Robust Consumption and Portfolio Rules with a New State Variable*

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Preliminary and Incomplete. Please do not circulate.

Abstract

We find a robust consumption and portfolio rules for an investor with max-min type utility suggested by Golboa and Schmeidler (1989). Following Hansen, Sargent, Turmuhambetova, and Williams (2006) suggestions, we employ a new state variable, continuation entropy, as a measure of the magnitude of investors pessimism toward information loss of the distribution of risky asset return. Numerical results tell us that the optimal consumption and portfolio rules can change dramatically according to the change of the level of the continuation entropy.

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1 Introduction

During past several decades, optimal consumption and investment problems have been an essential area of research in Finance. This kind of research was pioneered by Merton (1969) and Samuelson (1969), and they find consumption and portfolio rules of optimal investors with a time-additive utility under the conditions of constant stock returns and volatilities. These conventional set-up usually assumes that investors can be fully aware of return distributions of risky assets. This assumption, however, stands in sharp contrast to evidence from several empirical studies in Economics and Psychology.

The classical optimal consumption and investment problem assumes returns and volatilities of assets are known, so it is sensitive to the choice of particular point estimates and likely to be exposed to estimation errors. Merton (1991) and Cochrane (1997) discuss the difficulty in estimating the expected return of risky assets, and Welch (2000) reports the absence of consensus among researchers in Finance concerning expected risk premium of stocks. Thus, some people say that the problem should be reformed to be robust in choosing parameters.

Meanwhile, investor's *ambiguity aversion*, which is distinct from her risk aversion, was introduced by Ellsberg (1961). He asserts that investors usually have a tendency that they are unwilling to take a bet they do not know the distribution of outcome. Hence, it is reasonable that investor's preference toward risky assets might violate the expected utility hypothesis and the ambiguity aversion should be included in modelling investor's utility. As a matter of fact, lots of researchers follow him and, thus, they utilize investor's ambiguity aversion characteristics to explain well-known economic anomalies.

Following his idea, Gilboa and Schmeidler (1989) suggest the atemporal max-min expected utility. They consider an investor selecting a robust model of financial markets in the sense that she chooses the worst-case scenario among certain possible scenarios during a certain time period. They assume that each scenario can be represented by a probability measure used in calculating the expected max-min utility. Along the line of their research, Anderson *et al.* (2003) and Hansen and Sargent (2001) define and classify two robust control problems relative to ambiguity aversion by exploiting the concept of *relative entropy*. They make use of the relative entropy as a distance measure between a reference model (or a reference probability measure) and another model. They consider an investor who wants to find optimal consumption and investment rules under the circumstances where (1) she can choose the worst model of underlying assets (or a probability measure) representing the worst scenario and (2) the candidates of the model are restricted (called the *constraint* robust control problem) or the relative entropy is contained in the max-min utility as a penalty term (called the *penalty* robust control problem). In particular, they assert that the latter type of robust problem can be considered as an optimal consumption and investment problem for a risk-sensitive investor.

The penalty robust control problem do not take root in the ground of the ambiguity aversion of Ellsberg (1961), as already addressed by some researchers such as Pathak (2007). Roughly speaking, it is established from the psychological fact that investors concern the statistics of the continuation value of economic risks which they can face. Therefore, investors in the problem are assumed to be economic agents who want to minimize the value of risks they face when they choose their optimal consumption and investment policies. Some important results are found along this line. Maenhout (2004) find that considering the famous Equity Premium Puzzle can be resolved by the assumption of model misspecification, and Uppal and Wang (2003) find that the Under-diversification Puzzle also can be resolved using it.

On the other hand, it is well-known that there is a link between the two robust control problems described above. Hansen and Sargent (2001) show that the Lagrange multiplier theory can connect the two robust control problems. They show that the Lagrange multiplier in the penalty robust control problem should be constant in order that the two problems are considered as the same. However, since Maenhout (2004) and Uppal and Wang (2003) employ Lagrange multiplier changing according to investor's value function, it is ambiguous how we can interpret their outstanding results in the view point of Ellsberg.

Recently, Hansen, Sargent, Turmuhambetova, and Williams (2006) (hereafter, HSTW) suggest several types of robust control problems, which can be regarded as zero-sum games of two players. In their models, one player wants to maximize her expected utility by constructing optimal consumption and investment strategy, and the other player prevents her from doing that by creating the worst market situation. We call the latter as an evil market maker. HSTW address that, if we employ a new suitable state variable, there would not be any problems (e.g. the dynamic inconsistency issue indicated by Chen and Epstein, 2002) to construct optimal plans for the two players at the initial time. Following HSTW's suggestion, we employ a *continuation entropy* as the new state variable appropriate for our analysis.

In this paper, we introduce a modified version of the constraint robust control problem with a new state variable in the HSTW context. This new state variable, continuation entropy, can make us measure the confidence level of investors at any time, and also make our problem dynamically consistent. By applying dynamic programming approach, we induce a Hamilton-Jacobi-Bellman equation which is exactly the same with HSTW's. The solution to the HJB equation has an intuitive form with a function with respect to the level of initial entropy, which can be calculated by a simple numerical scheme.

Numerical results tell us that the optimal consumption and risky investment can change dramatically according to the change of initial entropy level. This result implies that investor's optimal consumption and portfolio rules can be significantly affected by the magnitude of her pessimism toward information loss of the distribution of risky asset return in the future. In addition, we present an explicit formula for a risk free rate in the Lucas equilibrium model and provide some economic implications for the investors described in our paper.

This paper is organized as follows. In Section 2, we state our portfolio choice problem, and in Section 3 we solve the problem. In Section 4 we give some examples and in Section 5 some interesting results in the Lucas equilibrium asset pricing model are presented. At last, Section 6 concludes.

2 The Problem

A given Brownian motion $\{B_t\}$ induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We call this probability measure the '*reference measure*' or '*reference model*' and we assume it is obtained from the historical market data.

We consider a financial market with a riskless asset S_t^0 (or a bond) and a risky asset S_t (or a stock). The prices of the two assets are evolved by the following equations:

$$\begin{cases} dS_t^0 = rS_t^0 dt, \\ dS_t = \mu S_t dt + \sigma S_t dB_t, \end{cases}$$

where r is a risk-free interest rate, μ is the rate of return of the stock, and σ is the volatility of the stock. Let $c_t \geq 0$ and π_t be the consumption rate and the stock-to-wealth ratio of an individual, respectively.[1] Hence, the individual's wealth process is governed by

$$dw_t = [(r + \pi_t(\mu - r))w_t - c_t]dt + w_t \pi_t \sigma dB_t, \qquad w_0 = w. \tag{1}$$

Now, consider an equivalent probability measure $\mathbb{Q}$ as a '*perturbed measure*' or '*perturbed model*' under which the process

$$d\tilde{B}_t = dB_t - h_t dt, \qquad \forall t \geq 0 \tag{2}$$

is an Brownian motion with a $\mathcal{F}_t$-adapted process h_t.[2] Then, it is well known that the exponential martingale

$$z_t := \exp\left[\int_0^t h_u dB_u - \int_0^t \frac{|h_u|^2}{2} du\right], \qquad \forall t \geq 0,$$

is the Radon-Nikodym derivative of the perturbed probability measure $\mathbb{Q}_t$ with respect to the

[1] As technical conditions, c_t and π_t should be $\mathcal{F}_t$-adapted and satisfy the following conditions:

$$\int_0^\infty c_s ds < \infty \qquad \text{a.s.,} \quad \text{and}$$

$$\int_0^\infty |\pi_s|^2 ds < \infty \qquad \text{a.s..}$$

[2] According to HSTW, the perturbed measure should be absolutely continuous with respect to the reference measure because this mathematical condition captures the difficulty in distinguishing two models with samples of finite length.

reference probability measure $\mathbb{P}_t$, $d\mathbb{Q}_t/d\mathbb{P}_t$, where $\hat{\mathbb{P}}_t$ is the restriction of a given probability measure $\hat{\mathbb{P}}$ on $\mathcal{F}_t$.

To quantify the distance between the new probability measure $\mathbb{Q}$ and the reference measure $\mathbb{P}$, the relative entropy is defined by

$$\mathcal{R}(\mathbb{Q}) := \mathbb{E}\left[\delta \int_0^\infty \exp(-\delta t)\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\log\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}dt\right] = \mathbb{E}^h\left[\delta \int_0^\infty \exp(-\delta t)\log z_t dt\right],$$

which is the sum of the discounted entropy of $\mathbb{Q}_t$ with respect to $\mathbb{P}_t$,[3] where $\mathbb{E}[\cdot]$ and $\mathbb{E}^h[\cdot]$ are the expectations with respect to the probability measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. HSTW show that

$$\mathcal{R}(\mathbb{Q}) = \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{|h_t|^2}{2}dt\right],$$

by the definition of the Radon-Nikodym derivatives z_t. The last equality implies that the quantitative discrepancy between two models, $\mathbb{Q}$ and $\mathbb{P}$, at time t is $|h_t|^2/2$.

With this relative entropy, HSTW propose two groups of robust control problems: the penalty robust control problem and the constraint robust control problem. The second class of robust control problem is specified through the control problem:

$$v(w, \eta) = \max_{(c,\pi)\in\mathcal{A}(w)} \min_{\mathbb{Q}\in\mathcal{Q}(\eta)} \mathbb{E}^h\left[\int_0^\infty e^{-\delta t}U(c_t)dt\right],$$

subject to

$$dw_t = [(r + \pi_t(\mu - r + \sigma h_t))w_t - c_t]dt + w_t\pi_t\sigma d\tilde{B}_t, \qquad w_0 = w,$$

and where

$$\mathcal{A}(w) = \{(c_t, \pi_t)|w_t \geq 0 \quad \text{for all } t \geq 0\},$$
$$\mathcal{Q}(\eta) = \{\hat{\mathbb{Q}} \in \mathcal{Q}|\mathcal{R}(\hat{\mathbb{Q}}) \leq \eta\},$$

and $\mathcal{Q}$ is the set of all perturbed measures which is equivalent to the reference measure $\mathbb{P}$.

[3] In statistics and physics, 'entropy' is considered as a measure of uncertainty and is also used to measure discrepancy between prior and posterior probabilities in Bayesian statistics.

Although the constraint robust control problem is linked to Gilboa-Schmeidler's original idea in a natural way, HSTW are interested in the first class of robust control problems, the penalty problem, because the problem is more naturally expressed as an equivalent recursive problem where dynamic programming tools can be used. We propose a new type of the robust control problem which appears to be intuitively related to Gilboa-Schmeidler's idea and to be easily converted to a recursive problem.

Problem 2.1. *To find*

$$V(w,\eta) \triangleq \max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right],$$

subject to

$$\begin{cases} dw_t = [(r+\pi_t(\mu-r+\sigma h_t))w_t - c_t]dt + w_t\pi_t\sigma d\tilde{B}_t, & w_0 = w, \\ de_t = (\delta e_t - \frac{h_t^2}{2})dt + g_t d\tilde{B}_t, & e_0 = \eta, \end{cases} \tag{3}$$

and where

$$\begin{aligned} \mathcal{A}(w) &= \{(c_t,\pi_t)|w_t \geq 0 \quad \textit{for all } t \geq 0\}, \\ \mathcal{B}(\eta) &= \{(h_t,g_t)|e_0 = \eta \textit{ and } e_t \geq 0 \quad \textit{for all } t \geq 0\}. \end{aligned} \tag{4}$$

As HSTW does, we introduce a new state variable e_t, a *continuation entropy*,to cope up with the dynamic inconsistency problem[4]: as time unfolds, the minimizing agent in robust control is not allowed freely to choose anew from among the original time 0 potential probability distortions. We consider the continuation entropy as the measure of how the investor is confident of the reference measure at time t; the larger value of the continuation entropy, the less does the investor rely on the reference measure. Recall that the the entropy of $\mathbb{Q}_t$ with respect to $\mathbb{P}_t$ is $|h_t|^2/2$ and that the relative entropy is the sum of discounted entropy. This fact implies that the drift of the continuation entropy should be $\delta e_t - |h_t|^2/2$. Thus, the

[4] This problem concerns Chen and Epstein (2002) and Epstein and Schneider (2003).

dynamics for the continuation entropy is given by

$$de_t = (\delta e_t - \frac{|h_t|^2}{2})dt + g_t d\tilde{B}_t,$$

with a control variable g_t for the volatility term of e_t.

With the continuation entropy process which could be controlled, the robust control problem can be regarded as the two player zero-sum game in which there are two player: one player is maximizer and the other is minimizer. The maximizer player in this case, chooses the optimal consumption and portfolio weights to maximize her expected utility and the other player aims to impede the maximizing player. So the latter player is sometimes called evil agent.

3 The Solution

Before starting this section, we need the following assumption.

Assumption 3.1. *The value function V(w,e) in Problem 2.1 is finite.*

This assumption is delivered to the restrictions of the coefficients in state processes for the finite value function. With the new state variable e_t and CRRA utility function, the value function defined in Problem 2.1 under measurability conditions of c, π, g and h, can be written recursively as

$$V(t, w_t, e_t) = \max_{c,\pi} \min_{h,g} \mathbb{E}_t^h \left[\int_t^\tau e^{-\delta(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du + e^{-\delta(\tau-t)} V(\tau, w_\tau, e_\tau) \right], \quad 0 \leq t < \tau.$$

This is the dynamic programming principle stated in Fleming and Souganidis (1989) and similar proof is also applicable. This principle leads to the following Hamilton-Jacobi-Bellman(HJB) equation by taking $dt \to 0$ where $\tau = t + dt$.

Proposition 3.2. *The value function $V(w, e)$ in Problem 2.1 is the solution of the following*

HJB equation:

$$\delta V = \max_{c,\pi}\min_{g,h}\Big(\frac{c^{1-\gamma}}{1-\gamma} + V_w(w(r+\pi(\mu-r+\sigma h))-c) + V_e(\delta e - \frac{h^2}{2})$$
$$+\frac{1}{2}V_{ww}(\pi w\sigma)^2 + \frac{1}{2}V_{ee}g^2 + V_{ew}\pi w\sigma g\Big). \tag{5}$$

Proof. In the limit of $dt \to 0$, The recursive relation becomes

$$\max_{c,\pi}\min_{g,h}\mathbb{E}^h\left[(\frac{c^{1-\gamma}}{1-\gamma} - \delta V)dt + dV\right] = 0.$$

By Ito's formula, the differential of $V(w,e)$ evolves into

$$\begin{aligned} dV(w,e) &= V_w dw + V_e de + \frac{1}{2}V_{ww}(dw)^2 + \frac{1}{2}(de)^2 + V_{we}(dw)(de) \\ &= \Big(V_w(w(r+\pi(\mu-r+\sigma h))-c) + V_e(\delta e - \frac{h^2}{2}) + \frac{1}{2}V_{ww}(\pi w\sigma)^2 \\ &\quad + \frac{1}{2}V_{ee}g^2 + V_{we}\pi w\sigma g\Big)\, dt + (V_w(w_t\pi_t\sigma) + V_e g)\, d\widetilde{B}_t \end{aligned}$$

Since the drift term also should be zero after taking expectation, we have

$$\begin{aligned} \delta V = \max_{c,\pi}\min_{g,h} \quad &\Big(\tfrac{c^{1-\gamma}}{1-\gamma} + V_w(w(r+\pi(\mu-r+\sigma h))-c) + V_e(\delta e - \tfrac{h^2}{2}) \\ &+\tfrac{1}{2}V_{ww}(\pi w\sigma)^2 + \tfrac{1}{2}V_{ee}g^2 + V_{we}\pi w\sigma g\Big), \end{aligned}$$

so the proof is complete. □

The following remark tells the properties of the value function which help further analysis.

Proposition 3.3. *The value function defined in Problem 2.1 is strictly increasing and strictly concave in w. In addition, it is also decreasing and convex in η.*

Proof. By the property of the power utility the value function is trivially increasing in w.

For a given $\hat{w}$ and $\tilde{w}$, if $(\hat{c},\hat{\pi}) \in \mathcal{B}(\hat{w})$ and $(\tilde{c},\tilde{\pi}) \in \mathcal{B}(\tilde{w})$ are solutions to each wealth dynamics (3) with initial condition $w_0 = \hat{w}$ and $w_0 = \tilde{w}$, respectively, then $(\theta\hat{c} + (1-\theta)\tilde{c}, \theta\hat{\pi} + (1-$

$\theta)\tilde{\pi}) \in \mathcal{B}(w_\theta)$ is a solution to the wealth dynamics with initial condition $w_0 = w_\theta$ where $w_\theta = \theta\hat{w} + (1-\theta)\tilde{w}$. This leads to

$$
\begin{aligned}
V(w_\theta, \eta) = & \max_{c,\pi\in\mathcal{A}(w_\theta)} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\int_0^\infty \exp\left(-\delta t\right) \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \\
\geq & \max_{\hat{c},\hat{\pi}\in\mathcal{A}(\hat{w}),\tilde{c},\tilde{\pi}\in\mathcal{A}(\tilde{w})} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\int_0^\infty \exp\left(-\delta t\right) \frac{(\theta\hat{c}_t + (1-\theta)\tilde{c}_t)^{1-\gamma}}{1-\gamma} dt \right] \\
> & \max_{\hat{c},\hat{\pi}\in\mathcal{A}(\hat{w}),\tilde{c},\tilde{\pi}\in\mathcal{A}(\tilde{w})} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\theta \int_0^\infty \exp\left(-\delta t\right) \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} dt + (1-\theta) \int_0^\infty \exp\left(-\delta t\right) \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \\
\geq & \theta \cdot \max_{\hat{c},\hat{\pi}\in\mathcal{A}(\hat{w})} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\int_0^\infty \exp\left(-\delta t\right) \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} dt \right] \\
& \quad + (1-\theta) \cdot \max_{\tilde{c},\tilde{\pi}\in\mathcal{A}(\tilde{w})} \min_{h,g\in\mathcal{B}(\eta)} \mathbb{E}^h \left[\int_0^\infty \exp\left(-\delta t\right) \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \\
= & \theta V(\hat{w}, \eta) + (1-\theta) V(\tilde{w}, \eta),
\end{aligned}
$$

and this implies the concavity of w. The second inequality is induced by Jensen's inequality. For the second variable, η, we first note that for $\tilde{\eta} > \eta$, $\mathcal{B}(\tilde{\eta}) \supset \mathcal{B}(\eta)$. Then the decreasing property in η is easily shown. This decreasing property of $V(w, \cdot)$ implies the following problem is same to the original Problem 2.1:

Problem 3.4. *To find*

$$
V(w, \eta) = \max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0), e_0\leq\eta} \mathbb{E}^h \left[\int_0^\infty \exp(-\delta t) \frac{c_t^{1-\gamma}}{1-\gamma} dt \right],
$$

subject to

$$
\begin{cases}
dw_t = [(r + \pi_t(\mu - r + \sigma h_t))w_t - c_t]dt + w_t \pi_t \sigma d\tilde{B}_t, & w_0 = w, \\
de_t = (\delta e_t - \frac{h_t^2}{2})dt + g_t d\tilde{B}_t,
\end{cases}
$$

where

$$
\mathcal{A}(w) = \{(c_t, \pi_t) |\ w_t \geq 0 \quad \textit{for all } t \geq 0\},
$$

$$
\mathcal{B}(e_0) = \{(h_t, g_t) |\ e_t \geq 0 \quad \textit{for all } t \geq 0, \quad \textit{with a given } e_0\}.
$$

To show the convexity, Let $\eta_\theta = \theta\hat{\eta} + (1-\theta)\tilde{\eta}$.

$$
\begin{aligned}
V(w,\eta_\theta) =& \max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(\eta_\theta)} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \\
=& \max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0),e_0\le\eta_\theta} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \\
=& \max_{\lambda\in\mathbb{R}_+}\left[\max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0),e_0} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] + \lambda(e_0-\eta_\theta)\right] \\
=& \max_{\lambda\in\mathbb{R}_+}\left[\max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0),e_0} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] + \theta\lambda(e_0-\hat{\eta}) + (1-\theta)\lambda(e_0-\tilde{\eta})\right] \\
\le& \theta\max_{\lambda\in\mathbb{R}_+}\left[\max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0),e_0} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] + \lambda(e_0-\hat{\eta})\right] \\
&+ (1-\theta)\max_{\lambda\in\mathbb{R}_+}\left[\max_{c,\pi\in\mathcal{A}(w)} \min_{h,g\in\mathcal{B}(e_0),e_0} \mathbb{E}^h\left[\int_0^\infty \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] + \lambda(e_0-\tilde{\eta})\right] \\
=& \theta V(w,\hat{\eta}) + (1-\theta)V(w,\tilde{\eta})
\end{aligned}
$$

This completes the proof. □

From now on, the optimal controls as well as the value function are determined from the HJB equation (5). The first order conditions (FOCs) for h and g yield, respectively, the optimal h^* and g^*:

$$h^* = \frac{V_w}{V_e}(\pi w\sigma), \quad g^* = -\frac{V_{ew}}{V_{ee}}(\pi w\sigma).$$

Then substituting these optimal values into the HJB equation (5) leads to the following reduced HJB equation as

$$\delta V = \max_{c,\pi}\Big(\frac{c^{1-\gamma}}{1-\gamma} + w(r+\pi(\mu-r+\sigma\frac{V_w(\pi w\sigma)}{V_e}))V_w - cV_w + \delta e V_e + \frac{(\pi w\sigma)^2}{2}\Big(\frac{V_w^2}{V_e} + V_{ww} - \frac{V_{ew}^2}{V_{ee}}\Big)\Big).$$

The FOCs for c and π also provide us with the optimal consumption and investment strategy,

$$c^* = (V_w)^{-\frac{1}{\gamma}}, \quad \pi^* = -\frac{V_w(\mu-r)}{w\sigma^2(V_{ww} + \frac{V_w^2}{V_e} - \frac{V_{ew}^2}{V_{ee}})}.$$

Then the value function $V(w,e)$ in (5) should be a solution of the following PDE (partial differential equation)

$$\delta V = \frac{\gamma}{1-\gamma}(V_w)^{-\frac{1-\gamma}{\gamma}} + wrV_w + \delta e V_e - \frac{V_w^2(\mu - r)^2}{2\sigma^2(V_{ww} + \frac{V_w^2}{V_e} - \frac{V_{ew}^2}{V_{ee}})}. \tag{6}$$

It is impossible to solve this PDE explicitly but a certain conjectural form help us to reduce the complexity. The next theorem is our main theorem in which the value function is represented.

Theorem 3.5. *The value function $V(w, e)$ is of the form*

$$V(w, e) = \frac{1}{1-\gamma} f(e) w^{1-\gamma}. \tag{7}$$

Here, $f(\cdot)$ satisfies

$$\frac{\delta}{1-\gamma} f(e) = \frac{\gamma}{1-\gamma} f(e)^{-\frac{1-\gamma}{\gamma}} + rf(e) + \frac{\delta e}{1-\gamma} f'(e) - \frac{(\mu - r)^2 f(e)^2}{2\sigma^2 \Big(-\gamma f(e) + (1-\gamma)\big(\frac{f(e)^2}{f'(e)} - \frac{f'(e)^2}{f''(e)}\big)\Big)}, \tag{8}$$

with conditions

$$f(0) = \Big(\delta + (\gamma - 1)r + \frac{(\gamma-1)(\mu - r)^2}{2\gamma\sigma^2}\Big)^{-\gamma} \cdot \gamma^\gamma, \quad \lim_{e\to\infty} f(e) = (\delta + (\gamma - 1)r)^{-\gamma} \cdot \gamma^\gamma.$$

Proof. The ordinary differential equation(ODE) for $f(e)$ can be derived by putting the conjectured form (7) into (6).

'$e = 0$' implies that the problem in the paper is converted into a problem with no gap between the reference measure $\mathbb{P}$ and new measure $\mathbb{Q}$, that is, the problem is equivalent to the Merton's (1969) problem. This fact yields the first condition.

The second condition for the case where $e = \infty$ is derived through the following idea: The infinity gap between reference measure and new measure implies that the constraint (4) is useless anymore. Thus the problem for this case is to find the value function

$$\max_{c,\pi} \min_{h} \mathbb{E}^h \left[\int_0^\infty \exp(-\delta t) \frac{c_t^{1-\gamma}}{1-\gamma} dt \right],$$

subject to the wealth process (3). Then the HJB equation should be

$$\delta V = \max_{c,\pi} \min_{h} \Big(\frac{c^{1-\gamma}}{1-\gamma} + w(r + \pi(\mu - r + \sigma h))V_w - cV_w + \pi w \sigma h V_w + \frac{1}{2}(\pi w \sigma)^2 V_{ww} \Big).$$

Since h is unbounded for this case, the optimal portfolio should be zero, namely, $\pi^* = 0$ if there exists a solution of the HJB equation. Thus the equation can be rewritten

$$\delta V = \max_{c} \Big(\frac{c^{1-\gamma}}{1-\gamma} + wrV_w - cV_w \Big),$$

and if we guess the value function to be

$$V(w) = \frac{\tilde{M}}{1-\gamma} w^{1-\gamma},$$

then $\tilde{M}$ is the solution to the following equation:

$$\frac{\delta \tilde{M}}{1-\gamma} = \frac{\gamma}{1-\gamma} \tilde{M}^{-\frac{1-\gamma}{\gamma}} + r\tilde{M}.$$

Thus the second condition is derived form the fact of

$$\lim_{e \to \infty} f(e) = \tilde{M}.$$

□

Since the ODE (8) also doesn't have closed-form solution, numerical task is required.

4 Examples

In this section we investigate how the investor's initial confidence level on the reference model, the initial continuation entropy value e_0, affects the optimal consumption and investment strategy. With the conjectural form in Theorem 3.5, the function $f(e)$ is shown by Figure 1 when $\gamma = 2, \mu = 0.15, r = 0.1, \delta = 0.1$, and $\sigma = 0.2$.[5] This says that the value

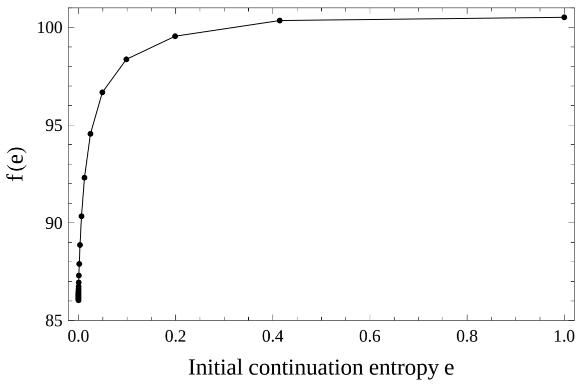

Figure 1: *The function $f(e)$ defined in the value function of Theorem 3.5.*

function decreases in the relative entropy and if the relative entropy is large enough, the value function converges.

The optimal consumption and portfolio can also be written using the function $f(e)$ above. The optimal consumption and portfolio are rewritten as

$$c^*(e, w) = (V_w)^{-\frac{1}{\gamma}} w = (f(e))^{-\frac{1}{\gamma}} w,$$

and

$$\pi^*(e) = \frac{\mu - r}{\gamma\sigma^2} \left(\frac{f(e)}{f(e) - \frac{(1-\gamma)f^2(e)}{\gamma f'(e)} - \frac{(1-\gamma)f'(e)^2}{\gamma f''(e)}} . \right)$$

Since the portfolio is defined by the rate of the wealth, the optimal portfolio rate only depends on the relative entropy. The optimal consumption rate also can be a function of the relative entropy only. Figure 2 and 3 shows the optimal consumption and investment strategy respectively with respect to the relative entropy. The thin lines in each figures 2 and 3 are

[5] We refer Jang *et al.* (2008) for the reasonable parameters.

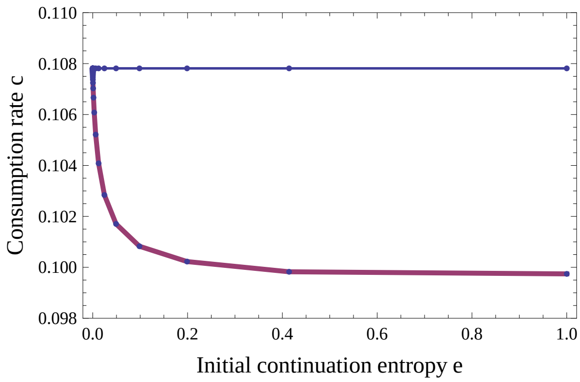

Figure 2: *The change of optimal consumption rate depending on the relative entropy. The blue line represents Merton's solution.*

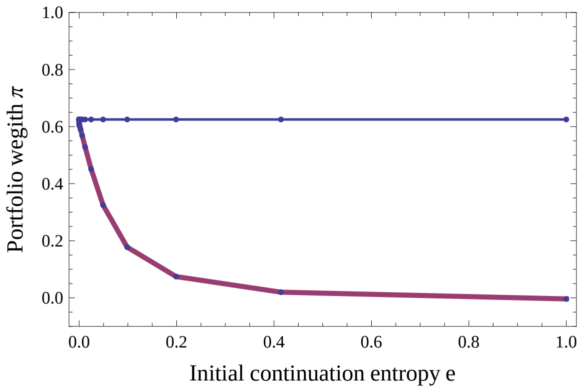

Figure 3: *The change of optimal portfolio depending on the relative entropy. The blue line represents Merton's solution*

the optimal consumption and portfolio rate of the classical Merton's problem. When there is ambiguity in the market, the consumption and portfolio are lower than those of the classical Merton's problem remarkably. Especially in the case of optimal portfolio, the agent reduces his investment to almost zero. We need to verify the appropriate value of the relative entropy in the financial market.

The other optimal controls for minimization are represented by

$$h^*(e,w) = \frac{V_w}{V_e}(\pi w \sigma) = (1-\gamma)\frac{f(e)}{f'(e)}\pi\sigma,$$

and

$$g^*(e,w) = -\frac{V_{ew}}{V_{ee}}(\pi w \sigma) = -(1-\gamma)\frac{f'(e)}{f''(e)}\pi\sigma.$$

Surprisingly, the optimal controls h^* and g^* are independent of the agent's wealth w_t and only depends on the relative entropy. Figure 4 and 5 describe the optimal controls as the function of the relative entropy.

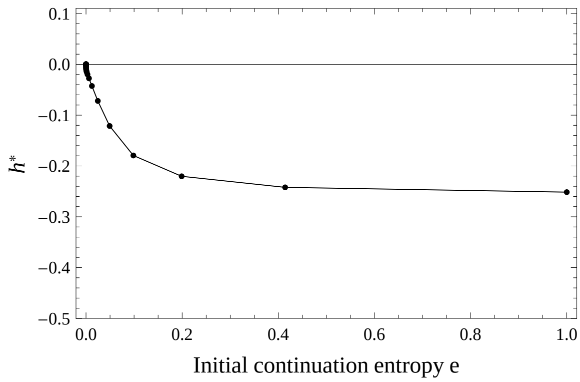

Figure 4: *The change of optimal control h^* depending on the relative entropy.*

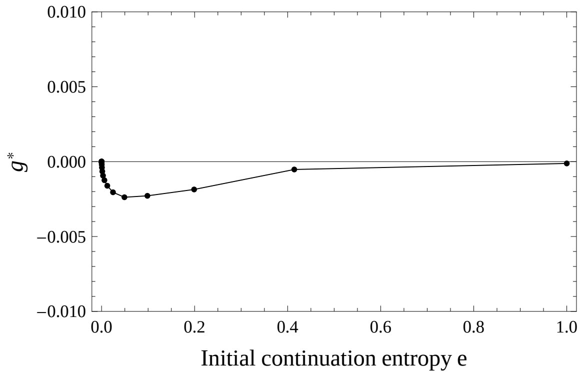

Figure 5: *The change of optimal control g^* depending on the relative entropy.*

5 Equilibrium Asset Pricing

As Meanhout (2004), let's think about the pure exchange economy of Lucas (1978). The dividend or endowment process which is given to the representative agent is assumed to follow the geometric Brwonian motion such as

$$dD_t = \mu_d D_t dt + \sigma_d D_t dB_t.$$

The representative agent can invest into two assets which are one risky asset S_t and a bond with interest rate r. Then her gain process is also assumed to evolve

$$\frac{dS_t + D_t dt}{S_t} = \mu_s dt + \sigma_s dB_t.$$

Therefore the representative agent's wealth process is given by

$$dW_t = \left[(r + \pi_t(\mu_s - r))W_t - c_t\right] dt + \pi_t W_t \sigma_s dB_t,$$

so that with the conjectural form in Theorem 3.5, the optimal consumption and investment strategy are obtained from

$$c_t^* = (V_w)^{-\frac{1}{\gamma}} = f^{-\frac{1}{\gamma}}(e) W_t, \tag{9}$$

and

$$\pi_t^* = \frac{(\mu_s - r)}{\sigma_s^2 \left\{\gamma + (1-\gamma)\left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)}\right)\right\}}. \tag{10}$$

Now we define the equilibrium of pure exchange economy.

Definition 5.1. *The equilibrium* (c^*, π^*, S_t, r) *is defined by the following two conditions:*

(1) $c_t^* = D_t^*$,

(2) $\pi_t^* = 1$.

In the pure exchange economy, the initial endowment or dividend is perishable so that it should be consumed. Furthermore, since there exists only one risky asset, the total rate of money invested in the risky asset should equal to one. We can find the price of risky asset and the equilibrium interest rate endogenously.

Proposition 5.2. *In the equilibrium, the excess return is governed by*

$$\frac{dS_t + D_t dt}{S_t} - rdt = \left(\frac{f'(e)}{\gamma f(e)} g + \sigma_d\right)^2 (\gamma + \theta(e))\, dt + \left(\frac{f'(e)}{\gamma f(e)} g + \sigma_d\right) dB_t, \tag{11}$$

and the equilibrium interest rate is determined from

$$r = \mu_d + (1 + P(e) + Q(e)\sigma_d)\, f(e)^{-\frac{1}{\gamma}} - \left(\frac{f'(e)}{\gamma f(e)} g + \sigma_d\right)^2 (\gamma + \theta(e))\,, \tag{12}$$

where $P(e)$ and $Q(e)$ are obtained from (14) and the function $\theta(e)$ is defined by

$$\theta(e) = (1-\gamma)\left(\frac{f'(e)^2}{f(e) f''(e)} - \frac{f(e)}{f'(e)}\right).$$

Proof. The equilibrium conditions imply that $W_t = S_t$ at equilibrium because with $c_t^* = D_t$ and $\pi_t^* = 1$ induce the following dynamics

$$\frac{dW_t + D_t dt}{W_t} = \mu_s dt + \sigma_s dB_t.$$

With the optimal controls (9) and (10) in equilibrium, the wealth process is rewritten as

$$\begin{aligned} dW_t = &\left[\left(r + \frac{f(e)(\mu_s - r)^2}{\left\{\gamma f(e) + (1-\gamma)\left(\frac{f'(e)^2}{f''(e)} - \frac{f(e)^2}{f'(e)}\right)\right\}\sigma_s^2}\right) W_t - f(e)^{-\frac{1}{\gamma}} W_t\right] dt \\ &+ \left(\frac{f(e)(\mu_s - r) W_t}{\left\{\gamma f(e) + (1-\gamma)\left(\frac{f'(e)^2}{f''(e)} - \frac{f(e)^2}{f'(e)}\right)\right\}\sigma_s}\right) dB_t \end{aligned}$$

Again, by the market clearing condition $\pi_t^* = 1$, we have

$$\mu_s - r = \sigma_s^2 \left(\gamma + (1-\gamma)\left(\frac{f'(e)^2}{f(e) f''(e)} - \frac{f(e)}{f'(e)}\right)\right),$$

so that the wealth process becomes

$$\frac{dW_t}{W_t} = \left\{ r + \sigma_s^2 \left(\gamma + (1-\gamma) \left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)} \right) \right) - f(e)^{-\frac{1}{\gamma}} \right\} dt + \sigma_s dB_t.$$

This implies that the excess return on the gain process is derived by

$$\frac{dS_t + D_t dt}{S_t} - rdt = \sigma_s^2 \left(\gamma + (1-\gamma) \left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)} \right) \right) dt + \sigma_s dB_t. \tag{13}$$

Furthermore, since $c_t^* = D_t = f(e)^{-\frac{1}{\gamma}} S_t$ in equilibrium, by Ito's formula, the dynamics of the risky asset S_t is determined from

$$dS_t = d\left(f(e)^{\frac{1}{\gamma}} D_t \right) = d\left(f(e)^{\frac{1}{\gamma}} \right) D_t + f(e)^{\frac{1}{\gamma}} dD_t + d\left(f(e)^{\frac{1}{\gamma}} \right) dD_t.$$

With the relation

$$\begin{aligned}
d\left(f(e)^{\frac{1}{\gamma}} \right) &= \frac{1}{\gamma} f(e)^{\frac{1-\gamma}{\gamma}} f'(e) de_t + \frac{1}{2} \left(\frac{1-\gamma}{\gamma^2} f(e)^{\frac{1-2\gamma}{\gamma}} f'(e)^2 + \frac{1}{\gamma} f(e)^{\frac{1-\gamma}{\gamma}} f''(e) \right) (de_t)^2 \\
&= \left\{ \frac{1}{\gamma} f(e)^{\frac{1-\gamma}{\gamma}} f'(e)(\delta e - \frac{h^2}{2} - gh) + \frac{1}{2} \left(\frac{1-\gamma}{\gamma^2} f(e)^{\frac{1-2\gamma}{\gamma}} f'(e)^2 + \frac{1}{\gamma} f(e)^{\frac{1-\gamma}{\gamma}} f''(e) \right) g^2 \right\} dt \\
&\qquad + \frac{1}{\gamma} f(e)^{\frac{1-\gamma}{\gamma}} f'(e) g dB_t \\
&\triangleq P(e)dt + Q(e)dB_t, \qquad (14)
\end{aligned}$$

the risky asset evolves

$$\begin{aligned}
dS_t &= P(e)D_t dt + Q(e)D_t dB_t + f(e)^{\frac{1}{\gamma}} \mu_d D_t dt + f(e)^{\frac{1}{\gamma}} \sigma_d D_t dB_t + Q(e) D_t \sigma_d dt \\
&= \left(P(e) + f(e)^{\frac{1}{\gamma}} \mu_d + Q(e)\sigma_d \right) D_t dt + \left(Q(e) + f(e)^{\frac{1}{\gamma}} \sigma_d \right) D_t dB_t,
\end{aligned}$$

so that

$$\frac{dS_t}{S_t} = \left(P(e) + f(e)^{\frac{1}{\gamma}} \mu_d + Q(e)\sigma_d \right) f(e)^{-\frac{1}{\gamma}} dt + \left(Q(e) + f(e)^{\frac{1}{\gamma}} \sigma_d \right) f(e)^{-\frac{1}{\gamma}} dB_t.$$

Since $W_t = S_t$ at the equilibrium, the simple comparison gives

$$r + \sigma_s^2 \left(\gamma + (1-\gamma) \left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)} \right) \right) - f(e)^{-\frac{1}{\gamma}} = \left(P(e) + f(e)^{\frac{1}{\gamma}} \mu_d + Q(e)\sigma_d \right) f(e)^{-\frac{1}{\gamma}},$$

$$\sigma_s = Q(e) f(e)^{-\frac{1}{\gamma}} + \sigma_d.$$

Then the equilibrium interest rate is characterized by

$$\begin{aligned} r &= \mu_d + (1 + P(e) + Q(e)\sigma_d) f(e)^{-\frac{1}{\gamma}} - \sigma_s^2 \left(\gamma + (1-\gamma) \left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)} \right) \right) \\ &= \mu_d + f(e)^{-\frac{1}{\gamma}} + \left\{ \frac{f'(e)}{\gamma f(e)} (\delta e - \frac{h^2}{2} - gh) + \frac{1}{2} \left(\frac{(1-\gamma) f'(e)^2}{\gamma^2 f(e)^2} + \frac{f''(e)}{\gamma f(e)} \right) g^2 + \frac{f'(e)}{\gamma f(e)} g \sigma_d \right\} \\ &\quad - \left(\frac{f'(e)}{\gamma f(e)} g + \sigma_d \right)^2 \left(\gamma + (1-\gamma) \left(\frac{f'(e)^2}{f(e)f''(e)} - \frac{f(e)}{f'(e)} \right) \right) \qquad (15) \end{aligned}$$

We complete the proof. □

6 Conclusion

We formulate a new version of the constraint robust control problem with a continuation entropy, which stands for the investor's confidence level with respect to the reference model at a certain time. Applying the dynamic programming approach, we obtain an intuitive form of the value function under our model, which can be calculated by an iterative numerical method. The numerical results tell us that the optimal consumption and risky investment can change dramatically according to the change of initial continuation entropy level. Therefore, we can conclude that investor's optimal consumption and portfolio rules can be significantly affected by the magnitude of her pessimism toward information loss of the distribution of risky asset return in the future. In addition, as Maenhout (2004) does, we present an explicit formula for a risk free rate in the Lucas equilibrium model.

Appendix

A Log Utility Case

Log utility function is a subclass of CRRA utility function with risk aversion $\gamma = 1$. Problem 2.1 is replaced by

Problem A.1. *To find*

$$V(w,\eta) \triangleq \max_{c,\pi} \min_{h,g} \mathbb{E}^h \left[\int_0^\infty \exp(-\delta t) \log c_t dt \right],$$

subject to

$$\begin{cases} dw_t = \left[(r + \pi_t(\mu - r + \sigma h_t))w_t - c_t \right] dt + w_t \pi_t \sigma d\widetilde{B}_t, & w_0 = w, \\ de_t = (\delta e_t - \frac{h_t^2}{2})dt + g_t d\widetilde{B}_t, & e_0 = \eta. \end{cases}$$

Since the state variables are same as those of previous section, the HJB equation is obtained from

$$\begin{aligned} \delta V = \max_{c,\pi} \min_{g,h} \Big(& \log c + V_w(w(r + \pi(\mu - r + \sigma h)) - c) + V_e(\delta e - \frac{h^2}{2}) \\ & + \frac{1}{2} V_{ww} (\pi w \sigma)^2 + \frac{1}{2} V_{ee} g^2 + V_{ew} \pi w \sigma g \Big). \end{aligned} \tag{16}$$

Then we can get the exactly same optimal controls except the optimal consumption such that

$$c^* = \frac{1}{V_w}.$$

With these optimal controls, the HJB equation (16) leads to the following PDE

$$\delta V = -\log V_w - 1 + wrV_w + \delta e V_e - \frac{\theta^2}{2} \frac{V_w^2}{\frac{V_w^2}{V_e} + V_{ww} - \frac{V_{ew}^2}{V_{ee}}}, \tag{17}$$

where $\theta = \frac{\mu - r}{\sigma}$.

Now we guess the solution to this PDE has the form of

$$V(w,e) = A\log w + \hat{f}(e),$$

where A is a constant. First note that

$$V_w = \frac{A}{w}, \quad V_{ww} = -\frac{A}{w^2}, \quad V_e = \hat{f}'(e), \quad V_{ee} = \hat{f}''(e), \quad V_{we} = 0.$$

If we substitute the value function into the PDE, we have

$$\begin{aligned}
\delta(A\log w + \hat{f}(e)) &= -\log\frac{A}{w} - 1 + \frac{A}{w}wr + \delta e\hat{f}'(e) - \frac{\theta^2}{2}\frac{(A/w)^2}{\frac{(A/w)^2}{\hat{f}'(e)} - \frac{A}{w^2}} \\
\Longrightarrow (\delta A - 1)\log w &= -\log A - \delta\hat{f}(e) - 1 + Ar + \delta e\hat{f}'(e) - \frac{\theta^2}{2}\frac{A\hat{f}'(e)}{A - \hat{f}'(e)}
\end{aligned}$$

For $A = 1/\delta$, we have an ODE for function $\hat{f}(e)$ such that

$$\begin{aligned}
0 &= -\log A - 1 + Ar - \delta\hat{f}(e) + \delta e\hat{f}'(e) - \frac{\theta^2}{2}\frac{\hat{f}'(e)/\delta}{1/\delta - \hat{f}'(e)} \\
\Longrightarrow 0 &= \left(\log\delta - 1 + \frac{r}{\delta}\right)\left(1 - \delta\hat{f}'(e)\right) - \delta\left(\hat{f}(e) + e\hat{f}'(e)\right)\left(1 - \delta\hat{f}'(e)\right) - \frac{\theta^2}{2}\hat{f}'(e) \qquad (18)
\end{aligned}$$

This ODE is also nonlinear and should be computed numerically.

Theorem A.2. *The value function is*

$$V(w,e) = \frac{1}{\delta}\log w + \hat{f}(e),$$

where $\hat{f}(e)$ is a continuously differentiable function which is the solution to the ODE in (18) with boundary conditions

$$\hat{f}(0) = \frac{1}{\delta}\left(\log\delta + \frac{r}{\delta} - 1 + \frac{(\mu - r)^2}{2\delta\sigma^2}\right),$$

$$\lim_{e\to\infty} \hat{f}(e) = \frac{1}{\delta}\left(\log\delta + \frac{r}{\delta} - 1\right).$$

Then in the pure exchange economy defined above, the equilibrium expected excess return and the equilibrium interest rate are derived by

$$\frac{dS_t + D_t dt}{S_t} - rdt = \sigma_d^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right)dt + \sigma_d dB_t,$$

$$r = \delta + \mu_d - \sigma_d^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right).$$

Proof. Similar to the power utility case, the representative agent's optimal controls are obtained from

$$c^* = \frac{1}{V_w}, \quad \text{and} \quad \pi^* = -\frac{V_w(\mu - r)}{w\sigma^2(V_{ww} + \frac{V_w^2}{V_e} - \frac{V_{ew}^2}{V_{ee}})} = \frac{\mu_s - r}{\sigma_s^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right)}.$$

Then the wealth dynamics are given by

$$\begin{aligned}\frac{dW_t}{W_t} &= \left(r - \delta + \frac{(\mu_s - r)^2}{\sigma_s^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right)}\right)dt + \sigma_s dB_t \\ &= \left(r - \delta + \sigma_s^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right)\right)dt + \sigma_s dB_t.\end{aligned}$$

Since $S_t = W_t$ at the equilibrium, we have

$$c_t^* = \delta W_t = \delta S_t = D_t.$$

This implies that the dynamics of risky asset has the same as that of dividend.

Again, simple comparison gives the following two conditions:

$$r - \delta + \sigma_s^2\left(1 - \frac{1}{\delta\hat{f}'(e)}\right) = \mu_d, \qquad \sigma_s = \sigma_d.$$

We complete the proof. □

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