Business Cycle and Commodity Futures*

Bong-Gyu Jang** and Hyeon-Wuk Tae**

Preliminary and Incomplete.

Abstract

We derive a closed-form pricing formula for commodity futures under a stochastically-changing market environment. We assume the commodity spot price follows a logarithmic mean-reverting process which suggested by Schwartz (1997) and the market environment changes according to two states of business cycles, economic expansions and economic recessions. We find a semi-closed form of the pricing formula for commodity future prices. The formula contains a multi-dimensional integral and, thus, we utilize a numerical quadrature method to get its approximate value.

*This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No.NRF-2012R1A1A2038735).

** Department of Industrial and Management Engineering, POSTECH. E-mail: bonggyujang@postech.ac.kr (Jang), taehy@postech.ac.kr (Tae)

1 Introduction

Recently, empirical researches have found that commodity prices has positive historical returns, but has a low correlation with equity and a positive correlation with inflation (Gorton and Rouwenhorst, 2006; Erb and Harvey, 2006) which is an attractive feature for financial investors. As a matter of fact, more and more financial investors has included commodity futures in their portfolios as explained by Buyuksahin and Robe (2012). Commodities attract more interest of the financial world as a research subject these days.

Many researchers have been developed stochastic models for commodity prices. Gibson and Schwartz (1990) use a stochastic convenience yield to model the oil price dynamics. Schwartz (1997) compares the performances of an one factor logarithmic mean-reverting model, a two factor model including a stochastic convenience yield as a factor, and a three factor model including a stochastic convenience yield and a stochastic interest rate as factors, using a Kalman filter method. Schwartz and Smith (2000) adopt a short-term deviation process and an equilibrium level price process to model a commodity underlying process. Cortazar and Schwartz (2003) modify the two factor model of Schwartz (1997) with fewer parameters, maintaining same explanation power.

Empirical studies show that commodity market is sensitive to a stochastic market environment. Fong and See (2002) implement a generalized regime-switching model to reflect such an environment of crude oil futures. De Jong (2006) applies a regime-switching approach for an electricity spot price process. Benz and Truck (2009) model dynamics of carbon emission allowance prices using a regime-switching model. These researches support that considering stochastic market environment is quite reasonable.

In this paper, we focus on the development of an analytic valuation method for commodity futures in a regime-switching market environment. Jang, Roh, and Yoon (2011) find an analytic valuation formula for path-dependent contingent claims. They utilize the expected value of the contingent claims when the occupation time of each regime is fixed. Since the same argument is not applicable to our model, we develop a new method for pricing commodity futures. We employ a numerical integration quadrature scheme to implement our method and compare the numerical results by our method with those by the Monte-Carlo simulation.

2 The Model

This paper develops an analytic representation of a contingent claim, where logarithm of the underlying asset price follows a regime-switching mean-reverting process. We assume a frictionless financial market, that is, we assume there is no tax, no transaction costs,and no short-sale restrictions in the financial market. We also assume that $(\Omega, \{\mathcal{F}_t\}\, \mathcal{F}, \mathbb{P})$ is a filtered probability space. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a Brownian motion W_t under a risk neutral measure $\mathbb{Q}$ and a continuous-time Markov chain $I(t)$. The underlying asset price process is given by

$$dS_t = \kappa(I(t))(\alpha(I(t)) - \ln S_t)S_t dt + \sigma(I(t))dW_t, \tag{1}$$

where $\kappa(I(t))$ is a rate of convergence, $\alpha(I(t))$ is a long term mean and $\sigma(I(t))$ is a volatility parameter at time t.[1] $\kappa(I(t)), \alpha(I(t))$, and $\sigma(I(t))$ are parameterized by two state continuous-time Markov chain $I(t)$ with infinitesimal generator Q,

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}, \tag{2}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are poisson arrival rates. There are two regimes classified according to the level of volatility, "low" regime (regime 1) and "high" regime (regime 2). The regime i shifts into regime $j(\neq i)$ after random time which follows an exponential distribution with mean $\frac{1}{\lambda_i}$, for $i, j \in \{1, 2\}$. In regime I(t)=i, $\kappa(I(t)) = \kappa_i$, $\alpha(I(t)) = \alpha_i$, and $\sigma(I(t)) = \sigma_i$. If we define

$$X_t \equiv \log S_t,$$

then we can obtain the following Ornstein-Uhlenbeck (OU) type stochastic process:

$$dX_t = \kappa(I(t))(\alpha(I(t)) - X_t)dt + \sigma(I(t))dW_t. \tag{3}$$

[1]Schwartz (1997) show that the non-regime-switching version of this logarithmic mean-reverting process might be suitable as a commodity price model.

We let N_T be the number of regime shifts in $[0, T]$, and $Y \equiv \{Y_i\}_{i=1}^{N_T}$ be a finite sequence of random variables of inter-arrival times between $(i-1)$-th regime shift and i-th regime shift. We assume time 0 is 0-th regime-switching time. We assume $i(0) = i_0$ is a known parameter. Let $\hat{i}_0$ be a regime other than i_0. Consequently for even number i, Y_i follows the exponential distribution with intensity $\lambda_{\hat{i}_0}$, for odd number i, Y_i follows the exponential distribution with intensity λ_{i_0}. Denote

$$
\begin{aligned}
F_n(y_1, y_2, ..., y_n) =& P(N_T = n, Y_1 \leq y_1, ..., Y_n \leq y_n) \\
=& P(Y_1 \leq y_1, ..., Y_n \leq y_n `|` N_T = n) \cdot P(N_T = n)
\end{aligned}
\tag{4}
$$

and

$$
f_n(y_1, ..., y_n) = \frac{\partial^n F_n(y_1, ..., y_n)}{\partial y_1 \partial y_2 ... \partial y_n} \tag{5}
$$

as a cumulative density function and a probability density function of $(Y_1, ..., Y_{N_T})$, given N_T times probability of $N_T = n$, respectively.

3 An Analytical Approach

In this section, we present a general analytic valuation method for contingent claim when the underlying asset follows equation (1).

Following Karatzas and Shreve (1991), the solution of the process (3) can be represented as

$$
X_t = Z_t\Big(X_0 + \int_0^t \frac{\kappa(I(u))\alpha(I(u))}{Z_u} du + \int_0^t \frac{\sigma(I(u))}{Z_u} dW_u\Big), \tag{6}
$$

where $Z_t \equiv e^{-\int_0^t \kappa(I(u))du}$.

Because regime changing is uncertain up to realization, obtaining the probability distribution of X_T directly is hardly possible. To address this issue, we firstly investigate the probability distribution of X_T conditioned on $N_T, Y_1, ..., Y_{N_T}$. This enables us to handle the probability distribution with a known parameter path. Because a random-variable within OU process follows a normal distribution, we expect a similar result to X_T with given $N_T, Y_1, ...,$ and Y_{N_T}.

Theorem 3.1 *Given $N_T = n, Y_1 = y_1, ..., Y_n = y_n$, X_T follows a normal distribution with mean m and variance v, where*

$$
\begin{aligned}
m \equiv& Z_T X_0 + \sum_{j=0}^{n} \alpha_j \frac{Z_T}{Z_{\tau_j}} (e^{k_j y_j} - 1), \\
v \equiv& \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{2\kappa_j Z_{\tau_j}^2} (e^{2\kappa_j y_j} - 1), \\
(\kappa_j, \alpha_j, \sigma_j) \equiv& \begin{cases} (\kappa_{i_0}, \alpha_{i_0}, \sigma_{i_0}) & \text{if } j \text{ is even,} \\ (\kappa_{\hat{i}_0}, \alpha_{\hat{i}_0}, \sigma_{\hat{i}_0}) & \text{if } j \text{ is odd.} \end{cases}
\end{aligned}
$$

Proof. We denote the time when the i-th regime switch occurs as

$$\tau_i \equiv \sum_{j=1}^{i} y_j,$$

for $1 \le i \le n$. Let $\tau_0 = 0$, $\tau_{n+1} = T$, and $y_{n+1} = \tau_{n+1} - \tau_n = T - \tau_n$. If we define

$$\Delta_j \equiv Z_T \Big(\int_{\tau_j}^{\tau_{j+1}} \frac{\kappa(I(u))\alpha(I(u))}{Z_u} du + \int_{\tau_j}^{\tau_{j+1}} \frac{\sigma(I(u))}{Z_u} dW_u \Big), \tag{7}$$

then we can easily verify that Δ_j follows a normal distribution. Note that parameters have a constant value on $[\tau_j, \tau_{j+1})$. Thus we can denote $\kappa(I(u)) = \kappa_j, \alpha(I(u)) = \alpha_j, \sigma(I(u)) = \sigma_j$.

We rewrite the first term of the right hand side in (7) as

$$
\begin{aligned}
Z_T \int_{\tau_j}^{\tau_{j+1}} \frac{\kappa(I(u))\alpha(I(u))}{Z_u} &= Z_T \int_{\tau_j}^{\tau_{j+1}} e^{\int_0^u \kappa(I(s))ds} \kappa(I(u))\alpha(I(u)) du \\
&= Z_T \int_{\tau_j}^{\tau_{j+1}} e^{\int_0^{\tau_j} \kappa(I(s))ds + \int_{\tau_j}^{u} \kappa(I(s))ds} \kappa_j \alpha_j du \\
&= \kappa_j \alpha_j \frac{Z_T}{Z_{\tau_j}} \int_{\tau_j}^{\tau_{j+1}} e^{\kappa_j (u - \tau_j)} du \\
&= \alpha_j \frac{Z_T}{Z_{\tau_j}} (e^{k_j y_{j+1}} - 1).
\end{aligned} \tag{8}
$$

Note also that the second term of the right hand side in (7) can be rewritten by

$$
\begin{aligned}
Z_T \int_{\tau_j}^{\tau_{j+1}} \frac{\sigma(I(u))}{Z_u} dW_u &= \sigma_j Z_T \int_{\tau_j}^{\tau_{j+1}} e^{\int_0^u \kappa(I(s))ds} dW u \\
&= \sigma_j Z_T \int_{\tau_j}^{\tau_{j+1}} e^{\int_0^{\tau_j} \kappa(I(s))ds + \int_{\tau_j}^{u} \kappa(I(s))ds} dW_u \\
&= \sigma_j \frac{Z_T}{Z_{\tau_i}} \int_{\tau_j}^{\tau_{j+1}} e^{\int_{\tau_j}^{u} \kappa_j ds} dW_u \\
&= \sigma_j \frac{Z_T}{Z_{\tau_i}} \int_{\tau_j}^{\tau_{j+1}} e^{\kappa_j (u - \tau_j)} dW_u.
\end{aligned} \tag{9}
$$

Since the first term is an Ito integral of a deterministic function, Δ_j follows the normal distribution. The variance of Δ_j is given by

$$\begin{aligned}\text{the variance of } Z_T \int_{\tau_j}^{\tau_{j+1}} \frac{\sigma(I(u))}{Z_u} dW_u &= \sigma_j^2 \frac{Z_T^2}{Z_{\tau_j}^2} \int_{\tau_j}^{\tau_{j+1}} e^{2\kappa_j (u-\tau_j)} dt \\ &= \frac{\sigma_j^2 Z_T^2}{2\kappa_j Z_{\tau_j}^2} (e^{2\kappa_j y_{j+1}} - 1).\end{aligned} \tag{10}$$

Therefore, we can conclude that Δ_j follows the normal distribution with mean $\alpha_j \frac{Z_T}{Z_{\tau_j}} (e^{k_j y_{j+1}} - 1)$ and variance $\frac{\sigma_j^2 Z_T^2}{2\kappa_j Z_{\tau_j}^2} (e^{2\kappa_j y_{j+1}} - 1)$. Consequently, we can obtain

$$X_T \Big|_{N=n, Y_1=y_1, ..., Y_n=y_n} \sim \aleph(Z_T X_0 + \sum_{j=0}^{n} \alpha_j \frac{Z_T}{Z_{\tau_j}} (e^{k_j y_{j+1}} - 1), \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{2\kappa_j Z_{\tau_j}^2} (e^{2\kappa_j y_{j+1}} - 1))) \tag{11}$$

from the relationship of $X_T = Z_T X_0 + \sum_{j=0}^{n} \Delta_j$. ■

Now, let $f : R \mapsto R$ be a Borel-measurable function. Then we can calculate the expectation of the payoff $f(X_T)$, $E[f(X_T)]$.

Theorem 3.2 *Suppose $f : R \mapsto R$ be a Borel-measurable function. Then the expectation of the payoff $f(X_T)$ is represented by*

$$\begin{aligned} E[f(X_T)] &= E[E[f(X_T)|N_T, Y_1, ..., Y_{N_T}]] \\ &= \sum_{n=0}^{\infty} A_n, \end{aligned}$$

where

$$\begin{aligned} A_n &= E[f(X_T)|N(t) = n] P(N_T = n) \\ &= \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} E[f(X_T)|N_T = n, Y_1 = y_1, ..., Y_n = y_n] \\ & e^{-\lambda_n (T-\tau_n)} (\prod_{i=0}^{n-1} \lambda_i e^{-\lambda_i y_{i+1}}) dy_n dy_{n-1} ... dy_1, \end{aligned} \tag{12}$$

$$\text{where } \lambda_i = \begin{cases} \lambda_{i_0} & \text{if } i \text{ is even,} \\ \lambda_{\hat{i}_0} & \text{if } i \text{ is odd,} \end{cases}$$

and

$$E[f(X_T)|N_T = n, Y_1 = y_1, ..., Y_n = y_n] = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi v}} e^{\frac{-(x-m)^2}{2v}} dx, \tag{13}$$

where m and v is the mean and variance described in (11).

Proof. By the tower property, we write

$$\begin{aligned} E[f(X_T)] &= E[E[f(X_T)|N(t)=n]] \\ &= \sum_{n=0}^{\infty} E[f(X_T)|N_T=n]P(N_T=n) \\ &= \sum_{n=0}^{\infty} E[E[f(X_T)|N_T=n, Y_1=y_1, ...Y_n=y_n]|N_T=n]]P(N_T=n). \end{aligned} \tag{14}$$

We note that $N_T = n$ is equivalent to

$$\sum_{i=1}^{n} Y_i < T \text{ and } \sum_{i=1}^{n+1} Y_i > T \tag{15}$$

and the probability density function of $Y_1, ..., Y_n$ conditioned on $N_T = n$ is given by

$$f_{Y_1,Y_2,...,Y_n,Y_{n+1}|N_t=n}(y_1, ..., y_n, y_{n+1}) = \prod_{i=0}^{n} \lambda_i e^{\lambda_i y_{i+1}}/P(N_T=n). \tag{16}$$

So, we can rewrite the conditional expectation in (14) as

$$\begin{aligned} &E[E[f(X_T)|N_T=n, Y_1=y_1, ...Y_n=y_n]|N_T=n]] \\ &= \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} \int_{T-\tau_n}^{\infty} E[f(X_T)|N_T=n, Y_1=y_1, ..., Y_n=y_n] \\ &f_{Y_1,Y_2,...,Y_n,Y_{n+1}|N_t=n}(y_1, ..., y_n, y_{n+1}) dy_{n+1}...dy_1 \\ &= \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} \int_{T-\tau_n}^{\infty} E[f(X_T)|N_T=n, Y_1=y_1, ..., Y_n=y_n] \\ &(\prod_{i=0}^{n} \lambda_i e^{-\lambda_i y_{i+1}})/P(N_T=n) dy_{n+1}...dy_1 \\ &= \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} E[f(X_T)|N_T=n, Y_1=y_1, ..., Y_n=y_n] \\ &e^{-\lambda_n(T-\tau_n)}(\prod_{i=0}^{n-1} \lambda_i e^{-\lambda_i y_{i+1}})/P(N_T=n) dy_n...dy_1. \end{aligned} \tag{17}$$

Using Theorem 3.1 and the equation (17), we get the desired result. ■

4 Commodity Future Prices

In this section, we find commodity future prices when underlying commodity prices follows the equation (3). Let $F(S, T)$ be the future price of the commodity with initial spot price S

and expiration date T. Then $F(S,T)$ is given by

$$F(S,T) = \mathbb{E}[S_T] = \mathbb{E}[e^{X_T}], \tag{18}$$

where $\mathbb{E}(\cdot)$ is expectation under risk-neutral measure $\mathbb{Q}$. Using Theorem 3.2 with $f(x) = e^x$, the expectation can be calculated. We can easily verify that $X_T\Big|_{N_T=n,Y_1=y_1,...,Y_n=y_n}$ follows a normal distribution with mean m and variance v where

$$\begin{aligned} X_0 =& \ln(S), \\ m =& Z_T X_0 + \sum_{j=0}^{n} \frac{\alpha_j Z_T}{Z_{\tau_j}} (e^{\kappa y_{j+1}} - 1), \\ v =& \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{2\kappa Z_{\tau_j}^2} (e^{2\kappa y_{j+1}} - 1). \end{aligned}$$

For a random variable X which follows a normal distribution with mean m and variance v, the following holds:

$$E[e^X] = e^{m+\frac{1}{2}v}.$$

Thus we get

$$E[e^{X_T}|N_T = n, Y_1 = y_1, ..., Y_n = y_n] = \exp(Z_T X_0 + \sum_{j=0}^{n} \frac{\alpha_j Z_T}{Z_{\tau_j}} (e^{\kappa y_{j+1}} - 1) + \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{4\kappa Z_{\tau_j}^2} (e^{2\kappa y_{j+1}} - 1)). \tag{19}$$

Then A_n is represented as follows:

$$\begin{aligned} A_n =& E[f(X_T)|N(t) = n]P(N_T = n) \\ =& \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} E[f(X_T)|N_T = n, Y_1 = y_1, ..., Y_n = y_n] \\ & e^{-\lambda_{n+1}(T-\tau_n)} (\prod_{i=1}^{n} \lambda_i e^{-\lambda_i y_i}) dy_n dy_{n-1} ... dy_1 \\ =& \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} \exp(Z_T X_0 + \sum_{j=0}^{n} \frac{\alpha_j Z_T}{Z_{\tau_j}} (e^{\kappa y_{j+1}} - 1) + \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{4\kappa Z_{\tau_j}^2} (e^{2\kappa y_{j+1}} - 1)) \\ & e^{-\lambda_{n+1}(T-\tau_n)} (\prod_{i=1}^{n} \lambda_i e^{-\lambda_i y_i}) dy_n dy_{n-1} ... dy_1. \end{aligned} \tag{20}$$

Then the expectation is given by

$$
\begin{aligned}
E[e^{X_T}] &= \sum_{n=0}^{\infty} E[e^{X_T}|N_T = n]P(N_T = n) \\
&= \sum_{n=0}^{\infty} A_n \\
&= \sum_{n=0}^{\infty} \int_0^T \int_0^{T-\tau_1} ... \int_0^{T-\tau_{n-1}} \exp(Z_T X_0 + \sum_{j=0}^{n} \frac{\alpha_j Z_T}{Z_{\tau_j}}(e^{\kappa y_{j+1}} - 1) + \sum_{j=0}^{n} \frac{\sigma_j^2 Z_T^2}{4\kappa Z_{\tau_j}^2}(e^{2\kappa y_{j+1}} - 1)) \\
& e^{-\lambda_{n+1}(T-\tau_n)}(\prod_{i=1}^{n} \lambda_i e^{-\lambda_i y_i}) dy_n dy_{n-1}...dy_1.
\end{aligned}
\tag{21}
$$

5 Numerical Implementation

Even though Theorem 3.2 is an explicit representation for calculating an expectation, it contains multi-dimensional integrations over some simplexes. In most cases, the calculation of a multiple integral over simplexes is a hard task. We implement a numerical integration method by Hammer and Stroud(1956) to get the numerical results.

Theorem 5.1 *Let S_n be an n-simplex with vertices V_0, V_1,...,V_n. An integration formula exact for the general cubic polynomial over S_n for $n \geq 1$ is given by*

$$\int_{S_n} f dv_n = a_n \sum_{i=0}^{n} f(U_i) + c_n f(C) \tag{22}$$

where

$$a_n = \frac{(n+3)^2}{4(n+1)(n+2)}\Delta_n, \qquad c_n = \frac{-(n+1)^2}{4(n+2)}\Delta_n, \qquad C = \sum_{i=1} V_i/(n+1), \tag{23}$$

and

$$U_i = \frac{2}{n+3}V_i + \frac{n+1}{n+3}C. \tag{24}$$

To find the value of the expectation of $E[e^{X_T}]$, we implement Theorem 5.1 to calculate A_n for each $n \geq 1$. The integration range of A_n is the n-simplex with vertices V_0=0 and $V_i = Te_i$ where $e_i = (0, ..., 1, ..., 0) \in R^{n+1}$ where e_i is i-th the standard basis vector of $\mathbb{R}^n$. For $n = 0$,

Parameter				Benchmark		Monte-Carlo		Analytic	
S	κ_2	α_2	λ_2	F_1	F_2	F_1	F_2	F_1	F_2
20	0.4	2	0.5	13.3518	12.4372	13.5093	12.3583	13.3621	12.4535
20	0.4	2	1.25	15.4474	14.7854	15.3655	14.9006	15.1889	14.5887
20	0.2	2	0.5	16.2550	15.4665	16.0106	15.3296	16.2800	15.4355
20	0.2	2	1.25	17.7380	17.2820	17.5106	17.2924	17.6318	17.1394
20	0.2	4	1.25	26.8971	27.8463	27.6733	27.8218	26.8512	27.8065
20	0.4	4	1.25	30.5324	31.9348	30.7696	31.9745	30.5708	31.9013
20	0.2	4	0.5	31.4942	33.8420	31.1856	33.0791	31.2679	33.4370
20	0.4	4	0.5	36.7297	39.4728	36.5929	28.8361	36.5734	39.2038
CPU time(sec)				628.08		25.74		1.06	

Table 1: **The values of commodity future prices by Monte-Carlo method and the analytic solution in Theorem 3.2.** Default parameters are $T = 5$, $\kappa_1 = 0.3$, $\alpha_1 = 3$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\lambda_1 = 0.75$, $S = 20$. whereas $\sigma_1 < \sigma_2$ is fixed. All possible cases such that $\kappa_1 > \kappa_2, \kappa_1 < \kappa_2$ and $\alpha_1 > \alpha_2, \alpha_1 < \alpha_2$, are displayed at 1-4 columns. F_i is future prices when initial underlying process is $S = 20$, and $T = 5$ with initial regime $i_0 = i$. The values in column 5-6 are obtained by the Monte-Carlo method with 365 time steps and 50000 trials, whereas the values in column 7-8 are obtained by 2000 trials, 365 time steps. Column 8-9 contain the values obtained by our analytic formula, calculated with quadrature for simplexes. All routines are programmed using MATLAB language and run on a 3.30-GHz, i-2500 computer.

we use the fact that

$$A_0 = E[e^{X_T}|N_T = 0]P(N_T = 0) = S_0^{e^{-\kappa_{i_0}T}} e^{\alpha_{i_0}(1-e^{-\kappa_{i_0}T}) + \frac{{\sigma_{i_0}}^2}{4\kappa_{i_0}}(1-e^{-2\kappa_{i_0}T})}$$

because Theorem 5.1 holds only for $n \geq 1$. A_0 is just the expected value of $f(x_T)$ in no regime-switching case multiplied by the probability that regime switch does not occur on time interval $[0, T]$. In Table 1, we present the expectation calculated by numerically integrating the valuation formula, by the Monte-Carlo simulation with 50000 trials (as the benchmark), and by the Monte-Carlo simulation with 2000 trials. We fix parameters $T = 5$, $\kappa_1 = 0.3$, $\alpha_1 = 3$, $\sigma_1 = 0.2$ $\sigma_2 = 0.4$, $\lambda_1 = 0.75$ and variate $\kappa_2, \alpha_2, \lambda_2$. Compared with the Monte-Carlo method, our analytic formula provides us more rapid valuation technique. The table shows that the result from the Monte Carlo method with 2000 trials are calculated 25.74 second, but the analytic solution takes just 1.06 second, which is about 24 times faster. There are slight differences between the benchmark and the analytic solution, but these differences are expected to due to numerical errors, and can be improved by implementing a more accurate quadrature. Silvester(1970) introduced general ways to find quadrature coefficients which is accurate for polynomial integrands with arbitrary degrees. We observe that our valuation method is superior to the Monte-Carlo method with 2000 trials.

6 Conclusion

We derive an analytic valuation formula for representing contingent claim prices when the logarithm of the underlying price follows a mean-reverting model with regime-switching parameters. The formula can be represented with infinite sum of integrations over simplexes. We calculate future prices of a commodity which follows the equation of (1). To practically evaluate the integration over simplexes, we implement the numerical quadrature scheme suggested by Hammer and Stroud(1956). We compare the result by the analytic formula with the results by the Monte-Carlo method. The Monte-Carlo methods are conducted with 50000 trials as the benchmark, and with 2000 trials for comparison purposes. It turns out to be our method is much more superior to the Monte-Carlo method.

References

Benz, Eva, and Stefan Trck. 2009. Modeling the price dynamics of CO2 emission allowances. *Energy Economics* **31.1** 4-15.

Buyuksahin, Bahattin, and Michel Robe. 2012. Does It Matter Who Trades Energy Derivatives?. *Review of Environment, Energy and Economics.*

Cortazar, Gonzalo, and Eduardo S. Schwartz. 2003. Implementing a stochastic model for oil futures prices. *Energy Economics* **25.3** 215-238.

De Jong, Cyriel. 2006 The nature of power spikes: A regime-switch approach. *Stud. Nonlinear Dyn. Econom.* **10.3** 3.

Erb, Claude B., and Campbell R. Harvey. 2006. The strategic and tactical value of commodity futures. *Financial Analysts Journal.* 69-97.

Fong, Wai Mun, and Kim Hock See. 2002. A Markov switching model of the conditional volatility of crude oil futures prices. *Energy Economics* **24.1** 71-95.

Gibson, Rajna, and Eduardo S. Schwartz. 1990. Stochastic convenience yield and the pricing of oil contingent claims. *The Journal of Finance* **45.3** 959-976.

Gorton, Gary, and K. Geert Rouwenhorst. 2004. Facts and fantasies about commodity futures. *National Bureau of Economic Research.* No.w10595.

Hammer, Preston C., and Arthur H. Stroud. 1956. Numerical integration over simplexes. *Mathematical tables and other aids to computation.* **10.55** 137-139.

Jang, Bong-Gyu, and Kum-Hwan Roh. 2009. Valuing qualitative options with stochastic volatility. *Quantitative Finance* **9.7** 819-825.

Jang, Bong-Gyu, Kum-Hwan Roh, and Ji Hee Yoon. 2011. An analytic valuation method for multivariate contingent claims with regime-switching volatilities. *Operations Research Letters* 39.3

Karatzas Ioannis Autor, and Steven Eugene Shreve. *Brownian motion and stochastic calculus*. Vol. 113. Springer, 1991.

Schwartz, Eduardo. 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging. *The Journal of Finance* **52.3** 923-973.

Schwartz, Eduardo, and James E. Smith. 2000. Short-term variations and long-term dynamics in commodity prices. *Management Science* **46.7** 893-911.

Silvester, P. 1970. Symmetric quadrature formulae for simplexes. *Mathematics of Computation* **24.109** 95-100.

Williams, David. 1991. *Probability with martingales*. Cambridge university press.