Optimal Portfolio Selection under Disappointment Averse Utility

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ABSTRACT

In this paper, we consider the portfolio choice problem for Gul(1991)'s disappointment averse investors in continuous time economy. Assuming a complete market and general geometric Brownian motions for asset prices, we provide the analytic method to derive the formulas for the optimal wealth and portfolio weight. In order to explore some important implications, we use the disappointment aversion preferences of Gul(1991) associated with the constant relative risk aversion utility and compare it to the standard CRRA utility. We show that the portfolio weight under the disappointment aversion model is less than under the standard CRRA model. This result partially explains the portfolio puzzle of Mankiw and Zeldes(1991). Also, we find that the portfolio weight under the disappointment aversion model is changed among the time horizon.

1 Introduction

The historical data show that realized premium of stock market return over the risk free rate is large. In the standard expected utility model, a large equity premium suggests large equity positions for most investors and low variation in portfolio choice. However, Mankiw and Zeldes(1991) points out the fact that there is a large variation in equity holdings and a large part of the population does not invest in stocks in United States. This portfolio puzzle is related to the *equity risk premium puzzle* introduced by Mehra and Prescott(1985). In order to explain the limited participation of investors, some researchers assume existence of transaction costs (Liu and Loewenstein(2002)) or introduce borrowing constraints in an overlapping-generations exchange economy (Constantinides et al(2002), Constantinides(2006)).

Another popular approach is to choose the heterogeneous preferences as investors' preferences, especially *loss averse utility* (LA utility) in prospect theory and *disappointment averse utility* (DA utility) introduced by Gul(1991). Under both of two utilities, an investor has different attitude to gain and loss. The main difference between them is that in the disappointment aversion model investor's expected of a future outcome, called the reference point, is determined endogenously, while in the loss aversion model it determined exogenously. A rapidly growing literature use the loss averse utility in the framework of Kahneman and Tversky(1979) and show that the introduction of loss aversion generates lower equity holdings than the standard expected models. (See Berkelaar et al.(2004) and Gomes(2005).) However Ang et al.(2005) mentioned some problems of the loss aversion model. They show that with loss averse utilities finite optimal solutions in the portfolio selection problems do not always exist. Also, it is not clear how to choose the loss aversion reference point and how to undate it in a dynamic setting.

Gul(1991)'s disappointment aversion model is an axiomatic model of preference over lotteries. This model is only one parameter extension of the expected utility framework and includes it as a special case. Moreover, it uses the certainty equivalent to distinguish outcomes into gain and loss, rather than exogenous reference point in the loss aversion model.

Gul offered the DA utility as a solution of the Allais paradox; people prefer p_1, a lottery which yields 200 dollars certainly, to a lottery p_2 that yields 300 dollars with probability 0.8 and 0 dollars with probability 0.2, but prefer $\overline{p}_2$ which yields 300 dollars with probability 0.4 and 0 dollars with probability 0.6 to $\overline{p}_1$ which yields 200 dollars with probability 0.5 and 0 dollars with probability 0.5. This behavior is not consistent with the independence axiom which is necessary condition for the expected utility model. The disappointment averse utility is consistent with Allais type behavior from replacing the independence axiom by a weaker axiom called the betweenness axiom. In addition to Allais paradox, the disappointment utility also provide a solution to Rabin's gamble problem. Rabin(2000) describe that under the expected model, if a person reject gambles where she loses 100 dollars or gains 110 dollars with equal probability then she will reject gambles where she loses 1000 dollars or gain any amount of money with equal probability. However, Ang et al.(2005) show that under the DA utility model, her acceptance of gambles depends on the amount of money she has chance to get. These two experiments support the disappointment averse utility as a good alternation of the expected utility. Furthermore, there are several researches which provide evidence for the usefulness of the disappointment averse utility by analyzing based on market data. Epstein and Zin(2001) find that using preference with disappointment aversion provides a substantial improvement in the empirical performance of a representative agent, intertemporal asset pricing model. Camerer and Ho(1994) and Morone and Schmidt(2008) also support the disappointment aversion model.

There are some literature which consider portfolio choice problems under the disappointment averse utility. Ang et al.(2005) find solutions of both static and dynamic portfolio choice under DA utility associated with the constant relative risk averse utility (CRRA utility) and provide the explicit form of non-participation region of investors. Epstein and Zin(2001) consider an infinite time horizon asset pricing model with recursive framework of DA utility. However, in my knowledge, no literature give a treatment of portfolio choice problem in the continuous time economy.

In this paper, I provide an analytic method to solve the optimal portfolio choice problem when a disappointment averse investor want to maximize her utility on terminal wealth. I assume that the market is complete and there are two assets, a risk-free asset and a risky asset, following general geometric Brownian motions. I derive formulas for the optimal wealth process and the optimal portfolio weight investing on the risky asset. The numerical results for a special utility function, for example, the disappointment averse utility function associated with the constant relative risk averse(CRRA) utility, provide mainly two important implications ; (1) lower investment on a risky asset which partially explain the portfolio puzzle I mentioned above, and (2) the change of strategies among the time horizon.

This paper is organized as follows. In Section 2, I describe my economy model and define the optimal portfolio choice problem under general utility models we cover. Section 3 give an analytic method to solve investment problems under both of the standard expected utility model and Gul(1991)'s disappointment averse utility model. An example is provided in Section 4. I derive the analytic formulas for the case of CRRA utility and of DA utility associated CRRA utility and exhibit some implications. At last, I conclude in Section 5.

2 Investment Problems

In this section, we describe our economy and investment problems. We consider a complete market during finite time horizon $[0, T]$ without tax, transaction costs, and difference between borrowing and lending interest rates. We assume that there are two assets, one risk-free asset and one risky asset. The time t price of a risk-free asset is denoted by $S_0(t)$ and the time t price of a risky asset is denoted by $S(t)$. Price processes are assumed to evolve the following stochastic differential equations.

$$\begin{cases} dS_0(t) &= rS_0(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dB_t \end{cases} \tag{2.1}$$

where r is the riskless return rate, μ and σ are return rate and volatility of risky asset respectively, and B_t is a Brownian motion under a filtered probability space $(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$. Here, the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be generated by B_t with usual conditions. Furthermore, r, μ, σ are constants and we assume that $\mu > r > 0, \sigma > 0$. From the market completeness assumption, there are the market price of risk, $\gamma = \dfrac{\mu - r}{\sigma}$ and the pricing kernel process,

$$H_t = \exp\left\{(r + \frac{1}{2}\gamma^2)t - \gamma B_t\right\},\ H_0 = 1.$$

During the finite time interval $[0, T]$, an investor invests on both of a risky asset and a risk-free asset. To maximize her utility on the wealth at the terminal time, she continuously choose a portfolio consisting of two assets. Let X_t be a wealth process of the investor with initial wealth $X_0 = x > 0$ and π_t is the proportion of her wealth invested in risky asset at time t. Then the wealth process is derived by following stochastic differential equation.

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t \tag{2.2}$$

The portfolio weight process π is called admissible if $X_t \geq 0$ for all $t \in [0, T]$.

We assume that the investor has the utility function $u : [0, \infty) \to \mathbb{R}$ on the certainty equivalent to her terminal wealth $\mu(X_T)$. The certainty equivalent is a function of distribution function F into $\mathbb{R}$ and equals to value which, if received with certainty, would be indifferent to receive random outcome with distribution F. Precisely, for a utility function u and for a distribution function F of any lottery X, the certainty equivalent $\mu(F)$(or denoted by $\mu(X)$) is defined by a constant such that

$$V(\delta_{\mu(F)}) = V(F)$$

where $\delta_{\mu(F)}(z) = \mathbf{I}_{\{z \geq \mu(F)\}}$ for $z \in \mathbb{R}$. By assuming the utility function u and the certainty equivalent μ, we can state the optimal investment problem for an investor who has the certain preference.

Problem 1 *When an investor has the utility $u(x)$ on $\mu(X_T)$ which is the certainty equivalent of her terminal wealth X_T, the optimal investment problem is to find the admissible portfolio strategy π_t such that*

$$\max_{\pi_t} \quad u(\mu(X_T))$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t, \quad X_0 = x$$

$$X(t) \geq 0, \text{ for all } t \in [0, T]$$

Investor's optimal strategies and optimal wealth depend on her utility function $u(x)$ and the certainty equivalent function $\mu(X)$. The next section will discuss about investor's preferences, especially the standard smooth utility and the disappointment averse utility of Gul(1991).

3 Optimal Strategies under Different Utility Assumptions

3.1 Standard Smooth Utility

Standard researches on the theory of optimal investment choice assume that an investor's utility function on her wealth is strictly increasing, concave, and differentiable on all points in $[0, \infty)$. In our notation in section 2, this assumption implies that $u(x)$ is a smooth function on $x \in [0, \infty)$ and the certainty equivalent $\mu(X)$ is defined by

$$u(\mu(X)) = E\left[u(X)\right] \tag{3.1}$$

for any random variable X. It implies that an investor who has the standard smooth utility has same attitude to gain or loss of her future wealth.

Using the martingale method introduced by Harrison and Kerps(1979), we can derive the formulas for optimal wealth process and optimal portfolio process for our problem 1. We simply summarize the well-known solutions without proof.

Consider the problem 1 with a smooth utility function u and the certainty equivalent described in equation (3.1). The optimal terminal wealth and intermediate wealth are

$$X_T^* = I(\lambda H_T), \quad X_t^* = \frac{1}{H_t} E\left[H_T I(\lambda H_T) | \mathcal{F}_t\right]$$

if there exist the optimal portfolio process π^* satisfying that the solution of equation (2.2) corresponding to π^*, X_t^{x,π^*}, is equal to X_t^*. Here, $I(y)$ is the inverse of the marginal utility function $u'(x)$ and λ is the unique solution of

$$E\left[H_T I(\lambda H_T)\right] = x$$

To obtain the optimal portfolio process π^*, we derive a stochastic differential equation which X_t^* satisfies and find π by using equation (2.2). A sufficient condition to exist the optimal portfolio π^* is provided by Karatzas and Wang(2001).

3.2 Disappointment Averse Utility

Disappointment averse utility introduced by Gul(1991) is one of the most popular generalization of the standard expected utility. Gul(1991) defined DA utility by replacing the independence axiom which generates expected utility by the betweenness axiom. Dekel(1986) showed that if the preference ordering satisfies betweenness assumption and some standard assumptions, then the certainty equivalent $\mu(F)$ of any distribution function F is defined implicitly as the unique solution to

$$\int H[x, \mu(F)]dF(x) = 0 \tag{3.2}$$

for each distribution function F in the domain of μ. The standard smooth utility in the previous section is obtained if H is specialized to $H(x, \mu) = u(x) - u(\mu)$. For the disappointment averse utility, as in Gul(1991), the certainty equivalent is defined by assuming that the function H in equation (3.2) is

$$H(x, y)^1 := \begin{cases} u(x) - u(y) & x \leq y \\ A\left(u(x) - u(y)\right) & x > y \end{cases}$$

where $A \in (0, 1]$ is the disappointment aversion coefficient. Obviously, when $A = 1$ we obtain the standard expected utility function. Smaller values for A reflect aversion to disappointment, which refer to an outcome as disappointing if it is worse than expected in the sense

[1]This definition of disappointment averse utility is little extension of Gul(1991)'s definition. He provided the restricted version by using constant relative risk aversion utility to $u(x)$. We will follow his definition later for an example.

of being smaller than the certainty equivalent of the lottery. This assumption give us to the definition of static disappointment utility for X in the following form.

Definition 3.1 *For an smooth utility function* $u : (0, \infty) \to [0, \infty)$*, which is strictly increasing, strictly concave, continuously differentiable and* $u'(0+) = \infty, u'(\infty) = 0$*, a certainty equivalent* μ *associated with* u *is implicitly defined by*

$$u(\mu(X)) = E\left[u(X) + (\frac{1}{A} - 1)(u(X) - u(\mu(X)))\mathrm{I}_{\{X<\mu(X)\}}\right] \tag{3.3}$$

for any nonnegative $\mathbb{R}$*-valued random variable* X*. Then* $u(\mu(X))$ *is the disappointment averse utility on wealth* X *of an investor whose disappointment aversion is* A *and associated utility function is* $u(x)$*.*

Under the disappointment averse utility in the definition 3.1, an investor want to solve the optimal investment problem stated in Problem 1 of section 2. We solve the investor's problem via two steps. Firstly, we solve the problem with assumption that the certainty equivalent of terminal wealth is given by θ. We will provide the formulas for optimal wealth and portfolio corresponding to any nonnegative θ by using the martingale method. Then we replace the optimal wealth and the optimal portfolio in the equation (3.3) by the obtained formulas to find a value of $\theta = \mu(X_T)$. These two steps can be represented as following:

Step 1 *Solve the following problem given* $\mu(X_T) = \theta$*,*

$$\max_{\pi_t} \; v(x, \pi; \theta) = \max_{\pi_t} \; E\left[u(X_T) + (\frac{1}{A} - 1)(u(X_T) - u(\theta))\mathrm{I}_{\{X_T<\theta\}}\right]$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t, \quad X_0 = x$$

$$X_t \geq 0, \quad \text{for all } t \in [0, T]$$

Step 2 *Let* $X_t^*(\theta)$ *and* $\pi_t^*(\theta)$ *be the optimal wealth process and the optimal portfolio process when* $\mu(X_T) = \theta$ *is given. Find the fixed point* θ*, in that the solution of the equation*

$$u(\theta) = v(x, \pi^*(\theta); \theta) = E\left[u(X_T^*(\theta)) + (\frac{1}{A} - 1)(u(X_T^*(\theta)) - u(\theta))\mathrm{I}_{\{X_T^*(\theta)<\theta\}}\right]$$

To solve the subproblem in step 1 with the fixed $\theta = \mu(X_T)$, let $U^\theta(x)$ be the modified utility function with disappointment aversion A,

$$U^\theta(x) = u(x) + (\frac{1}{A} - 1)(u(x) - u(\theta))I_{\{x<\theta\}} = u(x)I_{\{x\geq\theta\}} + (\frac{1}{A}u(x) + (1 - \frac{1}{A})u(\theta))I_{\{x<\theta\}}$$

The function $U^\theta(x)$ is strictly increasing, strictly concave, differentiable except on $x = \theta$, and kinked at $x = \theta$ as in figure 1.

[Insert Figure 1 here.]

Furthermore, the first derivative of $U^\theta(x)$ exists except on $x = \theta$ and is equal to

$$(U^\theta)'(x) = \begin{cases} \frac{1}{A}u'(x) & x < \theta \\ u'(x) & x > \theta \end{cases}$$

Then we consider $\widetilde{U}^\theta(y) : (0, \infty) \to (0, \infty)$ is the dual function of $U^\theta(x)$ and the function $I^\theta(y)$ is assumed to satisfy

$$\widetilde{U}^\theta(y) := \max_{x>0} [U^\theta(x) - xy] = U^\theta(I^\theta(y)) - yI^\theta(y)$$

From the first order condition for both cases $x < \theta$ and $x > \theta$, $I^\theta(y)$ is obtained as

$$I^\theta(y) = \begin{cases} (u')^{-1}(y) & if\ y < u'(\theta) \\ \theta & if\ u'(\theta) \leq y \leq \frac{1}{A}u'(\theta) \\ (u')^{-1}(Ay) & if\ \frac{1}{A}u'(\theta) < y \end{cases}$$

Here, $I^\theta(y) : (0, \infty) \to (0, \infty)$ is onto, continuous, differentiable except on $y = u'(\theta), \frac{1}{A}u'(\theta)$, and decreasing. Therefore, $U^\theta(x) = \min_{y>0} [\widetilde{U}^\theta(y) + xy]$.

Recall that $H_t := \exp\{-(r + \frac{1}{2}\gamma^2)t - \gamma B_t\}$ is the pricing kernel where $\gamma = \dfrac{\mu - r}{\sigma}$ is the market price of risk. From the above argument, any positive λ satisfies the next inequality,

$$v(x, \pi; \theta) = E\left[U^\theta(X_t^{x,\pi})\right] \leq E\left[\widetilde{U}^\theta(\lambda H_T)\right] + \lambda x \tag{3.4}$$

and the equality holds when $X_T^{x,\pi} = I^\theta(\lambda H_T), E[H_T X_T^{x,\pi}] = x$.

Define a function for any $\lambda > 0$, $\mathcal{X}^\theta(\lambda) = E[H_T I^\theta(\lambda H_T)]$. Then $\mathcal{X}^\theta(\lambda) : (0,\infty) \to (0,\infty)$ is onto, strictly decreasing, and continuous. Therefore, there exists an inverse function $\mathcal{Y}^\theta(x) : (0,\infty) \to (0,\infty)$. When x is the initial endowment for an investor, we know that the following equations hold.

$$\max_\pi v(x,\pi;\theta) \le E\left[\widetilde{U}^\theta(\mathcal{Y}^\theta(x)H_T)\right] + \mathcal{Y}^\theta(x)x$$
$$E\left[H_T I^\theta(\mathcal{Y}^\theta(x)H(T))\right] = x$$

If there exists a portfolio strategy $\pi_t^*(\theta)$ such that $X_T^{x,\pi^*(\theta)} = I^\theta(\mathcal{Y}^\theta(x)H_T)$, then

$$v(x,\pi^*(\theta);\theta) = \max_\pi v(x,\pi;\theta)$$

since the equality conditions of inequality (3.4) hold. Furthermore, from the above argument the optimal wealth process $X_t(\theta)$ is derived from

$$X_t^{x,\pi^*(\theta)} = \frac{1}{H_t} E[H_T X_T^{x,\pi^*(\theta)} | \mathcal{F}_t]$$

and we can obtain the optimal portfolio $\pi^*(\theta)$ from the diffusion term of a stochastic differential equation which $X_t^{x,\pi^*(\theta)}$ satisfies.

For the last step, we have to find the solution θ of the next equation to specify the actual reference level $\theta = \mu(X_T^{x,\pi^*})$.

$$u(\theta) = E\left[u(I^\theta(\mathcal{Y}^\theta(x)H_T)) + (\frac{1}{A} - 1)(u(I^\theta(\mathcal{Y}^\theta(x)H_T)) - u(\theta)) I_{\{I^\theta(\mathcal{Y}^\theta(x)H_T) < \theta\}}\right] \tag{3.5}$$

If θ which satisfies equation (3.5) exists and denote it by θ^*, then the optimal wealth and portfolio strategy of the investor who have disappointment averse utility are $X_t(\theta^*)$ and $\pi_t(\theta^*)$.

In section 4, we derive solutions of our investment problems under the constant relative risk aversion (CRRA) utility as an example for the standard smooth utility and under the disappointment averse utility associated with CRRA utility and positive disappointment aversion. These solutions give us some numerical implications which we are interested in.

4 The CRRA Utility Case and Numerical Results

4.1 Solutions under CRRA Utility

In this section, we apply our results in section 3 to a constant relative risk aversion(CRRA) utility. The CRRA utility function is defined by for any $x \in [0, \infty)$,

$$u(x) := \frac{1}{\alpha} x^{\alpha} \quad \text{if} \ \alpha \leq 1 \text{ and } \alpha \neq 0 \tag{4.1}$$

where $(1 - \alpha)$ is the relative risk aversion of an investor. Remind that the market price of risk is $\gamma := \frac{\mu - r}{\sigma}$ and the pricing kernel process is $H_t = \exp\{-(r + \frac{1}{2}\gamma^2)t - \gamma B_t\}$ for any $t \in [0, T]$.

Theorem 4.1 *When an investor has CRRA utility with risk aversion* $(1 - \alpha)$ *and her initial endowment is* $x > 0$, *her optimal portfolio is*

$$\pi_t^* = \frac{\mu - r}{\sigma^2(1 - \alpha)}, \quad \text{for } t \in [0, T]$$

and her optimal wealth process is

$$X_t^{x,\pi^*} = x H_t^{\frac{1}{\alpha - 1}} \exp\{\frac{\alpha}{\alpha - 1}(r - \frac{\gamma^2}{2(\alpha - 1)})t\}, \quad \text{for } t \in [0, T]$$

Proof. See Appendix A. □

From the above theorem, we know that the optimal portfolio weight for a CRRA investor is constant at any level of wealth and for any time horizon T. It is not natural because most people do not want to participate the stock market, as in literature, and people who is wealthy are more willing to take risk. Also, on the contrary to the result in theorem 4.1, it is natural that the more time until the terminal time an investor has, the more money she investa on a risky asset. We will discuss these two points by comparing results under the CRRA utility model to under the DA utility model.

4.2 Solutions under DA Utility

To define a disappointment averse utility, we have to specify both of the certainty equivalent $\mu(\cdot)$ and the smooth utility function $u(\cdot)$ as in the definition 3.1. In order to compare disappointment averse investors' behavior to CRRA type investor's, we define the disappointment averse utility associated with CRRA utility $u(x)$ in equation (4.1). The certainty equivalent for the DA utility is defined implicitly by

$$\frac{1}{\alpha}\mu(X)^\alpha = E\left[\frac{1}{\alpha}X^\alpha + (\frac{1}{A}-1)(\frac{1}{\alpha}X^\alpha - \frac{1}{\alpha}\mu(X)^\alpha)\mathrm{I}_{\{X<\mu(X)\}}\right] \tag{4.2}$$

where $(1-\alpha)$ is the investor's relative risk aversion coefficient for any nonzero constant $\alpha(\leq 1)$, and $A(\in (0,1])$ is her disappointment aversion coefficient. The utility of the investor on her wealth X_T at terminal time T is defined by $\frac{1}{\alpha}\mu(X_T)^\alpha$.

Now, consider an investor who has a disappointment averse utility with relative risk aversion $(1-\alpha)$ and disappointment aversion A. We assume that her initial endowment is $x > 0$. The following three theorems provide the solution of her optimal investment problem obtained by applying our method in the previous section. Theorems 4.2 and 4.3 are the solution of the subproblem described in Step 1 in section 3. All proofs are in Appendix.

Theorem 4.2 *Under the given value θ for the certainty equivalent of terminal wealth, the optimal wealth at terminal time T corresponding to θ is*

$$X_T^{x,\pi^*(\theta)} = \begin{cases} (\lambda^*(\theta)H_T)^{\frac{1}{\alpha-1}} & \text{if } \lambda^*(\theta)H_T < \theta^{\alpha-1} \\ \theta & \text{if } \theta^{\alpha-1} \leq \lambda^*(\theta)H_T \leq \frac{1}{A}\theta^{\alpha-1} \\ (A\lambda^*(\theta)H_T)^{\frac{1}{\alpha-1}} & \text{if } \frac{1}{A}\lambda^*(\theta)H_T < \theta^{\alpha-1} \end{cases}$$

where $\lambda^*(\theta)$ is the unique solution of the following equation.

$$\begin{aligned} x &= \theta e^{-rT}(N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+\gamma\sqrt{T}+B_1(\theta)) - N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+\gamma\sqrt{T}+B_2(\theta))) \quad (4.3)\\ &+ \lambda^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T\}N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1}-\frac{1}{\gamma\sqrt{T}}\ln(\lambda)-B_1(\theta)) \\ &+ (A\lambda)^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T\}N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1}+\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+B_2(\theta)) \end{aligned}$$

and

$$B_1(\theta) = -\frac{1}{\gamma\sqrt{T}}((r+\frac{1}{2}\gamma^2)T+(\alpha-1)\ln(\theta)),\ \ B_2(\theta) = -\frac{1}{\gamma\sqrt{T}}((r+\frac{1}{2}\gamma^2)T+(\alpha-1)\ln(\theta)-\ln(A))$$

Proof. See Appendix B. □

Theorem 4.3 *Under same assumptions of theorem 4.2, the investor's optimal wealth at time* $t \in [0, T)$ *is*

$$\begin{aligned} X_t^{x,\pi^*(\theta)} &= \theta e^{-r(T-t)} \left(N(\frac{K_1}{\gamma\sqrt{T-t}} + \gamma\sqrt{T-t}) - N(\frac{K_2}{\gamma\sqrt{T-t}} + \gamma\sqrt{T-t}) \right) \\ &\quad + (\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)} N(-\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t} - \frac{K_1}{\gamma\sqrt{T-t}}) \\ &\quad + (A\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)} N(\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t} + \frac{K_2}{\gamma\sqrt{T-t}}) \end{aligned} \tag{4.4}$$

and her optimal portfolio weight at time t is

$$\begin{aligned} \pi_t^*(\theta) &= \frac{1}{\sigma X_t^{x,\pi^*(\theta)}} \Bigg[\frac{\gamma}{1-\alpha} (\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)} \\ &\quad \times \left(N(-\frac{K_1}{\gamma\sqrt{T-t}} - \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) + A^{\frac{1}{\alpha-1}} N(\frac{K_2}{\gamma\sqrt{T-t}} + \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) \right) \\ &\quad - \frac{\theta}{\sqrt{2\pi(T-t)}} e^{-(r+\frac{1}{2}\gamma^2)(T-t)} (e^{-K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - e^{-K_2 - \frac{K_2^2}{2\gamma^2(T-t)}}) \\ &\quad + \frac{(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}}{\sqrt{2\pi(T-t)}} e^{-\frac{\alpha}{\alpha-1}(r+\frac{1}{2}\gamma^2)(T-t)} (e^{-\frac{\alpha}{\alpha-1}K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}K_2 - \frac{K_2^2}{2\gamma^2(T-t)}}) \Bigg] \end{aligned} \tag{4.5}$$

where

$$K_1 = \ln H_t + \ln \lambda^*(\theta) - (\alpha - 1)\ln\theta - (r + \frac{1}{2}\gamma^2)(T-t), \quad K_2 = K_1 + \ln A$$

Proof. See Appendix C. □

The last Theorem 4.4 is derived from the equation (3.5) and the definition of the disappointment averse utility associated with CRRA utility.

Theorem 4.4 *Let* $\overline{\theta}$ *be a constant defined by* $\overline{\theta} = x\exp\{(r - \frac{\gamma^2}{2(\alpha-1)})T\}$*. The value of the certainty equivalent,* $\mu(X_T^{x,\pi^*})$ *of an investor's optimal terminal wealth* X_T^{x,π^*} *is a solution in* $(0, \overline{\theta})$ *of the following equation.*

$$\begin{aligned} \theta^\alpha &= \theta^\alpha \Big(N(\frac{1}{\gamma\sqrt{T}} \ln(\lambda^*(\theta)) + B_1(\theta)) - \frac{1}{A} N(\frac{1}{\gamma\sqrt{T}} \ln(\lambda^*(\theta)) + B_2(\theta)) \Big) \\ &\quad + (\lambda^*(\theta))^{\frac{\alpha}{\alpha-1}} \exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\} N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}} \ln(\lambda^*(\theta)) - B_1(\theta)) \\ &\quad + A^{\frac{1}{\alpha-1}} (\lambda^*(\theta))^{\frac{\alpha}{\alpha-1}} \exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\} N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}} \ln(\lambda^*(\theta)) + B_2(\theta)) \end{aligned}$$

Moreover, the optimal wealth and portfolio weight are obtained by substituting θ in equations (4.4) and (4.5) by the solution $\mu(X_T^{x,\pi^*})$.

Proof. See Appendix D. □

Since the disappointment aversion A makes the certainty equivalent to decrease by weighting more on loss than on gain, the certainty equivalent $\mu(X_T)$ under the DA utility with $A < 1$ is less than under the standard CRRA utility, i.e. $A = 1$. In the above theorem, $\overline{\theta}$ is the value of certainty equivalent when $A = 1$. In the last part of Appendix D, we discuss about the existence and uniqueness of the solution θ in Theorem 4.4.

4.3 Numerical Implications

In order to study the effects of disappointment aversion, we explore the optimal investment on a risky asset under both of the standard CRRA utility and the disappointment averse utility corresponding to CRRA utility. Throughout this section, we consider an investor who invest her initial endowment $x = 1$ on a riskless asset and a risky asset and who want to maximize her utility on wealth at time $T = 1$. Two assets evolve stochastic differential equations (2.1) with parameters $r = 0.0408, \mu = 0.1063$, and $\sigma = 0.2193$ based on Ang et al.(2005). Also, the preference used for the example are $\alpha = 0.5$ for both models and additionally $A = 0.44$ for the disappointment averse utility model. The value we choose for the disappointment aversion coefficient is from Ang et al.(2005) and is consistent with the result of Tversky and Kahneman(1992).

[Insert Figure 2 here.]

[Insert Figure 3 here.]

Figure 2 displays the optimal terminal wealth of a disappointment averse investor(solid line) and of an investor having CRRA utility, as a benchmark(dashed line). Also, figure 3 is about the optimal intermediate wealth of both investors at time $t = 0.5$. Two figures are showed to be similar in their tendency. At both of terminal time and intermediate time, the optimal wealth profile for disappointment averse utility is similar to the benchmark CRRA case in good states of the world with low pricing kernel H_T. In intermediate states, the terminal wealth maintain at the level of $\mu(X_T)$. The interval where the optimal wealth is constant depends on disappointment aversion A, and it's length is larger when A is smaller. In bad states with high pricing kernel, the optimal wealth for DA utility approaches zero gradually. Our result is from disappointment averse investors' behavior that they give up some wealth in good states due to save for bad states.

[Insert Figure 4 here.]

[Insert Figure 5 here.]

Figure 4 compares the optimal investment amount on a risky asset by a disappointment averse investor (solid line) at an intermediate time $t = 0.5$ to the benchmark CRRA solution (dashed line). We obtain the fact that the optimal portfolio amount is U-shaped for a disappointment averse investor. In order to represent easily the difference between portfolio weights under the DA model and the CRRA model, figure 5 shows the optimal portfolio weight at time $t = 0.5$ among the wealth level by combining results in figures 3 and 4. In figure 5, we can see that the optimal portfolio weight for a disappointment averse investor is always less than the optimal weight for a standard CRRA type investor.

The disappointment averse investor behaves similarly to the CRRA case in good states. She does not concern significantly the loss of her wealth because her wealth level is far from the reference level which distinguishes between gain and loss. If her wealth level downs to the certainty equivalent of her terminal wealth, i.e. the reference level, it means that her wealth level goes from gain to loss. In that case, she invests less on the risky asset due to protect her current wealth. When her wealth is closer to the reference level, the proportion of her wealth invested on a risky asset is much smaller. With a large disappointment aversion and a small equity premium, it happens that the investor whose wealth level is near the reference level does not invest on a risky asset.(See non-participation puzzle of Mankiw and Zeldes(1991).) However, in bad states she abandons her precautious strategy and the portfolio weight of a risky asset increases. When her wealth level is sufficiently lower than the reference level, the increase of utility from preventing loss of her wealth is less than the decrease of utility from giving up possible gain. Therefore the investor is willing to take risk of loss and behaves similarly to the standard CRRA type investors.

[Insert Figure 6 here.]

[Insert Figure 7 here.]

Figure 6 and figure 7 represent the effects of the disappointment aversion on the reference level and on the initial portfolio weight, respectively. The more disappointment averse investor is, the less reference level she has and the less proportion of wealth she invests on a risky asset. When the disappointment aversion coefficient is $A = 0.44$, as in the previous example, the reference level is 1.3127, while it is 1.7314 for an investor who is not disappointed. The difference is 0.4187 which is surprisingly high for the initial wealth level 1. Also, the optimal initial portfolio weight decreases from 2.7239 to 1.2491 as the disappointment aversion coefficient A decreases from 1 to 0.44.

[Insert Figure 8 here.]

At last, we explore the change of the optimal initial portfolio weight when the investor's planning horizon is changed in figure 8. As we already mentioned in Section 4.1, the optimal portfolio weight under the standard CRRA utility model does not depend on the time horizon T. However, the figure 8 displays that the initial portfolio weight of a disappointment averse investor is increasing as the time horizon T increases. This result is from the intuitive aspect that people invest less in stocks as they get older and the planning horizon get nearer. Also, they are more willing to take risk when they have more time until the terminal time, since there are chances to enlarge their wealth or, at least, to restitute their wealth. From the theorem 4.3, we remark that the portfolio weight for a disappointment averse investor goes to the constant portfolio weight level for the CRRA case, as the time horizon goes to infinity.

5 Conclusion

In this paper, we provide an analytic method to solve the optimal investment problem for disappointment averse investors in a continuous time economy. Assuming a complete market and general geometric Brownian motion for asset prices, we solve the problem to maximize the disappointment averse investor's utility on her terminal wealth. In order to explore some important implications, we use the disappointment aversion preferences of Gul(1991) associated with the constant relative risk aversion utility and compare it to the standard CRRA utility, as an example of the standard expected utility.

The portfolio weight invested on a risk asset for a disappointment averse investor is lower than the portfolio weight for an investor who has the standard expected utility. Especially when her wealth level is near the reference level, i.e. the certainty equivalent of the terminal wealth, the portfolio weight is extremely small, because she want to maintain her wealth level in gain-states rather than to let it down to loss-states. The optimal portfolio weight is highly dependent on the disappointment aversion of the investor. The portfolio weight is much smaller when the investor is disappointed more. This result partially explains the portfolio puzzle of Mankiw and Zeldes(1991).

On the other hand, the portfolio weight under the disappointment aversion model is changed among the time horizon, while it is constant under the CRRA utility model. I find that as the terminal time goes to infinity, investment strategies for disappointment averse investors go up to the strategies for the CRRA type investors.

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Appendix

A Proof of Theorem 4.1

Section 3.1 give us formulas for the optimal terminal wealth and intermediate wealth and method to find the optimal portfolio weight. Let $I(y)$ is the inverse of the marginal utility function, then

$$I(y) = y^{\frac{1}{\alpha-1}}, \quad \text{for all } \ y > 0$$

Also, λ is the unique solution of the following equation.

$$x = E[H_T I(\lambda H_T)] = \lambda^{\frac{1}{\alpha-1}} \exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\}$$

Therefore, $\lambda = x^{\alpha-1} \exp\{\alpha(r - \frac{\gamma^2}{2(\alpha-1)})T\}$. The optimal terminal wealth process is

$$X_t^* = \frac{1}{H_t} E[H_T I(\lambda H_T)|\mathcal{F}_t] = x\, H_t^{\frac{1}{\alpha-1}} \exp\{\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})t\}, \qquad \text{for any } \ t \in [0, T]$$

Since the stochastic differential equation evolved by X_t^* is

$$dX_t^* = (r - \frac{\gamma^2}{\alpha - 1})X_t^* dt - \frac{\gamma}{\alpha - 1}X_t^* dB_t,$$

we can obtain the optimal portfolio weight process $\pi^* = \frac{\gamma}{\sigma(\alpha-1)}$ by comparing the above equation to equation (2.2).

B Proof of Theorem 4.2

From the section 3.2, when $\theta = \mu(X_T^{x,\pi})$ is given, the optimal wealth at final time T is

$$X_T^{x,\pi^*(\theta)} = I^\theta(\mathcal{Y}^\theta(x)H_T)$$

where

$$I^\theta(y) = \begin{cases} y^{\frac{1}{\alpha-1}} & if\ y < \theta^{\alpha-1} \\ \theta & if\ \theta^{\alpha-1} \le y \le \frac{1}{A}\theta^{\alpha-1} \\ (Ay)^{\frac{1}{\alpha-1}} & if\ \frac{1}{A}\theta^{\alpha-1} < y \end{cases}$$

and $\mathcal{Y}^\theta(x)$ is the inverse function of $\mathcal{X}^\theta(\lambda)$ which is defined on $(0,\infty)$ by

$$\mathcal{X}^\theta(\lambda) = E[H_T I^\theta(\lambda H_T)]$$

We know that $H_T = \exp\{-(r + \frac{1}{2}\gamma^2)T + \gamma B_T\}, \gamma = \frac{\mu - r}{\sigma}$ has lognormal distribution, in that,

$$(r + \frac{1}{2}\gamma^2)\frac{\sqrt{T}}{\gamma} + \frac{1}{\gamma\sqrt{T}}\log H_T \sim Normal(0,1)$$

Therefore, we can obtain the explicit form of $\mathcal{X}^\theta(\lambda)$ for any $\lambda \in (0,\infty)$. When $N : (-\infty,\infty) \to [0,1]$ is the cumulative distribution function of standard normal distribution,

$$\begin{aligned} \mathcal{X}^\theta(\lambda) &= \theta e^{-rT}(N(\frac{1}{\gamma\sqrt{T}}\ln\lambda + \gamma\sqrt{T} + B_1(\theta)) - N(\frac{1}{\gamma\sqrt{T}}\ln\lambda + \gamma\sqrt{T} + B_2(\theta))) \quad \text{(B.1)} \\ &+ \lambda^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\}N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}}\ln\lambda - B_1(\theta)) \\ &+ \lambda^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\}A^{\frac{1}{\alpha-1}}N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln\lambda + B_2(\theta)) \end{aligned}$$

where $B_1(\theta) = -\frac{1}{\gamma\sqrt{T}}((r+\frac{1}{2}\gamma^2)T + (\alpha-1)\ln\theta),\ B_2(\theta) = B_1(\theta) + \frac{1}{\gamma\sqrt{T}}\ln A.$

Let $\lambda^*(\theta)$ be a solution of $\mathcal{X}^\theta(\lambda) = x$ in equation (B.1), in that $\lambda^*(\theta) = \mathcal{Y}^\theta(x)$. We cab calculate $\lambda^*(\theta)$ associated with $\theta \in (0,\infty)$ implicitly. Then the optimal wealth at the terminal time T is

$$X_T^{x,\pi^*(\theta)} = I^\theta(\lambda^*(\theta)H_T)$$

C Proof of Theorem 4.3

The optimal wealth at time $t \in [0,T]$ is

$$\begin{aligned} X_t^{x,\pi^*(\theta)} &= \frac{1}{H_t}E[H_T I^\theta(\lambda^*(\theta)H_T)|\mathcal{F}_t] \qquad (C.1) \\ &= \theta e^{-r(T-t)}\left(N(\frac{K_1}{\gamma\sqrt{T-t}}+\gamma\sqrt{T-t}) - N(\frac{K_2}{\gamma\sqrt{T-t}}+\gamma\sqrt{T-t})\right) \\ &\quad +(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)}N(-\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}-\frac{K_1}{\gamma\sqrt{T-t}}) \\ &\quad +(A\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)}N(\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}+\frac{K_2}{\gamma\sqrt{T-t}}) \end{aligned}$$

where $K_1 = \ln H_t + \ln \lambda^*(\theta) - (\alpha - 1)\ln\theta - (r + \frac{1}{2}\gamma^2)(T-t)$, $K_2 = K_1 + \ln A$. Furthermore, stochastic differential equation evolved by $X_t^{x,\pi^*(\theta)}$ is

$$
\begin{aligned}
& dX_t^{x,\pi^*(\theta)} \\
= \; & dt\Bigg[r\theta^{-r(T-t)}\left(N(\frac{K_1}{\gamma\sqrt{T-t}} + \gamma\sqrt{T-t}) - N(\frac{K_2}{\gamma\sqrt{T-t}})\right) \\
& + (r - \frac{\gamma^2}{\alpha-1})(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}r(T-t) + \frac{\alpha\gamma^2}{2(\alpha-1)^2}(T-t)} \\
& \quad \times \left(N(-\frac{K_1}{\gamma\sqrt{T-t}} - \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) + A^{\frac{1}{\alpha-1}}N(\frac{K_2}{\gamma\sqrt{T-t}} + \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t})\right) \\
& - \frac{\gamma\theta}{\sqrt{2\pi(T-t)}}e^{-(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - e^{-K_2 = \frac{K_2^2}{2\gamma^2(T-t)}}) \\
& + \frac{\gamma(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}}{\sqrt{2\pi(T-t)}} e^{-\frac{\alpha}{\alpha-1}(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-\frac{\alpha}{\alpha-1}K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A^{\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}K_2 - \frac{K_2^2}{2\gamma^2(T-t)}})\Bigg] \\
- \; & dB_t\Bigg[\frac{\gamma}{\alpha-1}(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}r(T-t) + \frac{\alpha\gamma^2}{2(\alpha-1)^2}(T-t)} \\
& \quad \times \left(N(-\frac{K_1}{\gamma\sqrt{T-t}} - \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) + A^{\frac{1}{\alpha-1}}N(\frac{K_2}{\gamma\sqrt{T-t}} + \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t})\right) \\
& + \frac{\theta}{\sqrt{2\pi(T-t)}}e^{-(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - e^{-K_2 - \frac{K_2^2}{2\gamma^2(T-t)}}) \\
& - \frac{(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}}{\sqrt{2\pi(T-t)}} e^{-\frac{\alpha}{\alpha-1}(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-\frac{\alpha}{\alpha-1}K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A^{\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}K_2 - \frac{K_2^2}{2\gamma^2(T-t)}})\Bigg]
\end{aligned}
$$

Since $dX_t^{x,\pi^*(\theta)} = (r + (\mu - r)\pi^*(\theta))X_t dt + \sigma\pi^*(\theta)X_t dB_t$, the optimal portfolio strategy $\pi^*(\theta)$ can be derived as

$$
\begin{aligned}
\pi^*(\theta) = \; & \frac{1}{\sigma X_t^{x,\pi^*(\theta)}}\Bigg[\frac{\gamma}{1-\alpha}(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}r(T-t) + \frac{\alpha\gamma^2}{2(\alpha-1)^2}(T-t)} \\
& \quad \times \left(N(-\frac{K_1}{\gamma\sqrt{T-t}} - \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) + A^{\frac{1}{\alpha-1}}N(\frac{K_2}{\gamma\sqrt{T-t}} + \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t})\right) \\
& - \frac{\theta}{\sqrt{2\pi(T-t)}}e^{-(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - e^{-K_2 - \frac{K_2^2}{2\gamma^2(T-t)}}) \\
& + \frac{(\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}}}{\sqrt{2\pi(T-t)}} e^{-\frac{\alpha}{\alpha-1}(r+\frac{1}{2}\gamma^2)(T-t)}(e^{-\frac{\alpha}{\alpha-1}K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A^{\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}K_2 - \frac{K_2^2}{2\gamma^2(T-t)}})\Bigg]
\end{aligned}
\tag{C.2}
$$

D Proof of Theorem 4.4

If we find the optimal θ satisfying the identity (3.5) in step 2 of section 3 and denote it by θ^*, then $X_t^{x,\pi^*(\theta^*)}$ and $\pi^*(\theta^*)$ are the optimal wealth process and the optimal portfolio process of the investor who have the disappointment averse utility on wealth, respectively. To find θ^*, we use equations (C.1) and (C.2) to rearrange the identity (3.5) as

$$\begin{aligned} 0 &= -\frac{1}{\alpha}\theta^\alpha + \frac{1}{\alpha}\theta^\alpha\Big(N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda^*) + B_1(\theta)) - \frac{1}{A}N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda^*) + B_2(\theta))\Big) \\ &\quad + \frac{1}{\alpha}(\lambda^*)^{\frac{\alpha}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\}N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}}\ln(\lambda^*) - B_1(\theta)) \\ &\quad + \frac{1}{\alpha}A^{\frac{1}{\alpha-1}}(\lambda^*)^{\frac{\alpha}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T\}N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln(\lambda^*) + B_2(\theta)) \end{aligned} \tag{D.1}$$

This equation is the result in Theorem 4.4. By solving the equation (D.1) with respect to θ, we can obtain the value of $\theta^* = \mu(X_T^{x,\pi^*})$.

Now, we will discuss about the existence and the uniqueness of θ^*. In this part, we only give the proof of that the solution of equation (D.1) exists and it is unique in the case of $\alpha \in (0,1]$. It is not yet complete for us to show the existence of θ^* for negative α. In following, we assume that α is positive.

Consider the first derivative of right hand side in equation (D.1) with respect to θ.

$$\begin{aligned} \frac{\partial RHS}{\partial\theta} &= -\alpha\theta^{\alpha-1} + \alpha\theta^{\alpha-1}\left(N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_1) - \frac{1}{A}N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)\right) \\ &\quad + \theta^\alpha\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\lambda^*\gamma\sqrt{T}}\frac{\partial\lambda^*}{\partial\theta} - \frac{\alpha-1}{\theta\gamma\sqrt{T}}\right)\left(e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_1)^2} - \frac{1}{A}e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)^2}\right) \\ &\quad + \frac{\alpha}{\alpha-1}\lambda^{*\frac{1}{\alpha-1}}\frac{\partial\lambda^*}{\partial\theta}e^{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T} \\ &\qquad \times\left(N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}}\ln\lambda^* - B_1) + A^{\frac{1}{\alpha-1}}N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)\right) \\ &\quad - \frac{\lambda^{*\frac{\alpha}{\alpha-1}}}{\sqrt{2\pi}}e^{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})T}\left(\frac{1}{\lambda^*\gamma\sqrt{T}}\frac{\partial\lambda^*}{\partial\theta} - \frac{\alpha-1}{\theta\gamma\sqrt{T}}\right) \\ &\qquad \times\left(e^{-\frac{1}{2}(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_1)^2} - A^{\frac{1}{\alpha-1}}e^{-\frac{1}{2}(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)^2}\right) \end{aligned} \tag{D.2}$$

Here, we can find an implicit equation for $\dfrac{\partial\lambda^*}{\partial\theta}$ by differentiating both sides of equation (4.3) with respect to θ.

$$\begin{aligned} 0 &= e^{-rT}\left(N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_1) - N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_2)\right) \\ &+ \frac{1}{\alpha-1}\lambda^{*\frac{2-\alpha}{\alpha-1}}\frac{\partial\lambda^*}{\partial\theta}e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T} \\ &\quad\times\left(N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}}\ln\lambda^* - B_1) + A^{\frac{1}{\alpha-1}}N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)\right) \\ &+ \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\lambda^*\gamma\sqrt{T}}\frac{\partial\lambda^*}{\partial\theta} - \frac{\alpha-1}{\theta\gamma\sqrt{T}}\right) \\ &\quad\left\{\times\theta e^{-rT}\left(e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}+B_1)^2} - e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}+B_2)^2}\right)\right. \\ &\quad\left.-\lambda^{*\frac{1}{\alpha-1}}e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T}\left(e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}+B_1)^2} - A^{\frac{1}{\alpha-1}}e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}+B_2)^2}\right)\right\} \end{aligned} \tag{D.3}$$

From equation (D.3), the equation (D.2) is represented as

$$\begin{aligned} \frac{\partial RHS}{\partial\theta} &= -\alpha\theta^{\alpha-1}\left(N(-\frac{1}{\gamma\sqrt{T}}\ln\lambda^* - B_1) + \frac{1}{A}N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)\right) \\ &\quad -\lambda^* e^{-rT}\left(N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_1) - N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_2)\right) \\ &\quad +\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\lambda^*\gamma\sqrt{T}}\frac{\partial\lambda^*}{\partial\theta} - \frac{\alpha-1}{\theta\gamma\sqrt{T}}\right)\left\{\theta^\alpha(e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+B_1)^2} - \frac{1}{A}e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+B_2)^2})\right. \\ &\qquad\left. - \lambda^*\theta e^{-rT}(e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}B_1)^2} - \frac{1}{A}e^{-\frac{1}{2}(\frac{1}{\gamma\sqrt{T}}\ln\lambda^*+\gamma\sqrt{T}B_2)^2})\right\} \\ &= -\alpha\theta^{\alpha-1}\left(N(-\frac{1}{\gamma\sqrt{T}}\ln\lambda^* - B_1) + \frac{1}{A}N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + B_2)\right) \\ &\quad -\lambda^* e^{-rT}\left(N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_1) - N(\frac{1}{\gamma\sqrt{T}}\ln\lambda^* + \gamma\sqrt{T} + B_2)\right) \end{aligned}$$

Therefore, for all $\theta > 0$, $\frac{\partial RHS}{\partial\theta}$ is negative so that right hand side of equation (D.1) is decreasing as θ increases. Moreover, when $\theta = 0$ the value of right hand side of equation (D.1) is $x^\alpha e^{\alpha(r-\frac{\gamma^2}{2(\alpha-1)})T} > 0$ and when θ goes to ∞ it goes to $-\infty$. Therefore, there exists a unique solution of equation (D.1).

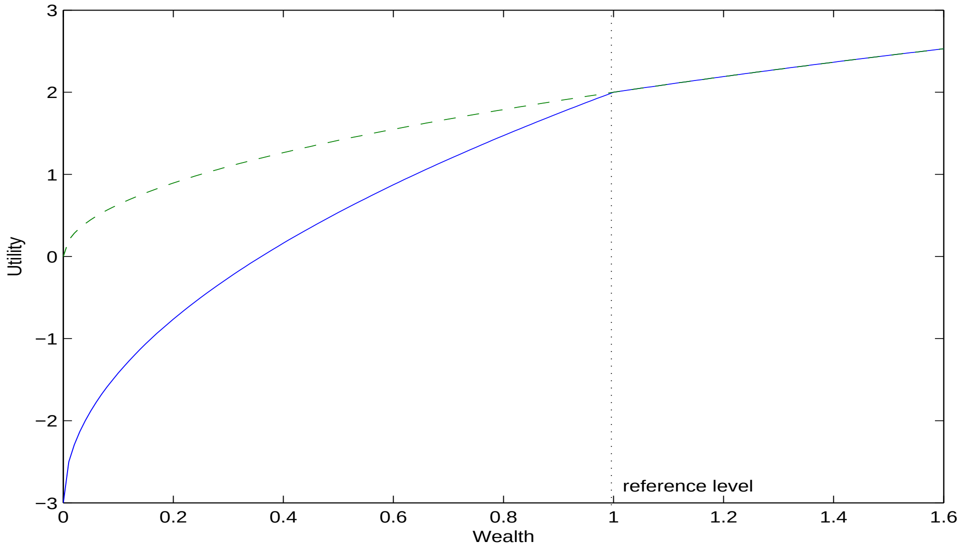

Figure 1: **Utility functions for wealth level**. The solid line is for the disappointment averse utility with $A = 0.4, \theta = 1$ and $u(x) = \frac{1}{0.5}x^{0.5}$ and the dashed line is for the smooth utility $u(x) = \frac{1}{0.5}x^{0.5}$.

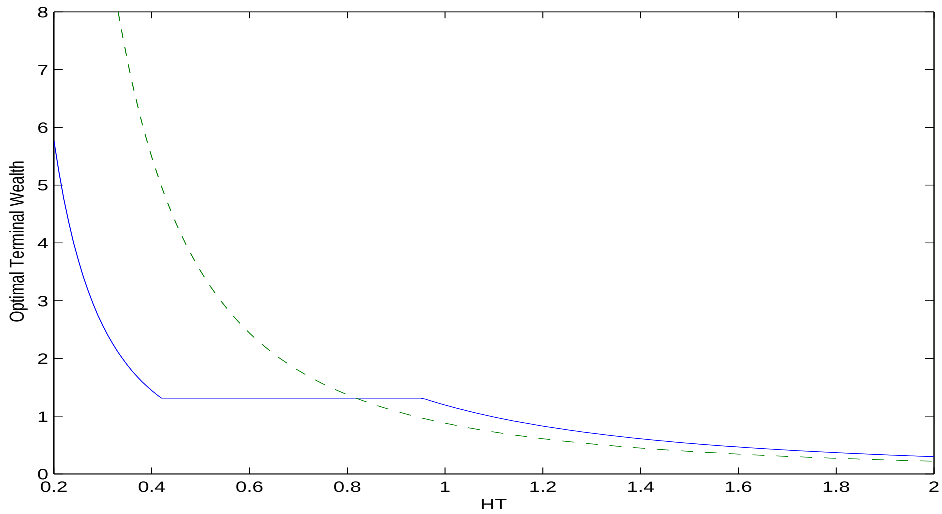

Figure 2: **The optimal terminal wealth among the realized value of pricing kernel H_T.** The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are $r = 0.0408, \mu = 0.1063$, and $\sigma = 0.2193$. Investor's initial wealth is $x = 1$ and length of life time is $T = 1$.

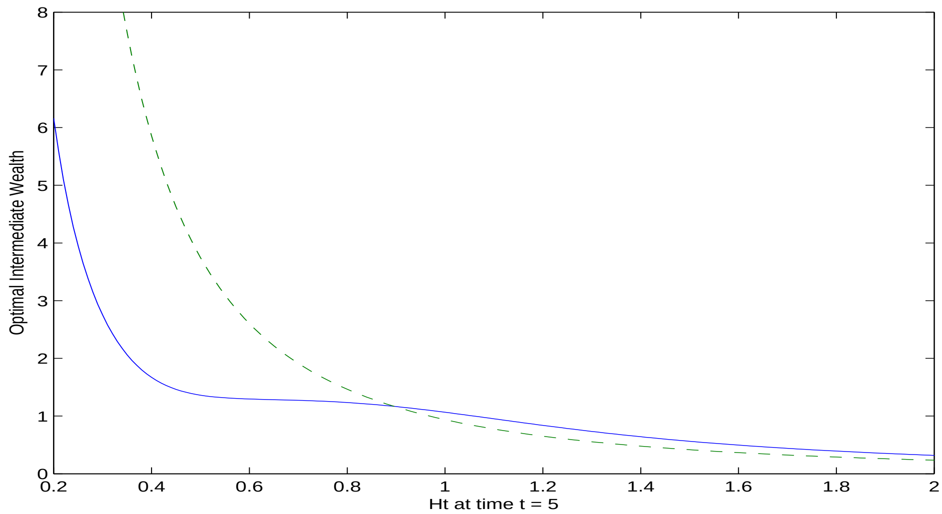

Figure 3: **The optimal intermediate wealth among H_t at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

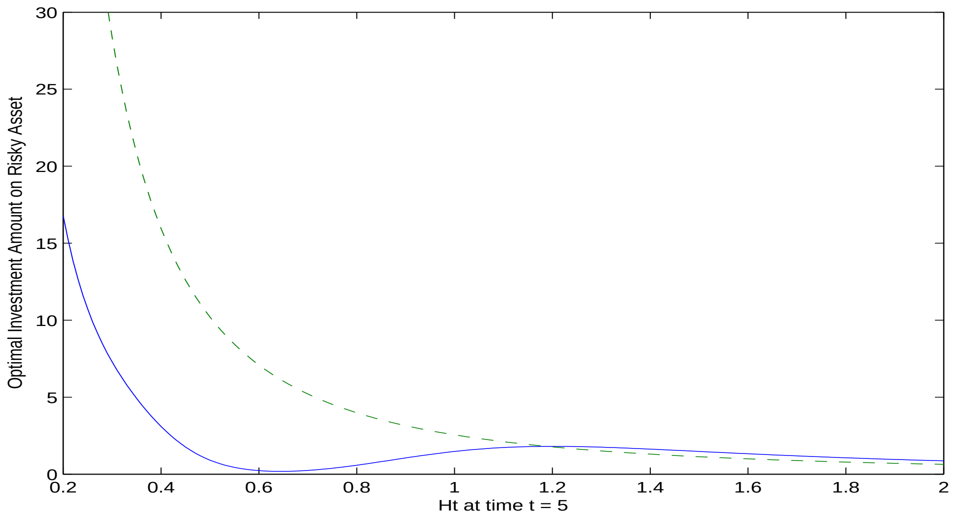

Figure 4: **The optimal portfolio amount among H_t at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

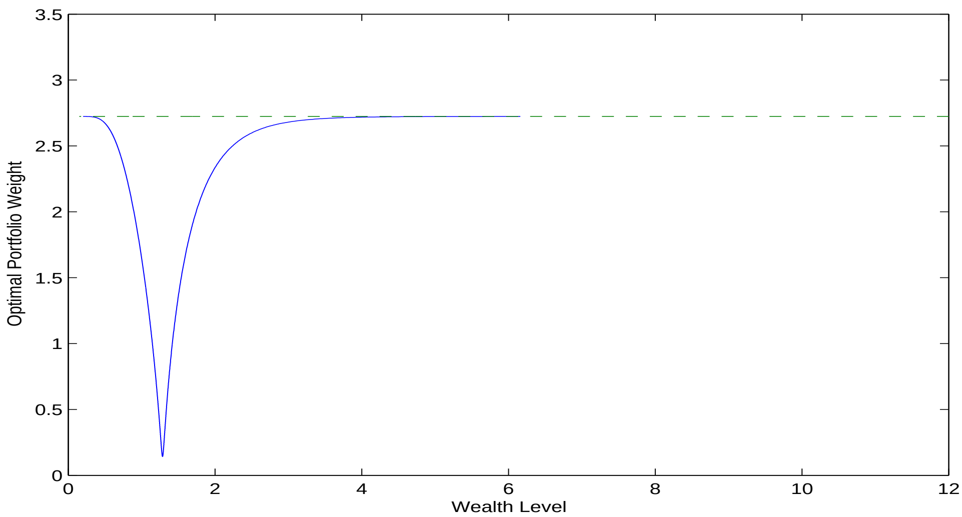

Figure 5: **The optimal portfolio weight among investor's wealth level at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

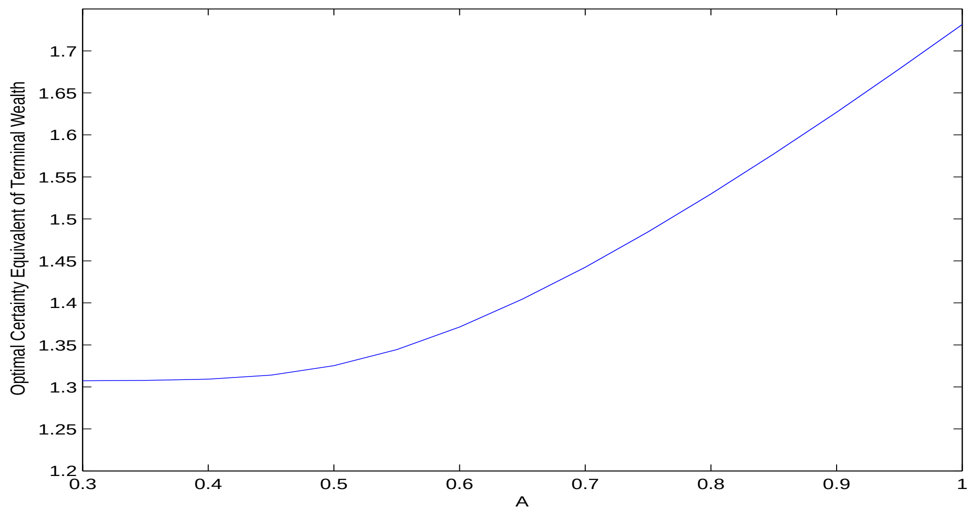

Figure 6: **The reference level $\mu(X_T^*)$ among disappointment aversion coefficient** A. We assume that all parameters except for A are as same as in figure 2.

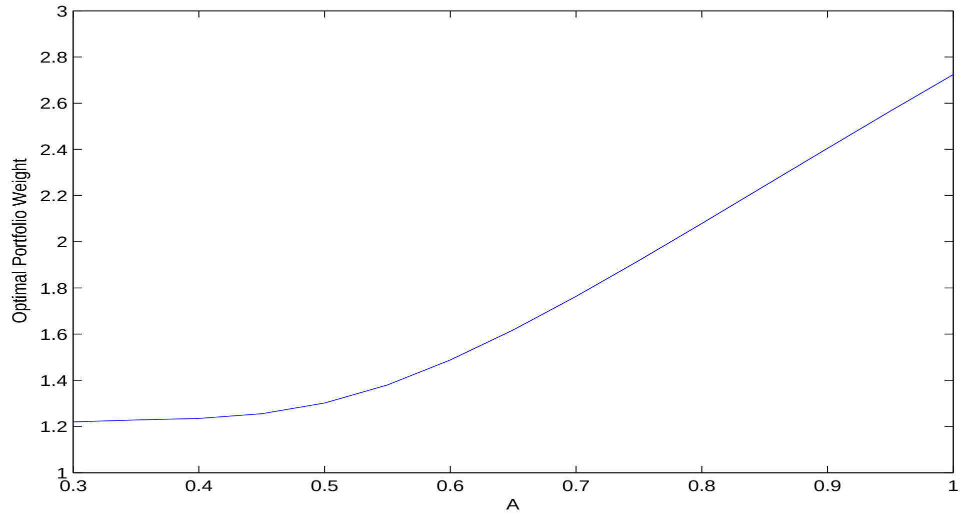

Figure 7: **The optimal initial portfolio wight among disappointment aversion coefficient** A. Initial wealth level is $x = 1$ and all parameters except for A are assumed as same as in figure 2.

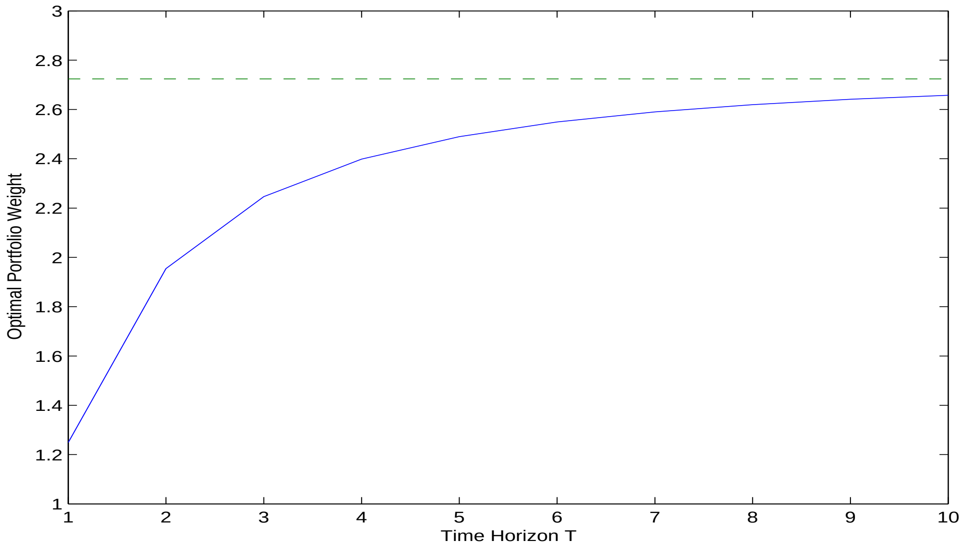

Figure 8: **The optimal initial portfolio weight among the length of life time T**. The disappointment aversion coefficient is $A = 0.44$ and relative risk aversion is $\alpha = 0.5$. The parameters are $r = 0.0408, \mu = 0.1063$, and $\sigma = 0.2193$. Investor's initial wealth is $x = 1$.