FDM Algorithm for pricing of ELS with Exit Probability

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1. Introduction

In this paper, we present a new finite difference method(FDM) for pricing the two-stock equity linked security(ELS) with knock-in barrier and early redemption conditions. Our new FDM algorithm consists of calculating the exit probability at each node and utilizing such probability with the FDM algorithm. In most ELS product in the market, barrier type structures are imbedded. Hence we first consider the standard barrier option to clarify the algorithm of utilizing the exit probability. After examining our method to the standard up and out call option, we apply our method to the two stock ELS product.

2. New F.D.M. algorithm for Up-and-Out Call

We define c(t,s) as the option value at time t and asset price s, where $t \in (0,2), s \in (0,4)$. The governing equation is $c_t - \frac{1}{2}\sigma^2 s^2 c_{ss} - rsc_s + rc = 0$ and its boundary condition is $c(t,0) = c(t,4) = 0$. And we assume the option parameters as follows: Exercise=1.0, Barrier=3.0, risk-free rate=0.05, Time to maturity=2 years, Rebate=0.0, div=0.0, volatility=0.2. It also has the following initial condition.

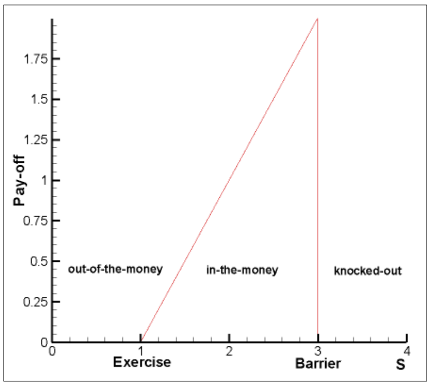


Figure 1. Initial Condition

We denote Ω as the computational domain and let $\Omega = \Omega_{KO} \cup \Omega_{ACTIVE}$, where $\Omega_{ACTIVE} = [0, B_U], B_U = 3.0$ and $\Omega_{KO} = \Omega \setminus \Omega_{ACTIVE}$. Then we implement the conventional FDM algorithm as follows:

1. $c^1(s)$: FDM solution with the initial condition $c^0(s)$

2. $f^1(s) = 1_{\{s \in \Omega_{ACTIVE}\}} c^1(s)$ **: Barrier checking process**

3. $c^{n+1}(s)$**: FDM solution at n+1 step from the force vector** $f^n(s)$

4. $f^{n+1}(s) = 1_{\{s \in \Omega_{ACTIVE}\}} c^{n+1}(s)$

Note that continuous monitoring for the underlying with respect to the upper barrier is performed in the conventional algorithm.

We add the probability of each nodes belonging to Ω_{ACTIVE} in the next time step, i.e., $\Pr\{s^{n+1} < B_U\}$ for $\forall s^n \in \Omega_{ACTIVE}$ to the conventional algorithm, since we know such probability is non-zero due to the assumption of Geometric Brownian Motion for the underlying process. We define Exit Probability for each node belonging to Ω_{ACTIVE} as the probability of hitting upper barrier at the next time step, denoted by $\Pr^{EXIT}(s, \triangle t)$, then we can calculate it for each node belonging to Ω_{ACTIVE} as follows:

From the assumption of asset dynamics, we have

$$S^{n+1} = S^n(1 + \mu \triangle t + \sigma \triangle B_t),$$

where μ:growth rate, σ:volatility, B_t:Brownian Motion

then, $\Pr\{S^{n+1} < B_U\} = \Pr\{Z < \frac{\frac{B_U}{S^n} - (1 + r\triangle t)}{\sigma \triangle t}\}$ for $\forall S \in \Omega_{ACTIVE}$, where $Z \sim N(0,1)$. Let $\alpha = \frac{\frac{B_U}{S^n} - (1 + r\triangle t)}{\sigma \triangle t}$. Then $\Pr\{S^{n+1} < B_U\} = \Phi(\alpha)$, where $\Phi(\cdot)$ is cumulative normal distribution function. Therefore, $\Pr^{EXIT}(s, \triangle t) = 1 - \Phi(\alpha_{s, \triangle t})$.

Now, we apply Exit probability to the conventional FDM algorithm as follows:

1. $c^1(s)$**: FDM solution with the initial condition** $c^0(s)$

2. $f^1(s) = \chi_{\{s \in \Omega_{ACTIVE}\}}(1 - \Pr^{EXIT}(s, \triangle t))c^1(s)$ **: Barrier checking process**

3. $c^{n+1}(s)$**: FDM solution at n+1 step from the force vector** $f^n(s)$

4. $f^{n+1}(s) = \chi_{\{s \in \Omega_{ACTIVE}\}}(1 - \Pr^{EXIT}(s, \triangle t))c^{n+1}(s)$

Here, we plot the numerical error calculated by comparing numerical solutions with closed-form solution for each node from those algorithms.

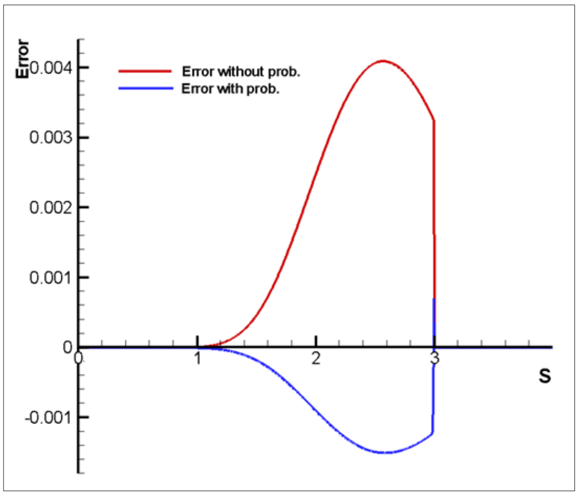


Figure 2. Error Plot

We can readily see in this plot that the absolute value of numerical error for each node from the new FDM algorithm is relatively smaller in general than that of the conventional FDM algorithm.

3. Its application to Reverse Convertible Note

In this section, we apply the new FDM algorithm to pricing of structured note, particularly, reverse convertible note. It is actually one of most popular types of ELS issued in Korea, and usually barrier option is imbedded in such type of ELS. Most of korean security firms has been issuing this type of ELS, and in this paper, we took one example of reverse convertible note issued by Kiwoom security firm.

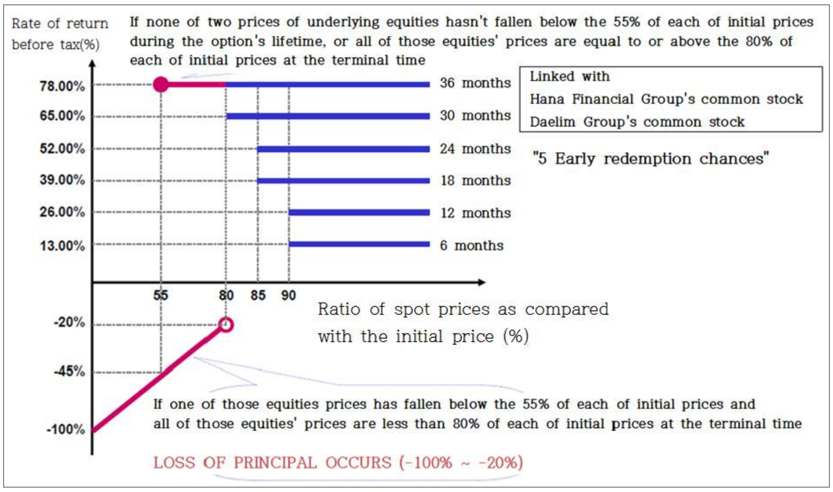
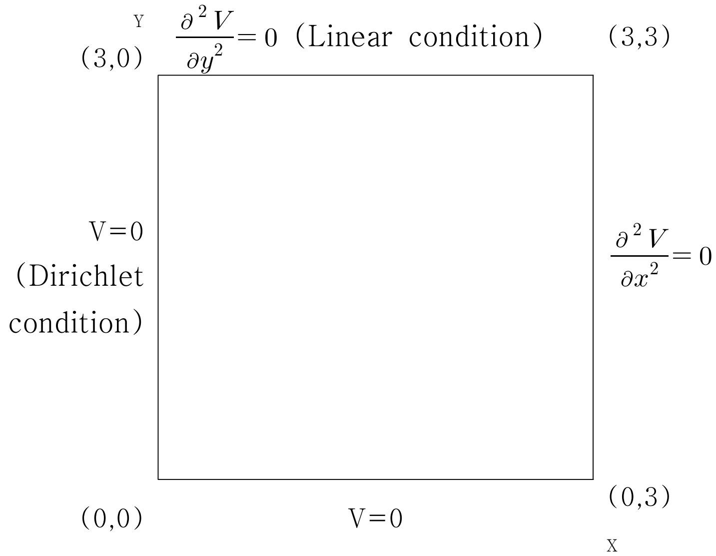


Figure 3. Specification of ELS Product

We define V(t,x,y) as the option value at time t and asset price x and y, where $t \in (0,3)$, $x \in (0,3)$, $y \in (0,3)$.

The governing equation is $V_t - \frac{1}{2}\sigma_x^2 x^2 V_{xx} - \frac{1}{2}\sigma_y^2 y^2 V_{yy} - \rho\sigma_x\sigma_y xy V_{xy} - r(xV_x + yV_y - V) = 0$ and its boundary condition is as follows:



And, its initial condition is as follows:

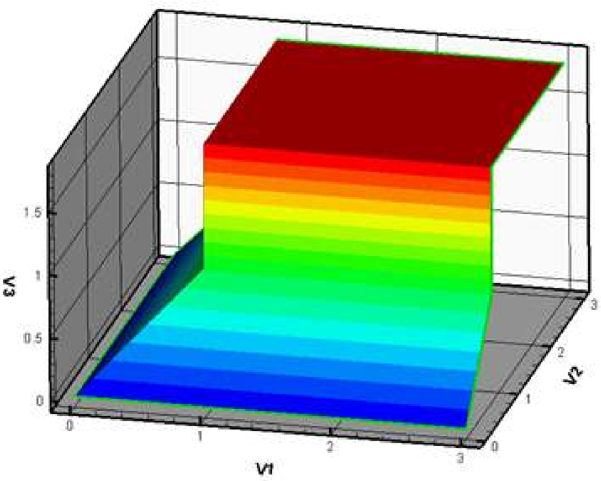


Figure 4. Knock-In event is assumed

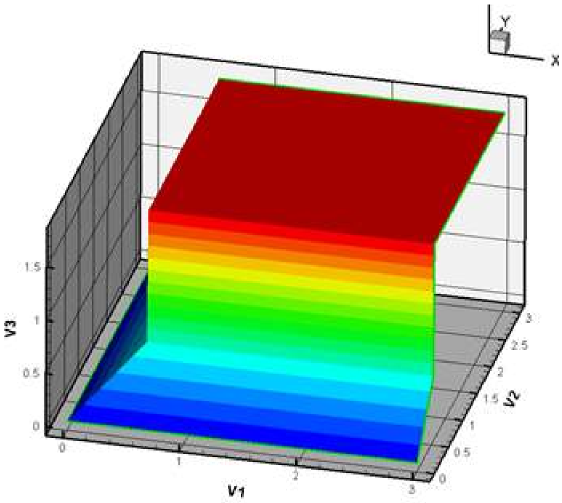


Figure 5. Knock-In event is not assumed

Note that it has two possible initial conditions due to the existence of imbedded down-in barrier put. When we implement FDM algorithm to price this product, we can conveniently regard an event of early redemption event as a knock-out event, and barrier checking process is discrete, since it has only 5 chance of early redemption during the lifetime of the note. Thus, we can think of it having 3 upper barriers and 1 lower barrier. According to the description of this product, we set the option parameters:

Upper Barrier1=0.8, Upper Barrier2=0.85, Upper Barrier3=0.9, Lower Barrier=0.55, Discrete Monitoring Time for Early Redemption=$0.5i, i=1,\ldots,5$, Time to maturity=3 years, div=0.0

We also assume the option parameters: risk-free rate=0.05, volatility=0.2, correlation coefficient=0.5

We denote $\Omega = [0,3] \times [0,3]$ as the computational domain and let $\Omega = \Omega_{KO} \cup \Omega_{NOKI}$, where $\Omega_{KI} = [0, B_D] \times [0, B_D]$, $B_D = 0.55$ and $\Omega_{NOKI} = \Omega \setminus \Omega_{KI}$. Then we implement the conventional FDM algorithm as follows:

1. $V_{KI}^1(x,y)$: **FDM solution with the initial condition** $V_{KI}^0(x,y)$

2. $V_{NOKI}^1(x,y)$: **FDM solution with initial condition** $V_{NOKI}^0(x,y)$

3. $V^1(x,y) = \chi_{\{(x,y) \in \Omega_{KI}\}} V_{KI}^1 + \chi_{\{(x,y) \in \Omega_{NOKI}\}} V_{NOKI}^1$

4. $V_{NOKI}^{n+1}(x,y)$: **FDM solution with previous time solution** V^n

5. $V^{n+1}(x,y) = \chi_{\{(x,y) \in \Omega_{KI}\}} V_{KI}^{n+1} + \chi_{\{(x,y) \in \Omega_{NOKI}\}} V_{NOKI}^{n+1}$

6. $V^{n+1}(x,y) = \chi_{\{(x,y) \geq (B_U, B_U)\}} ER_t + \chi_{\{(x,y) < (B_U, B_U)\}} V^{n+1}(x,y)$

: **Early Redemption Chance** $t = 0.5n, n = 1,\ldots,5 \, with \, ER_t = value \, of \, early redemption \, at \, t$

We add the probability of each set of nodes belonging to Ω_{NOKI} not to hit the knock-in barrier in the next time step, that is,

$$\Pr\{x^{n+1} > B_D \text{ and } y^{n+1} > B_D\} \text{ for } \forall \, (x^n, y^n) \in \Omega_{NOKI}$$

to the conventional algorithm. In this example, we define Exit Probability for each node belonging to Ω_{NOKI} as the probability of hitting lower barrier at the next time step, denoted by $\Pr^{EXIT}(x,y,\triangle t)$, then we can calculate it for each node belonging to Ω_{NOKI} as follows:

From the assumption of asset dynamics, we have

$$x^{n+1} = x^n (1 + \mu \triangle t + \sigma_1 \triangle B_t^1), \, y^{n+1} = y^n (1 + \mu \triangle t + \sigma_2 \triangle B_t^2),$$
$$where \, \frac{cov(\triangle B_t^1, \triangle B_t^2)}{\sqrt{Var(\triangle B_t^1)} \sqrt{Var(\triangle B_t^2)}} = \rho$$

where μ:growth rate, σ:volatility, B_t^1, B_t^2: Brownian Motions Then,

$$\Pr(x^{n+1} > 0.55) = \Pr(x^n(1 + \mu \triangle t + \sigma_1 \sqrt{\triangle t} Z_1) > 0.55)$$
$$= \Pr(\sigma_1 \sqrt{\triangle t} Z_1 > \frac{0.55}{x^n} - (1 + \mu \triangle t)) = \Pr(Z_1 > \frac{\frac{0.55}{x^n} - (1 + \mu \triangle t)}{\sigma_1 \sqrt{\triangle t}}) \, \text{for} \, each \, x^n > 0.55$$

Similarly, we have $\Pr(y^{n+1} > 0.55) = \Pr(Z_2 > \frac{\frac{0.55}{y^n} - (1 + \mu \triangle t)}{\sigma_2 \sqrt{\triangle t}}) \, \text{for} \, each \, y^n > 0.55$. If we let $\alpha_{x,\triangle t} = \frac{\frac{0.55}{x^n} - (1 + \mu \triangle t)}{\sigma_1 \sqrt{\triangle t}}, \beta_{y,\triangle t} = \frac{\frac{0.55}{y^n} - (1 + \mu \triangle t)}{\sigma_2 \sqrt{\triangle t}}$. Then for $\forall \, (x^n, y^n) \in \Omega_{NOKI}$, we have , $\alpha_{x,\triangle t} < 0, \beta_{y,\triangle t} < 0$. Thus, by symmetric property of cumulative bi-normal distribution function, we can get $\Pr(Z_1 > \alpha_{x,\triangle t}\ Z_2 > \beta_{y,\triangle t}) = \Pr(Z_1 < -\alpha_{x,\triangle t}\ Z_2 < -\beta_{y,\triangle t})$.

Therefore, $\Pr^{EXIT}(x,y,\triangle t) = CBND(-\alpha_{x,\triangle t}, -\beta_{y,\triangle t}, \rho)$.

Now, we apply Exit probability to the conventional FDM algorithm as follows:

1. $V_{KI}^1(x,y)$: **FDM solution with the initial condition** $V_{KI}^0(x,y)$

2. $V_{NOKI}^1(x,y)$: **FDM solution with initial condition** $V_{NOKI}^0(x,y)$

3. $V^1(x,y) = \chi_{\{(x,y)\in\Omega_{KI}\}} V_{KI}^1 + \chi_{\{(x,y)\in\Omega_{NOKI}\}} (1-\Pr^{EXIT}) V_{NOKI}^1$

4. $V_{NOKI}^{n+1}(x,y)$: **FDM solution with previous time solution** V^n

5. $V^{n+1}(x,y) = \chi_{\{(x,y)\in\Omega_{KI}\}} V_{KI}^{n+1} + \chi_{\{(x,y)\in\Omega_{NOKI}\}} (1-\Pr^{EXIT}) V_{NOKI}^{n+1}$

6. $V^{n+1}(x,y) = \chi_{\{(x,y)\geq(B_U,B_U)\}} ER_t + \chi_{\{(x,y)<(B_U,B_U)\}} V^{n+1}(x,y)$

: **Early Redemption Chance** $t = 0.5n, n = 1,...,5\, with\, ER_t = value\, of\, early\, redemption\, at\, t$

4.References

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