right] ds \right. \\
&\quad \left. - \int_0^T \left[\exp\left((-\rho + \mu_C - k\sigma_C)\, s \right) \int_0^s \exp\left((\mu_E - \mu_C - k(\sigma_E - \sigma_C))\, u \right) f_i(s,u) du \right] ds \right\},
\end{aligned} \tag{32}$$

where $i \in \{E,\ C\}$, and f_i are same as Equation (16), (17), respectively. For a given parameter set, this is constant which is slope of the line in Figure 5 and 7. Furthermore, the critical present profit flow π_i^* is in inverse proportion to the perceived level of ambiguity k as shown by Figure 6, 8.

6 Conclusion

We investigate about how the business cycle affects all or nothing problem whose profit flow is ambiguous. Numerical results show that the threshold value depends on the business cycle. When investment period is short, it is important to know whether it is an economic contraction or an economic expansion. Furthermore, the tendency of ambiguity seeking is mitigated by introducing regime-switching environment.

Appendix A. Proof of Theorem 1

Define

$$X(t) \triangleq \int_0^t I(s) dB(s)^\theta. \tag{33}$$

Lemma 1. *Given $0 \le T(t) = u \le t$, $X(t)$ is normally distributed with mean zero and variance u.*

Proof. *See Appendix of Jang and Roh (2007).*

□

Lemma 2. *Given* $0 \le T(t) = u \le t$*, the correlation between* $X(t)$ *and* $B(t)^\theta$ *is* $\sqrt{\frac{u}{t}}$.

Proof. *See Appendix of Jang and Roh (2007).*

□

The following Lemma is well known result.

Lemma 3. *Let* Y_1 *and* Y_2 *be standard normal variables with correlation coefficient* ρ*. Then for arbitrary constants* c, d*,*

$$E[e^{cY_1+dY_2}] = e^{(c^2+d^2+2\rho cd)/2}.$$

Lemma 4. *For any real constants* a*,* b*,* c

$$\begin{aligned}
&E^{Q^\theta}\left[\exp\left\{a\zeta(s) + bB(s)^\theta + cX(s)\right\}\middle|\,\sigma(0) = \sigma_i\right] ds \\
&= \int_0^s \exp\left\{\left(\frac{b^2}{2}\right)s + \left(a + bc + \frac{c^2}{2}\right)u\right\} f_i(s,u)du.
\end{aligned}$$

Proof. *Let* $f_i(s,u)$ *be a PDF of* $\zeta(s)$*, where* $\sigma(0) = \sigma_i$*, then we have*

$$\begin{aligned}
&E^{Q^\theta}\left[\exp\left\{a\zeta(s) + bB(s)^\theta + cX(s)\right\}\middle|\,\sigma(0) = \sigma_i\right] ds \\
&= \int_0^s E^{Q^\theta}\left[\exp\left\{au + b\sqrt{s}\frac{B(s)^\theta}{\sqrt{s}} + c\sqrt{u}\frac{X(s)}{\sqrt{u}}\right\}\middle|\,\sigma(0) = \sigma_i,\ \zeta(s) = u\right] f_i(s,u)ds.
\end{aligned}$$

By Lemma 1, $\frac{X(s)}{\sqrt{u}}$ *is a standard normal variable and the correlation between* $\frac{B(s)^\theta}{\sqrt{s}}$ *and* $\frac{X(s)}{\sqrt{u}}$ *is* $\sqrt{\frac{u}{s}}$ *by Lemma 2. Using Lemma 3, we obtain*

$$\begin{aligned}
&\int_0^s E^{Q^\theta}\left[\exp\left\{au + b\sqrt{s}\frac{B(s)^\theta}{\sqrt{s}} + c\sqrt{u}\frac{X(s)}{\sqrt{u}}\right\}\middle|\,\sigma(0) = \sigma_i,\ \zeta(s) = u\right] f_i(s,u)ds \\
&= \int_0^s \exp\left\{au + \frac{(b\sqrt{s})^2 + (c\sqrt{u})^2 + 2b\sqrt{s}\cdot c\sqrt{u}\cdot\sqrt{u/s}}{2}\right\} f_i(s,u)du \\
&= \int_0^s \exp\left\{\left(\frac{b^2}{2}\right)s + \left(a + bc + \frac{c^2}{2}\right)u\right\} f_i(s,u)du.
\end{aligned}$$

□

For any $(\theta(t)) \in [-k, k]$, it is true that

$$
\begin{aligned}
& E_0^{Q^\theta}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) - \theta(t)\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^\theta\right\} ds \middle| \sigma(0) = \sigma_i\right] \\
& = \int_0^T \exp\left\{\left(-\rho + \mu_C - \frac{1}{2}\sigma_C^2\right) s\right\} E^{Q^\theta}\left[\exp\left\{\left(\mu_E - \mu_C - \frac{1}{2}(\sigma_H^2 - \sigma_L^2)\right)\zeta(s)\right.\right. \\
& \left.\left. - \int_0^s \theta(t)\sigma(t)dt + \sigma_C B(s)^\theta + (\sigma_E - \sigma_C)\int_0^s I(t)dB(t)^\theta\right\} \middle| \sigma(0) = \sigma_i\right] ds \\
& \leq \int_0^T \exp\left\{\left(-\rho + \mu_C - \frac{1}{2}\sigma_C^2\right) s\right\} E^{Q^\theta}\left[\exp\left\{\left(\mu_E - \mu_C - \frac{1}{2}(\sigma_H^2 - \sigma_L^2)\right)\zeta(s)\right.\right. \\
& \left.\left. + \int_0^s k\sigma(t)dt + \sigma_C B(s)^\theta + (\sigma_E - \sigma_C)\int_0^s I(t)dB(t)^\theta\right\} \middle| \sigma(0) = \sigma_i\right] ds \\
& = \int_0^T \exp\left\{\left(-\rho + \mu_C - \frac{1}{2}\sigma_C^2 + k\sigma_C\right) s\right\} E^{Q^\theta}\left[\exp\left\{\left(\mu_E - \mu_C - \frac{1}{2}(\sigma_H^2 - \sigma_L^2) + k(\sigma_E - \sigma_C)\right)\zeta(s)\right.\right. \\
& \left.\left. + \sigma_C B(s)^\theta + (\sigma_E - \sigma_C)\int_0^s I(t)dB(t)^\theta\right\} \middle| \sigma(0) = \sigma_i\right] ds.
\end{aligned}
$$

Using the result of Lemma 4, we have

$$
\begin{aligned}
& \int_0^T \exp\left\{\left(-\rho + \mu_C - \frac{1}{2}\sigma_C^2 + k\sigma_C\right) s\right\} E^{Q^\theta}\left[\exp\left\{\left(\mu_E - \mu_C - \frac{1}{2}(\sigma_H^2 - \sigma_L^2) + k(\sigma_E - \sigma_C)\right)\zeta(s)\right.\right. \\
& \left.\left. + \sigma_C B(s)^\theta + (\sigma_E - \sigma_C)\int_0^s I(t)dB(t)^\theta\right\} \middle| \sigma(0) = \sigma_i\right] ds \\
& = \int_0^T \exp\left\{(-\rho + \mu_C + k\sigma_C) s\right\} \int_0^s \exp\left\{(\mu_E - \mu_C + k(\sigma_E - \sigma_C)) u\right\} f_i(s, u) du ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& E_0^{Q^{-k}}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) + k\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^{-k}\right\} ds \middle| \sigma(0) = \sigma_i\right] \\
& = \int_0^T \exp\left\{\left(-\rho + \mu_C - \frac{1}{2}\sigma_C^2 + k\sigma_C\right) s\right\} E^{Q^{-k}}\left[\exp\left\{\left(\mu_E - \mu_C - \frac{1}{2}(\sigma_H^2 - \sigma_L^2) + k(\sigma_E - \sigma_C)\right)\zeta(s)\right.\right. \\
& \left.\left. + \sigma_C B(s)^{-k} + (\sigma_E - \sigma_C)\int_0^s I(t)dB(t)^{-k}\right\} \middle| \sigma(0) = \sigma_i\right] ds \\
& = \int_0^T \exp\left\{(-\rho + \mu_C + k\sigma_C) s\right\} \int_0^s \exp\left\{(\mu_E - \mu_C + k(\sigma_E - \sigma_C)) u\right\} f_i(s, u) du ds,
\end{aligned}
$$

which is equal to (34). This implies

$$
\begin{aligned}
& E_0^{Q^\theta}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) - \theta(t)\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^\theta\right\} ds \middle| \sigma(0) = \sigma_i\right] \\
& \leq E_0^{Q^{-k}}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) + k\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^{-k}\right\} ds \middle| \sigma(0) = \sigma_i\right].
\end{aligned}
$$

Since $(\theta(t)) \in [-k, k]$ is arbitrary,

$$\begin{aligned}
&\sup_{Q^\theta \in \mathcal{P}} E^{Q^\theta}\left[\int_0^T e^{-\rho s}\ \pi(s)ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) \sup_{Q^\theta \in \mathcal{P}} E_0^{Q^\theta}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) - \theta(t)\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^\theta\right\} ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) E_0^{Q^{-k}}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) + k\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^{-k}\right\} ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) \int_0^T \exp\left\{(-\rho + \mu_C + k\sigma_C)\, s\right\} \int_0^s \exp\left\{(\mu_E - \mu_C + k(\sigma_E - \sigma_C))\, u\right\} f_i(s,u) du ds.
\end{aligned}$$

Using similar argument, we can prove that

$$\begin{aligned}
&\inf_{Q^\theta \in \mathcal{P}} E^{Q^\theta}\left[\int_0^T e^{-\rho s}\ \pi(s)ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) \inf_{Q^\theta \in \mathcal{P}} E_0^{Q^\theta}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) - \theta(t)\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^\theta\right\} ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) E_0^{Q^{k}}\left[\int_0^T \exp\left\{-\rho s + \int_0^s \left(\mu(t) - k\sigma(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^s \sigma(t) dB(t)^{k}\right\} ds\middle|\sigma(0)=\sigma_i\right] \\
&= \pi(0) \int_0^T \exp\left\{(-\rho + \mu_C - k\sigma_C)\, s\right\} \int_0^s \exp\left\{(\mu_E - \mu_C - k(\sigma_E - \sigma_C))\, u\right\} f_i(s,u) du ds.
\end{aligned}$$

References

Ang, A. and G. Bekaert(2002) "International asset allocation with regime shifts", *Review of Financial Studies* 15, 1137-1187.

Chen, Zengjing and Larry Epstein(2002) "Ambiguity, Risk, and Asset Returns in Continuous time", *Econometrica* Vol. 70, No. 4, 1403-1443.

Dixit, Avinash K. and Robert S. Pindyck (1994), *Investment under Uncertainty*, Pinceton University Press, Princeton.

Driffill, J., Raybaudi, M., and Martin Sola (2003) "Investment Under Uncertainty with Stochastically Switching Profit Streams : Entry and Exit over the business cycle" *Studies in Nonlinear Dynamics and Econometrics* Vol. 7, No. 1, Article 1.

Edwards, C.(2005) "Derivative Pricing Models with Regime Switching : A General Approach", *Journal of Derivative* Vol. 13, 41-47.

Ellsberg, E.(1961) "Risk, ambiguity and savage axioms", *Quarterly Journal of Economics* Vol. 25, 643-669.

Fuh, C.D., Hu, I. and S.K. Lin(2003) "Empirical Performance and Asset Pricing in Hidden Markov Models", *Communications in Statistics : Theory and Methods* Vol. 13, 2477-2512

Ghiradato, P., Maccheroni, F. and Massimo Marinacci (2004) "Differentiating ambiguity and ambiguity attitude", *Journal of Economic Thoery* 118, 133-173.

Gilboa, I. and D. Schmeidler(1989) "Maxmin expected utility with non-unique prior", *Journal of Mathematical Economics* 18, 141-153.

Hamilton, J.(1989) "A New Approach to the Economic Analysis of nonstationary Time Series and the business cycle", *Econometrica* 57, 357-384.

Hamilton, J.(1990) “Analysis of time series subject to changes in regime”, *Journal of Econometrics* 45, 39-70.

Hansen, L. P., Sargent, T. J., Turmuhambetova, G. and N. Williams(2006) “Robust control and model misspecification”, *Journal of Economic Theory* Vol. 128, 45-90.

Hansen, L. P. and T. J. Sargent (2001) “Robust Control and Model Uncertainty”, *The American Economic Review* Vol. 91, No. 2, 60-66.

Heath, Chip and Amos Tversky (1991) “Preference and belief : Ambiguity and competence in choice under uncertainty”, *Journal of Risk and Uncertainty* Vol. 4(1), 5-28.

Jang, B., Koo, H. K., Liu, H., and M. Loewenstein(2007) “Liquidity Premia and Transaction Costs”, *The Journal of Finance* Vol. 62, No. 5, 2329-2366.

Jang, B., K. Roh(2007) “ Valuing Qualitative Options with Stochastic Volatility”, *Quantative Finance* to apper

Kilka, M. and M. Weber(1998) “What determines the shape of the probability weighting function under uncertainty?”, *Management Science* Vol. 47, 1712-1726.

Knight, Frank H.(1921) “Risk, Uncertainty, and Profit”, Houghton Mifflin Company, Boston, MA.

Nishimura, Kiyohiko G. and Hiroyuki Ozaki(2007) “Irreversible investment and Knightian uncertainty”, *Journal of Economic Theory* Vol. 136, 668-694.

Øksendal, B., Stochastic Differential Equations An Introduction with Applications, Springer, 6th edition.

Olszewski, Wojciech (2007) “Preferences over sets of lotteries”, *Review of Economic Studies*, 74, Pathak, Parag A. (2002) "Notes on Robust Portfolio Choice", working paper.

Schröder, David (2007) "Investment under Ambiguity with the Best and Worst in Mind", working paper. 567-595.

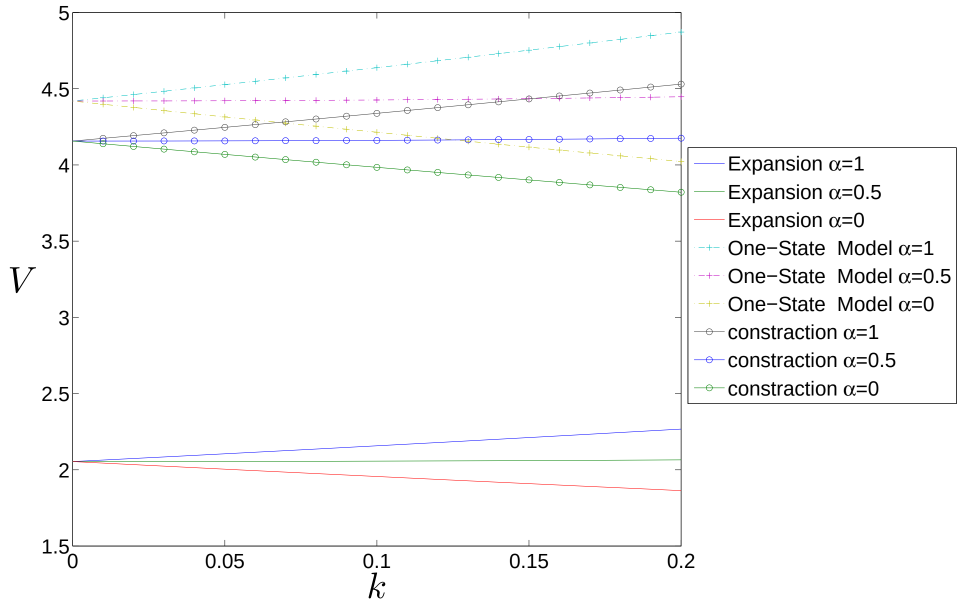

Figure 1: **The expected value of the investment $V(\pi_i(0))$ as a function of k varying α when $T = 5$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

T		10	30	50	150	250	450
$\alpha = 0$	$k = 0.1$	4.7291	10.9369	12.7982	13.5781	13.5799	13.5799
		7.1302	13.3535	15.1941	15.9654	15.9671	15.9671
	$k = 0.2$	4.3372	9.067	10.114	10.4059	10.406	10.406
		6.6223	11.3498	12.3808	12.6683	12.6684	12.6684
$\alpha = 0.25$	$k = 0.1$	4.9588	12.3482	15.2487	17.545	17.6281	17.6322
		7.4259	14.8504	17.7316	20.0169	20.0998	20.1038
	$k = 0.2$	4.7995	12.0167	15.5398	21.7949	23.1221	23.478
		7.2171	14.4759	17.9918	24.2539	25.5833	25.9398
$\alpha = 0.5$	$k = 0.1$	5.1885	13.7595	17.6993	21.5119	21.6763	21.6845
		7.7215	16.3473	20.269	24.0684	24.2324	24.2405
	$k = 0.2$	5.2617	14.9663	20.9655	33.184	35.8382	36.55
		7.8119	17.6019	23.6027	35.8395	38.4983	39.2113
$\alpha = 0.75$	$k = 0.1$	5.4182	15.1708	20.1498	25.4788	25.7245	25.7368
		8.0172	17.8442	22.8064	28.1199	28.365	28.3772
	$k = 0.2$	5.724	17.916	26.3913	44.573	48.5542	49.6219
		8.4067	20.7279	29.2136	47.4251	51.4132	52.4827
$\alpha = 1$	$k = 0.1$	5.6479	16.5821	22.6003	29.4457	29.7727	29.7891
		8.3128	19.3411	25.3439	32.1714	32.4976	32.5139
	$k = 0.2$	6.1862	20.8657	31.8171	55.962	61.2703	62.6939
		9.0015	23.854	34.8245	59.0108	64.3281	65.7542

Table 1: **The expected value of investment obtained by the semi-analytic solution in Corollary 1.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10$. The upper value corresponds when initial state is an economical expansion and the lower value corresponds when initial state is an economical recession.

T		10	30	50	150	250	450
$\alpha = 0$	$k = 0.1$	2.1146	0.9143	0.7814	0.7365	0.7364	0.7364
		1.4025	0.7489	0.6581	0.6264	0.6263	0.6263
	$k = 0.2$	2.3056	1.1029	0.9887	0.961	0.961	0.961
		1.5101	0.8811	0.8077	0.7894	0.7894	0.7894
$\alpha = 0.25$	$k = 0.1$	2.0166	0.8098	0.6558	0.57	0.5673	0.5671
		1.3466	0.6734	0.564	0.4996	0.4975	0.4974
	$k = 0.2$	2.0836	0.8322	0.6435	0.4588	0.4325	0.4259
		1.3856	0.6908	0.5558	0.4123	0.3909	0.3855
$\alpha = 0.5$	$k = 0.1$	1.9273	0.7268	0.565	0.4649	0.4613	0.4612
		1.2951	0.6117	0.4934	0.4155	0.4127	0.4125
	$k = 0.2$	1.9005	0.6682	0.477	0.3014	0.279	0.2736
		1.2801	0.5681	0.4237	0.279	0.2598	0.255
$\alpha = 0.75$	$k = 0.1$	1.8456	0.6592	0.4963	0.3925	0.3887	0.3885
		1.2473	0.5604	0.4385	0.3556	0.3525	0.3524
	$k = 0.2$	1.747	0.5582	0.3789	0.2244	0.206	0.2015
		1.1895	0.4824	0.3423	0.2109	0.1945	0.1905
$\alpha = 1$	$k = 0.1$	1.7706	0.6031	0.4425	0.3396	0.3359	0.3357
		1.203	0.517	0.3946	0.3108	0.3077	0.3076
	$k = 0.2$	1.6165	0.4793	0.3143	0.1787	0.1632	0.1595
		1.1109	0.4192	0.2872	0.1695	0.1555	0.1521

Table 2: **The critical present value of investment obtained by Equation (28).** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10$. The upper value corresponds when initial state is an economical expansion and the lower value corresponds when initial state is an economical recession.

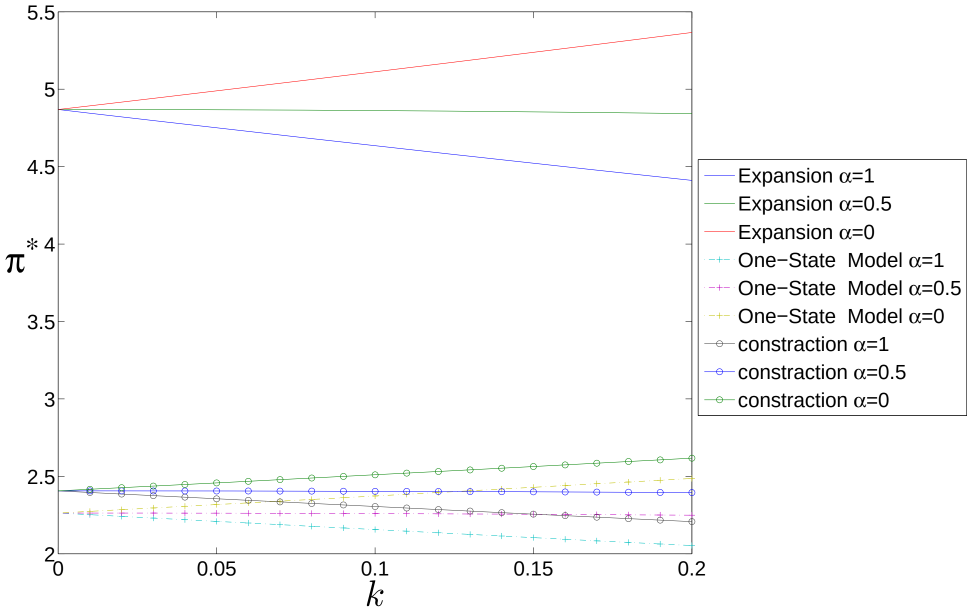

Figure 2: **The critical present value of the profit flow π_i^* as a function of k varying α when $T = 5$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

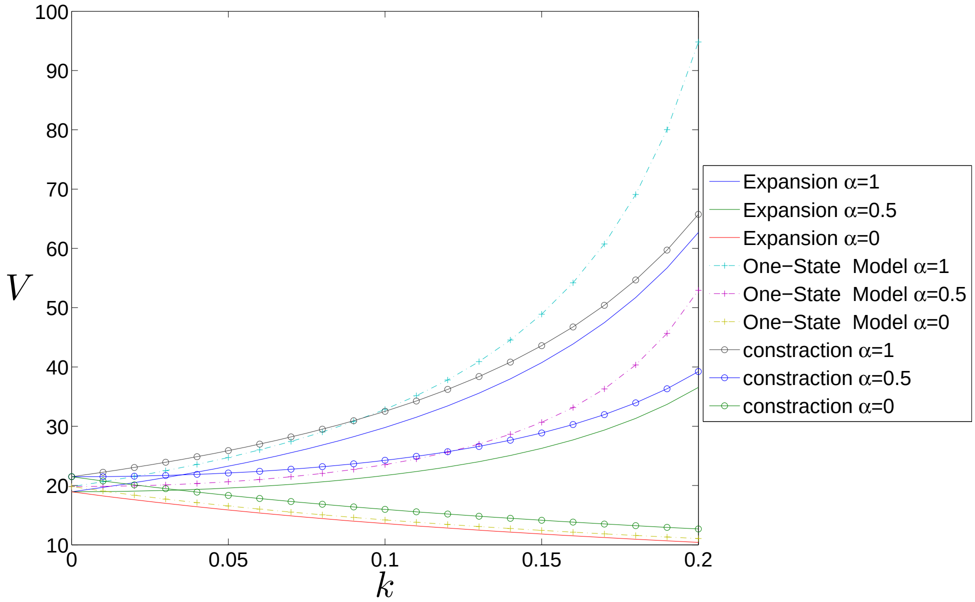

Figure 3: **The expected value of the investment $V(\pi_i(0))$ as a function of k varying α when $T = 450$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

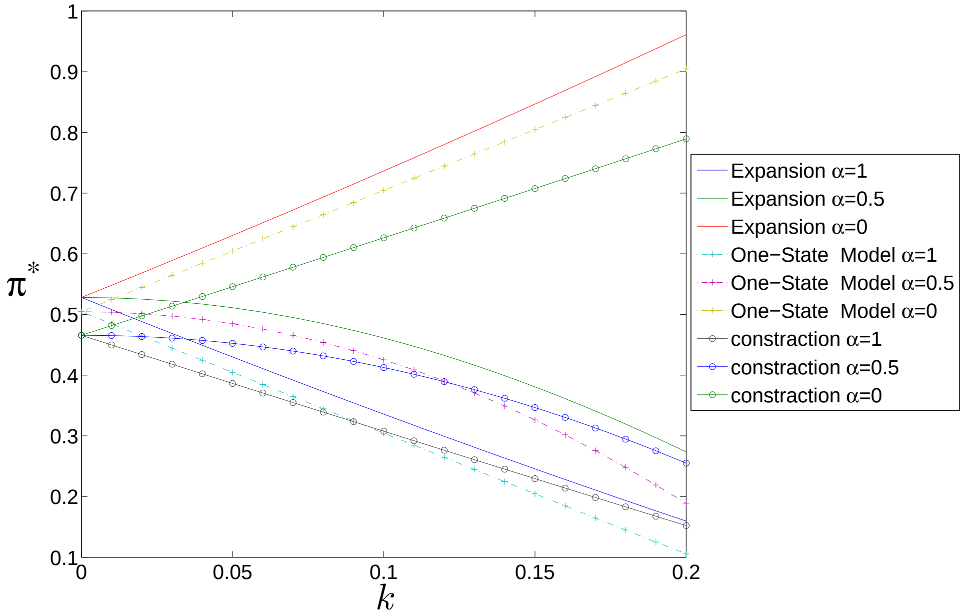

Figure 4: **The critical present value of profit flow π_i^* as a function of k varying α when $T = 450$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

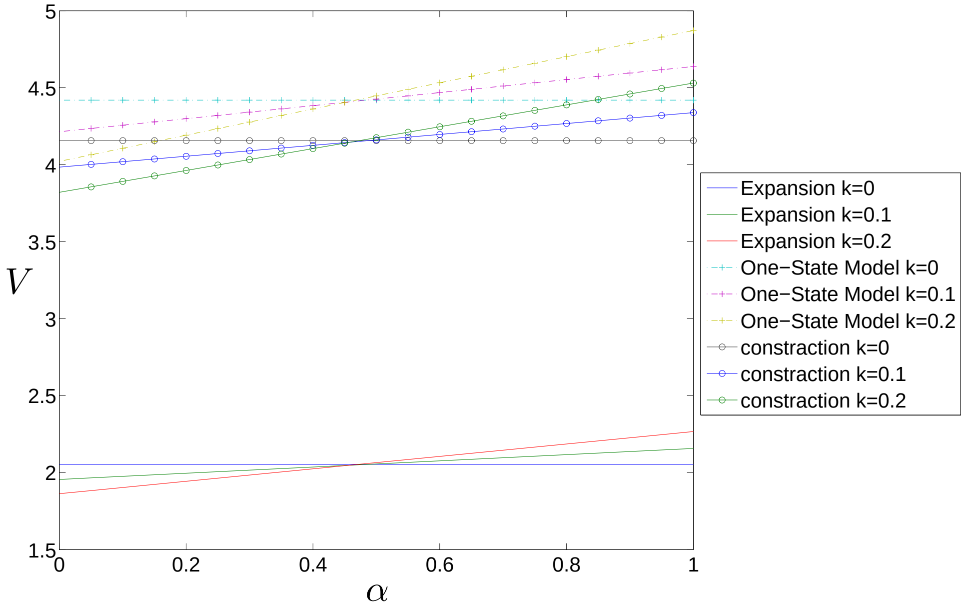

Figure 5: **The expected value of the investment $V(\pi_i(0))$ as a function of α varying k when $T = 5$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

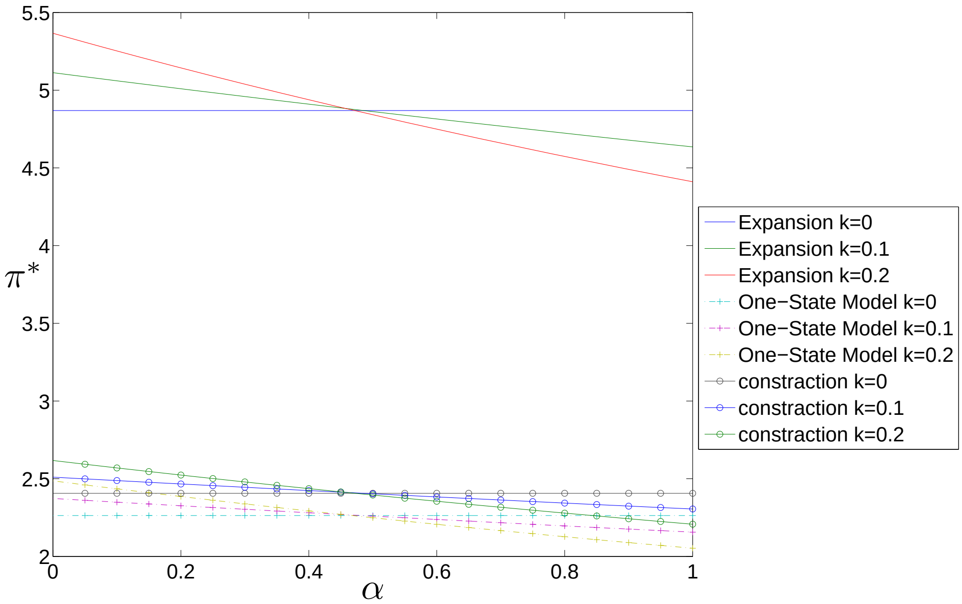

Figure 6: **The critical present value of profit flow π_i^* as a function of α varying k when $T = 5$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

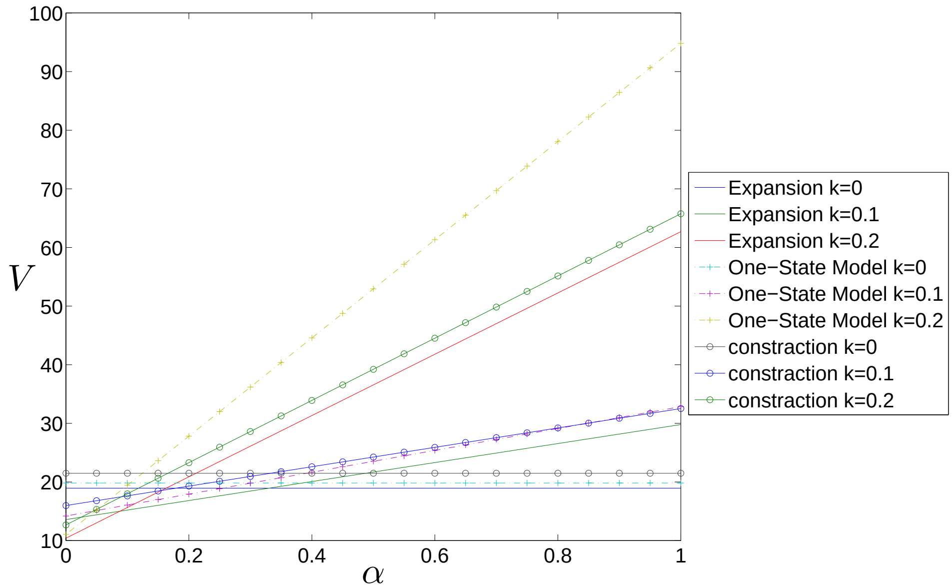

Figure 7: **The expected value of the investment $V(\pi_i(0))$ as a function of α varying k when $T = 450$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$

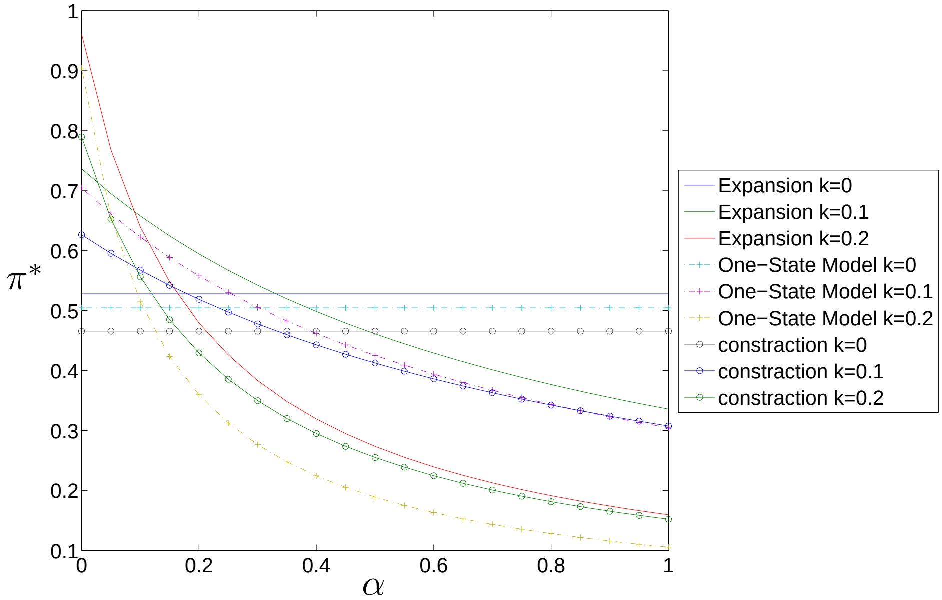

Figure 8: **The critical present value of profit flow π_i^* as a function of α varying k when $T = 450$.** Default parameters are $\lambda_E = 0.293,\ \lambda_C = 3.413,\ \mu_E = 0.056349723,\ \mu_C = 0.03069200,\ \sigma_E = 0.14,\ \sigma_C = 0.3,\ \rho = 0.1,\ \pi(0) = 1,\ I = 10,\ \mu = 0.049548513,\ \sigma = 0.2.$