Bayesian Calibration of Exponential Lévy Models

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Abstract

In this paper, a Bayesian estimation method for calibrating parameters of exponential Lévy models, used to predict distributions of option prices, is proposed. As real option prices are noisy, it is appropriate for option pricing models to provide confidence intervals for option prices. An algorithm for calibrating the parameter sets of exponential Lévy models is presented, and the performance of the proposed method is verified by comparing model-generated option prices with real market option prices such as KOSPI 200 index option prices. Simulation results show that the proposed method calibrates the parameter sets effectively, overcoming the ill-posed inverse problem of model calibration and enabling the construction of reasonable predicted distributions of option prices.

Key words:
Exponential Lévy models, Bayesian estimation method, Model calibration

1 Introduction

In the recent finance literature, the exponential Lévy process has been examined to gain an understanding of the intrinsic properties of asset returns and option prices (see Gutierrez (2008); Cont and Tankov (2004)). It is clear, based on considerable empirical research, that the Black-Scholes-Merton option pricing formula can not explain such properties of option prices in real markets as implied volatility smile behavior, whereas, by contrast, an option pricing theory based on the exponential Lévy process defined by its characteristic triplet (σ, ν, γ), can reproduce features of option prices traded in real

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markets (see Balland (2002); Cont and Fonseca (2002); Dumas et al. (1998)).

Although option pricing models that employ the Fourier analysis can be used to calculate option prices, the Fast fourier transform (FFT) method cannot be directly applied to the option pricing formula, due to the singularity of the payoff function at $v = 0$ (see Minenna and Verzella (2008)). To overcome this problem, Carr and Madan (1999) propose an approach that circumvents this problem through use of a damping parameter. We can easily estimate option prices, using Fourier transform if the characteristic function of the log prices in the risk-neutral measure is analytically known. In particular, the five Lévy process models, Merton (1976), Kou (2002), Variance gamma (see Madan (1998)), Normal inverse gaussian (see Barndorff (1997)), and CGMY (see Carr et al. (2003)) considered in this paper, can be expressed in analytic form. Using the FFT method to estimate option prices is thus made simple by way of the Lévy processes.

In most models with jumps, suitable choice of an equivalent martingale measure allows for the generation of any given price for a given European option: a range of possible prices obtained by picking various equivalent martingale measures is the maximal interval allowed by static arbitrage arguments.

Because the market is incomplete, knowledge of historical price processes alone does not allow for the computation of unique option prices. That is the main difficulty of using a historical approach to the identification of exponential-Lèvy models.

In incomplete market models, a risk-neutral measure $\mathbb{Q}$ bears only a weak relationship to the time series behavior described by $\mathbb{P}$: Q cannot be identified from $\mathbb{P}$ but only inherits some of its qualitative properties such as the presence of jumps, infinite or finite activity, and infinite or finite variation of the historical price process. Therefore, a natural approach, known as "implied" or risk-neutral modeling, is to directly model the risk-neutral dynamics of the asset by choosing a pricing measure $\mathbb{Q}$ that respects the qualitative properties of the asset price.

When option prices are quoted on the market, a market-consistent risk-neutral pricing model $\mathbb{Q}$ can not be obtained merely from an econometric analysis of the time series of the underlying returns but requires an examination of prices of contingent claims today ($t = 0$). One chooses a risk-neutral model so as to reproduce the prices of traded options, called **model calibration**, and then uses this model to price exotic, illiquid or OTC options and to compute hedge ratios.

Calibration Problem 1 *Given an exponential-Lèvy model* $(\sigma(\theta), \nu(\theta), \gamma(\theta))$ *and observed prices* C_i *of call options with maturities* T_i *and strike prices*

$K_i, i \in I$, find θ such that the discounted asset price $S_t \exp(-rt)$ is a martingale, and the observed option prices are given by their discounted risk-neutral expectations:

$$\forall i \in I, \quad C_i = e^{-rT} E^{\theta}[S_{T_i} - K_i)^+]$$

where E^{θ} denotes the expectation computed using the exponential-Lévy model with triplet $(\sigma(\theta), \nu(\theta), \gamma(\theta))$.

Because calibration of these models with market option data is usually formulated as a nonlinear optimization problem, such calibration often suffers from local minima problems, which can lead to poor performance or wrong information in analyzing derivatives markets. Therefore, to successfully apply a given model to the pricing of complex options and to analyze the behavior of cross-sectional option data, it is very important to obtain a parameter set that calibrates the market data well. In this paper, to overcome the ill-posed problem, we propose a Bayesian estimation approach with assumptions regarding the distribution of the parameters representing the prior distribution. To verify the performance of the proposed method, we conduct simulations of the model-generated option prices, KOSPI200 index option prices, which are traded in KRX(Korea Exchange).

The paper is structured as follows. In section II, we set out the five widely used exponential Lévy models and the FFT method of option pricing. In section III, we present a Bayesian estimation method to calibrate the five exponential Lévy models and provide a method for calculating the hyperparameters of the Bayesian estimation method. We then describe the calibration algorithm in section IV. We test the robustness of the proposed method with model-generated option prices in section V and conduct empirical tests, using the KOSPI200 index option prices, in section VI. Section VII concludes.

2 Exponential Lévy models

In this section we briefly describe exponential Lévy models, closely following the notation introduced by Cont and Tankov (2004); details can be found in that reference.

2.1 Review of Lévy processes

Definition 1 (Lévy process) *A cadlag stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\Re^d$ such that $X_0 = 0$, is called a* ***Lévy process*** *if it possesses* *the following properties:*

(1) Independent increments: for every increasing sequence of units of time $t_0, ..., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.
(2) Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.
(3) Stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

The third condition excludes processes with jumps at fixed (nonrandom) times, which can be regarded as "calendar effects": discontinuities occur at random times. The measure ν on $\Re^d$, defined by

$$\nu(A) = E[\#\{t \in [0,1] : \Delta X_t \neq 0,\ \Delta X_t \in A\}], \qquad A \in \mathcal{B}(\Re^d)$$

is called the **Lévy measure** of X: $\nu(A)$ is the expected number, per unit of time, of jumps whose sizes belong to A.

Theorem 2 *[Lévy-Itô decomposition] Let $(X_t)_{t\geq 0}$ be a Lévy process on $\Re^d$, where ν is its Lévy measure. Then,*

- *ν is a Radon measure on $\Re^d \setminus \{0\}$ and is verified as follows:*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \qquad \int_{|x|\geq 1} \nu(dx) < \infty$$

- *Its jump measure J_X is a Poisson random measure on $[0,\infty) \times \Re^d$, with intensity measure $\nu(dx)dt$. ($J_X([0,t] \times A)$ is the number of jumps of X, occurring between 0 and t, whose amplitudes belong to A.)*
- *There is a vector γ and a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ with covariance matrix A such that*

$$X_t = \gamma t + W_t + X_t^l + \lim_{\epsilon \downarrow 0} \tilde{X}_t^\epsilon \tag{1}$$

$$X_t^l = \int_{|x|\geq 1, s\in[0,t]} x J_X(ds \times dx) = \sum_{0\leq s\leq t}^{|\Delta X_s|\geq 1} \Delta X_s,$$

$$\tilde{X}_t^\epsilon = \int_{\epsilon\leq|x|<1, s\in[0,t]} x\{J_X(ds \times dx) - \nu(dx)ds\} = \int_{\epsilon\leq|x|<1, s\in[0,t]} x\tilde{J}_X(ds \times dx).$$

*The terms in (1) are independent, and the convergence in the last term is almost sure and uniform in t on $[0,T]$. The triplet (A,ν,γ) is called a **Lévy triplet** of the process X_t.*

Here, $\tilde{X}_t^\epsilon$ is a martingale and can be interpreted as an infinite superposition of independent compensated Poisson processes. This implies that every Lévy process can be approximated with arbitrary precision by a jump-diffusion process: the sum of a Brownian motion with drift and a compound Poisson process. Note that when ν has a singularity at zero, there can be infinitely many small jumps, and the sum in the uncompensated jump integral does not necessarily converge. This fact prevents us from making ϵ go to 0 directly, and we must replace it by its compensated version to obtain convergence.

Theorem 3 *[Lévy-Khinchin representation] Let $(X_t)_{t\geq 0}$ be a Lévy process on $\Re^d$, with Lévy triplet (A, ν, γ). Then its characteristic function Φ_X and characteristic exponent Ψ_X are given by*

$$\Phi_X(z) = \mathbb{E}[e^{iz^\top X_t}] = e^{t\Psi_X(z)}$$
$$\Psi_X(z) = -\frac{1}{2}z^\top Az + i\gamma^\top z + \int_{\Re^d}(e^{iz^\top x} - 1 - iz^\top x 1_{|x|\leq 1})\nu(dx)$$

$(X_t)_{t\geq 0}$ is a Lévy process of finite variation if and only if its Lévy triplet is given by $(0, \nu, \gamma)$, with $\int_{\Re^d} |x|\nu(dx) < \infty$. Its characteristic exponent is then

$$\Psi_X(z) = i\beta^\top z + \int_{\Re^d}(e^{iz^\top x} - 1)\nu(dx), \quad \text{where} \quad \beta = \gamma - \int_{|x|\leq 1} x\nu(dx).$$

2.2 Exponential Lévy processes

A tractable class of risk neutral models with jumps generalizing the Black-Scholes model can be obtained by the **exponential-Lévy model**, as in

$$S_t = S_0 \exp(rt + X_t)$$

where $(X_t)_{t\geq 0}$ is a Lévy process on $\Re$, with Lévy triplet (σ^2, ν, γ). Then, to guarantee that $e^{-rt}S_t$ is a martingale, we should impose additional restrictions on the Lévy triplet (σ^2, ν, γ) of X, specifically, that $E^{\mathbb{Q}}[e^{X_t}] < \infty$, i.e., $\int_{|x|\geq 1} e^x \nu(dx) < \infty$ (see Sato (1999), Theorem 25.17) and $E^{\mathbb{Q}}[e^{X_t}] = e^{t\Psi_X(-i)} = 1$ for all t under a risk neutral martingale measure $\mathbb{Q}$, i.e.,

$$\Psi_X(-i) = \gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty}(e^x - 1 - x1_{|x|\leq 1})\nu(dx) = 0.$$

Exponential-Lévy models enable the use of Fourier transform methods in option pricing due to the availability of closed-form expressions for characteristic functions of Lévy processes, which makes it possible to calibrate models to market option prices and reproduce implied volatility skews/smiles.

Since the introduction of the Black-Scholes option pricing model by Black, Scholes and Merton, alternative option pricing models have been studied in order to understand characteristic features of option prices in real markets, for example, the volatility smile effects. Well known and widely used, Lévy processes employed in the financial literature can be divided into two categories: jump-diffusion models and infinite activity models. Jump-diffusion models include diffusion processes and jumps that represent abnormal rare events for example, crashes or bubbles that occur in real markets at random intervals with known distributions of jump size. In contrast, infinite activity models are models with an infinite number of jumps in every interval, where distributions of jump sizes do not exist. In addition, such models do not necessarily contain diffusion processes because the dynamics of jumps are already sufficiently rich to provide realistic descriptions of the historical price process over various time scales.

In this paper, we consider a risk-neutral stock price process given by

$$S_t = S_0 \exp((r - q)t + X_t(\theta) + \omega t)$$

where r and q denote the constant continuously compounded interest rate and dividend yield, respectively, and $X_t(\theta)_{t\geq 0}$ is a parameterized Lévy process on $\Re$, with parameter set θ. Here, ω is introduced to guarantee the martingale property of the price process (Carr, 2003), i.e., $E^{\mathbb{Q}}[e^{X_t(\theta)+\omega t}] = e^{t\Psi_X(-i)}e^{\omega t} = 1$, where $\omega = -\Psi_X(-i)$.

Five widely used homogeneous exponential Lévy processes are employed in this paper and are briefly introduced in this section, with their parameter sets and characteristic functions. Their detailed statistical and computational properties are provided in the appendix section 8.1.

2.3 Lévy processes via jump-diffusion model

A Lévy process of the jump-diffusion type has the following form:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where $(N_t)_{t\geq 0}$ is a Poisson process of the number of jumps of X, and Y_i is jump size. Two widely used jump-diffusion models are the following:
1) **Merton model:** a jump-diffusion model with Gaussian jumps

- Four parameters $\theta = (\sigma, \lambda, \mu, \delta)$, where σ is the diffusion volatility, λ is the jump intensity, μ is mean jump size, and δ is the standard deviation of the jump size.

- Characteristic function of the log prices, $s_t = \ln(S_t/S_0)$:

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz\{(r-q+\omega_0)t\} + t\left\{-\frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{(i\mu z - \frac{1}{2}\delta^2 z^2)} - 1\right)\right\}\right]\end{aligned}$$

  where

$$\omega_0 = \omega + \gamma = -\frac{1}{2}\sigma^2 + \lambda\left(1 - \exp\left(\mu + \frac{1}{2}\delta^2\right)\right)$$

2) **Kou model:** a jump-diffusion model with double exponential jumps

- Five parameters $\theta = (\sigma, \lambda, \lambda_+, \lambda_-, p)$, where σ is the diffusion volatility, λ is the jump intensity, and λ_+, λ_-, p are parameters of jump size distribution.
- Characteristic function of the log prices, $s_t = \ln(S_t/S_0)$:

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= e^{iz(r-q+w_0)t + t\{-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda(p\frac{\lambda_+}{\lambda_+ - iz} + (1-p)\frac{\lambda_-}{\lambda_- + iz} - 1)\}}\end{aligned}$$

  where

$$\omega_0 = \omega + \gamma = -\frac{1}{2}\sigma^2 + \lambda\left(1 - p\frac{\lambda_+}{\lambda_+ - 1} - (1-p)\frac{\lambda_-}{\lambda_- + 1}\right)$$

2.4 Lévy processes via infinite activity models

2.4.1 Brownian subordinated Lévy processes

Let T_t be a subordinator, i.e. its trajectories are almost surely nondecreasing with the Laplace exponent $l(u) = \int_0^\infty (e^{ux} - 1)\rho(dx)$ where $\mathbb{E}[e^{uT_t}] = e^{t\cdot l(u)}$. An infinite activity Lévy process can be obtained by subordinating the Brownian motion by subordinator T_t as follows:

$$X_t = \gamma T_t + \sigma W(T_t),$$

Then, the characteristic exponent of X_t is given by $\Psi_X(z) = l(-z^2\sigma^2/2 + i\gamma z)$ and the Lévy triplet (A_X, ν_X, γ_X) of X_t is given by

$$A_X = 0,\ \nu_X(x) = \int_0^\infty p_s^T(x)\rho(ds),\ \gamma_X = \int_0^\infty \rho(ds)\int_{|x|\le 1} x p_s^T(dx)$$

where p_t^T is the probability distribution of the subordinator T_t. This process is a Brownian motion observed on a new time scale, for example, business time.

Widely used subordinators are α-stable processes with $\alpha \in [0, 1)$. Because a subordinator has no diffusion component, only positive jumps of finite variation and positive drift, the Lévy measure of a real-valued α-stable process is of the form $\frac{ce^{-\lambda x}}{x^{\alpha+1}} 1_{x>0}$; by tempering this Lévy measure, we obtain a tempered stable subordinator, which is a three-parameter process, with Lévy measure $\rho(x) = \frac{ce^{-\lambda x}}{x^{\alpha+1}} 1_{x>0}$ where $c > 0$ alters the intensity of jumps (i.e., the time scale of the process) of all sizes simultaneously, $\lambda > 0$ fixes the decay rate of big jumps, and $1 > \alpha \geq 0$ indicates the relative importance of small jumps in the path of the process. The probability density of the tempered stable subordinator is only known in explicit form for $\alpha = 0$ (variance gamma) and $\alpha = 1/2$ (normal inverse Gaussian); thus, the corresponding subordinated processes have been widely used because they are easier to simulate and more mathematically tractable.

3) **The Variance Gamma model:**

- Three parameters $\theta = (\gamma, \sigma, \kappa)$, where γ is diffusion drift, σ is diffusion volatility, and κ is the variance of the subordinator.
- Characteristic function of the log prices, $s_t = \ln(S_t/S_0)$:

$$\Phi_s(z) = \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)}$$
$$= e^{iz(r-q+\omega)t} \left(\frac{1}{1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2} \right)^{\frac{t}{\kappa}}$$

where

$$\omega = \frac{1}{\kappa} \ln \left(1 - \gamma\kappa - \frac{1}{2}\sigma^2\kappa \right)$$

4) **Normal Inverse Gaussian model:**

- Three parameters $\theta = (\gamma, \sigma, \kappa)$, where γ is diffusion drift, σ is diffusion volatility, and κ is variance of the subordinator.
- Characteristic function of the log prices, $s_t = \ln(S_t/S_0)$:

$$\Phi_s(z) = \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)}$$
$$= \exp\left[iz(r - q + \omega)t + t \left(\frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2i\gamma\kappa z + \sigma^2\kappa z^2} \right) \right]$$

where

$$\omega = \frac{1}{\kappa} \left(\sqrt{1 - 2\gamma\kappa - \sigma^2\kappa} - 1 \right)$$

2.4.2 Tempered stable Lévy processes

A tempered stable process is a Lévy process $\Re$ with no Gaussian component and is constructed by directly specifying a Lévy measure of the form

$$\nu(x) = \frac{c_-}{(-x)^{1+\alpha_-}} e^{\lambda_- x} 1_{x<0} + \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0}$$

where $c_\pm, \lambda_\pm > 0$ and $\alpha_\pm < 2$. The process can be represented as a time changed Brownian motion (with drift) if and only if $c_- = c_+$ and $\alpha_- = \alpha_+ = \alpha \geq -1$. This condition on the coefficients (e.g. a CGMY model with four parameters) implies that small jumps must be symmetric, whereas decay rates for big jumps may vary. Because the main impact on option prices is due to large jumps, the CGMY subclass is as flexible as the whole class of tempered stable processes that allow for asymmetry of small jumps.
5) **CGMY model:** (also called "truncated Lévy flights")

- Four parameters $\theta = (c, \alpha, \lambda_-, \lambda_+)$, where c determines the overall and relative frequency of jumps; α determines the local behavior of the process (how the price evolves between big jumps); and λ_-, λ_+ determines the tail behavior of the Lévy measure.
- Characteristic function of the log prices, $s_t = \ln(S_t/S_0)$:

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz \ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz(r-q+\omega)t + tc\Gamma(-\alpha)\{(\lambda_+ - iz)^\alpha - \lambda_+^\alpha + (\lambda_- + iz)^\alpha - \lambda_-^\alpha\}\right]\end{aligned}$$

where

$$\omega = c\Gamma(-\alpha)\{(\lambda_+ - 1)^\alpha - \lambda_+^\alpha + (\lambda_- + 1)^\alpha - \lambda_-^\alpha\}$$

3 Fourier transform methods for option pricing

The availability of an explicit formula for the characteristic functions of exponential Lévy models as in Section 2, led to the development of Fourier-based methods of option pricing. We now briefly review Carr-Madan's Fourier transform method, which we have employed to calibrate exponential Lévy models in this paper.

A necessary assumption of Carr-Madan's method is that the Lévy density of X_T satisfies:

$$\int_{|y|\geq 1} \nu(dy) e^{(1+\alpha)y} < \infty \tag{2}$$

which holds for all the five models in Section 2 when a constraint is placed on the exponential decay parameter for positive jumps (negative jumps do not affect it).

The European call option price of maturity T and strike price K is

$$C_T(k) = S_0 \cdot e^{-rT} E[(S_T/S_0 - K/S_0)_+] = S_0 \cdot \int_k^\infty e^{-rT}(e^s - e^k)\rho_T(s)ds. \tag{3}$$

where $s_t = \ln(S_t/S_0)$, $k = \ln(K/S_0)$, and ρ_T is the risk-neutral density of X_T.

Because $C(k,T)$ is not integrable (because $C(k,T) \to S_0$ as $k \to \infty$), we define the following adjusted call prices to compute its Fourier transform:

$$c_\alpha(k,T) \equiv e^{\alpha k} C(k,T)/S_0,$$

with the damping parameter $\alpha > 0$. Then, $c_\alpha(k,T)$ is square integrable in k over the whole real range of proper positive values of α, and we can compute the Fourier transform of $c_\alpha(k,T)$ as follows:

$$\begin{aligned}
\psi_T(z) &= \int_{-\infty}^{+\infty} e^{izk} c_\alpha(k,T)dk = \int_{-\infty}^{+\infty} e^{izk} e^{\alpha k} \int_k^{\infty} e^{-rT}(e^s - e^k)\rho_T(s)dsdk \\
&= \int_{-\infty}^{+\infty} e^{-rT}\rho_T(s)ds \int_{-\infty}^{s} (e^{\alpha k+s} - e^{(\alpha+1)k})e^{izk}dk \\
&= e^{-rT}\int_{-\infty}^{+\infty} \left[\frac{e^{(\alpha+1+iz)s}}{\alpha+iz} - \frac{e^{(\alpha+1+iz)s}}{\alpha+1+iz}\right]\rho_T(s)ds \\
&= \frac{e^{-rT}\Phi_s(z-(\alpha+1)i)}{\alpha^2+\alpha-z^2+i(2\alpha+1)z}.
\end{aligned}$$

where $\Phi_s(z) = \mathbb{E}[e^{izs_t}]$ is the characteristic function of the log prices $s_t = \ln(S_t/S_0)$. By choosing $\alpha > 0$, we can estimate the integrand at the singular point $\nu = 0$ because the denominator is then nonzero. In this paper, we will use $\alpha = 1.6$.

Option prices can now be found by inverting the Fourier transform:

$$C(k,T) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\psi_T(v)dv. \tag{4}$$

To compute (4) for N-log strike levels k ranging from $-b$ to b, i.e.

$$k_u = -b + \Delta_k(u-1), \quad \text{for} \quad u = 1, ..., N, \quad \Delta_k = 2b/N,$$

we apply the fast Fourier transform (FFT) method to (4) which is an $O(N \ln N)$ algorithm for computing

$$w(u) \approx \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} x(j) \quad \text{for} \quad u = 1, ..., N,$$

where N is a power of 2. Using the trapezoid rule for the integral part in (4) where the upper limit for the integration is set to $2\pi/\Delta_k$, and setting

$\eta = 2\pi/(N\Delta_k)$ and $v_j = \eta(j-1)$, we obtain the following equation:

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-iv_j k_u} \psi_T(v_j)\eta, \quad \text{for} \quad u = 1, ..., N$$
$$\approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} [e^{i\pi(j-1)} \psi_T(v_j)\eta]$$

FFT is very efficient in pricing several options with the same maturity.

4 Proposed Method

4.1 Problem formulation

Because exponential Lévy models correspond to incomplete markets, knowledge of the historical price process alone does not allow for computation of unique option prices. (See the appendix section 8.3.) To choose a market-consistent risk-neutral model Q, equivalent to the prior historical probability P, one should consider information both from the historical time series and from prices of traded options. To achieve this, one can restrict the choice of pricing rules to the class of martingale measures that are equivalent to the prior historical probability, either resulting from historical estimation or specified according to the views of a risk manager: the calibration procedure then updates the prior based on information in the market prices of options. The chosen model can then be used to price exotic, illiquid or OTC options and compute hedge ratios. The calibration problem can be formulated as follows:

Calibration Problem 2 *Given a prior model $(\sigma_0, \nu_0, \gamma_0)$ and the observed prices C_i for call options of maturities T_i and strike prices K_i, $i \in I$, find an exponential-Lèvy model $(\sigma_\theta, \nu_\theta, \gamma_\theta)$ such that the discounted asset price $S_t e^{-rt}$ is a martingale, the probability measure Q_θ generated by $(\sigma_\theta, \nu_\theta, \gamma_\theta)$ is equivalent to the prior P, and the observed option prices are given by discounted risk-neutral expectations under Q_θ :*

$$C_i = C(T_i, K_i; \theta) + \epsilon_i, \quad C(T_i, K_i; \theta) = e^{-rT} E^\theta[(S_{T_i} - K_i)^+ | S_0 = S], \ \forall i \in I,$$

where the noise term ϵ_i is due to the presence of observation errors, such as bid-as spreads or other market frictions, in the market data.

The Lèvy measure, calibrated via nonlinear least-squares, is very sensitive not only to the input prices but also to the numerical starting point in the minimization algorithm. Reformulating the calibration problem as a nonlinear least squares problem does not resolve the uniqueness and stability issues, even if the calibration is restricted to the class of measures equivalent to the prior: the inverse problem remains ill-posed. To resolve this problem, we use the Bayesian approach as follows (see Bishop (2006)).

4.2 Bayesian estimation

We assume that the conditional distribution of the output variable y is given by a deterministic and nonlinear parametric model $\hat{C}(T,K,\boldsymbol{\theta})$ with additive Gaussian noise, such that

$$p(C|T,K,\boldsymbol{\theta},\sigma_\epsilon^2) = \mathcal{N}(C|\hat{C}(T,K,\boldsymbol{\theta}),\sigma_\epsilon^2) \tag{5}$$

Let a prior probability distribution over the model parameter $\boldsymbol{\theta}$ be given by a Gaussian distribution of the form

$$p(\boldsymbol{\theta}|\sigma_\theta^2) = \mathcal{N}(\boldsymbol{\theta}|\boldsymbol{\theta}_0,\sigma_\theta^2\mathbf{I}) \tag{6}$$

Since $p(\{T_n,K_n,C_n\}_{n=1}^N|\boldsymbol{\theta}) = \prod_{n=1}^N \mathcal{N}(C_n|\hat{C}(T_n,K_n,\boldsymbol{\theta}),\sigma_\epsilon^2)$, the posterior distribution is given by

$$p(\boldsymbol{\theta}|\{T_n,K_n,C_n\}_{n=1}^N,\sigma_\epsilon^2,\sigma_\theta^2) \propto p(\boldsymbol{\theta}|\sigma_\theta^2)p(\{T_n,K_n,C_n\}_{n=1}^N|\boldsymbol{\theta},\sigma_\epsilon^2) \tag{7}$$

which, as a consequence of the nonlinear dependence of $\hat{C}(T,K,\boldsymbol{\theta})$ on $\boldsymbol{\theta}$, will be non-Gaussian. To obtain a Gaussian approximation of the posterior distribution, we use the Laplace approximation. To do so, we first find a maximum, say $\boldsymbol{\theta}_{\mathrm{MAP}}$, of the logarithm of the posterior as given by

$$\begin{aligned} \ln p(\boldsymbol{\theta}|\{T_n,K_n,C_n\}_{n=1}^N,\sigma_\epsilon^2,\sigma_\theta^2) = &-\frac{1}{2\sigma_\theta^2}(\boldsymbol{\theta}-\boldsymbol{\theta}_0)^T(\boldsymbol{\theta}-\boldsymbol{\theta}_0) \\ &-\frac{1}{2\sigma_\epsilon^2}\sum_{n=1}^N\{\hat{C}(T_n,K_n,\boldsymbol{\theta})-C_n\}^2 + \text{const.} \end{aligned} \tag{8}$$

by using a standard nonlinear optimization algorithm. Then the Gaussian approximation to the posterior distribution using the Laplace approximation takes the form

$$q(\boldsymbol{\theta}|\{T_n,K_n,C_n\}_{n=1}^N) = \mathcal{N}(\boldsymbol{\theta}_{\mathrm{MAP}},\boldsymbol{\Sigma}_N)$$

where

$$\mathbf{\Sigma}_N^{-1} = \frac{1}{\sigma_\theta^2}\mathbf{I} + \frac{1}{\sigma_\epsilon^2}\mathbf{H}$$

and where $\mathbf{H}$ is the Hessian matrix of the second derivatives of the sum-of-squares error function with respect to the components of $\boldsymbol{\theta}$. Note that the Hessian matrix of $h(\boldsymbol{\theta})$ can be approximated by using the finite difference

$$\begin{aligned}\frac{\partial^2 h}{\partial\theta_i\partial\theta_j} = &\frac{1}{4\varepsilon^2}\{h(\theta_i+\varepsilon,\theta_j+\varepsilon) - h(\theta_i+\varepsilon,\theta_j-\varepsilon)\\ &-h(\theta_i-\varepsilon,\theta_j+\varepsilon) + h(\theta_i-\varepsilon,\theta_j-\varepsilon)\} + O(\varepsilon^2)\end{aligned}$$

The *predictive distribution* is obtained by marginalizing with respect to the following posterior distribution:

$$p(C|\{T_n,K_n,C_n\}_{n=1}^N) = \int p(C|T,K,\boldsymbol{\theta})q(\boldsymbol{\theta}|\{T_n,K_n,C_n\}_{n=1}^N)d\boldsymbol{\theta}$$

where the dependence on the input data and $\sigma_\theta^2, \sigma_\epsilon^2$ is omitted, as the integration remains analytically intractable due to the nonlinearity of $\hat{C}(T,K,\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. To make progress, we now assume that the posterior distribution has a small variance compared with the characteristic scales of $\boldsymbol{\theta}$ over which $\hat{C}(T,K,\boldsymbol{\theta})$ varies. This assumption allows us to perform a Taylor series expansion of the function around $\boldsymbol{\theta}_{\text{MAP}}$ and retain only the linear terms

$$\hat{C}(T,K,\boldsymbol{\theta}) \simeq \hat{C}(T,K,\boldsymbol{\theta}_{\text{MAP}}) + \mathbf{g}^T(\boldsymbol{\theta} - \boldsymbol{\theta}_{\text{MAP}})$$

where we have defined $\mathbf{g} = \nabla\hat{C}(T,K,\boldsymbol{\theta})|_{\boldsymbol{\theta}=\text{MAP}}$. With this approximation, we have

$$p(C|\{T_n,K_n,C_n\}_{n=1}^N, \boldsymbol{\theta}, \sigma_\epsilon^2) \simeq \mathcal{N}(\hat{C}(T,K,\boldsymbol{\theta}_{\text{MAP}}) + \mathbf{g}^T(\boldsymbol{\theta} - \boldsymbol{\theta}_{\text{MAP}}), \sigma_\epsilon^2).$$

Hence the predictive distribution takes the form

$$p(C|\{T_n,K_n,C_n\}_{n=1}^N, \sigma_\theta^2, \sigma_\epsilon^2) \simeq \mathcal{N}(\hat{C}(T,K,\boldsymbol{\theta}_{\text{MAP}}), \sigma_N^2(T,K)) \tag{9}$$

where the variance of the predictive distribution is given by

$$\sigma_N^2(T,K) = \sigma_\epsilon^2 + \mathbf{g}^T\mathbf{\Sigma}_N\mathbf{g}$$

The variance consists of two terms. The first arises from the intrinsic noise of the target variable, while the second is a T,K-dependent term that expresses uncertainty of the interpolant due to uncertainty of the model parameters $\boldsymbol{\theta}$.

In a fully Bayesian treatment, we introduce hyperpriors over $\sigma_\theta^2, \sigma_\epsilon^2$ and the predictive distribution is obtained by marginalization over $\boldsymbol{\theta}, \sigma_\theta^2, \sigma_\epsilon^2$, so that

$$\begin{aligned} p(y|\mathbf{y}) &= \int p(y|\boldsymbol{\theta}, \sigma_\epsilon^2) p(\boldsymbol{\theta}|\mathbf{y}, \sigma_\theta^2, \sigma_\epsilon^2) p(\sigma_\theta^2, \sigma_\epsilon^2|\mathbf{y}) d\boldsymbol{\theta} d\sigma_\theta^2 d\sigma_\epsilon^2 \\ & \text{where } p(y|\boldsymbol{\theta}, \sigma_\epsilon^2) = \mathcal{N}(y|f(\mathbf{x}, \boldsymbol{\theta}), \sigma_\epsilon^2), \quad p(\boldsymbol{\theta}|\mathbf{y}, \sigma_\theta^2, \sigma_\epsilon^2) \sim \mathcal{N}(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N) \end{aligned}$$

Here the dependence on the input data $\mathbf{X}, \mathbf{x}$ is omitted. If the posterior distribution $p(\sigma_\theta^2, \sigma_\epsilon^2|\mathbf{y})$ is sharply peaked around the values $\hat{\sigma}_\theta^2, \hat{\sigma}_\epsilon^2$, then

$$p(y|\mathbf{y}) \simeq p(y|\mathbf{y}, \hat{\sigma}_0^2, \hat{\sigma}^2) = \int p(y|\boldsymbol{\theta}, \hat{\sigma}^2) p(\boldsymbol{\theta}|\mathbf{y}, \hat{\sigma}_0^2, \hat{\sigma}^2) d\boldsymbol{\theta}$$

From Bayes' theorem, the posterior distribution for $\sigma_\theta^2, \sigma_\epsilon^2$ is given by

$$p(\sigma_\theta^2, \sigma_\epsilon^2|\mathbf{y}) \propto p(\mathbf{y}|\sigma_\theta^2, \sigma_\epsilon^2) p(\sigma_\theta^2, \sigma_\epsilon^2).$$

In the evidence framework, the prior is assumed to be relatively flat; the values of $\hat{\sigma}_\theta^2, \hat{\sigma}_\epsilon^2$ are then obtained by maximizing the marginal likelihood function $p(\mathbf{y}|\sigma_\theta^2, \sigma_\epsilon^2)$, which is given by

$$p(\mathbf{y}|\sigma_\theta^2, \sigma_\epsilon^2) = \int p(\mathbf{y}|\boldsymbol{\theta}, \sigma_\epsilon^2) p(\boldsymbol{\theta}|\sigma_\theta^2) d\boldsymbol{\theta} = \left(\frac{1}{2\sigma_\epsilon^2 \pi}\right)^{N/2} \left(\frac{1}{2\sigma_\theta^2 \pi}\right)^{M/2} \int \exp\{-\mathcal{E}(\boldsymbol{\theta})\} d\boldsymbol{\theta}$$

where M is the dimensionality of $\boldsymbol{\theta}$. We define:

$$\begin{aligned} \mathcal{E}(\boldsymbol{\theta}) &= \frac{1}{\sigma_\epsilon^2} \mathcal{E}_D(\boldsymbol{\theta}) + \frac{1}{\sigma_\theta^2} \mathcal{E}_W(\boldsymbol{\theta}) = \frac{1}{2\sigma_\epsilon^2} \sum_{n=1}^N \{\hat{C}(T_n, K_n, \boldsymbol{\theta}) - C_n\}^2 + \frac{1}{2\sigma_\theta^2} \boldsymbol{\theta}^T \boldsymbol{\theta} \\ &= \frac{1}{2\sigma_\epsilon^2} \sum_{n=1}^N \{f(\mathbf{x}_n, \boldsymbol{\theta}_{\mathrm{MAP}}) - C_n\}^2 + \frac{1}{2\sigma_\theta^2} \boldsymbol{\theta}_{\mathrm{MAP}}^T \boldsymbol{\theta}_{\mathrm{MAP}} + \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}})^T \boldsymbol{\Sigma}_N^{-1} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}}) \\ &= \mathcal{E}(\boldsymbol{\theta}_{\mathrm{MAP}}) + \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}})^T \boldsymbol{\Sigma}_N^{-1} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}}) \end{aligned}$$

Then from

$$\begin{aligned} \int \exp\{-\mathcal{E}(\boldsymbol{\theta})\} d\boldsymbol{\theta} &= \exp\{-\mathcal{E}(\boldsymbol{\theta}_{\mathrm{MAP}})\} \int \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}})^T \boldsymbol{\Sigma}_N^{-1} (\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathrm{MAP}}) d\boldsymbol{\theta} \\ &= \exp\{-\mathcal{E}(\boldsymbol{\theta}_{\mathrm{MAP}})\} (2\pi)^{M/2} |\boldsymbol{\Sigma}_N^{-1}|^{-1/2} \end{aligned}$$

we can write the log of the marginal likelihood in the form

$$\ln p(\mathbf{y}|\sigma_\theta^2, \sigma_\epsilon^2) = -\mathcal{E}(\boldsymbol{\theta}_{\text{MAP}}) - \frac{1}{2}\ln|\boldsymbol{\Sigma}_N^{-1}| + \frac{M}{2}\ln\frac{1}{\sigma_\theta^2} + \frac{N}{2}\ln\frac{1}{\sigma_\epsilon^2} - \frac{N}{2}\ln(2\pi)$$

In the evidence framework, we make point estimates for $\hat{\sigma}_\theta^2$ and $\hat{\sigma}_\epsilon^2$ by maximizing $\ln p(\mathbf{y}|\sigma_\theta^2, \sigma_\epsilon^2)$.

If we let $\mathbf{H}\mathbf{u}_i = \lambda_i \mathbf{u}_i$ where $\mathbf{H}$ is the Hessian matrix of the second derivatives of the sum-of-squares error function evaluated at $\boldsymbol{\theta}_{\text{MAP}}$ and differentiate (10) with respect to $1/\sigma_\theta^2, 1/\sigma_\epsilon^2$, we obtain

$$\begin{aligned}
\hat{\sigma}_\theta^2 &= \frac{1}{\gamma}\boldsymbol{\theta}_{\text{MAP}}^T \boldsymbol{\theta}_{\text{MAP}} \\
\hat{\sigma}_\epsilon^2 &= \frac{1}{N-\gamma}\sum_{n=1}^{N}\{\hat{C}(T_n, K_n, \boldsymbol{\theta}_{\text{MAP}}) - C_n\}^2 \\
\gamma &= \sum_{i=1}^{M}\frac{\lambda_i}{\lambda_i + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\theta^2}
\end{aligned} \tag{10}$$

Note that these are implicit solutions for $\sigma_\theta^2, \sigma_\epsilon^2$ and can be solved by choosing initial values for $\sigma_\theta^2, \sigma_\epsilon^2$ and then using these values to calculate $\boldsymbol{\theta}_{\text{MAP}}$ and γ and then re-estimate them using these implicit equations, repeating until convergence is achieved. Using this method, we prefer a model that has a slightly poorer calibration quality but is similar to the prior to a model that reproduces option prices exactly but is very different from the prior.

4.4 Algorithm

The final calibration algorithm consists of the following steps:

(1) Choose an exponential Lévy model we want to find a parameter set $\boldsymbol{\theta}$.
(2) Calculate the first four moments (4 cumulants of log stock prices related to mean, variance, skewness, and kurtosis) of the historical stock prices data, which is denoted by $\mathbf{m}^{Mkt} = (c_1^{Mkt}, c_2^{Mkt}, c_3^{Mkt}, c_4^{Mkt})^T$. We choose a prior mean $\boldsymbol{\theta}_0$ in (6) that minimizes the difference between the historical moments $\mathbf{m}^{Mkt}$ and the model moments $\mathbf{m}(\boldsymbol{\theta}_0) = (c_1^{\boldsymbol{\theta}_0}, c_2^{\boldsymbol{\theta}_0}, c_3^{\boldsymbol{\theta}_0}, c_4^{\boldsymbol{\theta}_0})^T$ whose explicit formula are given in the appendix section 8.1. Choose a prior reflecting the users view of the model can also be preferred depending on the problems.
(3) Given a option market data set $\{T_n, K_n, C_n\}_{n=1}^N$, we augment the data set by using the bootstrap in such as way to have the similar number of data for each maturities or by reflecting the liquidity of a given option. This pre-processing step can help to enhance the calibration performance.

(4) Choose initial values for $\sigma_\theta^2, \sigma_\epsilon^2$ and then using these values, calculate $\boldsymbol{\theta}_{\text{MAP}}$ and γ by solving

$$\boldsymbol{\theta}_{\text{MAP}} = \arg\min_{\boldsymbol{\theta}} \left\{ (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + \frac{\sigma_\theta^2}{\sigma_\epsilon^2} \sum_{n=1}^{N} \{\hat{C}(T_n, K_n, \boldsymbol{\theta}) - C_n\}^2 \right\}$$

$$\gamma = \sum_{i=1}^{M} \frac{\lambda_i}{\lambda_i + \sigma_\epsilon^2/\sigma_\theta^2}$$

Then using these $\boldsymbol{\theta}_{\text{MAP}}$ and γ, re-estimate $\sigma_\theta^2, \sigma_\epsilon^2$ given by

$$\begin{aligned} \sigma_\theta^2 &= \frac{1}{\gamma} \boldsymbol{\theta}_{\text{MAP}}^T \boldsymbol{\theta}_{\text{MAP}} \\ \sigma_\epsilon^2 &= \frac{1}{N - \gamma} \sum_{n=1}^{N} \{\hat{C}(T_n, K_n, \boldsymbol{\theta}_{\text{MAP}}) - C_n\}^2 \end{aligned} \tag{11}$$

We repeat this process until convergence is achieved. To compute $\boldsymbol{\theta}_{\text{MAP}}$, we can use a local search method called the Nelder-Mead method which is one of the state-of-art, derivative-free, nonlinear optimization method with low precision to estimate the "model error". We present the Nelder-Mead method in the appendix section 8.4. Alternatively, we can run any gradient-based nonlinear optimization solver (e.g., the BFGS method or the trust-region method) by using the gradients of the chosen exponential Lévy model. The gradients of the exponential Lévy models used in this paper are explicitly given in the appendix section 8.2.

5 Model Robustness Test

In this section, the stability of the Bayesian method is verified by calibrating a parameter set of exponential Lévy models with model-generated option price data. For this verification, we simulate European call option prices for each of the five exponential Lévy models, using Fast fourier transforms (FFT), then calculating the implied volatility surface, using the Black-Scholes model. To calculate the option prices, we first set the dampening parameter α *to* 1.6.

First, we analyze the implied volatility surface calculated for European call options, using the Black-Scholes model, to test the usefulness of the five Lévy models. The estimated implied volatility surface is presented in Fig. 1. and Fig. 2. We find that the implied volatility surface, regardless of which of the five exponential Lévy models is used, can explain both smile and sneer effects observed in real markets.

Next, using the calibration function defined in the previous section 4.4, we analyze the calibration problem with respect to European option prices gen-

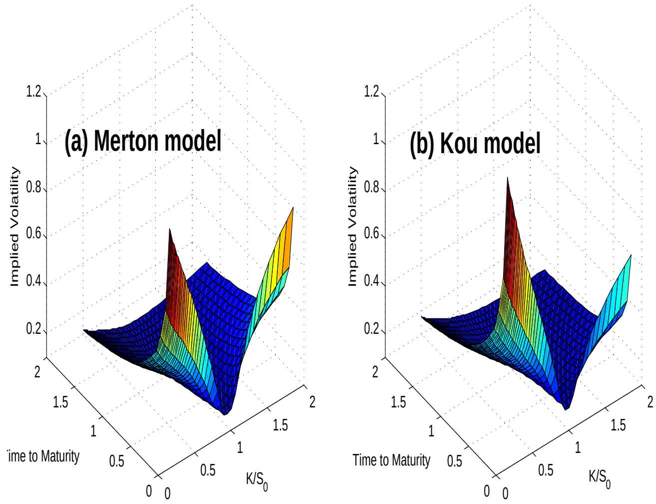

Fig. 1. (a) and (b) displays the implied volatility surface for the Merton and Kou models, respectively.

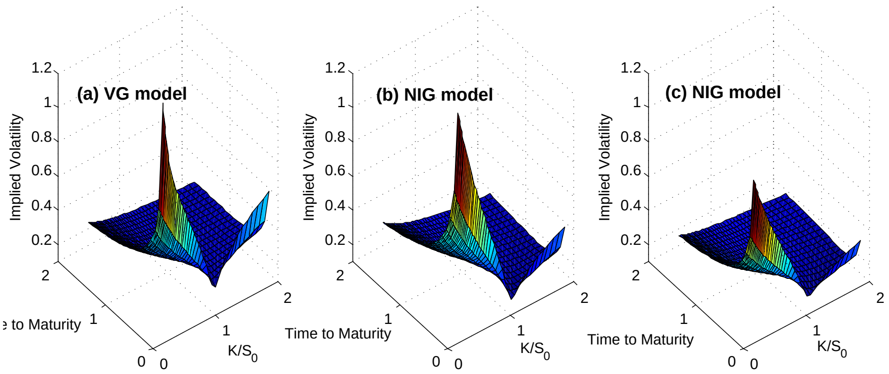

Fig. 2. (a), (b) and (c) show the implied volatility surfaces for the variance gamma, normal inverse Gaussian and CGMY models, respectively.

erated by the five exponential Lévy models. We establish the initial and true Lévy parameter sets for the five exponential Lévy models and calculate option prices for a diverse set of strike prices and times to maturity. We then calibrate the Lévy triplet, using a gradient-based optimization algorithm. We find that, for the five exponential Lévy models, the calibrated parameters are equal to the true parameter sets. These results are presented in Table 1.

Fig. 3. and Fig. 4. display the European call option prices, which are calculated by the FFT method, with an initial, true and calibrated parameter set.

The diamonds, circles and stars indicate initial, true and calibrated call option prices, respectively. In Fig. 3. and Fig. 4., we observe that, regardless of the model used, option prices calculated calibrated parameter are exactly equal

Table 1
The parameter calibration for five exponential Lévy model

Lévy Model	Parameter set	True	Calibrated	RMSE
Merton	$\{\sigma,\lambda,\mu,\delta\}$	{0.15, 0.1, 0.1, 0.3}	{0.15, 0.1, 0.1, 0.3}	0
Kou	$\{\sigma,\lambda,\lambda_+,\lambda_-,\text{p}\}$	{0.1, 1, 14, 8, 0.5}	{0.1, 1, 14, 8, 0.5}	0
VG	$\{\gamma, \sigma, kappa\}$	{-0.3, 0.3, 0.25}	{-0.3, 0.3, 0.25}	0
NIG	$\{\gamma, \sigma, kappa\}$	{-0.5, 0.2, 0.3}	{-0.5, 0.2, 0.3}	0
CGMY	$\{\text{c}, \alpha, \lambda_-, \lambda_+\}$	{3, 13, 52, 0.5}	{3, 13, 52, 0.5}	0

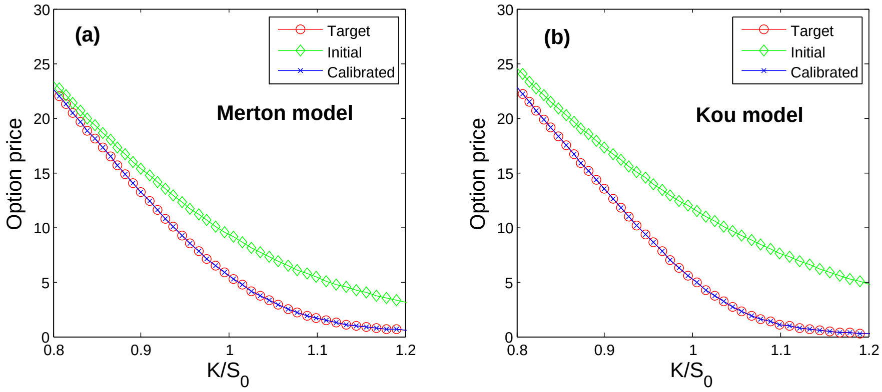

Fig. 3. (a) and (b) show European call option prices for given levels of moneyness, as determined by the Merton and Kou models. The red circles, green diamonds and blue stars indicate European call option prices estimated with true, initial and calibrated Lévy parameter sets, respectively.

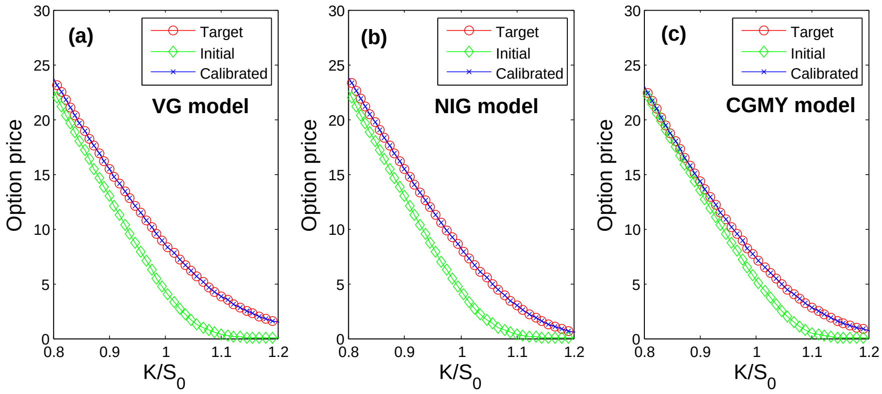

Fig. 4. (a), (b) and (c) show European call option prices for given levels of moneyness, as determined by the variance gamma, normal inverse Gaussian and CGMY models, respectively. The red circles, green diamonds and blue stars indicate European call option prices, estimated with true, initial and calibrated Lévy parameter sets, respectively.

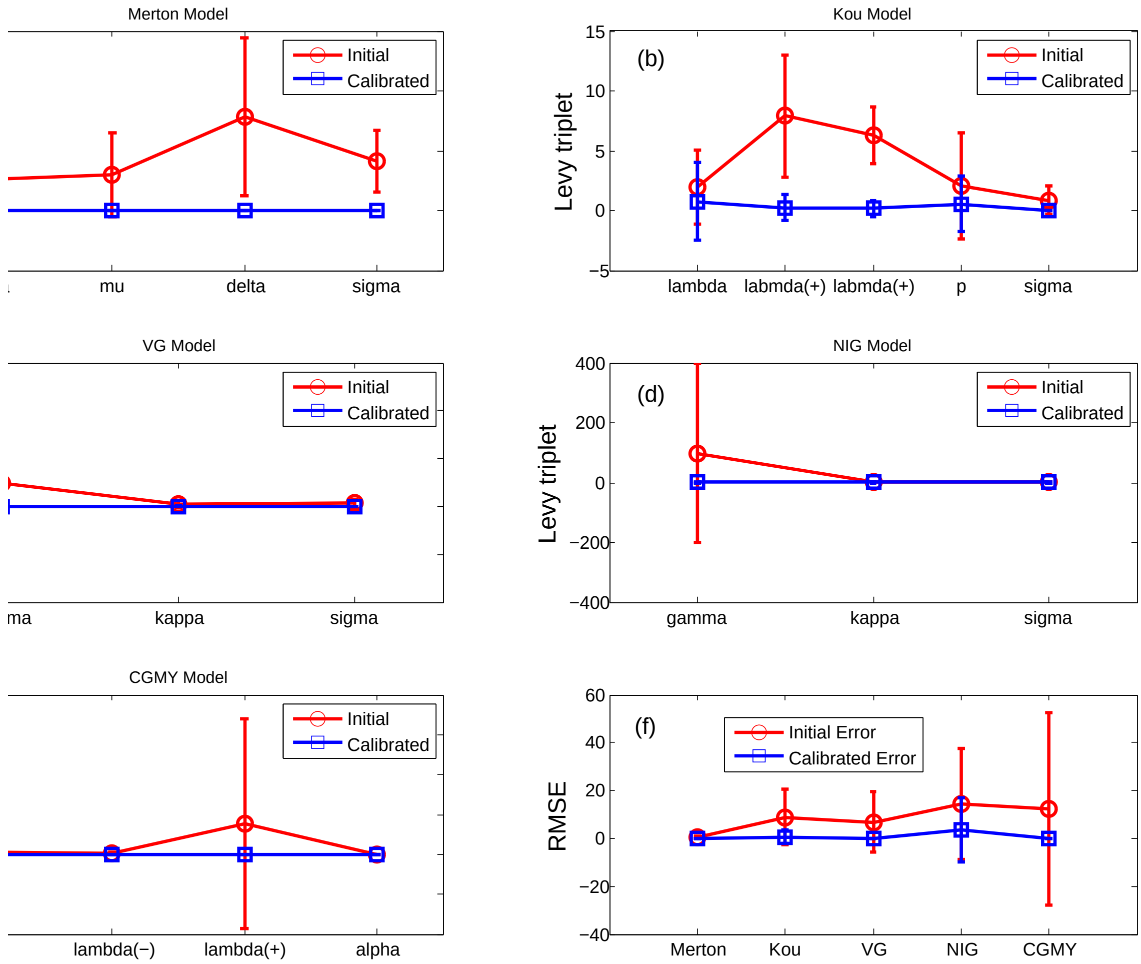

Fig. 5. The difference between the initial (or calibrated) and true Lévy parameters for the five exponential Lévy models (Merton, Kou, variance gamma, normal inverse Gaussian and CGMY models) are shown in the error-bar plot.

to those for the true parameter. To obtain the results observed in Table 1, we construct 100 Lévy parameter sets for each of the five exponential Lévy models and perform the calibration process. Fig. 5. displays the differences between the initial (or calibrated) and the true Lévy parameters in the error-bars for the five Lévy models.

The circles and squares correspond to the initial and calibrated parameter, respectively. Irrespective of which of the five exponential Lévy models is used, the difference of between calibrated and true parameters are approximately zero. Based on the above results, we may assert that the calibrations of European option prices, estimated using the five exponential Lévy models, are very effective. To apply the Bayesian calibration approach to option pricing

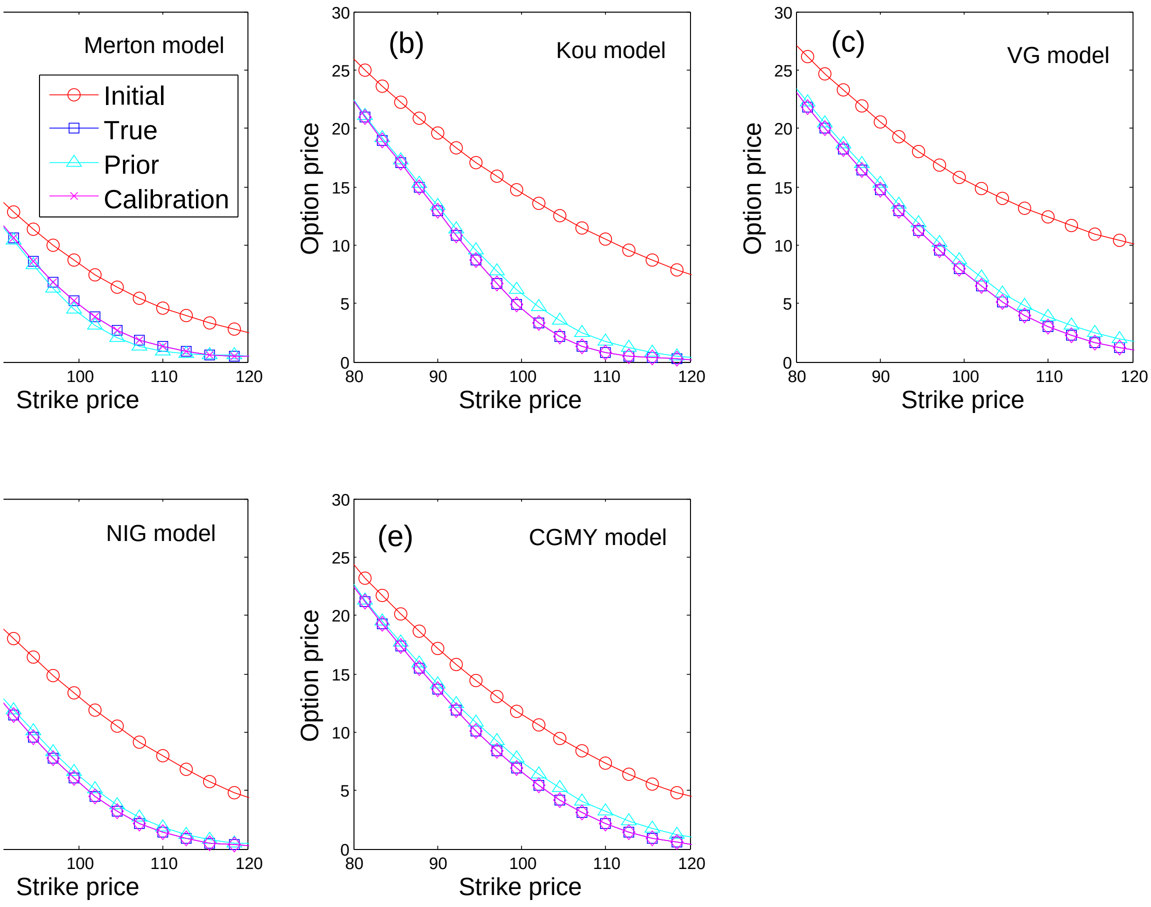

Fig. 6. The European call option prices: initial (red circles), true (blue squares), prior (cyan triangles) and calibration (pink diamonds). The five panels illustrate the (a) Merton, (b) Kou, (c) variance gamma, (d) normal inverse Gaussian, and (e) CGMY models.

calibration in the real market, we must know the true Lévy triplet. Unfortunately, because we can not estimate a true Lévy parameter in the real market, choosing the prior Lévy triplet in the penalty term of the calibration function is very important. We use the prior parameters instead of the true Lévy parameter used in the penalty term to analyze the effect of choosing the prior parameter in the Bayesian calibration. Fig. 6. shows European call option prices estimated using the FFT method, with initial, true, prior and calibrated parameters, respectively.

Although the prior parameter set is not equivalent to the true parameter, we find that the option prices estimated using the calibrated parameter are very similar to those estimated using the true parameter, regardless of which of the five exponential Lévy models is used. This result suggests that, although

Table 2
Calibration results for five exponential Lévy models based on KOSPI 200 index option prices on January 1st, 2005

Lévy Model	Parameter set	Calibrated	RMSE
Merton	$\{\sigma,\lambda,\mu,\delta\}$	{0.1956, 2.464e-7, -0.0433, 1.498e-4}	0.1726
Kou	$\{\sigma,\lambda,\lambda_+,\lambda_-,p\}$	{0.1975, 2.39e-14, 38.6249, 2.0972, 0.9974}	0.1724
VG	$\{\gamma, \sigma, \kappa\}$	{0.6189, 0.1956, 0.0200}	0.1947
NIG	$\{\gamma, \sigma, \kappa\}$	{0.0508, 0.1936, 0.0053}	0.1748
CGMY	$\{c, \alpha, \lambda_-, \lambda_+\}$	{5.79e-4, 1.9765, 44.2601, 45.2601}	0.1565

we do not know the true Lévy parameter, our Bayesian calibration approach performs well in European option pricing calibration, if we choose a proper prior parameter set.

6 Empirical Results

6.1 Data Description

6.2 Simulation Results

We also check the performance of the proposed method, using KOSPI 200 index option prices. First, we conduct simulations of the real market option prices on January 1st, 2005. The following table 2 shows the calibration results for the five exponential Lévy models.

We calculate the predictive distribution of option prices with the proposed method using the calibration results in Table 2, and the distribution of option prices, using the calibration result of the least-squares method, the most widely used calibration method in the finance literature. We plot the 95% confidence intervals for option prices, based on the predictive distribution with the proposed method and with the least squares method, comparing each with KOSPI 200 index option prices. Figs 7 - 11 show the simulation results.

As seen in Fig 7 to Fig 11, almost all actual option prices, represented by the black dots, are within the 95% confidence interval (represented by the red lines) of the predictive distribution, where the latter is calculated using the proposed method. (Mean option prices are represented by blue lines.) We can also verify that option prices computed using the least squares method only fit the real option prices well at short maturities, while the option prices

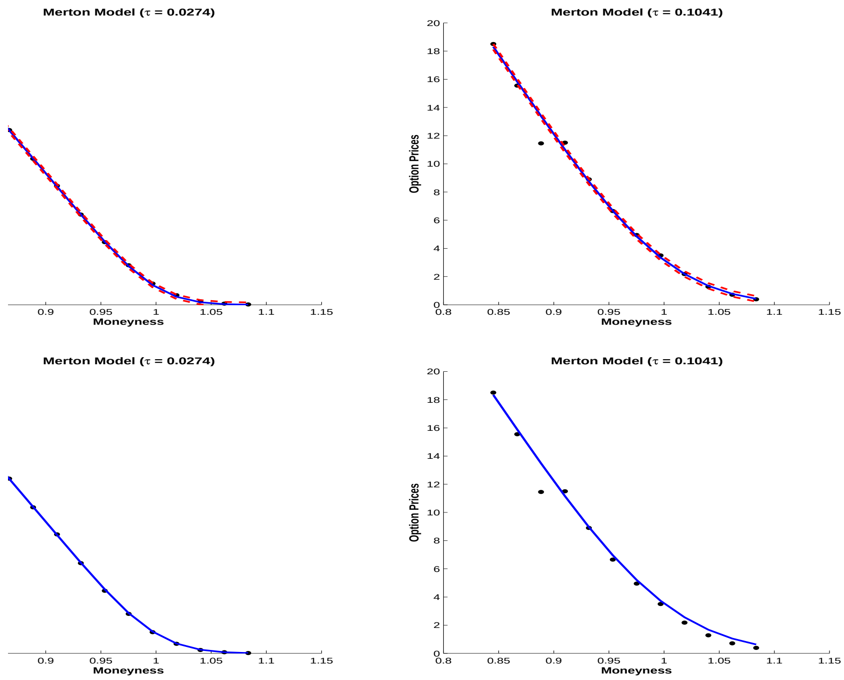

Fig. 7. The panels in the first row represent the KOSPI 200 index option prices and Merton model's option prices with the calibration result obtained using the proposed method, while the panels in the second row represent the calibration result obtained using the least squares method.

calculated using the proposed method capture the real option prices well, regardless of maturities.

To consider the robustness of the proposed method we calibrate model parameters with 3 months of KOSPI 200 index option prices, from January 1st, 2007 to March 31th, 2007. We calibrate the parameter sets for the five exponential Lévy models for each day, with randomly chosen initial parameter sets. First, based on with the calibration results for the 3 months period, we compute the in-sample errors for each day. Table **??** shows the mean squared errors of the exponential Lévy models for each day over one month. The mean squared errors are shown in Fig **??**.

In this paper, we present mean squared errors for only one month: January. The mean squared errors for the other month are similar to those found in Table 3. In Table 3, we can verify that the mean squared errors for the proposed method are always smaller than those for the least squares method. Hence, the calibration results based on the proposed method provide a superior representation of KOSPI 200 index option prices. Based on the mean squared errors for each day, we average all mean squared errors according to the exponential Lévy models. Fig 12 shows the average in-sample errors.

It is shown that average in-sample errors under the proposed method are also

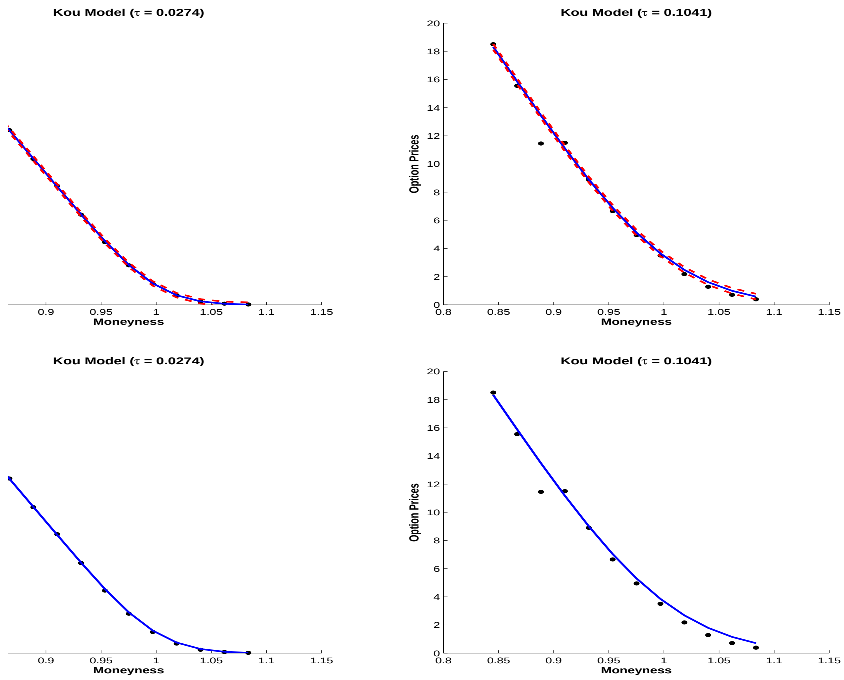

Fig. 8. The panels in the first row represent the KOSPI 200 index option prices and Kou model's option prices, based on the calibration result obtained using the proposed method. The panels in the second row are based on the calibration results of the least squares method.

smaller than those under the least squares method, as shown in Fig 12. Among the exponential Lévy models, the CGMY model shows the best performance, as it has the lowest average mean squared errors. The Jump-diffusion models, i.e., the Merton and Kou models, have similar average mean squared errors, and the infinity activity models, i.e., the Variance Gamma and the Normal Inverse Gaussian models, also have similar average mean squared errors.

7 Conclusion

In this paper, we have proposed a Bayesian estimation method to calibrate the parameter sets of five exponential Lévy models of European option prices. Using implied volatility surfaces, it is shown that the five exponential Lévy models can explain smile or sneer patterns in real market data. To verify the performance of the proposed method, we use model-generated option prices, finding that for the five exponential Lévy models, the calibrated parameter sets obtained using the proposed method perform well, as almost every calibrated parameter set converges to the true parameter set, regardless of the initial value of the parameters. The proposed method is shown to render the inverse calibration problems well-posed and to ease the burden associated with the

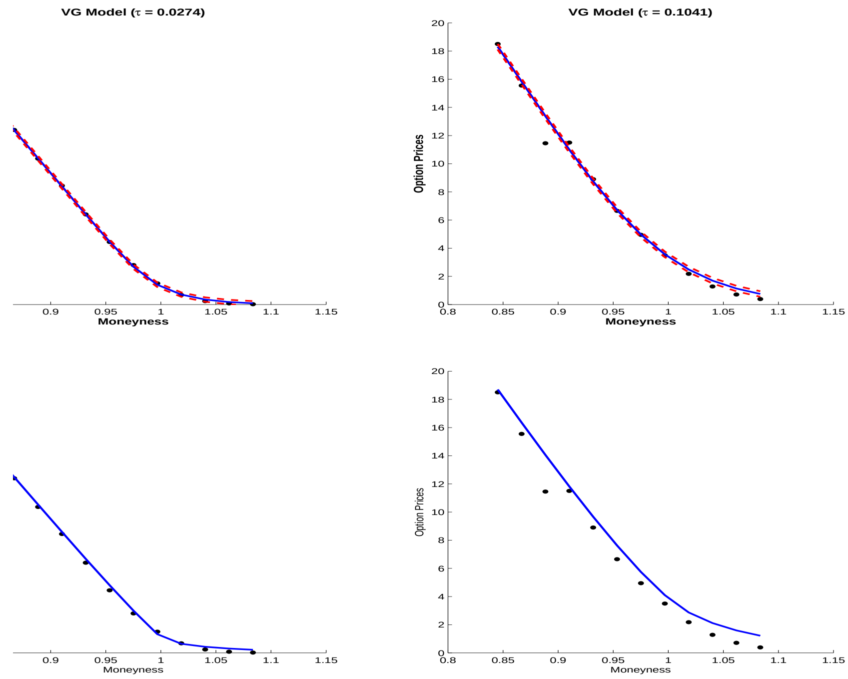

Fig. 9. The panels in the first row represent the KOSPI 200 index option prices and the VG model's option prices, based on the calibration result of the proposed method. The panels in the second row are based on the calibration result of the least squares method.

selection of priors . In further studies, we will evaluate the performance of the proposed method with real market option price data and compare its performance with other methods.

8 Appendix

8.1 Statistical and Computational Properties of six exponential Lévy models

8.1.1 Lévy processes via jump-diffusion model

1) **Merton model:** In the Merton model, jump sizes follow a Gaussian distribution $\mathcal{N}(\mu, \delta^2)$ with a Lévy measure:

$$\nu(x) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\delta^2}\right)$$

- Four parameters $\theta = (\sigma, \lambda, \mu, \delta)$: σ-diffusion volatility, λ-jump intensity, μ-mean jump size, and δ-standard deviation of jump size.

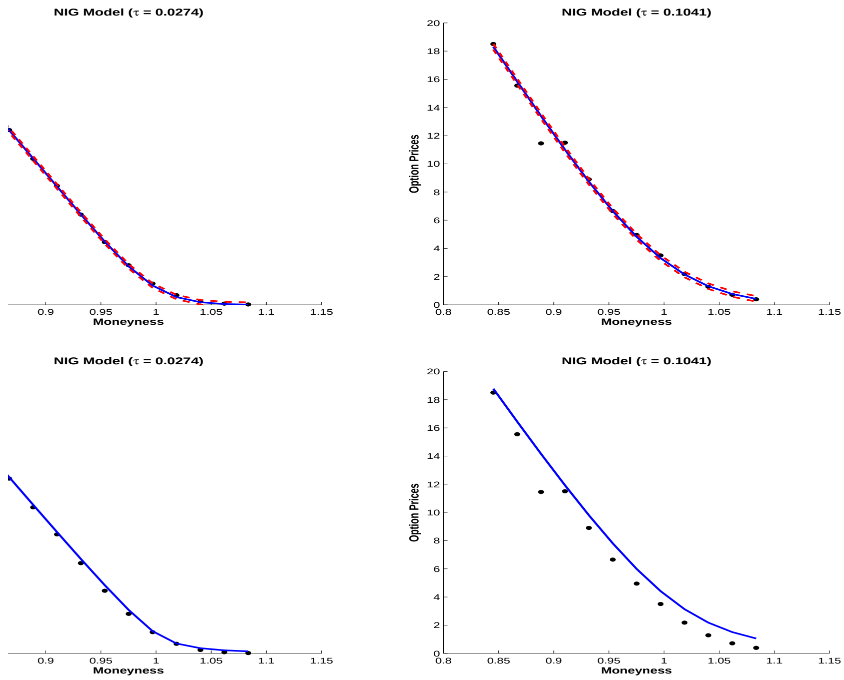

Fig. 10. The panels in the first row represent the KOSPI 200 index option prices and the NIG model's option prices, based on the calibration result of the proposed method. The panels in the second row are based on the calibration result of the least squares method.

- The probability density function $p_t(x)$ of X_t satisfies

$$p_t(x) = \sum_{n=0}^{\infty} \mathbb{P}[X_t = x | N_t = n]\mathbb{P}[N_t = n] = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n \exp\left\{-\frac{(x-\gamma t - n\mu)^2}{2(\sigma^2 t + n\delta^2)}\right\}}{n!\sqrt{2\pi(\sigma^2 t + n\delta^2)}}$$

- The characteristic function $\Phi_{X_t}(z)$ of X_t (obtained using the Levy-Khinchin formula), is

$$\begin{aligned}\Phi_{X_t}(z) &= \mathbb{E}[\exp izX_t] = \sum_{n=0}^{\infty} \mathbb{E}[e^{izX_t} | N_t = n]\mathbb{P}\{N_t = n\} \\ &= \sum_{n=0}^{\infty} e^{\left\{iz(\gamma t + \mu n) - \frac{1}{2}z^2(\sigma^2 t + \delta^2 n)\right\}}\mathbb{P}\{N_t = n\} = e^{t\left\{-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda(e^{i\mu z - \frac{1}{2}\delta^2 z^2} - 1)\right\}}\end{aligned}$$

and the characteristic exponent $\Psi_X(z)$ of X_t is therefore

$$\Psi_X(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda\left(e^{i\mu z - \frac{1}{2}\delta^2 z^2} - 1\right)$$

- The characteristic function of the log prices, $s_t = \ln(S_t/S_0)$, used to calcu-

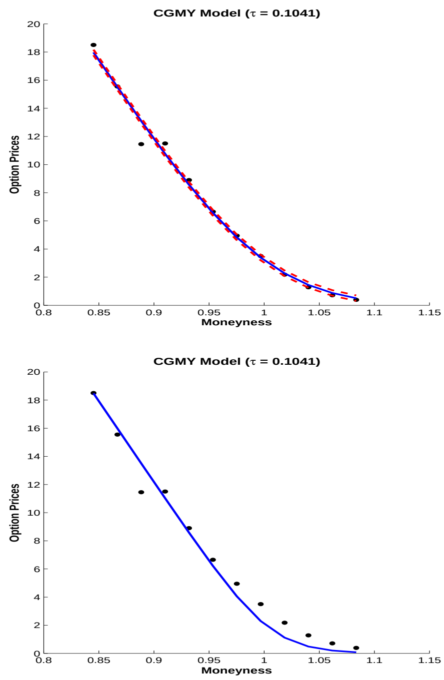

Fig. 11. The panels in the first row represent the KOSPI 200 index option prices and the CGMY model's option prices, based on the calibration result of the proposed method. The panels in the second row represent option prices, based on the calibration result of the least squares method.

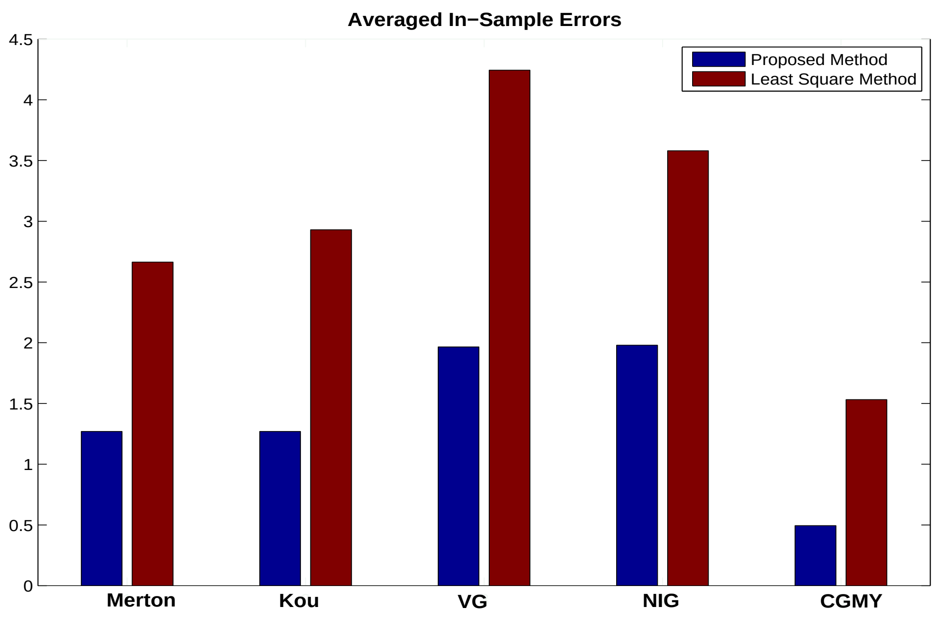

Fig. 12. The average in-sample errors with the calibration results.

Table 3
Mean Squared Errors of KOSPI index option prices during one month

Date	01/03/2005		01/04/2005		01/05/2005		01/06/2005	
Method	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square
Merton	0.165932	0.327045	0.015977	0.293418	0.077368	0.386332	0.293668	0.541619
Kou	0.171431	0.205929	0.013139	0.153184	0.042407	0.377745	0.151886	0.299115
VG	0.183418	0.485891	0.019737	0.4496	0.084782	0.934149	0.179759	0.606762
NIG	0.169229	0.467822	0.017887	0.484214	0.106094	0.658352	0.166863	0.672723
CGMY	0.160638	0.476006	0.016797	0.500304	0.041782	0.686419	0.158383	0.667299
Date	01/07/2005		01/10/2005		01/11/2005		01/12/2005	
Method	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square
Merton	0.061127	0.297139	0.04236	0.402962	0.185008	0.702211	0.059412	0.157767
Kou	0.061269	0.153452	0.084399	0.131125	0.365053	0.711451	0.044033	0.083929
VG	0.094701	0.699621	0.06968	0.509414	0.31954	0.846022	0.063589	0.532787
NIG	0.062217	0.513871	0.071155	0.787575	0.324209	1.444105	0.070837	0.260466
CGMY	0.060292	0.521332	0.071802	0.818535	0.322659	1.43285	0.08276	0.274362
Date	01/13/2005		01/14/2005		01/17/2005		01/18/2005	
Method	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square
Merton	0.041933	0.397133	1.717278	3.785101	0.503736	2.581022	0.436372	1.785231
Kou	0.053415	0.4045	3.701637	4.059861	2.153646	3.246182	1.259151	2.116589
VG	0.050703	0.487509	3.010606	10.46842	1.459468	4.125919	0.89871	1.841884
NIG	0.052394	0.775006	2.978464	7.919491	1.411878	5.043737	1.484798	5.056782
CGMY	0.053828	0.784285	3.117864	7.862287	1.484798	5.056782	0.862571	3.574706
Date	01/19/2005		01/20/2005		01/21/2005		01/24/2005	
Method	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square
Merton	0.158755	0.243832	0.199152	0.22124	0.169785	2.055862	0.405911	1.514049
Kou	0.015948	0.02288	0.12634	2.55483	0.092912	0.202147	0.129242	0.297898
VG	0.140617	0.932025	0.209176	0.778848	0.176793	3.009115	0.844011	2.201172
NIG	0.124017	0.71742	0.217378	0.28461	0.17001	3.36001	0.851239	3.132825
CGMY	0.39231	0.76384	0.664184	0.69748	0.098842	3.528859	0.858076	3.141008
Date	01/25/2005		01/26/2005		01/27/2005		01/28/2005	
Method	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square	Proposed	Least Square
Merton	0.130536	0.401842	0.221471	0.849423	0.056891	0.468822	0.055425	0.654364
Kou	0.129242	0.297898	0.356001	0.92294	0.071996	0.098465	0.10967	0.671842
VG	0.136363	1.435118	0.277013	1.041807	0.065952	0.520824	0.078456	1.224546
NIG	0.136718	0.564191	0.282418	1.608061	0.075622	0.988802	0.079495	1.365129
CGMY	0.109024	0.579529	0.308942	1.618982	0.075504	1.0152	0.085004	1.380085

late the option prices is expressed as

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz\{(r-q+\omega_0)t\} + t\left\{-\frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{(i\mu z - \frac{1}{2}\delta^2 z^2)} - 1\right)\right\}\right]\end{aligned}$$

where

$$\omega_0 = \omega + \gamma = -\frac{1}{2}\sigma^2 + \lambda\left(1 - \exp\left(\mu + \frac{1}{2}\delta^2\right)\right)$$

Note that γ is not present in the above characteristic function; thus, the number of parameters is reduced to 4. The characteristic exponent $\Psi_s(z)$ of $s_t = \ln(S_t/S_0)$ is therefore

$$\Psi_s(z) = (r - q + \omega_0)iz + \left\{-\frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{(i\mu z - \frac{1}{2}\delta^2 z^2)} - 1\right)\right\}$$

- The 4 cumulants of X related to mean, variance, skewness, and kurtosis are (by using $c_n^X = t \cdot \frac{1}{i^n}\frac{\partial^n \Psi_X(z)}{\partial z^n}|_{z=0}$) given by

$$\begin{aligned} c_1^X &= \mathbb{E}(X_t) = t(\gamma + \lambda\mu) \\ c_2^X &= \text{Var}(X_t) = t(\sigma^2 + \lambda\delta^2 + \lambda\mu^2) \\ c_3^X &= t\lambda(3\delta^2\mu + \mu^3) \\ c_4^X &= t\lambda(3\delta^4 + 6\mu^2\delta^2 + \mu^4) \end{aligned}$$

where $s = c_3^X/(c_2^X)^{1.5}$ and $\kappa = c_4^X/(c_2^X)^2$ are the skewness and the kurtosis of X_t respectively. The corresponding 4 cumulants and the skewness and the kurtosis of $s_t = \ln(S_t/S_0)$ are the same as those of X_t except c_1 since $\Psi_s(z) = \Psi_X(z) + (r - q + \omega)iz$. The first cumulant c_1^s for s_t is $c_1^X + (r - q + \omega)t$, i.e.

$$c_1^s = \mathbb{E}(s_t) = t(r - q + \lambda\mu)$$

Note that γ is not present in c_1^s.

2) **Kou model:** In the Kou model, the distribution of jump sizes is asymmetric exponential with a Lévy measure:

$$\nu(x) = \lambda \cdot \left(p\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1 - p)\lambda_- e^{\lambda_- x} 1_{x<0}\right)$$

where $\lambda_\pm > 0$ determines the tail behavior of the distribution of positive and negative jump sizes and $p \in [0, 1]$ represents the probability of an upward jump. For this process, the probability distribution of returns has an exponential tail. Due to the "memorylessness" of exponential random variables, analytical expressions for expectations involving first passage times can be obtained.

- five parameters $\theta = (\sigma, \lambda, \lambda_+, \lambda_-, p)$: σ-diffusion volatility, and λ-jump intensity, λ_+, λ_-, p-parameters of the jump size distribution.
- The characteristic function $\Phi_{X_t}(z)$ of X_t is given by

$$\begin{aligned} \Phi_{X_t}(z) &= \mathbb{E}[\exp izX_t] = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{izX_t}|N_t = n\right]\mathbb{P}\{N_t = n\} \\ &= \sum_{n=0}^{\infty} e^{iz\gamma t - \frac{1}{2}z^2\sigma^2 t}\left(p\frac{\lambda_+}{\lambda_+ - iz} + (1 - p)\frac{\lambda_-}{\lambda_- + iz}\right)^n \mathbb{P}\{N_t = n\} \\ &= e^{t\left\{-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda\left(p\frac{\lambda_+}{\lambda_+ - iz} + (1-p)\frac{\lambda_-}{\lambda_- + iz} - 1\right)\right\}} \end{aligned}$$

and the characteristic exponent $\Psi_X(z)$ is therefore

$$\Psi_X(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda\left(p\frac{\lambda_+}{\lambda_+ - iz} + (1-p)\frac{\lambda_-}{\lambda_- + iz} - 1\right)$$

- The characteristic function of the log prices, $s_t = \ln(S_t/S_0)$, is given by

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz(r-q+w_0)t + t\left\{-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \lambda\left(p\frac{\lambda_+}{\lambda_+ - iz} + (1-p)\frac{\lambda_-}{\lambda_- + iz} - 1\right)\right\}\right]\end{aligned}$$

where

$$\omega_0 = \omega + \gamma = -\frac{1}{2}\sigma^2 + \lambda\left(1 - p\frac{\lambda_+}{\lambda_+ - 1} - (1-p)\frac{\lambda_-}{\lambda_- + 1}\right)$$

- The 4 cumulants of X related to the mean, variance, skewness, and kurtosis are

$$\begin{aligned} c_1^X &= t\left(\gamma + \frac{\lambda p}{\lambda_+} - \frac{\lambda(1-p)}{\lambda_-}\right) \\ c_2^X &= t\left(\sigma^2 + \frac{\lambda p}{{\lambda_+}^2} + \frac{\lambda(1-p)}{{\lambda_-}^2}\right) \\ c_3^X &= t\lambda\left(\frac{p}{{\lambda_+}^3} - \frac{1-p}{{\lambda_-}^3}\right) \\ c_4^X &= t\lambda\left(\frac{p}{{\lambda_+}^4} + \frac{1-p}{{\lambda_-}^4}\right)\end{aligned}$$

8.1.2 Lévy processes via infinite activity models

A. Brownian subordinated Lévy processes

3) **The Variance Gamma model:** The variance gamma process is a finite variation process with infinite but relatively low activity of small jumps obtained by evaluating Brownian motion with drift γ and volatility σ at an independent gamma time, i.e. $X_t = \gamma T_t + \sigma W(T_t)$, where T_t is a gamma process with mean rate t and variance rate κt and the density function of the gamma time change g over a finite interval t is given by

$$p_t^T(g) = e^{-\frac{g}{\kappa}} g^{\frac{t}{\kappa}-1} / (\Gamma(\frac{t}{\kappa})\kappa^{\frac{t}{\kappa}})$$

The Lévy measure of T_t, the rate of arrival as a function of the jump size x, is given by $\rho(x) = \frac{1}{\kappa}\frac{e^{-x/\kappa}}{x}1_{x>0}$ with Laplace exponent $l(u) = -\frac{1}{\kappa}\ln(1-\kappa u)$.

- 3 parameters $\theta = (\gamma, \sigma, \kappa)$: γ-diffusion drift, σ-diffusion volatility, κ-variance of the subordinator.
- The probability density function $p_t(x)$ of X_t is obtained by integrating the probability of X conditional on the realization of gamma time change g over the probability of g lying in the interval $[g \quad g + dg]$:

$$
\begin{aligned}
p_t(x) &= \int_0^\infty p_t(x|g) p_t^T(g) dg = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} e^{-\frac{(x-\gamma g)^2}{2\sigma^2 g}} \frac{1}{\Gamma\left(\frac{t}{\kappa}\right)\kappa^{\frac{t}{\kappa}}} e^{-\frac{g}{\kappa}} g^{\frac{t}{\kappa}-1} dg \\
&= \frac{1}{\sigma\sqrt{2\pi}\Gamma\left(\frac{t}{\kappa}\right)\kappa^{\frac{t}{\kappa}}} e^{\frac{\gamma x}{\sigma^2}} \int_0^\infty e^{-\left\{\frac{1}{2}\left(\frac{\gamma^2}{\sigma^2}+\frac{2}{\kappa}\right)g + \frac{x^2}{2\sigma^2}\frac{1}{g}\right\}} g^{-\left\{1-\left(\frac{t}{\kappa}-\frac{1}{2}\right)\right\}} dg \\
&= \frac{1}{\sigma\sqrt{2\pi}\Gamma\left(\frac{t}{\kappa}\right)\kappa^{\frac{t}{\kappa}}} e^{\frac{\gamma x}{\sigma^2}} \times 2 \left(\frac{|x|}{\sigma\sqrt{\frac{\gamma^2}{\sigma^2}+\frac{2}{\kappa}}}\right)^{\frac{t}{\kappa}-\frac{1}{2}} \mathcal{K}_{\frac{t}{\kappa}-\frac{1}{2}}\left(\frac{|x|}{\sigma}\sqrt{\frac{\gamma^2}{\sigma^2}+\frac{2}{\kappa}}\right) \\
&= \sqrt{\frac{2}{\pi\sigma^2\kappa}} \frac{(\gamma^2\kappa^2 + 2\sigma^2\kappa)^{\frac{1}{4}-\frac{t}{2\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)} |x|^{\frac{t}{\kappa}-\frac{1}{2}} e^{\frac{\gamma}{\sigma^2}x} \mathcal{K}_{\frac{t}{\kappa}-\frac{1}{2}}\left(\frac{\sqrt{\gamma^2+\frac{2\sigma^2}{\kappa}}}{\sigma^2}|x|\right) \\
&= C|x|^{\frac{t}{\kappa}-\frac{1}{2}} e^{Ax} \mathcal{K}_{\frac{t}{\kappa}-\frac{1}{2}}(B|x|) \ ,
\end{aligned}
$$

  where $\mathcal{K}$ is the modified Bessel function of the second kind and

$$
A = \frac{\gamma}{\sigma^2}, \quad B = \frac{\sqrt{\gamma^2 + \frac{2\sigma^2}{\kappa}}}{\sigma^2}, \quad C = \sqrt{\frac{2}{\pi\sigma^2\kappa}} \frac{(\gamma^2\kappa^2 + 2\sigma^2\kappa)^{\frac{1}{4}-\frac{t}{2\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}.
$$

- The characteristic exponent $\Psi_X(z)$ is given by

$$
\Psi_X(z) = l(-z^2\sigma^2/2 + i\gamma z) = -\frac{1}{\kappa}\ln\left(1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2\right)
$$

  and the characteristic function $\Phi_{X_t}(z)$ of X_t is therefore

$$
\Phi_{X_t}(z) = \mathbb{E}[\exp izX_t] = e^{t\Psi_X(z)} = e^{-\frac{t}{\kappa}\ln(1-i\gamma\kappa z+\frac{1}{2}\sigma^2\kappa z^2)}
$$

- The Lévy measure ν_X of X_t is given by

$$
\begin{aligned}
\nu_X(x) &= \int_0^\infty p_s^{BS}(x)\rho(s) ds = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(x-\gamma s)^2}{2\sigma^2 s}} \frac{e^{-\frac{s}{\kappa}}}{\kappa s} ds \\
&= \frac{1}{\kappa|x|} e^{Ax - B|x|} \qquad \text{where} \quad A = \frac{\gamma}{\sigma^2},\ B = \frac{\sqrt{\gamma^2+\frac{2\sigma^2}{\kappa}}}{\sigma^2}.
\end{aligned}
$$

- The characteristic function of the log prices, $s_t = \ln(S_t/S_0)$, is given by

$$
\begin{aligned}
\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\
&= e^{iz(r-q+\omega)t} \left(\frac{1}{1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2}\right)^{\frac{t}{\kappa}}
\end{aligned}
$$

where

$$\omega = \frac{1}{\kappa}\ln\left(1 - \gamma\kappa - \frac{1}{2}\sigma^2\kappa\right)$$

- The 4 cumulants of X related to mean, variance, skewness, and kurtosis are

$$\begin{aligned}
c_1^X &= t\gamma \\
c_2^X &= t(\gamma^2\kappa + \sigma^2) \\
c_3^X &= t(2\gamma^3\kappa^2 + 3\gamma\sigma^2\kappa) \\
c_4^X &= t(6\gamma^4\kappa^3 + 12\gamma^2\sigma^2\kappa^2 + 3\sigma^4\kappa)
\end{aligned}$$

4) **Normal Inverse Gaussian model:** The normal inverse Gaussian process is an infinite variation process with stable-like behavior of small jumps constructed from Brownian subordination at an independent inverse Gaussian time, i.e. $X_t = \theta T_t + \sigma W(T_t)$, where T_t is an independent inverse Gaussian process with mean t and variance κt and the density function of the inverse Gaussian time change g over a finite interval t is given by

$$p_t^T(g) = \frac{t}{\sqrt{2\pi\kappa}} e^{\frac{t}{\kappa}} e^{(-\frac{1}{2\kappa}g - \frac{t^2}{2\kappa}\frac{1}{g})} g^{-\frac{3}{2}}$$

The Lévy measure of T_t, the rate of arrival as a function of the jump size x, is given by $\rho(x) = \frac{t}{\sqrt{2\pi\kappa}}\frac{e^{-x/2\kappa}}{x^{\frac{3}{2}}} 1_{x>0}$ with Laplace exponent $l(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2\kappa u}$.

- 3 parameters $\theta = (\gamma, \sigma, \kappa)$: γ-diffusion drift, σ-diffusion volatility, κ-variance of the subordinator.
- The probability density function $p_t(x)$ of X_t is

$$\begin{aligned}
p_t(x) &= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi s}} e^{-\frac{(x-\gamma s)^2}{2\sigma^2 s}} \frac{t}{\sqrt{2\pi\kappa}} e^{\frac{t}{\kappa}} e^{\left(-\frac{1}{2\kappa}s - \frac{t^2}{2\kappa}\frac{1}{s}\right)} s^{-\frac{3}{2}} ds \\
&= \frac{t}{2\pi\sigma\sqrt{\kappa}} e^{\frac{t}{\kappa}} e^{\frac{\gamma}{\sigma^2}x} \int_0^\infty e^{-\left(\frac{\gamma^2}{2\sigma^2} + \frac{1}{2\kappa}\right)s - \left(\frac{x^2}{2\sigma^2} + \frac{t^2}{2\kappa}\right)\frac{1}{s}} \frac{1}{s^2} ds \\
&= \frac{t}{\pi} e^{\frac{t}{\kappa}} \sqrt{\frac{\gamma^2}{\sigma^2\kappa} + \frac{1}{\kappa^2}} e^{\frac{\gamma}{\sigma^2}x} \frac{\mathcal{K}_1\left(\frac{\sqrt{\gamma^2 + \frac{\sigma^2}{\kappa}}}{\sigma^2}\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}}\right)}{\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}}} \\
&= Ce^{Ax} \frac{\mathcal{K}_1\left(B\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}}\right)}{\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}}}
\end{aligned}$$

where

$$A = \frac{\gamma}{\sigma^2}, \qquad B = \frac{\sqrt{\gamma^2 + \frac{\sigma^2}{\kappa}}}{\sigma^2}, \qquad C = \frac{t}{\pi} e^{\frac{t}{\kappa}} \sqrt{\frac{\gamma^2}{\sigma^2\kappa} + \frac{1}{\kappa^2}}$$

- The characteristic exponent $\Psi_X(z)$ is given by

$$\Psi_X(z) = l(-z^2\sigma^2/2 + i\gamma z) = e^{t(\frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1-2i\gamma\kappa z+\sigma^2\kappa z^2})}$$

and the characteristic function $\Phi_{X_t}(z)$ of X_t is therefore

$$\Phi_{X_t}(z) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2i\gamma\kappa z + \sigma^2\kappa z^2}$$

- The Lévy measure ν_X of X_t is given by

$$\begin{aligned}\nu_X(x) &= \int_0^\infty p_s^{BS}(x)\rho(s)ds = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(x-\gamma s)^2}{2\sigma^2 s}} \frac{e^{-\frac{s}{2\kappa}}}{\sqrt{2\pi\kappa} s^{\frac{3}{2}}} ds \\ &= \frac{D}{|x|} e^{Ax}\mathcal{K}_1(B|x|) \qquad \text{where} \quad A = \frac{\gamma}{\sigma^2},\ B = \frac{\sqrt{\gamma^2 + \frac{2\sigma^2}{\kappa}}}{\sigma^2},\ D = \frac{\sqrt{\gamma^2 + \frac{\sigma^2}{\kappa}}}{2\pi\sigma\sqrt{\kappa}}\end{aligned}$$

- The characteristic function of the log prices, $s_t = \ln(S_t/S_0)$, is given by

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz(r-q+\omega)t + t\left(\frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2i\gamma\kappa z + \sigma^2\kappa z^2}\right)\right]\end{aligned}$$

where

$$\omega = \frac{1}{\kappa}\left(\sqrt{1 - 2\gamma\kappa - \sigma^2\kappa} - 1\right)$$

- The 4 cumulants of X related to mean, variance, skewness, and kurtosis are

$$\begin{aligned}c_1^X &= t\gamma \\ c_2^X &= t(\gamma^2\kappa + \sigma^2) \\ c_3^X &= t(3\gamma^3\kappa^2 + 3\gamma\sigma^2\kappa) \\ c_4^X &= t(15\gamma^4\kappa^3 + 18\gamma^2\sigma^2\kappa^2 + 3\sigma^4\kappa)\end{aligned}$$

B. Tempered stable Lévy processes

5) **CGMY model:** The CGMY process (also called "truncated Lévy flights") is an infinite activity tempered stable process given by

$$\nu(x) = \frac{c}{(-x)^{1+\alpha}} e^{\lambda_- x} 1_{x<0} + \frac{c}{x^{1+\alpha}} e^{-\lambda_+ x} 1_{x>0}$$

It is of finite variation if $0 \leq \alpha < 1$ and of infinite variation if $\alpha \geq 1$.

- 4 parameters $\theta = (c, \alpha, \lambda_-, \lambda_+)$: c determine the overall and relative frequency of jumps; α determine the local behavior of the process (how the price evolves between big jumps); λ_-, λ_+ determine the tail behavior of the Lévy measure.

- The characteristic function $\Phi_{X_t}(z)$ of a CGMY process X_t is given by

$$\Phi_{X_t}(z) = \mathbb{E}[\exp izX_t] = e^{\left\{tc\Gamma(-\alpha)((\lambda_+ - iz)^\alpha - \lambda_+^\alpha + (\lambda_- + iz)^\alpha - \lambda_-^\alpha)\right\}}$$

and the characteristic exponent $\Psi_X(z)$ is therefore

$$\Psi_X(z) = c\Gamma(-\alpha)[(\lambda_+ - iz)^\alpha - \lambda_+^\alpha + (\lambda_- + iz)^\alpha - \lambda_-^\alpha]$$

where $0 < \alpha < 1$ or $\alpha > 1$
- The characteristic function of the log prices, $s_t = \ln(S_t/S_0)$, is given by

$$\begin{aligned}\Phi_s(z) &= \mathbb{E}[\exp(iz\ln(S_t/S_0))] = e^{iz((r-q)t+\omega t)} \times e^{t\Psi_X(z)} \\ &= \exp\left[iz(r - q + \omega)t + tc\Gamma(-\alpha)\{(\lambda_+ - iz)^\alpha - \lambda_+^\alpha + (\lambda_- + iz)^\alpha - \lambda_-^\alpha\}\right]\end{aligned}$$

where

$$\omega = c\Gamma(-\alpha)\{(\lambda_+ - 1)^\alpha - \lambda_+^\alpha + (\lambda_- + 1)^\alpha - \lambda_-^\alpha\}$$
- The 4 cumulants of X related to mean, variance, skewness, and kurtosis are

$$\begin{aligned}c_1^X &= t\Gamma(1-\alpha)[-\lambda_+^{\alpha-1} + \lambda_-^{\alpha-1}] \\ c_2^X &= t\Gamma(2-\alpha)[-\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}] \\ c_3^X &= t\Gamma(3-\alpha)[-\lambda_+^{\alpha-3} + \lambda_-^{\alpha-3}] \\ c_4^X &= t\Gamma(4-\alpha)[-\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}]\end{aligned}$$

8.2 Gradients for five Lévy models

To apply the gradient-based nonlinear optimization solver, which is the trust-region method, we derive the gradient of the posterior distribution, $\ln p(\boldsymbol{\theta}|\sigma_\theta^2)$, presented in (8):

$$\frac{\partial \ln p(\boldsymbol{\theta}|\sigma_\theta^2)}{\partial \boldsymbol{\theta}} = -\frac{1}{\sigma_\theta^2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0) - \frac{1}{\sigma_\epsilon^2}\sum_{n=1}^{N}\left\{\hat{C}(T_n, K_n, \boldsymbol{\theta}) - C_n\right\}\frac{\partial \hat{C}(T_n, K_n, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}.$$

Using (4), the gradient of the option price is as follows:

$$\frac{\partial \hat{C}(T_n, K_n, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\frac{\partial \psi_T(v)}{\partial \boldsymbol{\theta}}dv.$$

where $\frac{\partial \psi_T(v)}{\partial \boldsymbol{\theta}} = \frac{e^{-rT}}{\alpha^2+\alpha-z^2+i(2\alpha+1)z}\frac{\partial \Phi_s(z-(\alpha+1)i)}{\partial \boldsymbol{\theta}}$. By computing the gradient of the characteristic function, we can obtain the gradient of the posterior distribution and can use the gradient-based nonlinear optimization solver.

1) **Merton model:**

The gradient of the characteristic function of log prices with respect to the Lévy triplet$(\sigma, \lambda, \mu, \delta)$, used to obtain the Hessian, is expressed as

$$\begin{aligned}
\frac{\partial \Phi}{\partial \sigma} &= -(i+z)z\sigma t \times \Phi \\
\frac{\partial \Phi}{\partial \lambda} &= \Big((iz-1)t + t\exp(i\mu z - \frac{1}{2}\delta^2 z^2) - izt\exp(\mu + \frac{1}{2}\delta^2)\Big) \times \Phi \\
\frac{\partial \Phi}{\partial \mu} &= \Big(-iz\lambda\exp(\mu + \frac{1}{2}\delta^2)t + it\lambda z\exp(i\mu z - \frac{1}{2}\delta^2 z^2)\Big) \times \Phi \\
\frac{\partial \Phi}{\partial \delta} &= \Big(-iz\lambda\exp(\mu + \frac{1}{2}\delta^2)t - t\lambda z^2\exp(i\mu z - \frac{1}{2}\delta^2 z^2)\Big)\, \delta \times \Phi
\end{aligned}$$

2) **Kou model:**

The gradient of the characteristic function of the log prices with respect to the Lévy triplet$(\sigma, \lambda, \lambda_+, \lambda_-, p)$ is given by

$$\begin{aligned}
\frac{\partial \Phi}{\partial \sigma} &= -(i+z)zt\sigma \times \Phi \\
\frac{\partial \Phi}{\partial \lambda} &= \left[-iz\left(p\frac{\lambda_+}{\lambda_+ - 1} + (1-p)\frac{\lambda_-}{\lambda_- + 1} - 1\right) + \left(p\frac{\lambda_+}{\lambda_+ - iz} + (1-p)\frac{\lambda_-}{\lambda_- + iz} - 1\right)\right] t \times \Phi \\
\frac{\partial \Phi}{\partial \lambda_+} &= \left[iz\left(\frac{-1}{\lambda_+ - 1} + \frac{\lambda_+}{(\lambda_+ - 1)^2}\right) + \left(\frac{1}{\lambda_+ - iz} - \frac{\lambda_+}{(\lambda_+ - iz)^2}\right)\right] \lambda p t \times \Phi \\
\frac{\partial \Phi}{\partial \lambda_-} &= \left[iz\left(\frac{-1}{\lambda_- + 1} + \frac{\lambda_-}{(\lambda_- + 1)^2}\right) + \left(\frac{1}{\lambda_- + iz} - \frac{\lambda_-}{(\lambda_- + iz)^2}\right)\right] \lambda(1-p) t \times \Phi \\
\frac{\partial \Phi}{\partial p} &= \left[iz\left(\frac{-\lambda_+}{\lambda_+ - 1} + \frac{\lambda_-}{\lambda_- + 1}\right) + \left(\frac{\lambda_+}{\lambda_+ - iz} - \frac{\lambda_-}{\lambda_- + iz}\right)\right] t\lambda \times \Phi
\end{aligned}$$

3) **Variance Gamma model**

The gradient of the characteristic function of the log prices with respect to the Lévy triplet(θ, σ, κ) is given by

$$\begin{aligned}
\frac{\partial \Phi}{\partial \gamma} &= \left[\frac{-iz}{(1-\gamma\kappa - 1/2\sigma^2\kappa)} + \frac{iz}{(1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2)}\right] t \times \Phi \\
\frac{\partial \Phi}{\partial \kappa} &= \left[\log\left(1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2\right) - iz\log\left(1 - \gamma\kappa - \frac{1}{2}\sigma^2\kappa\right)\right] \frac{t}{\kappa^2} \times \Phi \\
&\quad - \left[\frac{\kappa\left(\frac{1}{2}\sigma^2 z^2 - i\gamma z\right)}{1 - i\gamma\kappa z + \frac{1}{2}\sigma^2\kappa z^2} + \frac{\kappa\left(\gamma + \frac{1}{2}\sigma^2\right) iz}{1 - \gamma\kappa - \frac{1}{2}\sigma^2\kappa}\right] \frac{t}{\kappa^2} \times \Phi \\
\frac{\partial \Phi}{\partial \sigma} &= \left[\frac{-iz\sigma}{(1-\gamma\kappa - 1/2\sigma^2\kappa)} - \frac{\sigma z^2}{(1 - i\gamma\kappa z + 1/2\sigma^2\kappa z^2)}\right] t \times \Phi
\end{aligned}$$

4) **Normal Inverse Gaussian model**

The gradient of the characteristic function of the log prices with respect to the Lévy triplet(θ, σ, κ) is given by

$$\begin{aligned}
\frac{\partial \Phi}{\partial \gamma} &= \left(\frac{1}{\sqrt{1-2i\gamma\kappa z+\sigma^2\kappa z^2}} - \frac{1}{\sqrt{1-2\gamma\kappa-\kappa\sigma^2}}\right) tiz \times \Phi \\
\frac{\partial \Phi}{\partial \kappa} &= \left(-\frac{1}{\kappa^2} + \frac{\sqrt{1-2i\gamma\kappa z+\sigma^2\kappa z^2}}{\kappa^2} + \frac{2\gamma iz-\sigma^2 z^2}{2\kappa\sqrt{1-2i\gamma\kappa z+\sigma^2\kappa z^2}}\right) t \times \Phi \\
&+ \left(\frac{1-\sqrt{1-2\gamma\kappa-\kappa\sigma^2}}{\kappa^2} - \frac{\sigma^2+2\gamma}{2\kappa\sqrt{1-2\gamma\kappa-\kappa\sigma^2}}\right) tiz \times \Phi \\
\frac{\partial \Phi}{\partial \sigma} &= \left(-\frac{z}{\sqrt{1-2i\gamma\kappa z+\sigma^2\kappa z^2}} - \frac{i}{\sqrt{1-2\gamma\kappa-\kappa\sigma^2}}\right) t\sigma z \times \Phi
\end{aligned}$$

5) **CGMY model**

The gradient of the characteristic function of the log prices with respect to the Lévy triplet$(c, \lambda_-, \lambda_+, \alpha)$ is given by

$$\begin{aligned}
\frac{\partial \Phi}{\partial c} &= \left(\frac{izw}{c} + \Gamma(-\alpha)((\lambda_+ - iz)^\alpha - \lambda_+^\alpha + (\lambda_- + iz)^\alpha - \lambda_-^\alpha)\right) t \times \Phi \\
\frac{\partial \Phi}{\partial \lambda_-} &= \left((\alpha(\lambda_- + iz)^{\alpha-1} - \alpha\lambda_-^{\alpha-1}\right) + iz\left(\alpha\lambda_-^{\alpha-1} - \alpha\left(\lambda_- + 1\right)^{\alpha-1}\right) ct\Gamma(-\alpha) \times \Phi \\
\frac{\partial \Phi}{\partial \lambda_+} &= \left((\alpha(\lambda_+ - iz)^{\alpha-1} - \alpha\lambda_+^{\alpha-1}\right) + iz\left(\alpha\lambda_+^{\alpha-1} - \alpha\left(\lambda_+ - 1\right)^{\alpha-1}\right) ct\Gamma(-\alpha) \times \Phi \\
\frac{\partial \Phi}{\partial \alpha} &= \left(\log(\lambda_- + iz)(\lambda_- + iz)^\alpha + \log(\lambda_+ - iz)(\lambda_+ - iz)^\alpha - \lambda_-^\alpha \log(\lambda_-) - \lambda_+^\alpha \log(\lambda_+)\right) ct\Gamma(-\alpha)\Phi \\
&+ \left((\lambda_-^\alpha \log(\lambda_-) + \lambda_+^\alpha \log(\lambda_+) - log(\lambda_- + 1)(\lambda_- + 1)^\alpha - \log(\lambda_+ - 1)(\lambda_+ - 1)^\alpha\right) izct\Gamma(-\alpha)\Phi \\
&+ \left(\lambda_- + 1)^\alpha(\lambda_+ - 1)^\alpha - \lambda_-^\alpha - \lambda_+^\alpha\right) \psi^0(-\alpha)ct\Gamma(-\alpha)\Phi \\
&+ \left(\lambda_-^\alpha + \lambda_+^\alpha - (\lambda_- + iz)^\alpha - (\lambda_+ - iz)^\alpha\right) \psi^0(-\alpha)ct\Gamma(-\alpha)\Phi
\end{aligned}$$

where $\psi^0(z)$ represents the polygamma function. $\psi^0(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

8.3 Equivalence of measures for Lévy processes

Recall that if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures, then there is a positive random variable, called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, denoted $\frac{d\mathbb{Q}}{d\mathbb{P}}$, such that for any random variable Z

$$E^{\mathbb{Q}}[Z] = E^{\mathbb{P}}\left\{Z\frac{d\mathbb{Q}}{d\mathbb{P}}\right\}.$$

The next result shows that in the presence of jumps, the class of equivalent probability measures is surprisingly large. Unless a diffusion component is present, we cannot change the drift but can obtain a much greater variety of equivalent measures by altering the distribution of jumps.

Proposition 1 (Sato (1999), Theorems 33.1 and 33.2) *Let (X_t, P) and (X_t, P') be two Lévy processes on $\mathbb{R}$ with characteristic triplets (σ^2, ν, γ) and $(\sigma'^2, \nu', \gamma')$. Then $P|_{\mathcal{F}_t}$ and $P'|_{\mathcal{F}_t}$ are equivalent for all t(equivalently for one $t > 0$) if and only if the three following conditions hold:*

(1) $\sigma = \sigma'$.
(2) ν *and* ν' *are equivalent with* $\int_{-\infty}^{\infty}(e^{phi(x)/2} - 1)^2\nu(dx) < \infty$, *where* $\phi(x) = \ln(\frac{d\nu'}{d\nu})$.
(3) If $\sigma = 0$ *then we must in addition have*

$$\gamma' - \gamma = \int_{-1}^{1} x(\nu' - \nu)(dx). \tag{12}$$

When P and P' are equivalent, the Radon-Nikodym derivative is

$$\frac{dP'}{dP}\Big|_{\mathcal{F}_t} = e^{U_t} \tag{13}$$

with

$$U_t = \eta X_t^c - \frac{\eta^2\sigma^2 t}{2} - \eta\gamma t + \lim_{\varepsilon\downarrow 0}\left(\sum_{0\le s\le t}^{|\Delta X_s|>\varepsilon}\phi(\Delta X_s) - t\int_{|x|>\varepsilon}(e^{\phi(x)} - 1)\nu(dx)\right)$$

In this expression, (X_t^c) is the continuous part of (X_t) and η is such that

$$\gamma' - \gamma - \int_{-1}^{1} x(\nu' - \nu)(dx) = \sigma^2\eta$$

if $\sigma > 0$ and zero if $\sigma = 0$. U_t is a Lévy process with a characteristic triplet $(\sigma_U^2, \nu_U, \gamma_U)$ given by

$$\sigma_U^2 = \sigma^2\eta^2, \quad \nu_U = \nu\phi^{-1}, \quad \gamma_U = -\frac{1}{2}\sigma^2\eta^2 - \int_{-\infty}^{\infty}(e^y - 1 - y1_{|y|\le 1})(\nu\phi^{-1})(dy).$$

and $E^P[e^{U_t}] = 1$.

Example 1 *(i) (Equivalence of measures for Brownian motions with drift): If (X_t, P) and (X_t, P') are two Brownian motions with volatilities $\sigma > 0$ and*

$\sigma' > 0$ and drifts μ and μ'. P and P' are equivalent if $\sigma = \sigma'$. When they are equivalent, the Radon-Nikodym derivative is

$$\frac{dP'}{dP} = \exp\left\{\frac{\mu'-\mu}{\sigma^2}X_T - \frac{1}{2}\frac{(\mu')^2-(\mu)^2}{\sigma^2}T\right\} = \exp\left\{\frac{\mu'-\mu}{\sigma^2}W_T - \frac{1}{2}\frac{(\mu'-\mu)^2}{\sigma^2}T\right\}.$$

where $W_t = (X_t - \mu t)/\sigma$ is a standard Brownian motion under P.
(ii) (Equivalence of measures for compound Poisson processes): If (X_t, P) and (X_t, P') are two compound Poisson processes with Lévy measures ν and ν', P and P' are equivalent if and only if ν and ν' are equivalent. In this case, the Radon-Nikodym derivative is

$$\frac{dP'}{dP} = \exp\left(T(\lambda-\lambda') + \sum_{s\le T}\phi(\Delta X_s)\right), \tag{14}$$

where $\lambda \equiv \nu(\mathbb{R})$ and $\lambda' \equiv \nu'(\mathbb{R})$ are the jump intensities of the two processes and $\phi \equiv \ln\left(\frac{d\nu'}{d\nu}\right)$.

Let (X, P) be a Lévy process with a characteristic triplet (σ^2, ν, γ). If the trajectories of X are neither almost surely increasing nor almost surely decreasing, then the exponential-Lévy model given by $S_t = e^{rt+X_t}$, is arbitrage-free: there is a probability measure P' equivalent to P, such that $(e^{-rt}S_t)_{t\in[0,T]}$ is a P'-martingale, where r is the interest rate. Thus, the exponential-Lévy model is arbitrage-free in the following(not mutually exclusive) cases: (i) X has a nonzero Gaussian component, $\sigma > 0$, or (ii) X has infinite variation $\int_{-1}^{1}|x|\nu(dx) = \infty$, or (iii) X has both positive and a negative jumps, or (iv) has positive jumps and negative drift or negative jumps and a positive drift.

An Esscher transform can be applied in constructing P' equivalent to P: (i) when $\sigma > 0$, P' can be obtained by changing the drift as in the Black-Scholes without changing the Lévy measure, by Proposition 1; (ii) when $\sigma = 0$, from the Esscher transform with $\phi(x) = \theta x$, we obtain an equivalent probability under which X is a Lévy process with a characteristic triplet $(0, \tilde{\nu}, \tilde{\gamma})$, where $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$ and drift $\tilde{\gamma} = \gamma + \int_{-1}^{1} x(e^{\theta x}-1)\nu(dx)$ by Proposition 1. By choosing θ satisfying

$$\gamma + \int_{-1}^{1} x(e^{\theta x}-1)\nu(dx) + \int_{-\infty}^{\infty}(e^x - 1 - x1_{|x|\le 1})e^{\theta x}\nu(dx) = 0,$$

e^{X_t} becomes a martingale under P', and the Radon-Nikodym derivative is

$$\frac{dP'}{dP}\Big|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{E[e^{\theta X_t}]} = e^{\theta X_t + \gamma(\theta)t}, \qquad \gamma(\theta) = -\ln E[e^{\theta X_1}]$$

Preservation properties under equivalent changes of measures

- *Preserved:* continuity/discontinuity of sample paths, cadlag property for sample paths, quadratic variation of sample paths, absence of arbitrage, finite/infinite jump rate, range of jump sizes, finite/infinite variation, presence and volatility of diffusion component, etc.
- *Non-preserved:* distribution of returns, heavy tails of increments, independence of increments, Markov property, intensity of jumps, etc.

8.4 Nelder-Mead Simplex Method

Four scalar parameters must be specified to define a complete Nelder-Mead method: coefficients of reflection ρ, expansion χ, contraction γ, and shrinkage σ. These parameters are chosen to satisfy

$$\rho > 0, \chi > 1, 0 < \gamma < 1 \quad \textit{and} \quad 0 < \sigma < 1$$

The Nelder-Mead method consists of the following steps:

1. Order. Order and re-label the $n+1$ vertices as $x_1, x_2, \cdots, x_{n+1}$ so that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{n+1})$. Because we want to minimize f, we refer to x_1 as the best vertex or point and to x_{n+1} as the worst point.

2. Reflect. Compute the reflection point x_r by

$$x_r = \overline{x} + \rho(\overline{x} - x_{n+1})$$

where $\overline{x}$ is the centroid of the n best points, i.e., $\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n}$. Evaluate $f(x_r)$. If $f(x_1) \leq f(x_2) < f(x_n)$, replace x_{n+1} with the reflected point x_r and go to step 6.

3. Expand. If $f(x_r) < f(x_1)$, compute the expansion point x_e by

$$x_e = \overline{x} + \chi(x_r - \overline{x}).$$

Evaluate $f(x_e)$. If $f(x_e) < f(x_r)$ replace x_{n+1} with x_e and go to step 6; otherwise replace x_{n+1} with x_r and go to step 6.

4. Contract. If $f(x_r) \geq f(x_n)$, perform a contraction between $\overline{x}$ and the better of x_{n+1} and x_r.

4.1. Outside. If $f(x_n) \leq f(x_r) < f(x_{n+1})$ (i.e., x_r is strictly better than x_{n+1}), perform an outside contraction. Calculate

$$x_{oc} = \overline{x} + \gamma(x_r - \overline{x}).$$

Evaluate $f(x_{oc})$. If $f(x_{oc}) \leq f(x_r)$, replace x_{n+1} with x_{oc} and go to step 6; otherwise, go to step 5.

Fig. 13. The reflection, expansion, contraction and shrinkage points for a simplex in two dimensions.

4.2. Inside. If $f(x_r) \geq f(x_{n+1})$, perform an inside contraction: Calculate

$$x_{ic} = \overline{x} + \gamma(x_{n+1} - \overline{x}).$$

Evaluate $f(x_{ic})$. If $f(x_{ic}) \leq f(x_{n+1})$, replace x_{n+1} with x_{ic} and go to step 6; otherwise, go to step 5.

5. Shrink. Evaluate f at the n new vertices

$$x_i' = x_1 + \gamma(x_i - x_1), \quad i = 2, 3, \cdots, n+1$$

Replace the vertices $x_2, \cdots, x_{n+1}$ with the new vertices $x_2', \cdots, x_{n+1}'$.

6. Stopping Condition. Order and re-label the vertices of the new simplex as $x_1, x_2, \cdots, x_{n+1}$ such that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{n+1})$. If $f(x_{n+1}) - f(x_1) < \varepsilon$, then stop, where $\varepsilon > 0$ is a small predetermined tolerance. Otherwise go to step 2.

Fig 1 shows the effects of reflection, expansion, contraction and shrinkage for a simplex in two dimensions using the standard values of the coefficients

$$\rho = 1, \chi = 2, \gamma = \frac{1}{2} \;\; and \;\; \sigma = \frac{1}{2}$$

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