Technical Analysis, Momentum and Autocorrelation

K. J. Hong[a] and S.Satchell[b]

Working Paper Apr 04, 2012

[a] *Korea Capital Market Institute*
Tel: 82 10 2730 4083 E-mail: kjh47@kcmi.re.kr
Address: 143 Uisadang-daero, Youngdeungpo-gu, Seoul, Korea

[b] *Trinity College, University of Cambridge*
Tel: 44 12233 38409 E-mail: ses999gb@yahoo.co.uk
Address: Trinity College, Cambridge, CB2 1TQ, U.K.

Abstract

This article assumes general stationary processes for prices and derives the autocorrelation function for a general Moving Average (MA) trading rule to investigate why this rule is used. The result shows that the MA rule is popular because it can identify the price momentum and is a simple way of tracing and exploiting the price autocorrelation structure without actually knowing it. We focus on analyzing the impact of price momentum on the profitability of the MA rule because the price momentum effect tends to be stronger and longer-lived than the return momentum effect.

1 Introduction

While fundamental analysts study a company's underlying indicators of profit such as earnings, dividends, new products and R&D, technical analysts focus mainly on price and return but, unwittingly or otherwise, also examine psychological aspects in the demand for a company's stock. Technicians employ many techniques, one of which is the use of charts. Using charts, technical analysts seek to identify price patterns and market trends in financial markets and attempt to exploit these patterns. Traders and portfolio managers continue to use technical analysis that forecast stock price movements using historical prices to formulate buy and sell decisions. Empirical studies report supporting evidence for the profitability of technical analysis, including Sweeney (1986, 1988), Brock et al. (1992), Blume et al. (1994), Chan, Jegadeesh, and Lakonishok (1996), Gencay (1996, 1998, 1999), Neely et al. (1997), Brown et al. (1998), Rouwenhorst (1998), Allen and Karjalainen (1999), Chang and Osler (1999), Neely and Weller (1999), Chan et al. (2000), Shleifer (2000), and Lo et al. (2000). These results suggest that technical analysis beat the market in risk neutral terms, hence its popularity.

One of the simplest and most widely used trading strategies based on technical analysis is the Moving Average (MA) rule, which is an objective rule-based trading strategy in which buy and sell signals are determined by the relative magnitudes of short and long term MAs. An objective rule-based trading system has well-defined and indisputable buy and sell signals following a decision rule based on past data. The MA rule often leads investors to invest with or against the trend, or momentum since it assumes that prices trend directionally. The profitability of the MA rule has been heavily investigated in the previous literature. Brook, Lakonishock and LeBaron (1992) suggested evidence that technical trading rules have predictive ability with respect to market indices in the USA in both conditional means and variances. Further, they showed that these results were relatively stable over their 90 year sample period. Hudson et al. (1996) did the same for market indices in the UK. Gunasekarage and Power (2000) indicated that the employment of these techniques also generates excess returns to investors in South Asian markets. Reitz (2006) shows that self-fulfilling expectations and herd behavior not only re-confirm the profitability of technical trading rules, but also put forward a sensible explanation as to why such rules may actually work.

The MA rule takes advantage of price trend, which is closely linked with the rather nebulous idea of price momentum. The profitability of price momentum strategies, initially investigated by Jegadeesh and Titman (1993), has received the most attention among all the anomalies examined by Fama and French (1996). It is the fourth anomaly included in the Carhart four-factor model. Jegadeesh and Titman (2001) argue that their 1993 results are not due to data mining since the monthly momentum strategy profit is about 1% in the 1990s. Furthermore, Rouwenhorst (1998) confirms the robustness of this strategy by showing that between 1980 and 1995 an internationally diversified portfolio of past medium-term Winners outperformed a portfolio of medium-term Losers after correcting for risk by more than one percent per month. Specifically, Rouwenhorst (1998) finds significantly positive momentum payoffs in 12 countries that he investigated. (Price) Momentum is defined by the above authors as the product of the relative prices over the holding period and a past winners strategy is to buy the top n-th of stocks with this characteristic in a portfolio. A careful reading of the momentum literatures fails to find clarification over whether momentum is present when the price of the asset follows (a) a random walk (b) a random walk with drift or (c) a mean reverting process around a linear trend. The issue is further complicated by different treatments for prices, returns and earnings. One of the purposes of this paper is to try and understand the time series basis of a momentum strategy. Unlike previous momentum literature that focuses on the relative performance of securities in the cross-section, we emphasize that this paper intends to investigate momentum within asset price process. Moskowitz et al (2010) also look at momentum for a single asset and call it 'time series momentum'. This single asset momentum focuses purely on a security's own past process. We assume general stationary processes for prices and derive the autocorrelation function for a general MA rule. We focus on analyzing the impact of price momentum, instead of returns or earnings momentum, on the profitability of the MA rule because (1) most of, if not all, trading rules using technical analysis are based on past prices, and (2) the price momentum effect tends to be stronger and longer-lived than the return momentum effect. Most previous momentum studies typically include a large cross-section of stocks weighted by the previous period return. There exists another type of price trend besides the momentum discussed, however. Equity prices tend to show a long run market-wide trend. This can be dues to inflation or other economic conditions. We define this market-wide trend by the parameter, μ. We remove the market-wide trend from our price process in order to focus on price momentum only.

This paper presents a simple and straightforward evaluation of the MA rule with one investor and one asset. To evaluate the MA rule, an exact definition of momentum is required, which is presented in Section 2.2 with a statistical model. We find that that there are two reasons why the MA rule is popular. First, the MA rule can identify the price momentum (trend), which comfortably confirms the results of previous momentum literature. Second, the MA rule is a simple way of tracing and exploiting the price autocorrelation structure without actually knowing it. It is very hard, if not impossible, to know the real structure of the price autocorrelation of an asset, and the MA rule provides a simple and clear methodology that can take advantage of the price autocorrelation structure. The main contribution of this paper is that we suggest two reasons why the technical trading strategy, or Chartism, is popular and this leads to price autocorrelation. If a conservatism bias exists in a market, we can confirm that the MA rule is profitable. A conservatism bias indicates that investors are too slow (too conservative) in updating their beliefs in response to recent evidence. This means that they might initially underreact to news about a firm, meaning that prices will fully reflect new information only gradually. Such a bias would give rise to momentum (price autocorrelation) in the stock market. Using AR (1) and MA (1) processes, we take the discussion away from short term price trend continuation or long term price reversal. In short, it is not just about buying past losers or winners but is also about identifying price trends and the autocorrelation structure. This paper does not intend to explain all the reasons for the predictability nor does it attempt to explain why arbitrage fails to eliminate mispricing.

The contribution of the paper is clear. Unlike previous literatures that mainly investigate cross sectional momentum among various equities, this paper investigates single asset price momentum to suggest that momentum trading strategy is popular not only because it takes advantage of trend within the times series data but also it amplifies the autocorrelation. This shows that a momentum trading strategy is essentially a simple way of conditioning on past price information. The paper further investigates the structural change of autocorrelation to identify the pattern of the amplification. This is important because such increase in autocorrelation is not captured by the standard risk measures such as VaR. The paper provides a guideline of how much amplification is occurring. The paper also provides theoretical justification to wide use of longer period MA for price momentum and shorter period MA for return momentum.

The rest of the paper is organized as follows. Section 2 explains the trading rule and how it is related to price momentum. Section 3 presents the general model. Section 4 shows the autocorrelation structure of the trading strategy. Section 5 presents empirical evidence and Section 6 concludes the paper.

2 Trading Rule and Momentum

2.1 Trading Rules Definition

We employ and expand the definition of the MA oscillator by Brock et al. (1992), which states: "According to the moving average trading rule, buy and sell signals are generated by two moving averages of the level of the index – a long-period average and a short-period average. In its simplest form this strategy is expressed as buying (or selling) when the short-period moving average rises above (or falls below) the long-period moving average. The idea behind computing moving averages it to smooth out an otherwise volatile series. When the short-period average exceeds the long-period moving average, a trend is considered to be initiated. A very popular moving average rule is 1-200, where the short period is one day and the long period is 200 days. While numerous variations of this rule are used in practice, we attempted to select several of the most popular ones: 1-50, 1-150, 5-150, 1-200, and 2-200." However, we can also take into consideration the flip side of the strategy, where we buy when short-period MA falls below the long-period MA and we sell when the opposite happens.

Let *l* stands for the time period that the MA is computed for a long position and *s* for the time period that the MA is computed for a short position. Hence $s > l$ and they overlap. Denote SMA as MA computed over time period *s* and LMA as MA computed over time period *l*. For example at time t, if $s = 7$ and $l = 3$, then SMA is computed over time t – 6 to t and LMA is computed over time t – 2 to t. We classify two opposite MA rules with SMA and LMA, MAbull(*l*,*s*) and MAbear(*l*,*s*), and define the MAbull(*l*,*s*) rule as taking a long position in the asset when SMA > LMA and as taking no position in the asset when the SMA < LMA and MAbear(*l*,*s*) rule does the opposite.

Definition 1: *The MAbull(l,s) and MAbear(l,s) position can be represented as*

$$MAbull(l,s) = \max(\delta(t),0) \qquad MAbear(l,s) = \max(1-\delta(t),0)$$

where $\delta(t) = \begin{cases} 1 \text{ if } LMA(t) - SMA(t) > 1 \\ 0 \text{ if } LMA(t) - SMA(t) < 1 \end{cases}$

and $SMA(t) = \frac{\sum_{i=1}^{s} P(t-i+1)}{s}$ and $LMA(t) = \frac{\sum_{i=1}^{l} P(t-i+1)}{l}$

The logic behind the MAbull rule is well known, when a price penetrates the MA from below, the bull trend is believed to be established and a trader wants to take advantage of this in the expectation that there will be further upward movements in prices. The MAbear rule tells us the opposite story, short term price reversal. When the price falls enough to penetrate the MA from the above, this might be due to the market over-reaction; hence a trader would take a long position, expecting that the price would bounce immediately back.

2.2 Momentum Definition and a Statistical Model

It is very hard to find a formal definition of momentum that has a statistical basis. By this we mean a definition of momentum defined as a property or properties of a stochastic process. Much previous literature such as Brook et al. (1992), Barberis et al. (1998), Rowenhorst (1998), Daniel et al. (1998), Hong and Stein (1999) Jegadeesh and Titman (2001) and Fong and Yong (2005) explain momentum as an upward trend in a time series, which can be assets' prices, earnings or returns. But they concentrate only on historical data and do not give explanations in terms of underlying processes. Technical analysts interpret momentum in terms of moving averages; when the current price rises above its moving average it is interpreted as an upward (bullish) trend , while a downward (bearish) trend emerges when the current price falls below its moving average. These interpretations do not really clarify if momentum is related to trend or to autocorrelation; both ideas having precise definitions in a statistical sense.

We turn to the practitioner literature to see if matters are resolved. The definition of momentum in Wikipedia is "Momentum investing, also sometimes known as "Fair Weather Investing", is a system of buying stocks or other securities that have had high returns over the past three to twelve months, and selling those that have had poor returns over the same period." And in Investopedia, momentum is defined as "An investment strategy that aims to capitalize on the continuance of existing trends in the market. The momentum investor believes that large increases in the price of a security will be followed by additional gains and vice versa for declining values." Achelis (2000) defines "The Momentum Indicator is the ratio of today's price compared to the price n periods ago: Momentum = (Close / Close n periods ago) * 100. Therefore Momentum Investing is investing in securities with levels of Momentum Indicator above certain threshold." As quoted in Schwager (1992), Richard Driehaus (who is widely regarded as the father of momentum investing) once said "far more money is made buying high and selling at even higher prices" (as opposed to "buy low sell high").

The only definition of the above that has a clear mathematical structure is that of Achelis. We shall take his definition and see if it can be consistent with standard dynamic stochastic processes and, as a consequence, what the determinants of momentum will be. The obvious candidate is the log Ornstein – Uhlenbeck (log OU) process since it has been successfully used by many others such as Bergstrom and Lo and Wang in investigating the behaviour of financial asset price processes. Although our work focuses on price momentum, the same approach can be applied to other time series such as returns and earnings.

Whilst momentum is attributed to behavioural explanations, typically herding and the conservatism bias, we shall not look at equilibrium models with momentum investors in. On the basis of previous definitions, momentum manifests itself in statistical models as possibly either trend or autocorrelation or both. Therefore, we advocate the log Ornstein – Uhlenbeck model to demonstrate this. This model can be thought of as a stationary process concerning a trend or a random walk with drift or an explosive process about a trend. Such versatility should allow us to gain some insights into the properties of momentum, as defined by Achelis. Assume that the logarithm of the asset price $\log(P(t))$ has linear trends, μt. We consider the process

$$q(t) := \log P(t) - \mu t$$

Assume a log price process,

$$dq(t) = -\theta q(t)dt + \sigma dW(t) \quad (1)$$

The solution for equation (1) is well known,

$$\ln P(t+h) - \ln P(t) = \mu h + \left(\exp\left(-h\theta\right) - 1\right)\ln P(t) + \sigma\int_{t}^{t+h} \exp\left(-\theta(t-\mu\right)dW(u) \quad (2)$$

h is the holding period, which is always positive. In this model, $\exp(-h\theta)$ can be interpreted as the degree of price autocorrelation. In fact, the model admits a autoregressive (AR(1)) representation in which this is the autoregressive coefficient. From well-known properties of the AR(1) model , we know that stationarity and the existence of a steady-state solution require $\theta > 0$. Indeed, we can think of two different processes for equation (2) depending on the magnitude of the autoregressive coefficient.. First, when equation (2) is mean reverting, then $\exp(-h\theta) < 1$ (i.e. positive autocorrelation) and hence $\theta > 0$. Second, when equation (2) is explosive, then $\exp(-h\theta) > 1$, there is neither stationarity nor a steady-state solution and hence $\theta < 0$

According to the definition of momentum proposed by Achelis, momentum can be interpreted as the sensitivity of the expected price ratio with respect to today's price. If this sensitivity is positive then there is a bullish momentum and if negative there is a bearish momentum.

Proposition 1: *The derivative of conditional population momentum with respect to previous period's price can be expressed as*

$$\frac{\partial \ln E\left(\frac{P(t+h)}{P(t)}\right)}{\partial \ln P(t)} = \exp(-h\theta) - 1 \quad (3)$$

The proof of Proposition 1 is in Appendix A1. This clearly shows that, for log OU, momentum is about autocorrelation. Equation (3) shows that when the price process is mean reverting, the elasticity of momentum is negative while it is positive if the price is explosive. This is exactly what we would anticipate. In other words, given the high past price, the future price is low if the price process reverts around a mean but the future price is high if the price process is explosive. If we were to interpret efficient markets as a log random walk with drift, then the elasticity would become zero. Finally, the elasticity is independent of the trend.

2.3 The MA Rule and Price Momentum

The relationship between the MA rule and price momentum is well explained in Fong and Yong (2005). "In the simplest form, a MA crossover rule operates on the assumption that buy signals are generated when the current stock price crosses its moving average from below while sell signals are generated when the stock price crosses its moving average from above. The rationale for this interpretation is that a trend is said to have emerged when the stock price penetrates the moving average. Specifically, an upward (bullish) trend emerges when the price rises above its moving average, while a downward (bearish) trend emerges when the price falls below its moving average." Therefore, the MAbull rule takes advantage of the bullish trend while the MAbear rule takes advantage of the bearish trend. MAbull strategy can be thought as buying a past loser because it takes a long position when the asset price is relatively low and liquidates the position when the asset price is relatively high.

3 The Model

3.1 General Model

Let the de-meaned log price process $q(t) = \log P(t) - \mu t$ has distribution $q(t) \sim \left(0, \sigma^2\right)$. Also denote the autocorrelation between $q(t)$ and $q(t-h)$ as $\rho_A(h)$. Since we are using log prices, it would be sensible to use geometric price MAs which are equivalent to arithmetic log price MAs.

$$SMA(t) = \exp\left(\frac{\sum_{i=1}^{s} q(t-i+1)}{s}\right) \text{ and } LMA(t) = \exp\left(\frac{\sum_{i=1}^{l} q(t-i+1)}{l}\right) \text{ where } s > l$$

Under such a framework, the return at time t is defined as
$q(t) - q(t-1) = \log P(t) - \log P(t-1) - \mu$

Definition 2: *The critical value is defined as*

$$C(s,l;t) = \frac{LMA(t)}{SMA(t)} = \exp\left(w'Q(t)\right) \quad (4)$$

where $w' = \left(\left(\frac{1}{l} - \frac{1}{s}\right) \cdots \left(\frac{1}{l} - \frac{1}{s}\right) \; -\left(\frac{1}{s}\right) \cdots -\left(\frac{1}{s}\right)\right)$ and

$$Q(t)' = \begin{pmatrix} q(t) & q(t-1) & \cdots & q(t-l+1) & \cdots & q(t-s+1) \end{pmatrix}$$

The derivation of definition 2 is in Appendix A2. Take a long position if C(s,l;t) becomes more than 1 and liquidate the position if C(s,l;t) becomes less than 1. When the MA rule is applied based on a positive trending price process, the exit point (where the manager liquidates a long position due to the current price penetrating the MA from above) will always be higher than the entry point hence the return is always positive. However, the manager is simply picking up the price trend and the same return can also be made with a buy and hold strategy. This is less interesting. The importance of the autocorrelation within a financial time series is also noted in Moskowitz et al (2010), as they state "We find that positive auto-covariance in our futures contracts' return drives most of the time series and cross-sectional momentum effects we find in the data. The contribution of the other two return components – serial cross-correlations and variation in mean returns – is small". In order to separate the impact of momentum from the market-wide trend, we use the detrended log price, which also has useful statistical properties for estimation. The 'detrended' MA rule also has a simple price interpretation as shown below. Since MAbear rule is simply the opposite of the MAbull rule, we only look at the MAbull hereafter. However the MAbear rule would be useful as one of benchmarks in evaluating the profitability of MAbull rule.

Assuming stationarity in the second moments, the bivariate distribution of C(t) and C(t-1) can be expressed as,

$$\begin{pmatrix} w'Q(t) \\ w'Q(t-m) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} w'\Omega w & w'\Omega_m w \\ w'\Omega_m w & w'\Omega w \end{pmatrix} \right)$$

where Ω is the (s × s) covariance matrix of Q(t) and Ω_C is (s × s) covariance matrix between Q(t) and Q(t-1). For now, we do not assume anything about the covariance matrix but simply note that the autocovariance is not zero.

Lemma 1: *The MAbull (l,s) rule based on a detrended log price is equivalent to buying a unit of an asset when*

$$P(t) > \exp\left(\frac{\mu m}{2}\right)\left(\prod_{j=1}^{m-1} P(t-j)\right)^{\frac{1}{m-1}} \quad (5)$$

The derivation of Lemma 1 is in Appendix A3. C(*s*,*l*;t) becomes log normal distribution.

$$C(s,l;t) = \exp(w' Q(t)) \qquad w' Q(t) \sim N(0, w'\Omega w)$$

Although we apply this model to the price process only, the application of the model does not need to be restricted to price process only. It can also be applied to returns or earnings or to any process that would have momentum (autocorrelation / trend). This point will be discussed again later in the paper.

3.2 The Profitability of the MA Rule

Definition 3: *Define* $\alpha(t)$ *as*

$$\alpha(t) = \begin{cases} 1 \text{ if } \mathrm{C}(t) > 1 \Leftrightarrow w' Q(t) > 0 \\ 0 \text{ if } \mathrm{C}(t) < 1 \Leftrightarrow w' Q(t) < 0 \end{cases} \quad (6)$$

hence $E[\alpha(t)] = prob(C(t) > 1)$

Remark 1: *The unconditional and conditional probability of* $\alpha(t)$ *taking the value of 1 under normality in prices can be expressed as*

$$\text{Unconditional: } prob(\alpha(t) = 1) = 0.5$$

$$\text{Conditional: } prob(\alpha(t) = 1 | \xi) = 1 - \Phi\left(\frac{-\rho_T(1)(q_{t-1} - SMA(t-1))}{\sqrt{(1-\rho_T^2(1))w'\Omega w}}\right) \quad (7)$$

where $\rho_T(1)$: the first order of the trading strategy autocorrelation and

$\xi = w' Q(t-1) = q_{t-1} - SMA(t-1)$

The proof of Remark 1 is in Appendix A4.

Proposition 2: *The h period return from the MAbull rule from time k,* $r_T(k+h,k)$ *can be expressed as*

$$r_T(s,l;k+h,k)=\sum_{i=1}^{h}\left(\alpha(s,l;k+i-1|k)\times r_A(k+i,k+i-1)\right) \quad (8)$$

where $r_A(k+h,k)$ is the h period asset return from time k. If $r_T(k+h,k)>r(k+h,k)$, then the MA rule is profitable. This is only possible if $\alpha(t)$ is more likely to take a value of 1 when $r_A(t)>0$. This provides with us a condition that would make the MAbull rule profitable, when there is a positive price trend and if one could exploit it. A positive trend means $\rho_T(1)>0$ and $\xi>0$ and this makes the conditional probability of α being equal to one larger than 0.5 when $q_t>q_{t-1}$, hence the MAbull rule is profitable.

Analytically analyzing equation (8) is too complicated hence simulation would be useful. We noted that price autocorrelation is important in determining the conditional probability of $\alpha(t)$. Thus an assumption regarding price autocorrelation needs to be made and we assume that the asset price follows the AR(1) process. We use Goldman Sachs (GS) data from May 4, 1999 to May 31, 2011 for estimation. There is no special reason for selecting such data and time period. The estimation is only to locate some sensible parameters. The simulation process is summarized in Appendix A5. This paper is concerned with price (level), which tends to have very high autocorrelation. Hence an investigation of high level autocorrelations would be more meaningful. For convenience, we examine the MA(1,10) rule.

Table 1: Simulated Profitability of the MA(1,10) Rule: Simulation Results

Annual Profit				
Rho	0.9900	0.9925	0.9950	0.9975
MA bull	13.19%	13.45%	13.94%	15.09%
MA bear	-15.54%	-15.01%	-13.92%	-11.73%
Buy and Hold	1.79%	2.40%	3.55%	6.54%

Table 1 presents the simulated results. The error terms are simulated to have a normal distribution. The simulated result shows that the conditioning on previous price information affects the profitability of the trading strategy when there is positive price autocorrelation. The MAbull rule outperforms the other strategies because there is a positive trend in the simulated log price process hence $prob\left(\alpha(t)=1\middle|\xi\right)>0.5$ when the asset return is positive. This confirms the results of previous literature including Fong and Yong (2005) and is what we exactly anticipated. However the result also highlights that the degree of autocorrelation affects the performance of the trading strategy as well.

The portfolio position is determined by the sign of C(t), therefore a further investigation of C(t) transition with respect to time would be meaningful. To gain insight into the properties of the MA(*1*, *s*) rule, we calculate a transition matrix. This calculates different values for the autocorrelation probability of C(t) that is how it changes from one sign to the other. If the price process was random and symmetric, this would be a quarter in all regions.

Table 2: C(t) Transition Vector of the Simulated Results of the MAbull(1, 10) rule

	0.99	0.9925	0.995	0.9975
R1 (+, +)	44.44%	45.25%	49.60%	54.04%
R2 (+, -)	7.68%	8.89%	9.49%	7.37%
R3 (-, +)	7.58%	8.99%	9.60%	7.47%
R4 (-, -)	40.30%	36.87%	31.31%	31.11%

* R1: C(t-1) > 0 and C(t) > 0 R2: C(t-1) > 0 and C(t) < 0
R3: C(t-1) < 0 and C(t) > 0 R4: C(t-1) < 0 and C(t) < 0

The probabilities of region 2 and 3 should be symmetric by the construction of the MA rule. When C(t) is in Region 1, the MAbull rule is likely to be profitable while when it is in Region 4, the MAbear rule is likely to be profitable. When in Region 2 (3), the MAbull rule will take (liquidate) a position in an asset and the MAbear rule will liquidate (take) a position in an asset. With a positive trend and autocorrelation in price, we see that the asset is more likely to be in Region 1 than in Region 4, which explains why the MAbull rule makes the most profit. The probability of Region 1 increases as price autocorrelation increases, which also corresponds to what we have stated previously. The larger the magnitude of the positive autocorrelation, the longer the positive trend will remain, hence both the MA (both bull and bear) rules should be more profitable. This is also shown in Table 2.

4 Autocorrelation

From the model, more specifically from equation (7), and the simulated results we have seen that price autocorrelation is an important factor in the profitability of the MA rule. Hence this section investigates autocorrelation of the trading strategy.

4.1 General Autocorrelation Structure

Let $\rho(k)$ be asset price autocorrelation in log prices between k periods. Then it can be expressed as

$$\Omega_m = \sigma^2 \begin{bmatrix} \rho_A(m) & \rho_A(m+1) & \cdots & \rho_A(m+s-1) \\ \rho_A(m-1) & \rho_A(m) & & \\ & & \ddots & \vdots \\ \vdots & & \rho_A(m) & \\ & & & \ddots \\ \rho_A(m-s+1) & \cdots & & \rho_A(m) \end{bmatrix}$$

$\rho_A(0) = 1$ and $\rho_A(k) = \rho_A(-k)$ since we assume stationarity in second moments.

Proposition 3: *The variance for C(s,l;t) and the covariance between C(s,l;t) and C(s,l;t-m) can be expressed as*

$$\gamma(s,l;m) = w'\Omega_m w = \sigma^2 \begin{pmatrix} a^2\left(l\rho_A(m) + \sum_{i=1}^{l-1}\left(\rho_A(m+i) + \rho_A(m-i)\right)\right) \\ + b^2\left((s-l)\rho_A(m) + \sum_{i=1}^{s-l-1}(s-i-1)\left(\rho_A(m+i) + \rho_A(m-i)\right)\right) \\ + ab\left(\sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho_A(m+l+i-j)\right) + ab\left(\sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho_A(m-l-i+j)\right) \end{pmatrix} \quad (9)$$

where $a = \left(\frac{1}{l} - \frac{1}{s}\right)$ and $b = -\left(\frac{1}{s}\right)$

Proof of Proposition 3 is in Appendix A6. When the MA rule is practically applied, managers often take the shorter time period as 1 (l =1). When l = 1, it is referred to as the "MA crossover rule". This case can be treated as a special case since it makes the equation significantly simpler by getting rid of the first of both double summation terms.

Remark 2: *The autocorrelation function between C(t) = w'Q(t) and C(t-m) = w'Q(t-m), when l = 1, can be expressed as*

$$\rho_T(m) = \frac{w'\Omega_m w}{w'\Omega w} = \frac{\left(\begin{array}{l} a^2\rho(m) + b^2\left((s-1)\rho_A(m) + \sum_{i=1}^{s-2}(s-i-1)\left(\rho_A(m+i) + \rho(m-i)\right)\right) \\ + ab\left(\sum_{i=0}^{s-2}\rho_A(m+i+1) + \sum_{i=0}^{s-2}\rho_A(m-i-1)\right) \end{array} \right)}{\left(a^2 + b^2\left((s-1) + \sum_{j=1}^{s-2}\left(\rho_A(j) + \rho_A(-j)\right)\right) + ab\left(\sum_{i=0}^{s-2}\rho_A(i+1) + \sum_{i=0}^{s-2}\rho_A(-i-1)\right)\right)} \tag{10}$$

The proof of Remark 2 is omitted.

4.2 Calculation Check

In this section, the previous result is checked with a simulation result assuming one of the most popular strategies, 1 – 10 (l = 1 and s = 10) and that the autocorrelation function, $\rho(m)$, follows the AR(1) process. 2000 sets of 5000 observations of q(t) are simulated where $q(t) = \rho q(t-1) + \varepsilon_t$, $\varepsilon \sim N(0,1)$ and $\rho = 0.95$. LMA(t) and SMA(t) are computed from the simulated q(t), then subtracted to get $w'Q(t)$. The autocorrelation of $w'Q(t)$ is compared with (4) with l = 1 and s = 10 and the percentage differences are reported.

Table 3: Simulation vs. Theoretical Result:
Autocorrelation of $w'Q(t)$ when l = 1 and s = 10

	Simulated AR(1)	Theoretical AR(1)	% Difference
wOmw[1]	0.8127	0.8131	0.0415%
wOmw[2]	0.6321	0.6325	0.0629%
wOmw[3]	0.4611	0.4619	0.1736%
wOmw[4]	0.3039	0.3051	0.3909%
wOmw[5]	0.1643	0.1658	0.9555%
wOmw[6]	0.0461	0.0479	3.8566%
wOmw[7]	-0.0469	-0.0448	-4.5953%
wOmw[8]	-0.1110	-0.1083	-2.4332%
wOmw[9]	-0.1417	-0.1387	-2.1642%
wOmw[10]	-0.1348	-0.1317	-2.3007%
wOmw[11]	-0.1282	-0.1251	-2.3803%
wOmw[12]	-0.1216	-0.1189	-2.2386%
wOmw[13]	-0.1152	-0.1129	-1.9212%
wOmw[14]	-0.1089	-0.1073	-1.5143%
wOmw[15]	-0.1029	-0.1019	-0.9676%
wOmw[16]	-0.0968	-0.0968	0.0307%
wOmw[17]	-0.0910	-0.0920	1.1268%
wOmw[18]	-0.0856	-0.0874	2.0700%
wOmw[19]	-0.0808	-0.0830	2.7619%
wOmw[20]	-0.0763	-0.0789	3.3288%
wOmw[21]	-0.0727	-0.0749	3.0680%
wOmw[22]	-0.0693	-0.0712	2.6541%
wOmw[23]	-0.0661	-0.0676	2.3749%
wOmw[24]	-0.0629	-0.0642	2.1334%
wOmw[25]	-0.0600	-0.0610	1.7012%
wOmw[26]	-0.0573	-0.0580	1.1180%
wOmw[27]	-0.0546	-0.0551	0.8748%
wOmw[28]	-0.0518	-0.0523	0.9222%
wOmw[29]	-0.0491	-0.0497	1.2625%
wOmw[30]	-0.0466	-0.0472	1.3013%

Table 3 shows that our computation is correct.

Choosing a particular MA(l,s) rule depends upon the autocorrelation properties of prices. When s is fixed, $w'\Omega w$ is a constant. Therefore for $s > l$, the sensitivity of trading strategy autocorrelation with respect to m is

$$\frac{\partial \rho_T(m)}{\partial m} = \frac{\partial w'\Omega_m w}{\partial m} = \begin{pmatrix} \left(a^2 + b^2(s-1)\right)\frac{\partial \rho_A(m)}{\partial m} + b^2 \sum_{i=1}^{s-2}(s-i-1)\left(\frac{\partial \rho_A(m+i)}{\partial m} + \frac{\partial \rho_A(m-i)}{\partial m}\right) \\ + ab\left(\sum_{i=0}^{s-2}\frac{\partial \rho_A(m+i+1)}{\partial m} + \frac{\partial \rho_A(m-i-1)}{\partial m}\right) \end{pmatrix}$$

If $\frac{\partial \rho_A(m)}{\partial m} < 0$, we have $\frac{\partial \rho_T(m)}{\partial m} < 0$

It may well be reasonable to design optimal properties. We do not pursue this further but instead look at the links between the Autocorrelation Function (ACF) of the MA rule and some simple time series models.

In the next two sections, the autocorrelation structure of the 1 – 10 strategy is investigated. Equation (9) is computed m = 30 and ρ_A incrementing from 0.05 to 0.95 with an interval of 0.05.

4.3 When ACF follows AR(1)

(Insert Table 4)

(Insert Figure 1)

4.4 When ACF follows MA(1)

(Insert Table 5)

(Insert Figure 2)

Since the tables and figures are too large to fit here, they have been placed on the following pages. The autocorrelation of the 1 – 10 strategy becomes 0 for m > 10. Figure 2 and 3 are consistent with behavioural literatures in that they show initial under-reaction and delayed over-reaction or price reversal. We should note that this seemingly behavioral result is driven without assuming any behavioral aspect in determining asset price process. This might suggest that such initial under reaction and price reversal might be a product of underlying security's autocorrelation. This would be an interesting future research topic.

4.5 The MA(*l*,*s*) vs. Price Autocorrelation

The benefit of calculating the AF of the MA(l,s) rule is that we can see if the imposition of the rule leads to an increase / decrease in autocorrelation in the money amounts associated with trading. To do this we need an overall measure of autocorrelation and a natural one is the population analogous to the Ljung-Box statistic, $LB = \sum_{j=0}^{\infty} \rho_j^2$. For a simple model this can be calculated analytically; for an AR(1) with correlation coefficient ρ, ($P_t = \beta + \rho P_{t-1} + v_t$), $LB = 1/(1-\rho^2); |\rho| < 1$ whilst for an MA(1), $P_t = \alpha + v_t + \theta v_t$, $LB = (1+2\theta^2)/(1+\theta^2); |\theta| < 1$. This section presents relative Ljung Box statistics based on price autocorrelation, which is assumed to follow the AR(1) process. By investigating the statistics of the MAbull(l,s) autocorrelation structure we can gain a good idea of how the autocorrelation structure of the MAbull(l,s) rule amplifies that of the price itself.

Moskowitz et. al (2010) investigate the correlation structure of the momentum strategy to find “Time series momentum strategies are positively correlated within an asset class, but less so than passive long strategies.” At first glance, it looks like their result is contradicting our result. However, close investigation of their trading strategy reveals that it is consistent with our result. Their strategy is stated as follows. “We consider whether the excess return over the past k months is positive or negative and go long the contract if positive and short if negative, holding the position for h months.” This is ‘buying the past winner’ strategy. The MAbull rule investigated in our paper, as previously discussed, does exactly the opposite. Although we have not explicitly shown this, extension of our result should predict that the impact of the trading strategy used in Moskowitz et al (2010) would decrease the amount of autocorrelation, and this is what they find. Hence their result can be seen as a possible empirical confirmation of our theoretical result.

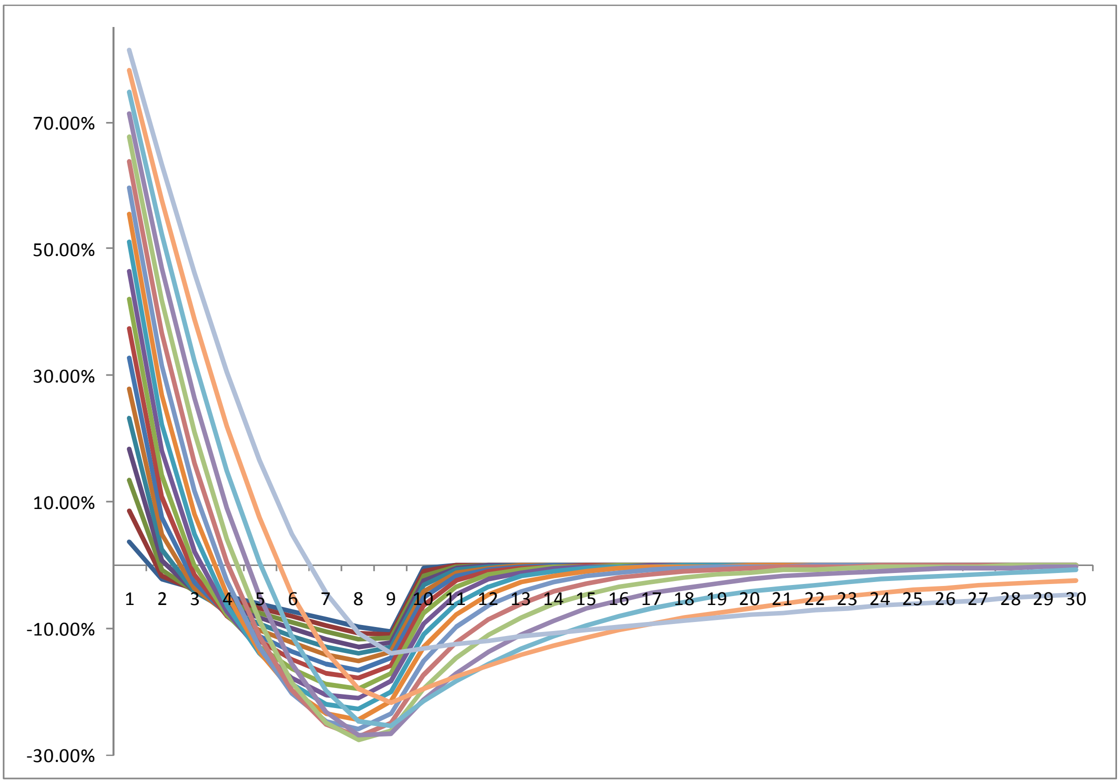
Table 4: The MA(1,10) Rule Autocorrelation Structure under AR(1) Asset Price Autocorrelation, ρ_A increases from 0.05 to 0.95 and m = 30

	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
1	3.77%	8.64%	13.49%	18.32%	23.13%	27.91%	32.65%	37.35%	41.98%	46.55%	51.03%	55.40%	59.63%	63.72%	67.64%	71.36%	74.89%	78.20%	81.31%
2	-2.21%	-1.72%	-0.77%	0.64%	2.51%	4.82%	7.55%	10.70%	14.23%	18.12%	22.34%	26.85%	31.61%	36.58%	41.72%	46.98%	52.34%	57.77%	63.25%
3	-3.68%	-3.99%	-4.19%	-4.23%	-4.05%	-3.57%	-2.75%	-1.54%	0.11%	2.24%	4.87%	8.04%	11.76%	16.05%	20.90%	26.34%	32.36%	38.97%	46.19%
4	-4.92%	-5.44%	-5.99%	-6.55%	-7.09%	-7.55%	-7.87%	-7.99%	-7.83%	-7.30%	-6.31%	-4.76%	-2.55%	0.42%	4.26%	9.09%	14.99%	22.10%	30.51%
5	-6.15%	-6.81%	-7.54%	-8.36%	-9.24%	-10.18%	-11.13%	-12.06%	-12.88%	-13.51%	-13.82%	-13.68%	-12.92%	-11.35%	-8.77%	-4.94%	0.38%	7.46%	16.58%
6	-7.38%	-8.17%	-9.05%	-10.04%	-11.14%	-12.34%	-13.65%	-15.02%	-16.42%	-17.76%	-18.94%	-19.82%	-20.20%	-19.84%	-18.48%	-15.76%	-11.29%	-4.62%	4.79%
7	-8.61%	-9.52%	-10.53%	-11.63%	-12.84%	-14.17%	-15.61%	-17.15%	-18.77%	-20.43%	-22.05%	-23.51%	-24.64%	-25.19%	-24.86%	-23.22%	-19.74%	-13.77%	-4.48%
8	-9.81%	-10.77%	-11.78%	-12.85%	-13.99%	-15.22%	-16.54%	-17.96%	-19.49%	-21.11%	-22.79%	-24.46%	-25.97%	-27.13%	-27.62%	-26.99%	-24.59%	-19.59%	-10.83%
9	-10.48%	-11.00%	-11.59%	-12.24%	-12.97%	-13.79%	-14.73%	-15.80%	-17.02%	-18.40%	-19.95%	-21.63%	-23.36%	-24.97%	-26.20%	-26.57%	-25.40%	-21.65%	-13.87%
10	-0.52%	-1.10%	-1.74%	-2.45%	-3.24%	-4.14%	-5.15%	-6.32%	-7.66%	-9.20%	-10.97%	-12.98%	-15.18%	-17.48%	-19.65%	-21.26%	-21.59%	-19.49%	-13.17%
11	-0.03%	-0.11%	-0.26%	-0.49%	-0.81%	-1.24%	-1.80%	-2.53%	-3.45%	-4.60%	-6.03%	-7.79%	-9.87%	-12.24%	-14.74%	-17.01%	-18.35%	-17.54%	-12.51%
12	0.00%	-0.01%	-0.04%	-0.10%	-0.20%	-0.37%	-0.63%	-1.01%	-1.55%	-2.30%	-3.32%	-4.67%	-6.41%	-8.57%	-11.05%	-13.60%	-15.60%	-15.78%	-11.89%
13	0.00%	0.00%	-0.01%	-0.02%	-0.05%	-0.11%	-0.22%	-0.40%	-0.70%	-1.15%	-1.83%	-2.80%	-4.17%	-6.00%	-8.29%	-10.88%	-13.26%	-14.21%	-11.29%
14	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.08%	-0.16%	-0.31%	-0.58%	-1.00%	-1.68%	-2.71%	-4.20%	-6.22%	-8.71%	-11.27%	-12.79%	-10.73%
15	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.06%	-0.14%	-0.29%	-0.55%	-1.01%	-1.76%	-2.94%	-4.66%	-6.97%	-9.58%	-11.51%	-10.19%
16	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.06%	-0.14%	-0.30%	-0.61%	-1.14%	-2.06%	-3.50%	-5.57%	-8.14%	-10.36%	-9.68%
17	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.07%	-0.17%	-0.36%	-0.74%	-1.44%	-2.62%	-4.46%	-6.92%	-9.32%	-9.20%
18	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.04%	-0.09%	-0.22%	-0.48%	-1.01%	-1.97%	-3.57%	-5.88%	-8.39%	-8.74%
19	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.02%	-0.05%	-0.13%	-0.31%	-0.71%	-1.48%	-2.85%	-5.00%	-7.55%	-8.30%
20	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.08%	-0.20%	-0.49%	-1.11%	-2.28%	-4.25%	-6.79%	-7.89%
21	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.02%	-0.05%	-0.13%	-0.35%	-0.83%	-1.83%	-3.61%	-6.12%	-7.49%
22	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.09%	-0.24%	-0.62%	-1.46%	-3.07%	-5.50%	-7.12%
23	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.02%	-0.06%	-0.17%	-0.47%	-1.17%	-2.61%	-4.95%	-6.76%
24	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.04%	-0.12%	-0.35%	-0.93%	-2.22%	-4.46%	-6.42%
25	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.02%	-0.08%	-0.26%	-0.75%	-1.89%	-4.01%	-6.10%
26	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.02%	-0.06%	-0.20%	-0.60%	-1.60%	-3.61%	-5.80%
27	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.04%	-0.15%	-0.48%	-1.36%	-3.25%	-5.51%
28	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.11%	-0.38%	-1.16%	-2.92%	-5.23%
29	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.02%	-0.08%	-0.31%	-0.98%	-2.63%	-4.97%
30	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.06%	-0.25%	-0.84%	-2.37%	-4.72%

Table 5: The MA(1,10) Rule Autocorrelation Structure under MA(1) Asset Price Autocorrelation, ρ_A increases from 0.05 to 0.95 and m = 30

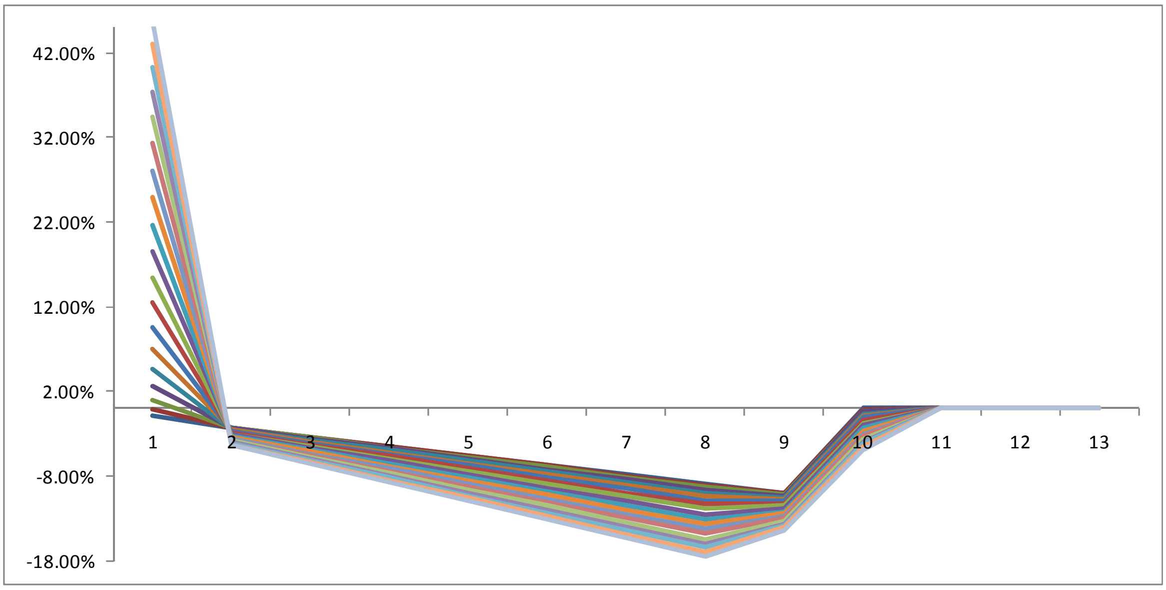
	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
1	-0.87%	-0.14%	1.04%	2.65%	4.65%	6.98%	9.58%	12.41%	15.41%	18.53%	21.71%	24.92%	28.12%	31.27%	34.36%	37.37%	40.28%	43.07%	45.75%
2	-2.23%	-2.27%	-2.32%	-2.40%	-2.49%	-2.59%	-2.71%	-2.84%	-2.98%	-3.13%	-3.27%	-3.42%	-3.57%	-3.71%	-3.85%	-3.99%	-4.12%	-4.25%	-4.38%
3	-3.35%	-3.40%	-3.48%	-3.59%	-3.73%	-3.89%	-4.07%	-4.27%	-4.47%	-4.69%	-4.91%	-5.13%	-5.35%	-5.57%	-5.78%	-5.99%	-6.19%	-6.38%	-6.57%
4	-4.47%	-4.53%	-4.64%	-4.79%	-4.97%	-5.19%	-5.43%	-5.69%	-5.96%	-6.25%	-6.54%	-6.84%	-7.13%	-7.42%	-7.71%	-7.98%	-8.25%	-8.51%	-8.75%
5	-5.58%	-5.67%	-5.80%	-5.99%	-6.22%	-6.48%	-6.78%	-7.11%	-7.45%	-7.81%	-8.18%	-8.55%	-8.91%	-9.28%	-9.63%	-9.98%	-10.31%	-10.63%	-10.94%
6	-6.70%	-6.80%	-6.96%	-7.19%	-7.46%	-7.78%	-8.14%	-8.53%	-8.95%	-9.38%	-9.81%	-10.26%	-10.70%	-11.13%	-11.56%	-11.97%	-12.37%	-12.76%	-13.13%
7	-7.82%	-7.93%	-8.12%	-8.38%	-8.70%	-9.08%	-9.50%	-9.95%	-10.44%	-10.94%	-11.45%	-11.97%	-12.48%	-12.99%	-13.49%	-13.97%	-14.44%	-14.89%	-15.32%
8	-8.93%	-9.07%	-9.28%	-9.58%	-9.95%	-10.38%	-10.86%	-11.38%	-11.93%	-12.50%	-13.09%	-13.68%	-14.26%	-14.84%	-15.41%	-15.97%	-16.50%	-17.01%	-17.51%
9	-10.02%	-10.09%	-10.20%	-10.35%	-10.54%	-10.75%	-11.00%	-11.26%	-11.54%	-11.83%	-12.13%	-12.43%	-12.72%	-13.02%	-13.31%	-13.59%	-13.86%	-14.12%	-14.37%
10	-0.02%	-0.10%	-0.22%	-0.38%	-0.59%	-0.83%	-1.09%	-1.38%	-1.69%	-2.01%	-2.33%	-2.66%	-2.99%	-3.31%	-3.63%	-3.94%	-4.23%	-4.52%	-4.79%
11	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
12	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
13	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

Figure 1: Trading Strategy Autocorrelation Structure under AR(1) Asset Price Autocorrelation



x – axis: lag time period, m y – axis: autocorrelation

Figure 2: Trading Strategy Autocorrelation Structure under MA(1) Asset Price Autocorrelation



x – axis: lag time period, m y – axis: autocorrelation

Table 6: Absolute Ljung Box Statistics for Simulated Results

Autocorrelation	Asset	MA(1,5)	MA(1,10)	MA(1,20)	MA(1,50)	MA(1,100)
0.05	0.0025	0.0831	0.0429	0.0228	0.0106	0.0066
0.1	0.0101	0.0957	0.0565	0.0342	0.0199	0.0150
0.15	0.0230	0.1122	0.0762	0.0516	0.0349	0.0290
0.2	0.0417	0.1322	0.1024	0.0757	0.0561	0.0490
0.25	0.0667	0.1553	0.1357	0.1072	0.0843	0.0757
0.3	0.0989	0.1810	0.1769	0.1473	0.1205	0.1100
0.35	0.1396	0.2087	0.2267	0.1976	0.1662	0.1534
0.4	0.1905	0.2380	0.2860	0.2600	0.2234	0.2078
0.45	0.2539	0.2684	0.3557	0.3375	0.2951	0.2758
0.5	0.3333	0.2996	0.4366	0.4338	0.3854	0.3614
0.55	0.4337	0.3313	0.5292	0.5540	0.5004	0.4701
0.6	0.5625	0.3630	0.6338	0.7051	0.6492	0.6108
0.65	0.7316	0.3946	0.7501	0.8962	0.8464	0.7973
0.7	0.9608	0.4258	0.8773	1.1387	1.1161	1.0529
0.75	1.2857	0.4564	1.0138	1.4457	1.5007	1.4205
0.8	1.7778	0.4863	1.1578	1.8299	2.0795	1.9862
0.85	2.6036	0.5154	1.3066	2.2991	3.0104	2.9509
0.9	4.2632	0.5435	1.4577	2.8501	4.5992	4.8720
0.95	9.2564	0.5838	1.6118	3.4647	7.2945	9.6521

Table 7: Relative Ljung Box Statistics

Autocorrelation	Asset	MA(1,5)	MA(1,10)	MA(1,20)	MA(1,50)	MA(1,100)
0.05	1.0000	33.1600	17.1296	9.0909	4.2439	2.6233
0.1	1.0000	9.4696	5.5937	3.3821	1.9728	1.4897
0.15	1.0000	4.8740	3.3098	2.2419	1.5169	1.2618
0.2	1.0000	3.1739	2.4578	1.8163	1.3468	1.1767
0.25	1.0000	2.3300	2.0362	1.6082	1.2641	1.1355
0.3	1.0000	1.8298	1.7887	1.4897	1.2180	1.1126
0.35	1.0000	1.4947	1.6239	1.4154	1.1902	1.0991
0.4	1.0000	1.2492	1.5015	1.3652	1.1729	1.0909
0.45	1.0000	1.0571	1.4009	1.3292	1.1622	1.0862
0.5	1.0000	0.8989	1.3098	1.3014	1.1563	1.0841
0.55	1.0000	0.7639	1.2203	1.2775	1.1538	1.0840
0.6	1.0000	0.6454	1.1268	1.2536	1.1542	1.0859
0.65	1.0000	0.5394	1.0253	1.2250	1.1570	1.0898
0.7	1.0000	0.4432	0.9131	1.1851	1.1617	1.0959
0.75	1.0000	0.3550	0.7885	1.1244	1.1672	1.1048
0.8	1.0000	0.2736	0.6512	1.0293	1.1697	1.1173
0.85	1.0000	0.1979	0.5019	0.8831	1.1562	1.1334
0.9	1.0000	0.1275	0.3419	0.6685	1.0788	1.1428
0.95	1.0000	0.0631	0.1741	0.3743	0.7880	1.0428

Figure 3: Absolute Ljung Box Statistics for Asset and Various the MA(l,s) Rules

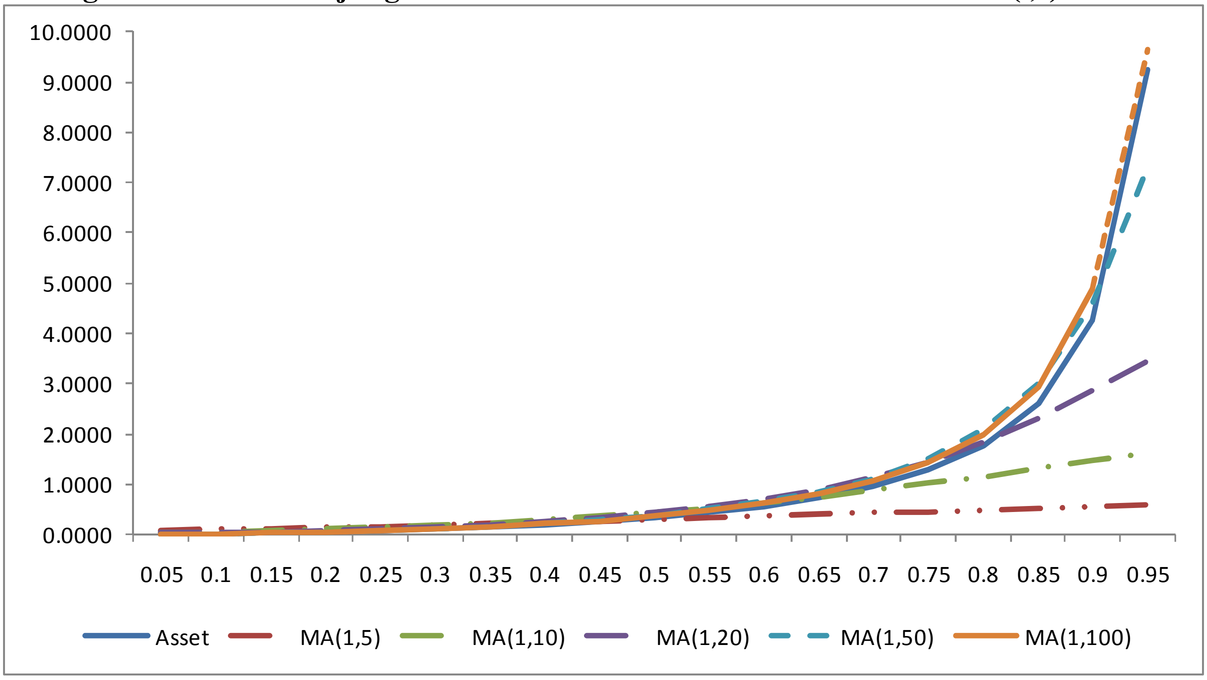
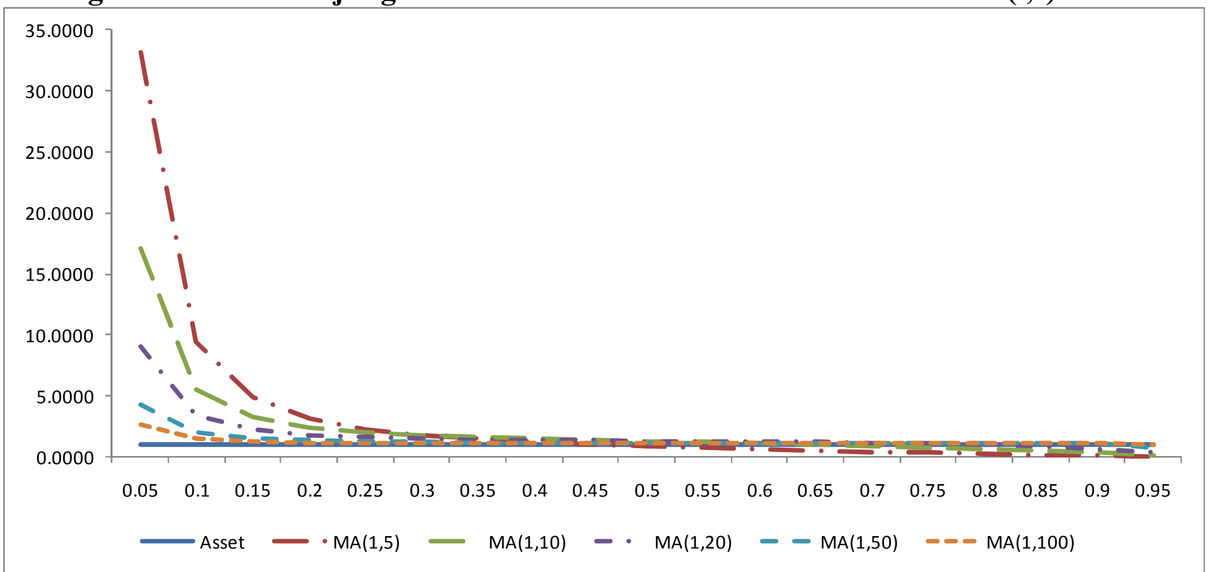


Figure 4: Relative Ljung Box Statistics for Asset and Various the MA(l,s) Rules



As s increases, for l =1, the autocorrelation of the MAbull amplifies see figures 3 and 4 and tables 6 and 7. This result highlights the fact that the MA rule traces the price autocorrelation structure and amplifies it, although the rule does not explicitly identify the structure. Looking at Figure 3, Figure 4, Table 6 and Table 7, we see that for plausible price values, the strategy has led to increased LB for 'short' rules and low autocorrelation whilst it has increased autocorrelation for 'long' rules (1 – 50 or 1 – 100) and higher autocorrelation (prices). What these results indicate is that when time series (price, earnings, returns) autocorrelation is low an MA rule with a small *s* should be used whilst an MA rule with a large *s* should be used with a highly autocorrelated process. As we discussed earlier, the application of our current model is not necessarily restricted to the price process. Based on our results in this section, when the MA rule is employed with a price MA, where price autocorrelation is very high, an MA rule with a large *s* would be appropriate. And when the MA rule is applied to a return process, which has lower autocorrelation than the price process in general, a smaller *s* would be suitable.

5 Empirical Evidence

5.1 Empirical Details of Autocorrelation Amplification

The previous section has focused on analyzing how the MA rule identifies the price autocorrelation structure and amplifies it in a simulated setting where we know the actual price autocorrelation structure. It would be meaningful to apply the MA rule to real historical data. Before we conduct a formal, thorough empirical test on the main theoretical result of the paper, autocorrelation structural amplification, it would be meaningful to provide some empirical observations first. The objective of section 5.1 is to empirically provide processing figures used in section 3 in deriving the main result of the paper on selected indices. These 'processing figures' include the detrended log price first order autocorrelation, index price trend, critical value and its transition matrix as shown in Table 2, and error term statistics. We also conducted Dickey Fuller test result to show that the innate autocorrelation in index price process is statistically significant; therefore the result of the paper is relevant in the real world. Since this is only an observation, we chose 6 indices (3matured and 3 emerging markets) as an example.

Table 8: Detrended Log Price First Order Autocorrelation and Trends of International Indices

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
Rho	0.9995	0.9962	0.9961	0.9961	0.9960	0.9972
Trend	0.046%	0.025%	0.088%	-0.001%	0.001%	-0.007%

* Sample date from January 4, 2000 to June 1, 2011

In Table 8 we see a very high first order autocorrelation in detrended log price process in major indices: We employ Dickey Fuller tests to see whether a unit root is present

Table 9: Price trend, Dickey Fuller Test and the Profitability of the MA(1,10) Rule Applied to Various Indies. Historical Daily Data from Jan 4, 2000 to May 31, 2011

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
Mu	0.00046	0.00025	0.00088	-0.00001	0.00001	-0.00007
Rho	0.9955	0.9971	0.9980	0.9966	0.9967	0.9976
DF Test Stat	-3.71	-2.59	-3.15	-2.43	-2.51	-2.13
MA bull	7.30%	6.29%	13.25%	-4.02%	-4.10%	-2.38%
MA bear	-2.28%	-2.73%	-6.30%	4.68%	4.71%	-1.34%
Buy and Hold	5.25%	3.13%	9.66%	-0.90%	-0.97%	-4.76%

* Dickey Fuller critical value: -2.58 at 1% and -1.95 at 5%

Table 9 presents price trend, Dickey Fuller test results and profitability measures of the MA(1,10) rule when applied to the daily historical data of various indices; a description of the trading strategy is detailed in Section 2.2. The sample data are from January 4, 2000 to May 31, 2011. First we identify the positive trend in the emerging market indices and the negative trend in the advanced market indices, which comfortably conforms to general expectations. The Dickey Fuller test result shows that the emerging markets parameters are significant at 1%, while the parameters of the advanced markets are significant at 5% but not at 1%. This reflects and conforms to the common belief that advanced markets are more "efficient" (in the sense that log prices follow a random walk) than emerging markets. Considering that emerging markets have had strong positive price trends, which is also reflected in the profitability of the buy and hold strategy, it is not surprising that the MAbull rule performs best during the sample period. Again, we identify a negative long run price trend in advanced markets; hence the MAbear rule performs best. Table 9 tells us that identifying the price trend is critical to profitability of a given trading strategy in the sample period.

Table 10: C(t) Transition Matrix of Indices of the MA(1,10) Rule

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
R1 (+, +)	47.49%	44.13%	47.12%	46.36%	45.40%	43.07%
R2 (+, -)	8.11%	7.95%	6.85%	8.92%	9.79%	8.54%
R3 (-, +)	8.11%	7.91%	6.85%	8.95%	9.79%	8.54%
R4 (-, -)	35.94%	39.66%	38.82%	35.42%	34.68%	39.49%

*R1: $C(t-1) > 0$ and $C(t) > 0$ R2: $C(t-1) > 0$ and $C(t) < 0$
R3: $C(t-1) < 0$ and $C(t) > 0$ R4: $C(t-1) < 0$ and $C(t) < 0$

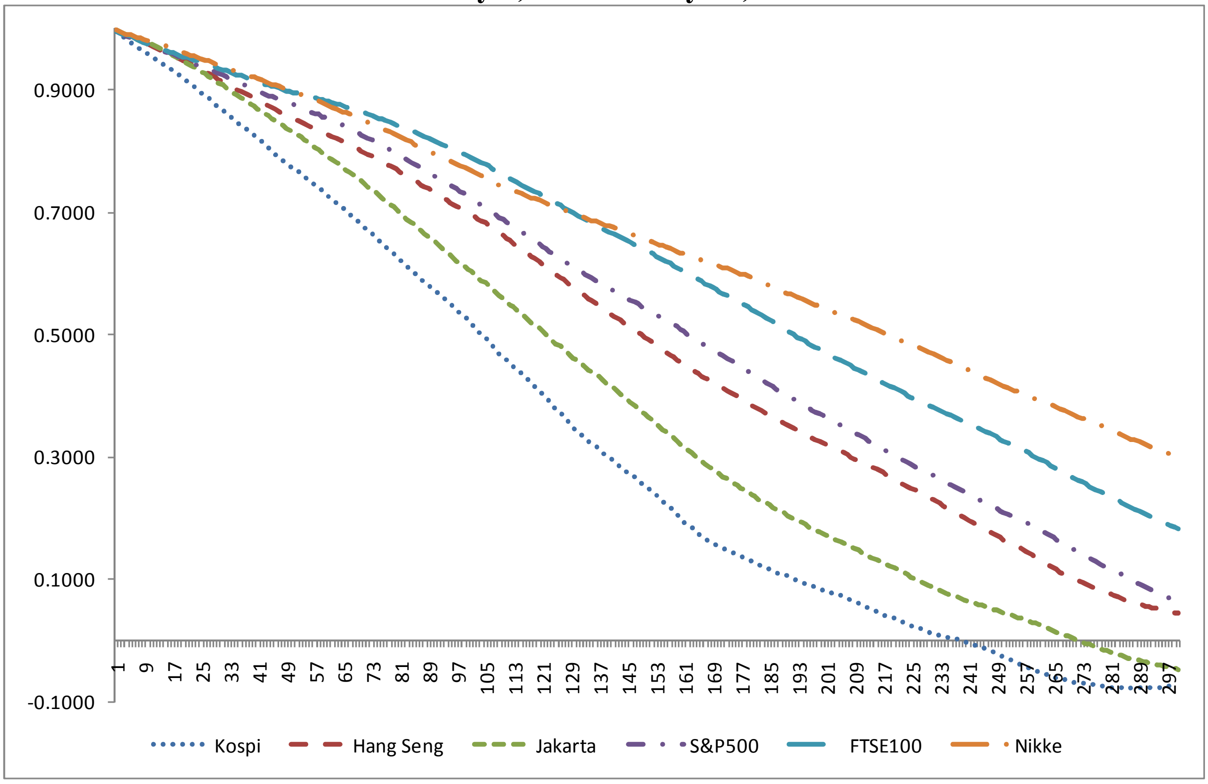
Table 10 presents the transition matrix of C(t). Since details of how to interpret the transition matrix are presented at the end of Section 3.2, there is no need to reiterate them here. There is asymmetry between R1 and R4 and we believe the asymmetry is coming from the non-normality of the error terms in the historical data. To check this, the error term is backed out from the historical detrended log prices of the index historical data assuming they follow the AR(1) process.

Table 11: Error Term Statistics

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
Mean	-0.0002	-0.0001	-0.0003	0.0000	-0.0001	-0.0002
Stdev	0.0179	0.0164	0.0151	0.0136	0.0131	0.0162
Skewness	-0.5559	-0.0591	-0.6588	-0.2027	-0.1865	-0.4302
Kurtosis	4.9721	7.8572	5.7923	7.6380	6.0143	6.5963

Table 11 shows that they are not normally distributed as expected. We see that two factors decide the profitability of the MA rule, the price trend and the autocorrelation structure of the strategy. Since we investigated index level trend of international indices, an investigation of autocorrelation structure of the international indices should follow.

Figure 5: Historical Daily International Indices Autocorrelation Structure: from January 4, 2000 to May 31, 2011



x – axis: lag time period, m   y – axis: autocorrelation

Ljung Box test statistics are computed, they are truncated at the 300th autocorrelation.

Table 12: Ljung-Box Test Statistics for Indices (Prices) and the MA(1,250) Strategy from January 4, 2000 to May 31, 2011

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
1st order Autocorrel	0.9955	0.9971	0.9980	0.9966	0.9967	0.9976
log Price LB	70.22	103.57	85.26	112.08	133.02	141.40
MA(1,250) LB	127.22	137.52	127.57	147.73	160.46	182.01

Table 13: Ljung-Box Test Statistics for Indices (Returns) and the MA(1,10) Strategy from January 4, 2000 to May 31, 2011

	Kospi	Hang Seng	Jakarta	S&P500	FTSE100	Nikke
1st order Autocorrel	0.0155	-0.0216	0.1209	-0.0852	-0.0628	-0.0310
log Price LB	0.1404	0.2041	0.1589	0.1654	0.1897	0.1424
MA(1,10) LB	0.2087	0.2919	0.1884	0.1871	0.2189	0.1627

Table 14: Simulated Ljung-Box Test Statistics for the AR(1) Model

Asset rho	0.99	0.9925	0.995	0.9975
LB	49.25	65.92	99.25	199.25

Table 12 presents the LB statistics for international indices price and the MA(1,250) strategy, and Table 13 presents the LB statistics for international indices returns and the MA(1,10) strategy. The number in Table 12 may look very high compared to previous simulated results of trading strategy autocorrelations below 0.95. Hence, the LB statistics of an asset when it follows the AR(1) process with a very high ρ are presented in 14 to confirm the reasonableness of the numbers in Table 12. We have seen that when the price autocorrelation is very high, a strategy with a large *s* becomes effective. In this empirical test of index price series, we will use 1 year MA, hence the MA(1, 250) rule. 250 days were chosen for no specific reason but to ensure large enough s. We could have used (1, 100) or (1, 500) and the result would be the same. We also argued that when ρ is very low, a strategy with small *s* is effective. Hence for index return series we will use the MA(1, 10) rule, although any small *s* would have equivalent results. The result clearly shows that the MA strategy amplifies the level of autocorrelation in the index level.

5.2 A Formal Test on Autocorrelation Amplification

In this section, we present a formal empirical test of our finding. We use 11 international equity indices from January 4, 2001 to December, 2011. Table 14 contains the list of the indices and the corresponding countries.

Table 14: List of the Indices and the Corresponding Countries

Matured		Emerging	
Country	Index	Country	Index
Canada	S&P/TSX	Brazil	Bovespa
France	CAC40	China	Hang Seng
Germany	DAX30	Indonesia	IHSG (Jakarta)
Japan	Nikkei 225	Korea	KOSPI
UK	FTSE100	Singapore	FSSTI
US	S&P500	Russia	RTS

We chose major indices around the world and we expect the similar empirical result can be driven from different sample. However, for the purpose of the current paper, we believe the sample is sufficiently large and representative.

The hypothesis we intend to test is "When the MA rule is applied to an index; amplification of autocorrelation is significant and depends on the degree of autocorrelation of the original price process". Define MA(1, *s*) as the Moving Average with l and s as defined in Lemma 1 where where s = 5, 10, 20, 30, 60 let MA(1, *s*)LB be Ljung-Box test statistics for MA(1, s) strategy and let PLB be Ljung-Box test statistics for log index price. We compute daily MA(1, *s*) for the sample period then non-overlapping 6-month MA(1, *s*)LB is computed from 2001 to 2011 (10 years). I intend to use non-overlapping measures because underlying index prices are serially correlated. This serial correlation will automatically make MA(1, *s*)LB serially correlated, which is not desirable. There will be 20 observations of MA(1, *s*)LB for each of s (= 5, 10, 20, 30, 60). 6-month PLB is computed from 2001 to 2011. Then the formal test can be done by running 5 regressions of MA(1, *s*)LBt = β_s $PLB_t + \varepsilon_t$ for each index, where t = 1 … 22. If $\beta_s > 1$ and both are statistically significant, it means that there exists amplification of autocorrelation and the degree of the amplification depends on the level of autocorrelation of the original index price process. Table 15 and 16 summarizes the test result.

Table 15: βs Estimates in Matured Markets

	Matured				
s	5	10	20	30	60
S&P/TSX	1.113	1.194	1.279	1.344	1.158
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
CAC40	1.208	1.409	1.722	1.94	1.607
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
DAX30	1.156	1.313	1.555	1.687	1.306
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Nikkei 225	1.146	1.269	1.46	1.549	1.344
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
FTSE100	1.175	1.307	1.491	1.652	1.447
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
S&P500	1.143	1.262	1.41	1.488	1.206
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

Table 16: βs Estimates in Emerging Markets

Emerging					
s	5	10	20	30	60
Bovespa	1.131	1.24	1.381185	1.505178	1.477067
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Hang Seng	1.105	1.193	1.296333	1.363696	1.359526
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
IHSG (Jakarta)	1.088	1.169	1.267139	1.29838	1.139249
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
KOSPI	1.129	1.246	1.414025	1.485353	1.200378
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
FSSTI	1.078	1.141	1.232656	1.281892	1.095732
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
RTS	1.114	1.228	1.415172	1.55133	1.392335
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

Empirical test shows all β_s are larger than 1 and statistically significant. From this formal test, we can conclude that the amplification of the overall correlation in log price occurs in both Matured and Emerging markets when the MA rule is applied. We believe this can explain why the momentum trading strategy is still valid and widely used in matured markets where there has been no significant upward trend in the markets.

6 Conclusion

This paper first presents a technical definition of momentum in order to investigate how technical analysis, especially the MA rule, becomes profitable and theoretically demonstrates that the trading rule is profitable. The theoretical examination revealed that, in addition, price autocorrelation might be another fundamental reason for profitability, and the further investigation of the autocorrelation structure of the MA rule showed that the MA rule reflects the price autocorrelation structure, without identifying it. This analysis sheds some light on why analysts use technical trading rules. There are three main findings in this paper:

1. The MA rule identifies and takes advantage of price trend
2. 2. The MA rule is a simple and straightforward way of exploiting price autocorrelation without actually knowing it

3. 3. In the case of MA(1, *s*) rules, where s is a time period for the short position's MA, the amplification of the autocorrelation structure is more prominent for a small *s* when the level of autocorrelation is low and for a large *s*, when the level of autocorrelation is high

The first finding is already well known and discussed in the earlier literature. We confirm it by constructing a simple model with a single asset. The second finding has not been investigated in the previous literature due to lack of a precise technical definition of momentum. We show this with simulated results and empirical data from 6 international indices from developing and mature markets. The third finding suggests what length of s should be applied in employing the MA rule.

These findings should be considered important in themselves since they identify why the MA rule is used. However, the practical application of this is also interesting. One of many natural questions that can be asked is how well would the MA rule would perform if the full price autocorrelation structure was known. After all, we saw that the MA rule is all about simplification of conditioning on the past price information. Therefore, if we use the entire past price information, this should outperform the MA rule. However, in practice the tradeoff between the inefficiency and the cost of knowing the full autocorrelation would justify the use of the MA rule. Nevertheless the extent of the efficiency gain would be a worthy area for future research.

Reference

Achelis, S. (2000): "Technical Analysis from A to Z", 2nd Edition, McGraw-Hill Companies

Allen, F. and R. Karjalainen. (1999): "Using Genetic Algorithms to Find Technical Trading Rules." Journal of Financial Economics 51, 245–271.

Blume, L., Easley, D. and O'Hara, M. B. (1994): "Market Statistics and Technical Analysis: The Role of Volume." Journal of Finance 49, 153–181.

Brock, W., Lakonishok, J., and LeBaron, B. (1992): "Simple Technical Trading Rules and the Stochastic Properties of Stock Returns", *The Journal of Finance*, Vol. 47, No. 5, December 1992, pp. 1731 – 1764.

Brown, S. J., W. N. Goetzmann, and A. Kumar. (1998): "The Dow Theory: William Peter Hamilton's Track Record Reconsidered." Journal of Finance 53, 1311–1333.

Carhart, M.M. (1997): "On persistence in mutual fund performance", Journal of Finance 52, 57-82.

Chan, K., Hameed, A. and Tong, W. (2000): "Profitability of Momentum Strategies in the International Equity Markets." Journal of Financial and Quantitative Analysis 35, 153–172.

Chan, K., Jegadeesh, N. and Lakonishok, J. (1996): "Momentum Strategies", Journal of Finance, Vol. 51 No. 5.

Chang, P. H. K. and Osler, C. L. (1999): "Methodological Madness: Technical Analysis and the Irrationality of Exchange-Rate Forecasts." Economic Journal 109, 636–661.

Fama, E.F. and French, K.R. (1996): "Multifactor explanations of asset pricing anomalies", Journal of Finance 51, 55-84.

Fong, W. M. and Yong, H. M. (2005): "Chasing trends: recursive moving average trading rules and internet stocks", Journal of Empirical Finance, Volume 12, Issue 1, January 2005, Pages 43-76.

Gencay, R. (1996): "Non-Linear Prediction of Security Returns with Moving Average Rules", Journal of Forecasting 15, 165–174.

Gencay, R. (1998): "The Predictability of Security Returns with Simple Technical Trading Rules." Journal of Empirical Finance 5, 347–359.

Gencay, R. (1999): "Linear, Non-Linear and Essential Foreign Exchange Rate Prediction with Simple Technical Trading Rules", Journal of International Economics 47, 91– 107.

Gunasekarage, A. and Power, D.M. (2000): "The profitability of moving average trading rules in South Asian stock markets", Emerging Markets Review 2 (2001) 17-33.

Hudson, R., Dempsey, M. and Keasey, K. (1996): "A note on the weak form efficiency of capital markets: the application of simple technical trading rules to UK stock prices - 1935 to 1994", Journal of Banking and Finance 20, 1121-1132.

Jegadeesh N. and Titman S. (1993): "Returns to buying winners and selling losers: Implications for stock market efficiency", Journal of Finance 48, 65–91.

Jegadeesh N. and Titman S. (2001): "Profitability of momentum strategies: an evaluation of alternative explanations", Journal of Finance 56:699–720.

Lo, A. W., H. Mamaysky, and J. Wang. (2000): "Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation." Journal of Finance 55, 1705–1765.

Moskowitz T., Ooi, Y. H., and Pedersen L.H. (2010): "Time Series Momentum", Working paper

Neely, C. J. and Weller. P. (1999): "Technical Trading Rules in the European Monetary System." Journal of International Money and Finance 18, 429–458

Neely, C. J., Weller, P. and Dittmar, R. (1997): "Is Technical Analysis in the Foreign Exchange Market Profitable? A Genetic Programming Approach." Journal of Financial and Quantitative Analysis 32, 405–426.

Reitz, S., (2006): "On the predictive content of technical analysis", The North American Journal of Economics and Finance, 17(2), 121-137.

Rouwenhorst, K. G. (1998): "International Momentum Strategies", Journal of Finance, Vol. 53, No. 1, pp. 267-284.

Schwager, J. D. (1992) The New Market Wizards: Conversations With America's Top Traders. John Wiley and Sons, (pg. 224)

Shleifer, A. (2000): "Inefficient Markets", Oxford University Press.

Sweeney, R. J. (1986): "Beating the Foreign Exchange Market." Journal of Finance 41, 163–182.

Sweeney, R. J. (1988): "Some New Filter Rule Tests: Methods and Results." Journal of Financial and Quantitative Analysis 23, 285–300.

Appendix

A1 Proof of Proposition 1

The ratio of future price in period h to today's price can be represented as

$$\frac{P(t+h)}{P(t)} = P(t)^{(\exp(-h\theta)-1)} \exp\left(\sigma \int_t^{t+h} \exp\left(-\theta(t-\mu\right) dW(u)\right) \exp(\mu h)$$

Hence the expected value of this is

$$E\left(\frac{P(t+h)}{P(t)}\right) = P(t)^{(\exp(-h\theta)-1)} \exp\left(\frac{1}{4}\sigma^2 \frac{\exp(2h\theta)-1}{\theta}\right) \exp(\mu h)$$

This can be thought of as the population momentum measure conditioned on time t. Taking logs, we get

$$\ln E\left(\frac{P(t+h)}{P(t)}\right) = \left(\exp(-h\theta)-1\right) \ln P(t) + \frac{1}{4}\sigma^2 \frac{\exp(2h\theta)-1}{\theta} + \mu h$$

Therefore we have

$$\frac{\partial \ln E\left(\frac{P(t+h)}{P(t)}\right)}{\partial \ln P(t)} = \exp(-h\theta) - 1$$

A2 Derivation of Definition 2

$$C(t) = \exp\left(\frac{\sum_{i=1}^{l} q(t-i+1)}{l}\right) \Bigg/ \exp\left(\frac{\sum_{i=1}^{s} q(t-i+1)}{s}\right)$$

$$= \exp\left(\frac{\sum_{i=1}^{l} q(t-i+1)}{l} - \frac{\sum_{i=1}^{s} q(t-i+1)}{s}\right)$$

$$= \exp\left(\sum_{i=1}^{l}\left(\frac{1}{l} - \frac{1}{s}\right) q(t-i+1) + \sum_{i=1}^{s} \frac{q(t-i+1)}{s}\right) = \exp\left(w' Q(t)\right)$$

A3 Derivation of Lemma 1

$$\log(P(t)) - \mu t > \frac{\sum_{j=0}^{m-1}(\log(P(t-j)) - \mu(t-j))}{m}$$

$$\Rightarrow \log(P(t)) > \frac{\sum_{j=0}^{m-1}\log(P(t-j))}{m} + \frac{\sum_{j=0}^{m-1}\mu j}{m}$$

$$\Rightarrow \log(P(t)) > \frac{\sum_{j=0}^{m-1}\log(P(t-j))}{m} + \frac{\mu(m-1)}{2}$$

$$\Rightarrow \log(P(t)) > \frac{1}{m}\log\left(\prod_{j=0}^{m-1}P(t-j)\right) + \frac{\mu(m-1)}{2}$$

$$\Rightarrow P(t) > \exp\left(\frac{\mu(m-1)}{2}\right)\left(\prod_{j=0}^{m-1}P(t-j)\right)^{\frac{1}{m}}$$

$$\Rightarrow (P(t))^{\frac{m-1}{m}} > \exp\left(\frac{\mu(m-1)}{2}\right)\left(\prod_{j=1}^{m-1}P(t-j)\right)^{\frac{1}{m}}$$

Therefore,

$$P(t) > \exp\left(\frac{\mu m}{2}\right)\left(\prod_{j=1}^{m-1}P(t-j)\right)^{\frac{1}{m-1}}$$

A4 Proof of Remark 1

Since $E[X|Y] = E[X] + \rho\frac{\sigma_X}{\sigma_Y}(Y - E[Y])$ and $\sigma_{X|Y}^2 = (1-\rho^2)\sigma_X^2$, the unconditional and the conditional probability distribution of C(t) can be written as

$$prob(C(t) > 1) = prob(w'Q(t) > 0) = 0.5$$

$$pdf(w'Q(t)|w'Q(t-1)) \sim N(\rho(1)w'Q(t-1), (1-\rho^2(1))w'\Omega w)$$

$$prob(w'Q(t) > \eta|\xi) = 1 - \Phi\left(\frac{\eta - \rho(1)(q(t-1) - SMA(t-1))}{\sqrt{(1-\rho^2(1))w'\Omega w}}\right)$$

where $\xi : w'Q(t-1) = q(t-1) - SMA(t-1)$

Since $\alpha(t) = \begin{cases} 1 \text{ if } \mathrm{C}(t) > 1 \Leftrightarrow w'Q(t) > 0 \\ 0 \text{ if } \mathrm{C}(t) < 1 \Leftrightarrow w'Q(t) < 0 \end{cases}$ hence $E[\alpha(t)] = prob(C(t) > 1)$

Unconditionally, $prob(\alpha(t) = 1) = 0.5$

Conditionally, $prob(\alpha(t)=1|\xi)=1-\Phi\left(\frac{-\rho(1)(q(t-1)-SMA(t-1))}{\sqrt{(1-\rho^2(1))w'\Omega w}}\right)$

A5 Simulation Procedure

1. We have $q(t)=\log P(t)-\mu t-c$, assume $q(t)=\gamma+\rho_A q(t-1)+\varepsilon_t$
   From GS data (May 4, 1999 - May 31, 2011), estimate c, and σ_ε

   $c=4.182011$, $\sigma_\varepsilon=0.002687$, $\gamma=0.0008$ and set $\mu=0$

   Let % trading cost = 0%, annual risk free rate: 2% ➔ daily risk free rate: 0.008%

2. Simulate $\varepsilon \sim N(0,\sigma_\varepsilon^2)$

3. $q(t)=\gamma+\rho_A q(t-1)+\varepsilon_t$, $\log P(t)=q(t)+\mu t+c$, $C(t)=q(t)/SMA(t)$
   $SMA(t)=\exp\left((1/s)\sum_{i=1}^{s}q(t-i+1)\right)$, $\alpha(t)=\begin{cases}1 \text{ if } C(t)>1 \Leftrightarrow w'Q(t)>0\\ 0 \text{ if } C(t)<1 \Leftrightarrow w'Q(t)<0\end{cases}$

   $R_A(t)=\log P(t)-\log P(t-1)$

4. % Return from the MA rule:
   $R_T(t+1)=\alpha(t)R_A(t+1)+(1-\alpha(t))(r_f)-TC(t+1)$

A6 Proof of Proposition 3

Decompose Ω into four pieces where Ω_{11} is $l\times l$, Ω_{12} is $l\times(s-l)$, Ω_{21} is $(s-l)\times l$ and Ω_{22} is $(s-l)\times(s-l)$, also decompose $w'=(w_1' \quad w_2')$ where w_1' is $1\times s$ and w_1' is $1\times(s-l)$

Let $a=\left(\frac{1}{l}-\frac{1}{s}\right)$ and $b=\left(\frac{1}{s}\right)$

$$\underset{l\times l}{\Omega_{m11}} = \sigma^2 \begin{bmatrix} \rho(m) & \rho(m+1) & \cdots & \rho(m+l-1) \\ \rho(m-1) & \rho(m) & \cdots & \rho(m+l) \\ & & \ddots & \vdots \\ \vdots & & \rho(m) & \\ & & & \ddots \\ \rho(m-l+1) & & \cdots & \rho(m) \end{bmatrix}$$

$$\underset{l\times(s-l)}{\Omega_{m12}} = \sigma^2 \begin{bmatrix} \rho(m+l) & \rho(m+l+1) & \cdots & \rho(m+s-1) \\ \rho(m+l-1) & \rho(m+l) & \cdots & \rho(m+s-2) \\ \vdots & & & \vdots \\ & & \ddots & \\ \rho(m+1) & & \cdots & \rho(m+s-l) \end{bmatrix}$$

$$\underset{(s-l)\times l}{\Omega_{m21}} = \sigma^2 \begin{bmatrix} \rho(m-l) & \rho(m-l+1) & \cdots & \rho(m-1) \\ \rho(m-l+1) & \rho(m-l) & & \\ \vdots & \vdots & \ddots & \vdots \\ \rho(m-s+1) & \rho(m-s+2) & \cdots & \rho(m-s+l) \end{bmatrix}$$

$$\underset{(s-1)\times(s-1)}{\Omega_{m22}} = \sigma^2 \begin{bmatrix} \rho(m) & \rho(m+1) & \cdots & \rho(m+s-l-1) \\ \rho(m-1) & \rho(m) & \cdots & \rho(m+s-l-2) \\ & \ddots & & \vdots \\ \vdots & & \rho(m) & \\ & & & \ddots \\ \rho(m-s+l-1) & & \cdots & \rho(m) \end{bmatrix}$$

$$\Omega_m = \begin{bmatrix} \Omega_{m11} & \Omega_{m12} \\ \Omega_{m21} & \Omega_{m22} \end{bmatrix}, \; w_1' = \begin{pmatrix} a & \cdots & a \end{pmatrix} \text{ and } w_2' = \begin{pmatrix} b & \cdots & b \end{pmatrix}$$

$$w_1'\Omega_{m11}w_1 = a^2\sigma^2\left(l\rho(m) + \sum_{i=1}^{l-1}\big(\rho(m+i)+\rho(m-i)\big)\right)$$

$$w_2'\Omega_{m22}w_2 = b^2\sigma^2\left((s-l)\rho(m) + \sum_{i=1}^{s-l-1}(s-i-1)\big(\rho(m+i)+\rho(m-i)\big)\right)$$

$$w_1'\Omega_{m12}w_2 = ab\sigma^2\left(\sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho(m+l+i-j)\right)$$

$$w_2'\Omega_{m21}w_1 = ab\sigma^2\left(\sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho(m-l-i+j)\right)$$

$$
\begin{aligned}
w'\Omega_m w &= w_1{}'\Omega_{m11}w_1 + w_1{}'\Omega_{m12}w_2 + w_2{}'\Omega_{m21}w_1 + w_2{}'\Omega_{m22}w_2 \\
&= \sigma^2 \left(\begin{aligned} & a^2\left(l\rho(m) + \sum_{i=1}^{l-1}\big(\rho(m+i)+\rho(m-i)\big)\right) \\ & + b^2\left((s-l)\rho(m) + \sum_{i=1}^{s-l-1}(s-i-1)\big(\rho(m+i)+\rho(m-i)\big)\right) \\ & + ab\left(\sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho(m+l+i-j) + \sum_{j=0}^{l-1}\sum_{i=0}^{s-l-1}\rho(m-l-i+j)\right) \end{aligned} \right)
\end{aligned}
$$