Long-Term Growth Rate of Expected Utility for Leveraged ETFs: Martingale Extraction Approach*†

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Abstract

This paper studies the long-term growth rate of expected utility or expected return from holding a leveraged exchanged-traded fund (LETF), which is a constant proportion portfolio of the reference asset. We develop a martingale extraction approach to tackle the path-dependence in the expectation and determine the long-term growth rate through the eigenpair associated with the infinitesimal generator of a time-homogeneous Markovian diffusion. The long-term growth rates are derived explicitly under a number of models for the reference asset, including the geometric Brownian motion model, GARCH model, inverse GARCH model, extended CIR model, 3/2 model, quadratic model, as well as the Heston and 3/2 stochastic volatility models. We also investigate the impact of stochastic interest rate such as the Vasicek model and the inverse GARCH short rate model. Additionally, we determine the optimal leverage ratio that maximizes the long-term growth rate, and examine the effects of model parameters.

*The authors acknowledge the helpful remarks from two anonymous referees, and feedback from the participants of the 2016 SIAM Conference on Financial Math & Engineering and the 1st Eastern Conference on Mathematical Finance.

†This is an electronic reprint of the original article published in *International Journal of Theoretical and Applied Finance, 2017, Vol. 20, No. 6.* This reprint differs from the original in pagination and typographic detail.

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1 Introduction

Exchange-traded funds (ETFs) are popular financial products designed to track the value of a reference asset or index. With over \$2 trillion of assets under management, ETFs are traded on major exchanges like stocks, even if the reference itself may not be traded. Within the growing ETF market, leveraged ETFs (LETFs) are created to generate a constant multiple β, called leverage ratio, of the daily returns of a reference index. For example, the ProShares Ultra S&P 500 (SSO) offers to generate twice ($\beta = 2$) the daily returns of the S&P 500 index. In the LETF market, the most common leverage ratios are $\beta \in \{1, 2, 3\}$ and $\beta \in \{-1, -2, -3\}$. In particular, investors can take a bearish position on the reference by taking a long position in an LETF with $\beta < 0$ without the need of borrowing shares or a margin account. For many speculative investors, LETFs are highly accessible and liquid instruments that give a leveraged exposure, and particularly attractive during periods of large momentum.

For LETF holders and potential investors, it is crucial importance to understand the price dynamics and the impacts of leverage ratio on the risk and return of each LETF. A number of market observations suggest that LETFs suffer from the volatility decay effect, which reflects the value erosion proportional to the realized variance of the reference index. Recent studies, including Avellaneda and Zhang (2010), Cheng and Madhavan (2009), Leung and Ward (2015), and Leung and Santoli (2016), present discrete-time and continuous-time stochastic frameworks to illustrate the path dependence of LETFs on the reference, including the volatility decay effect that is exacerbated over time. To address this issue, Leung and Santoli (2012) derived the admissible holding horizons for LETFs with respect to different risk measures. In fact, SEC issued in 2009 an alert announcement regarding the riskiness of LETFs, especially when holding them long-term.[1] This raises the questions: how do LETF returns behave under different models in the long run? In addition, is there an optimally leveraged ETF that is best for a risk-averse investor in the long run? The answers to these questions will not only provide insight to investors on the model dependence of an LETF's long-term performance, but will also help address the regulator's concerns. All these motivate us to analyze the long-term growth rate of expected return or expected utility from holding an LETF. Examining the dependence of the growth rate on the leverage ratio β, we also determine the optimal leverage ratio under a variety of stochastic models.

In this paper, we investigate the long-term growth rate of the expected utility from holding a LETF. Specifically, we consider different stochastic models for the LETF price, denoted by L_t, and analyze the long-term growth rate represented by the limit:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[u(L_t)] \,, \tag{1.1}$$

where $u(\cdot)$ is the investor's utility of the power form: $u(w) = w^\alpha$ with $0 < \alpha \leq 1$. As such, the coefficient of relative risk aversion is given by $\varrho := -wu''(w)/u'(w) = 1 - \alpha$, $\forall w > 0$. The determination of the long-term growth rate of expected utility belongs to the *risk-sensitive* approach (see Fleming and Sheu (1999); **?**) to evaluate the long-term performance of LETFs, and the resulting limit is called the *risk-adjusted* growth rate. When $\alpha = 1$, corresponding to zero relative risk aversion, the limit is the long-term growth rate of expected return of the LETF. Hence, analyzing (1.1) allows us to understand the long-term growth rates of expected utility and expected return that are useful for risk-averse and risk-neutral investors, respectively.

[1] See the SEC alert on http://www.sec.gov/investor/pubs/leveragedetfs-alert.htm.

The expectation of the utility or return of an LETF is typically non-trivial as it involves an exponential function of the integrated drift and integrated variance of the reference asset price process. One of main contributions of this paper is to present a novel approach to determine the expectation and the associated limit explicitly under various models. For this purpose, we employ the method of *martingale extraction*, and turn a path-dependent expectation into a path-independent one that is significantly more amenable to analysis and computation. Moreover, the original probabilistic problem of finding the long-term growth rate of expected utility (or expected return) is transformed into a deterministic eigenpair problem of a second-order differential operator that is associated with the infinitesimal generator of the reference price process. In each of the single-factor and multi-factor models studied herein, the associated eigenpair (eigenvalue and eigenfunction), as well as the limit of the growth rate are derived explicitly.

Our results allow us to determine the optimal leverage ratio for the long-term risk-averse investor. For the β-LETF with price denoted by $L_t \equiv L_t(\beta)$, we find the optimal leverage ratio β^* that maximizes the long-term growth rate, that is,

$$\beta^* = \arg\max_{\beta\in\mathbb{R}} \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[u(L_t(\beta))] \,. \tag{1.2}$$

Furthermore, we examine through our explicit expressions the combined effects of risk aversion and model parameters on the optimal choice of leverage.

There are a number of related studies on the long-term growth rate of expected utility. The seminal work by Fleming and Sheu (1999) investigated the optimal growth rate of expected utility of wealth. The utility was of hyperbolic absolute risk aversion (HARA) type, and dynamic programming scheme was developed for different HARA parameters and policy constraints. Akian et al. (1999) studied the optimal investment strategies with transaction costs with the objective to maximize the long-term average growth rate under logarithmic utility. In another related study, Zhu (2014) also examined the long-term growth rate of expected power utility from a nonleveraged portfolio with a fixed fraction of wealth in the single risky asset, and derived explicit limits under some models. In comparison, we study *leveraged* portfolios under additional single-factor and multi-factor diffusion models.

Christensen and Wittlinger (2012) considered the growth rate maximization problem based on impulse control strategies with limited number of trades per unit time and proportional transaction costs. Guasoni and Mayerhofer (2016) analyzed the optimal strategy to maximize the long-term return given average volatility under the Black-Scholes model with proportional costs. Hata and Sekine (2006) studied a long-term optimal investment problem with an objective of maximizing the probability that the portfolio value would exceeed a given level in a market with Cox-Ingersoll-Ross interest rate. Applying the theory of large deviation, Pham (2003) derived the optimal long-term investment strategy under the CARA utility, and Pham (2015) examined the long-term asymptotics for optimal portfolios that involved maximizing the probability for a portfolio to outperform a target growth rate.

The martingale extraction method is a relatively new mathematical technique that has been used to investigate a number of financial and economic problems. Among our main references, Hansen and Scheinkman (2009) and Hansen (2012) developed the martingale extraction method to study the long-term risk in continuoue-time Markovian markets. Borovicka et al. (2011) utilized the martingale extraction method to examine the shock exposure in terms of shock elasticity that measures the impact of shock. In these studies, the authors decompose a pricing operator into three components: an exponential term, a martingale and a transient term, each of which carries a financial interpretation depending on the context of problem. Park (2016) studied sensitivities of long-term cash flows with respect to perturbations of underlying processes by using the martingale extraction method. Qin and Linetsky (2015) further analyzed the Hansen-Scheinkman factorization (martingale extraction) for positive eigenfunctions of Markovian pricing operators. Our contribution on this front is to be the first to apply the martingale extraction technique to compute explicitly the long-term growth rate of expected utility.

The rest of this paper proceeds as follows. In Section 2, we discuss our martingale extraction approach for LETFs. In Section 3, we solve the long-term growth rate problem when the reference price follows a one-dimensional Markov diffusion. Sections 4, 5 and 6 are dedicated to, respectively, stochastic volatility models, interest rate models, and quadratic models. We compute the long-term growth rates and determine the optimal leverage ratios. Section 7 summarizes this paper.

2 Martingale extraction approach for LETFs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\mathbb{P}$ is the subjective probability measure. Denote by $\mathbb{F} \equiv (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by a d-dimensional standard Brownian motion B. Consider a reference index, such as the S&P500 index, whose price process X is a one-dimensional positive diffusion process satisfying [2]

$$\frac{dX_t}{X_t} = \mu_t\, dt + \sigma_t \cdot dB_t\,, \qquad t \geq 0\,, \tag{2.1}$$

where the drift process μ_t and vector volatility process σ_t are both $\mathbb{F}$-adapted. At this point, we do not specify a parametric stochastic drift or volatility model, though many well-known models, such as the Heston model as well as other stochastic or local volatility models, also fit within the above framework. In addition, the risk-free rate process is denoted by $(r_t)_{t\geq 0}$ which may be constant or stochastic depending on the model.

2.1 LETF price dynamics

A leveraged ETF is a constant proportion portfolio in the reference X. A long-LETF based on X has a leverage ratio $\beta \geq 1$. At any time t, the cash amount of βL_t (β times the fund value) is invested in X and the amount $(\beta-1)L_t$ is borrowed at the risk-free rate r_t. Strictly speaking, for $\beta \in [0,1)$, the fund is not leveraged since only a fraction of the fund value is invested in the risky asset, and no money is borrowed. For a short-LETF with ratio $\beta < 0$, a short position of the amount $|\beta|L_t$ is taken on X while the amount $(1-\beta)L_t$ is kept in the money market account at the risk-free rate r_t. As a result, the LETF price satisfies

$$\begin{aligned}\frac{dL_t}{L_t} &= \beta\left(\frac{dX_t}{X_t}\right) - ((\beta-1)r_t)\, dt \\ &= (\beta\mu_t - (\beta-1)r_t)\, dt + \beta\sigma_t \cdot dB_t\,.\end{aligned}$$

[2] Throughout, we use the dot notation $\cdot$ for the multiplication of column vectors, and omit the dot for the matrix multiplication.

Without loss of generality, we set $L_0 = X_0 = 1$.

The LETF value at time t admits the expression

$$
\begin{aligned}
L_t &= X_t^{\beta} e^{\int_0^t (-(\beta-1)r_s - \frac{1}{2}\beta(\beta-1)|\sigma_s|^2)\, ds} \\
&= e^{\int_0^t (\beta\mu_s - (\beta-1)r_s - \frac{1}{2}\beta^2|\sigma_s|^2)\, ds + \beta\int_0^t \sigma_s \cdot dB_s} ,
\end{aligned}
\tag{2.2}
$$

where $|\cdot|$ is the usual d-dimensional norm.

The investor's risk preference is modeled by the power utility function

$$
u(w) = w^{\alpha}, \qquad \text{for } w > 0, \qquad \text{with } \ 0 < \alpha \le 1.
$$

As such, the coefficient of relative risk aversion is given by $\varrho := -wu''(w)/u'(w) = 1-\alpha$, $\forall w > 0$. The expected utility from holding the LETF up to time t is given by

$$
\mathbb{E}^{\mathbb{P}}[L_t^{\alpha}] = \mathbb{E}^{\mathbb{P}}\big[X_t^{\alpha\beta} e^{\int_0^t (-\alpha(\beta-1)r_s - \frac{1}{2}\alpha\beta(\beta-1)|\sigma_s|^2)\, ds}\big] \tag{2.3}
$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{P}}\big[e^{\int_0^t (\alpha\beta\mu_s - \alpha(\beta-1)r_s - \frac{1}{2}\alpha\beta^2|\sigma_s|^2)\, ds + \alpha\beta\int_0^t \sigma_s \cdot dB_s}\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[H_t\, e^{\int_0^t (\alpha\beta\mu_s - \alpha(\beta-1)r_s - \frac{1}{2}\alpha(1-\alpha)\beta^2|\sigma_s|^2)\, ds}\big] ,
\end{aligned}
\tag{2.4}
$$

where we have defined the stochastic exponential

$$
H_t := e^{\alpha\beta\int_0^t \sigma_s \cdot dB_s - \frac{1}{2}\alpha^2\beta^2\int_0^t |\sigma_s|\, ds} . \tag{2.5}
$$

In particular, when $\alpha = 1$, the risk aversion ϱ is zero so that the expectation (2.3) is the expected return from holding the LETF L over $[0,t]$.

Suppose that a local martingale H_t in (2.5) is a martingale. Then, we can define a new measure $\hat{\mathbb{P}}$ via

$$
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = H_t . \tag{2.6}
$$

By Girsanov theorem, the process $\hat{B}$ defined by

$$
\hat{B}_t := -\alpha\beta\int_0^t \sigma_s\, ds + B_t \quad \text{for} \quad t \ge 0 \tag{2.7}
$$

is a standard Brownian motion under $\hat{\mathbb{P}}$. Applying (2.7) to (2.1) and (2.4), we get

$$
\frac{dX_t}{X_t} = (\mu_t + \alpha\beta|\sigma_t|^2)\, dt + \sigma_t \cdot d\hat{B}_t
$$

and

$$
\mathbb{E}^{\mathbb{P}}[L_t^{\alpha}] = \mathbb{E}^{\hat{\mathbb{P}}}\left[e^{\int_0^t (\alpha\beta\mu_s - \alpha(\beta-1)r_s - \frac{1}{2}\alpha(1-\alpha)\beta^2|\sigma_s|^2)\, ds}\right] . \tag{2.8}
$$

To analyze the expected utility, we employ the *martingale extraction method*, which will be described in Section 2.2. This method allows us to express the expected utility in a form that is more amenable for analysis and computation.

2.2 Martingale extraction

We now discuss the martingale extraction method with a generic multi-dimensional time-homogeneous Markov diffusion process G_t on $(\Omega, \mathcal{F}, \mathbb{P})$ with drift $b(G_t)$ and volatility $\sigma(G_t)$. In the SDE form, we can write by

$$dG_t = b(G_t)\,dt + v(G_t)\,dB_t\ ,$$

where b is d-dimensional column vector and v is a $d \times d$ matrix. The components of b and σ are differentiable functions and assume that the SDE has a strong solution.

The d-dimensional process G_t may represent multiple components of the model, such as the reference, stochastic volatility, stochastic interest rate, or other stochastic factors. Fix a continuously differentiable multi-variate function $k(\cdot)$. Denote by $\mathcal{L}$ the infinitesimal generator of G_t with killing rate k. Suppose that (λ, ϕ) is an eigenpair corresponding to

$$\mathcal{L}\phi = -\lambda\phi\,, \tag{2.9}$$

where $\lambda \in \mathbb{R}$ and ϕ is a positive continuous twice-differentiable function. It can be shown that

$$M_t := e^{\lambda t - \int_0^t k(G_s)\,ds}\,\phi(G_t)\,\phi^{-1}(G_0) \tag{2.10}$$

is a local martingale by checking that the dt-term of dM_t is zero (see Hurd and Kuznetsov (2008)).

Definition 2.1. *Let* (λ, ϕ) *be an eigenpair of* $-\mathcal{L}$ *satisfying* (2.9). *When the process* M_t *defined in equation* (2.10) *is a martingale, we say that the pair* (λ, ϕ) *admits the martingale extraction of* $e^{-\int_0^t k(G_s)\,ds}$. *In this case, the eigenpair* (λ, ϕ) *is called an* admissible *eigenpair.*

In this case, we can express equation (2.10) as

$$e^{-\int_0^t k(G_s)\,ds} = M_t\,e^{-\lambda t}\,\phi^{-1}(G_t)\,\phi(G_0)\,,$$

and interpret it as the martingale M_t being *extracted* from $e^{-\int_0^t k(G_s)\,ds}$. With each admissible eigenpair (λ, ϕ), one can define a new measure $\mathbb{Q}^\phi$ by

$$\mathbb{Q}^\phi(A) := \int_A M_t\,d\mathbb{P} = \mathbb{E}^{\mathbb{P}}\left[\mathbb{I}_A M_t\right] \qquad \text{for } \ A \in \mathcal{F}_t\,. \tag{2.11}$$

This measure $\mathbb{Q}^\phi$ is called the *transformed measure* from $\mathbb{P}$ with respect to (λ, ϕ). For convenience, we use notation $\mathbb{Q}$ instead of $\mathbb{Q}^\phi$. In turn, we apply a change of measure from $\mathbb{P}$ to $\mathbb{Q}$ to express the expectation

$$\mathbb{E}^{\mathbb{P}}[e^{-\int_0^t k(G_s)ds} f(G_t)] = \mathbb{E}^{\mathbb{Q}}[(\phi^{-1}f)(G_t)] \cdot e^{-\lambda t}\,\phi(G_0)\,. \tag{2.12}$$

In many cases, the right-hand side is more amenable to computation and analysis. For instance, the expectation $\mathbb{E}^{\mathbb{Q}}[(\phi^{-1}f)(G_t)]$ depends on the marginal distribution of G_t at time t, whereas $\mathbb{E}^{\mathbb{P}}[e^{-\int_0^t k(G_s)ds} f(G_t)]$ depends on the whole path of $(G_s)_{0\le s\le t}$. This observation is particularly useful for our analysis of LETFs since they are also path-dependent.

The dynamic of G_t is also altered under the transformed measure $\mathbb{Q}$. To see this, we define the *Girsanov kernel* associated with M_t by

$$\varphi := v^\top \cdot \frac{\nabla \phi}{\phi}\,, \tag{2.13}$$

then the martingale M_t satisfies

$$\frac{dM_t}{M_t} = \varphi(G_t)\,dB_t\,.$$

According to the Girsanov theorem, the process defined by

$$W_t := B_t - \int_0^t \varphi(G_s)\,ds, \qquad t \geq 0, \tag{2.14}$$

is a standard Brownian motion under $\mathbb{Q}$. As a result, given an admissible eigenpair (λ, ϕ), the process G evolves under $\mathbb{Q}$ according to

$$dG_t = (b + v\varphi)(G_t)\,dt + v(G_t)\,dW_t.$$

As expected, the eigenfunction ϕ arises in the drift adjustment of G_t, but does not affect the diffusion term.

Furthermore, if the density function of G_t under $\mathbb{Q}$ is also available in closed form, one can compute and analyze the expectation $\mathbb{E}^{\mathbb{Q}}[(\phi^{-1}f)(G_t)]$. Not all but for many cases, we will choose an admissible eigenpair such that the term $\mathbb{E}^{\mathbb{Q}}[(\phi^{-1}f)(G_t)]$ converges to a non-zero constant. From this we derive the long-term growth rate of the expected utility of LETFs.

Proposition 2.1. *Let (λ, ϕ) be an admissible eigenpair of $\mathcal{L}$, and $\mathbb{Q}$ be the corresponding transformed measure. If $\mathbb{E}^{\mathbb{Q}}[(\phi^{-1}f)(G_t)]$ converges to a nonzero constant as $t \to \infty$, then the limit*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[e^{-\int_0^t k(G_s)ds} f(G_t)] = -\lambda$$

holds.

3 Univariate processes

We now demonstrate how the martingale extraction technique can be applied to analyze the growth rate of expected utility for LETFs. In this section, the reference asset X_t is a one-dimensional Markov diffusion process that satisfies

$$\frac{dX_t}{X_t} = \mu(X_t)\,dt + \sigma(X_t)\,dB_t\,, \qquad X_0 = 1\,, \tag{3.1}$$

where B is a one-dimensional standard Brownian motion under the subjective measure $\mathbb{P}$. The coefficients μ and σ are continuously differentiable functions such that SDE (3.1) has a strong solution. Throughout this section, the short interest rate is a constant $r > 0$.

According to (2.3), the expected utility from holding the LETF is given by

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{P}}[X_t^{\alpha\beta}\, e^{-\frac{1}{2}\alpha\beta(\beta-1)\int_0^t \sigma^2(X_u)\,du}]\, e^{r\alpha(1-\beta)t}\ .$$

To utilize the martingale extraction method, we can view X_t as playing the role of the process G_t in Section 2.2. Define $\mathcal{L}$ as the infinitesimal generator of X_t with killing rate $-\frac{1}{2}\alpha\beta(\beta-1)\sigma^2(\cdot)$. As such, we have

$$\mathcal{L}\phi(x) = \frac{1}{2}x^2\sigma^2(x)\phi''(x) + x\mu(x)\phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2(x)\phi(x)\ .$$

A key step in our approach is to find, as explicitly as possible, an eigenpair (λ,ϕ) of $\mathcal{L}\phi = -\lambda\phi$ with positive ϕ. According to (Pinsky, 1995, Theorem 4.3.3), there always exists such a solution pair if $\beta(\beta-1)\geq 0$. For any $\beta\in\mathbb{R}$, the martingale extraction method is applicable as long as one can find an admissible eigenpair. Indeed, for each case studied herein we will find an explicit solution by direct calculation

Given that there exists an eigenpair (λ,ϕ) which admits the martingale extraction of $e^{-\frac{1}{2}\alpha\beta(\beta-1)\int_0^t \sigma^2(X_s)\,ds}$, the expected utility can be expressed as

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}\,\phi^{-1}(X_t)]\, e^{(r\alpha(1-\beta)-\lambda)t}\phi(1)\ , \tag{3.2}$$

where $\mathbb{Q}$ is the corresponding transformed measure. Since the term $\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}\,\phi^{-1}(X_t)]$ depends only on the value X_t at time t, rather than its whole path, this significantly simplifies the analysis of $\mathbb{E}^{\mathbb{P}}[L_t^\alpha]$, as we present in the following models.

Applying Proposition 2.1, we obtain the long-term growth rate of expected utility from holding the LETF in this univariate framework. Precisely, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}\,\phi^{-1}(X_t)] + r\alpha(1-\beta) - \lambda\,, \tag{3.3}$$

and if $\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}\,\phi^{-1}(X_t)]$ converges to a nonzero constant as $t\to\infty$, then the limit in (3.3) reduces to

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = r\alpha(1-\beta) - \lambda\,.$$

In particular, we recover the long-term growth rate of expected return for the LETF by setting $\alpha = 1$ corresponding to zero risk aversion. Again, the eigenvalue plays a crucial role in the long-term growth rate, along with the first term that depends explicitly on the interest rate r, risk aversion parameter α, and the leverage ratio β. It is important to note that the eigenvalue λ also depends on α, β, $\mu(\cdot)$ and $\sigma(\cdot)$, but not r.

3.1 The GBM model

As a warm-up exercise, we present the long-term growth rate of expected utility in the geometric Brownian motion (GBM) model

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t\,,\ t\geq 0$$

with $\sigma \neq 0$. The corresponding generator is

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + \mu x \phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2\phi(x)\,.$$

To apply martingale extraction, we find the corresponding eigenpair

$$(\lambda, \phi(x)) = (-\alpha\beta\mu + \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2, x^{\alpha\beta})\,.$$

We obtain the expected utility

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{Q}}[1]\, e^{(\alpha\beta\mu - \alpha(\beta-1)r - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2)t} = e^{(\alpha\beta\mu - \alpha(\beta-1)r - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2)t}\,.$$

This implies the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \alpha(1-\beta)r + \alpha\beta\mu - \alpha(\beta-1)r - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2\,. \tag{3.4}$$

The right-hand side consists of two parts: the factor $\alpha(1-\beta)r$ and the negative eigenvalue $-\lambda$. Moreover, the long-term growth rate is quadratic in β. Using this result, we maximize the long-term growth rate in equation (3.4) over $\beta \in \mathbb{R}$ to obtain the optimal leverage ratio

$$\beta^* = \frac{\mu - r}{(1-\alpha)\sigma^2}. \tag{3.5}$$

From (3.5), we see that the optimal leverage ratio is wealth independent, proportional to the Sharpe ratio, but inversely proportional to the coefficient of relative risk aversion $\varrho = 1-\alpha$. The investor should select a positive (resp. negative) β^* if and only if $\mu > r$ (resp. $\mu < r$). It resembles the optimal strategy in the classical Merton portfolio optimization problem.

3.2 The GARCH model

In this section, we consider a positive mean-reverting model for the reference price process X_t. Specifically, it satisfies the continuous-time GARCH diffusion model (see Lewis (2000)):

$$dX_t = (\theta - aX_t)\,dt + \sigma X_t\,dB_t\,, \tag{3.6}$$

with $a, \theta, \sigma > 0$. The GARCH diffusion model is sometimes referred to as the *inhomogeneous geometric Brownian motion* (see e.g. Zhao (2009)). The corresponding generator is

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + (\theta - ax)\phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2\phi(x)\,.$$

To apply martingale extraction, we solve the eigenpair problem $\mathcal{L}\phi = -\lambda\phi$ to obtain the eigenpair

$$(\lambda, \phi(x)) = \Big(\frac{1}{2}\alpha\beta(\beta-1)\sigma^2, 1\Big)\,.$$

Since the eigenfunction $\phi(x) = 1$ is just a constant, the transformed measure $\mathbb{Q}$ coincides with the original measure $\mathbb{P}$ (see (2.13)-(2.14)). Following from (3.2), the expected utility is

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}]\, e^{(r\alpha(1-\beta) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2)t}\,. \tag{3.7}$$

To evaluate (3.7), we first deduce that

$$\begin{cases} \lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = (\text{positive constant}) & \text{if} \quad -\alpha\beta + \frac{2a}{\sigma^2} + 1 > 0\,, \\ \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = \infty & \text{otherwise}\,. \end{cases} \tag{3.8}$$

The proof is as follows. The process $Y_t := \frac{2\theta}{\sigma^2 X_t}$ converges to the Gamma random variable with parameter $\gamma = \frac{2a}{\sigma^2} + 1$, that is, the density function $p(y;t)$ of Y_t converges to $p(y;\infty) := \frac{1}{\Gamma(\gamma)} y^{\gamma-1} e^{-y}$ as $t \to \infty$ (Theorem 2.5 in Zhao (2009)). We obtain the above result by considering the density function $p(y;t)$ and the limiting density function $p(y;\infty)$ above. The asymptotic behaviors of $p(y,t)$ near $y = 0$ and $y = \infty$ are as follows. For fixed t and any small $\epsilon > 0$,

$$\begin{aligned} &y^{\frac{2a}{\sigma^2}} \lesssim p(y;t) \lesssim y^{\frac{2a}{\sigma^2}-\epsilon} && \text{as} \;\; y \to 0 \\ &p(y;t) \lesssim e^{(-1+\epsilon)y} && \text{as} \;\; y \to \infty\,. \end{aligned}$$

Here, for two positive functions $p(y)$ and $q(y)$, we denote by

$$p(y) \lesssim q(y)$$

if there exists a positive constant c such that $p(x) \le c \cdot q(x)$. Refer to Section 6.5.4 in Linetsky (2004) for the density funtion $p(y;t)$. If $-\alpha\beta + \frac{2a}{\sigma^2} + 1 > 0$, then

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = \left(\frac{2\theta}{\sigma^2}\right)^{\alpha\beta} \mathbb{E}^{\mathbb{Q}}[Y_t^{-\alpha\beta}] &= \left(\frac{2\theta}{\sigma^2}\right)^{\alpha\beta} \int_0^\infty y^{-\alpha\beta} p(y;t)\, dy \\ &\to \left(\frac{2\theta}{\sigma^2}\right)^{\alpha\beta} \frac{1}{\Gamma(\gamma)} \int_0^\infty y^{-\alpha\beta + \frac{2a}{\sigma^2}} e^{-y}\, dy\,, \qquad \text{as } t \to \infty\,, \end{aligned} \tag{3.9}$$

which is finite. Otherwise,

$$\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = \left(\frac{2\theta}{\sigma^2}\right)^{\alpha\beta} \mathbb{E}^{\mathbb{Q}}[Y_t^{-\alpha\beta}] = \left(\frac{2\theta}{\sigma^2}\right)^{\alpha\beta} \int_0^\infty y^{-\alpha\beta} p(y;t)\, dy = \infty$$

since $y^{\frac{2a}{\sigma^2}} \lesssim p(y;t)$ near $y = 0$. In conclusion, we obtain the following long-term growth rate.

Proposition 3.1. *Let $\beta \in \mathbb{R}$. Assume that $L = (L_t)_{t\ge 0}$ is the LETF whose reference price X satisfies the GARCH model* (3.6). *Then,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \begin{cases} r\alpha(1-\beta) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2 & \text{if} \;\; \frac{2a}{\sigma^2} + 1 > \alpha\beta\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{3.10}$$

This result implies two distinct scenarios. When $\frac{2a}{\sigma^2} + 1 > \alpha\beta$, there is a finite long-term limit of the growth rate. Interestingly the long-term limit is linear in α and decreasing in σ^2, but does not depend on θ. When $\frac{2a}{\sigma^2} + 1 \le \alpha\beta$, the long-term limit is infinitely large. The limit also applies when $\alpha = 1$, in which case the condition $\beta < \frac{2a}{\sigma^2} + 1$ represents an upper bound on the leverage ratio in order to obtain a finite long-term growth rate of return.

By Proposition 3.1 and direct calculation, we maximize

$$\Lambda(\beta) := r\alpha(1-\beta) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2$$

to obtain the optimal leverage ratio β^* for a long-term investor

$$\beta^* = \frac{1}{2} - \frac{r}{\sigma^2}. \tag{3.11}$$

Surprisingly, in contrast to the GBM model, the optimal leverage ratio in (3.11) under the GARCH model is independent of α, which means that under this model investors with different risk aversion coefficients, including zero risk aversion, will have the same optimal leverage ratio β^*. In fact, β^* only depends on the interest rate r and volatility parameter σ. It is also notable that the GARCH model is reduced to the GBM model as $\theta \to 0$; however, not only the optimal growth rate but also the optimal leverage ratio β^* do not converge to those of the geometric Brownian motion as $\theta \to 0$. It is because the path behaviors and other qualitative features of the GARCH model differ significantly from the GBM model.

3.3 The inverse GARCH model

As an alternative to the GARCH model, suppose now the reference price X_t follows the inverse GARCH diffusion model, which is also referred to as the Pearl-Verhulst logistic process in Tuckwell (1974):

$$dX_t = (\theta - aX_t)X_t\,dt + \sigma X_t\,dB_t\,, \tag{3.12}$$

with $a, \sigma > 0$ and $\theta > \sigma^2$. Both the GARCH and inverse GARCH models are positive and mean-reverting. The process X_t is called the inverse GARCH model because its inverse process $Y_t := 1/X_t$ follows the GARCH model:

$$dY_t = (a - (\theta - \sigma^2)Y_t)\,dt - \sigma Y_t\,dB_t\,.$$

The infinitesimal generator of X_t is

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + (\theta - ax)x\phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2\phi(x)\,.$$

By direct substitution, we verify that

$$(\lambda, \phi(x)) = \Big(\frac{1}{2}\alpha\beta(\beta-1)\sigma^2, 1\Big)$$

is an admissible eigenpair to $\mathcal{L}\phi = -\lambda\phi$. Since the eigenfunction $\phi(x) = 1$ is a constant, the corresponding transformed measure $\mathbb{Q}$ is identical to the original measure $\mathbb{P}$ (see (2.13)-(2.14)). Following from (3.2), the expected utility is

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}]\,e^{(r\alpha(1-\beta) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2)t}\,.$$

Since $Y_t = 1/X_t$ is the GARCH model, we observe from (3.8) that

$$\begin{cases} \lim\limits_{t\to\infty}\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = \lim\limits_{t\to\infty}\mathbb{E}^{\mathbb{Q}}[Y_t^{-\alpha\beta}] = (\text{positive constant}) & \text{if} \quad \alpha\beta + \dfrac{2\theta}{\sigma^2} > 1\,, \\ \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta}] = \infty & \text{otherwise}\,. \end{cases}$$

This leads to the long-term limit summarized as follows.

Proposition 3.2. *Let $\beta \in \mathbb{R}$. Assume that $L = (L_t)_{t\geq 0}$ is the LETF whose reference price X follows the inverse GARCH model* (3.12)*. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \begin{cases} r\alpha(1-\beta) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2 & \text{if } \alpha\beta + \frac{2\theta}{\sigma^2} > 1\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{3.13}$$

Therefore, the long-term growth rate of expected utility can be finite or infinite, depending on the leverage ratio β, risk aversion parameter α, and model parameters (θ, σ) but *not* a. Interestingly, the limits in (3.10) and (3.13), respectively, for the GARCH and inverse GARCH models are the same, except for the conditions for the finiteness of the limits.

By direct calculation, the optimal leverage ratio for the long-term investor is $\beta^* = \frac{1}{2} - \frac{r}{\sigma^2}$ when the long-term growth rate is finite. While β^* does not depend explicitly on α, but α plays a role in determining the finite/infinite growth rate scenario. As $a \to 0$, the inverse GARCH model reduces to the GBM model. Nevertheless, the optimal growth rate and optimal leverage ratio β^*, being independent of a, do not converge to those in the GBM model as $a \to 0$. The same phenomenon was observed in the GARCH model case in Section 3.2.

3.4 The extended CIR model

We now turn to the extended Cox-Ingersoll-Ross (CIR) model proposed by Cox et al. (1985):

$$dX_t = (\theta + \mu X_t)\,dt + \sigma\sqrt{X_t}\,dB_t\,, \tag{3.14}$$

with parameters $\mu, \sigma > 0$ and $\theta \geq \sigma^2$. This process is a transient process diverging to infinity given $\mu > 0$. The corresponding infinitesimal generator is given by

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x\,\phi''(x) + (\theta + \mu x)\,\phi'(x) - \frac{1}{2}\alpha\beta(\beta-1)\sigma^2 \frac{1}{x}\,\phi(x)\,.$$

Set

$$\kappa := \sqrt{\left(\frac{1}{2} - \frac{\theta}{\sigma^2}\right)^2 + \alpha\beta(\beta-1)} + \frac{1}{2} - \frac{\theta}{\sigma^2}\,.$$

The first square-root term is real because $\theta \geq \sigma^2$. It can be verified by direct substitution that

$$(\lambda, \phi(x)) := \left(\mu\kappa + \frac{2\theta\mu}{\sigma^2}\,,\, e^{-\frac{2\mu x}{\sigma^2}} x^\kappa\right)$$

is an admissible eigenpair of $\mathcal{L}$ according to equation (2.9). Under the transformed measure $\mathbb{Q}$ with respect to this eigenpair, the process X_t follows

$$dX_t = (\theta + \kappa\sigma^2 - \mu X_t)\,dt + \sigma\sqrt{X_t}\,dW_t\,, \tag{3.15}$$

where W_t is a $\mathbb{Q}$-Brownian motion. We note that this is a standard mean-reverting CIR process and the Feller condition is satisfied, thus 0 is an unattainable boundary.

The expected utility is given by

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta-\kappa}\, e^{\frac{2\mu}{\sigma^2}X_t}]\, e^{(r\alpha(1-\beta)-\mu\kappa-\frac{2\theta\mu}{\sigma^2})t-\frac{2\mu}{\sigma^2}}\,. \tag{3.16}$$

For the RHS of (3.16), we obtain the long-term limit (see Appendix A):

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta-\kappa}e^{\frac{2\mu}{\sigma^2}X_t}] = \begin{cases} \Big(\alpha\beta+\dfrac{2\theta}{\sigma^2}+\kappa\Big)\mu & \text{if } \alpha\beta+\dfrac{2\theta}{\sigma^2}+\kappa>0\,, \\ \infty & \text{if } \alpha\beta+\dfrac{2\theta}{\sigma^2}+\kappa\le 0\,. \end{cases} \tag{3.17}$$

In turn, we obtain the long-term growth rate of expected utility.

Proposition 3.3. *Let $\beta\in\mathbb{R}$. Suppose that the reference price process X satisfies the extended CIR model* (3.14). *Then, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}\left[L_t^{\alpha}\right] = \begin{cases} \alpha r+\alpha\beta(\mu-r) & \text{if } \alpha\beta+\dfrac{2\theta}{\sigma^2}+\kappa>0\,, \\ \infty & \text{if } \alpha\beta+\dfrac{2\theta}{\sigma^2}+\kappa\le 0\,. \end{cases}$$

This result has a number of implications. First, the long-term growth rate is affine in the leverage ratio β and excess return $(\mu-r)$, and linear in α, but it does not depend on the model parameters θ and σ explicitly other than in the condition separating the two scenarios. In the scenario with $\alpha\beta+\frac{2\theta}{\sigma^2}+\kappa>0$, denote the limit as a function of β: $\Lambda(\beta):=\alpha r+\alpha\beta(\mu-r)$. When $\beta=0$, it follows that the long-term growth rate $\Lambda(0)=\alpha r$. This is because the resulting "leveraged" ETF portfolio is simply growing deterministically at rate r, and the utility is $e^{\alpha rt}$ at time $t\ge 0$. Second, the function $\Lambda(\beta)$ reveals the optimal choice β^* for a static investor. In a bullish market with $\mu>r$, a higher leverage ratio is preferred, though in practice the available leverage ratios are capped at +3. In contrast, in a bearish market with $\mu<r$, then a more negative leverage ratio is better, and in practice the most negative leverage ratio available among LETFs is −3.

3.5 The 3/2 model

We now consider the 3/2 model for the reference price X_t of the form:

$$dX_t=(\theta-aX_t)X_t\,dt+\sigma X_t^{3/2}\,dB_t\,, \tag{3.18}$$

with $a,\theta,\sigma>0$. This is a positive mean-reverting model that has been used to model interest rates and volatility (see Ahn and Gao (1999), Carr and Sun (2007)), so this model would be appropriate for fixed-income and volatility LETFs with a mean-reverting reference price.

The infinitesimal generator corresponding to (3.18) is

$$\mathcal{L}\phi(x)=\frac{1}{2}\sigma^2x^3\,\phi''(x)+(\theta-ax)x\,\phi'(x)-\frac{1}{2}\alpha\beta(\beta-1)\sigma^2x\,\phi(x)\,.$$

Denoting

$$\kappa:=\sqrt{\left(\frac{1}{2}+\frac{a}{\sigma^2}\right)^2+\alpha\beta(\beta-1)}-\left(\frac{1}{2}+\frac{a}{\sigma^2}\right)\,,$$

we find that

$$(\lambda,\phi(x)):=\big(\theta\kappa\,,\,x^{-\kappa}\big)$$

is an admissible eigenpair of $\mathcal{L}$. Under the transformed measure $\mathbb{Q}$, the reference price X_t follows

$$dX_t = (\theta - (a + \sigma^2\kappa)X_t)X_t\,dt + \sigma X_t^{3/2}\,dW_t\ ,$$

where $dW_t = dB_t + \sigma\kappa X_t^{1/2}dt$ is a Brownian motion under $\mathbb{Q}$. Notice that X_t satisfies a re-parametrized 3/2 model under $\mathbb{Q}$.

The expected utility from holding an LETF can be expressed under the transformed measure $\mathbb{Q}$ by

$$\mathbb{E}^{\mathbb{P}}[L_t^{\alpha}] = \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta+\kappa}]\,e^{(r\alpha(1-\beta)-\theta\kappa)t}\ .$$

We show that

$$\begin{cases} \lim\limits_{t\to\infty}\mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta+\kappa}] = (\text{positive constant}) & \text{if } \dfrac{2a}{\sigma^2} + \kappa - \alpha\beta + 2 > 0\,, \\ \mathbb{E}^{\mathbb{Q}}[X_t^{\alpha\beta+\kappa}] = \infty & \text{otherwise}\ . \end{cases}$$

The proof is as follows. Define $Y_t := 1/X_t$. Then Y_t is a CIR process with

$$dY_t = (a + \sigma^2(\kappa+1) - \theta Y_t)\,dt - \sigma\sqrt{Y_t}\,dW_t\ .$$

By considering the density function of the CIR process, which is given in equation (A.1), we obtained the desired result. In conclusion, we obtain the following long-term growth rate.

Proposition 3.4. *Let $\beta \in \mathbb{R}$. Assume that L is the LETF whose reference price X satisfies the* 3/2 *model* (3.18)*. Then, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}\left[L_t^{\alpha}\right] = \begin{cases} r\alpha(1-\beta) - \theta\kappa & \text{if } \dfrac{2a}{\sigma^2} + \kappa - \alpha\beta + 2 > 0\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

In general, the sign of $\frac{2a}{\sigma^2} + \kappa - \alpha\beta + 2$ depends on the model parameters (θ, a, σ), risk aversion coefficient α, and leverage ratio β. Nevertheless, we find that for $|\beta| \le 3$, which holds for market-traded LETFs, the condition $\frac{2a}{\sigma^2} + \kappa - \alpha\beta + 2 > 0$ is satisfied.

Next, we investigate the optimal leverage ratio β^* for a static investor (see (1.2)). In the scenarios with $\frac{2a}{\sigma^2} + \kappa - \alpha\beta + 2 > 0$, we define

$$\Lambda(\beta) := r\alpha(1-\beta) - \theta\kappa\ .$$

Next, we determine the critical points of Λ. Differentiation yields that

$$\Lambda'(\beta) = -r\alpha - \frac{\theta\alpha(2\beta-1)}{2\sqrt{\left(\frac{1}{2} + \frac{a}{\sigma^2}\right)^2 + \alpha\beta(\beta-1)}}\ .$$

When $\alpha \ge \frac{\theta^2}{r^2}$, the equation $\Lambda'(\beta) = 0$ has no solutions and $\Lambda'(\beta) < 0$ for all β. Therefore, $\Lambda(\beta)$ is a decreasing function of β. In practice, $\beta^* = -3$ is the optimal strategy. On the other hand, when $\alpha < \frac{\theta^2}{r^2}$, by considering the equation $\Lambda'(\beta) = 0$, we conclude that the maximum of $\Lambda(\beta)$ is attained at

$$\beta^* = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\left(1 + \frac{2a}{\sigma^2}\right)^2 - \alpha}{\frac{\theta^2}{r^2} - \alpha}}\ . \tag{3.19}$$

Note that the number inside the square root is positive because $\alpha \le 1 < (1+2a\sigma^{-2})^2$. Moreover, the optimal value in (3.19) satisfies

$$\beta^* = 0 \qquad \text{if and only if} \qquad 1 + \frac{2a}{\sigma^2} = \frac{\theta}{r}.$$

4 Stochastic volatility models

In this section, we analzye the martingale extraction method for LETFs under stochastic volatility models. Let B_t be a standard Brownian motion under $\mathbb{P}$. The reference price X_t satisfies the SDE

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma(Y_t)\cdot dB_t$$

with a constant μ, a d-dimensional column vector $\sigma(\cdot)$ and a Markov diffusion process Y_t as the driver of the stochastic volatility. Throughout this section, the interest rate is a constant $r > 0$.

As discussed in Section 2, define a new measure $\hat{\mathbb{P}}$ by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{\alpha\beta\int_0^t \sigma(Y_s)dB_s - \frac{1}{2}\alpha^2\beta^2\int_0^t |\sigma|^2(Y_s)\,ds}\,.$$

Then, the process defined by

$$\hat{B}_t := -\alpha\beta\int_0^t \sigma(Y_s)\,ds + B_t$$

is a standard Brownian motion under $\hat{\mathbb{P}}$. From equation (2.8), it follows that

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2\int_0^t |\sigma|^2(Y_s)\,ds}]\,e^{\alpha(r+\beta(\mu-r))t}\,.$$

Following (2.12) with the stochastic volatility driver Y_t here playing the role of G_t in Section 2.2, the main idea is to apply the martingale extraction of $e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2\int_0^t |\sigma|^2(Y_s)\,ds}$ and compute the expected utility explicitly.

4.1 The Heston model

We now present an example in which the reference price follows the Heston model (see Heston (1993))

$$\begin{aligned} dX_t &= \mu X_t\,dt + \sqrt{v_t}X_t\,dB_t\,, \\ dv_t &= (\theta - av_t)\,dt + \delta\sqrt{v_t}\,dZ_t\,, \end{aligned} \tag{4.1}$$

where B_t and Z_t are two correlated Brownian motions with $\langle Z, W\rangle_t = \rho t$ and correlation parameter $\rho \in [-1, 1]$. This model assumes that $\mu, \theta, a, \delta > 0$ and $2\theta > \delta^2$.

Define the measure $\hat{\mathbb{P}}$ via (2.6)-(2.7) in Section 2 so that the process defined by

$$\hat{B}_t = -\alpha\beta\int_0^t \sqrt{v_s}\,ds + dB_t$$

is a $\hat{\mathbb{P}}$-Brownian motion. By (2.8), we express the expected utility under the measure $\hat{\mathbb{P}}$ as

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2\int_0^t v_s\,ds}]\,e^{\alpha(r+\beta(\mu-r))t}\,.$$

The stochastic volatility process v_t is a re-parametrized CIR process

$$dv_t = (\theta - (a - \alpha\beta\delta\rho)v_t)\,dt + \delta\sqrt{v_t}\,d\hat{Z}_t\,,$$

where $\hat{Z}_t$ is another Brownian motion under $\hat{\mathbb{P}}$.

We now explore the martingale extraction of $e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2\int_0^t v_s\,ds}$. To this end, we consider stochastic volatility process v_t as playing the role of the process G_t discussed in Section 2.2. The infinitesimal generator $\mathcal{L}$ of v_t with killing rate $\frac{1}{2}\alpha(1-\alpha)\beta^2 v_t$ is

$$\mathcal{L}\phi(v) = \frac{1}{2}\delta^2 v\,\phi''(v) + (\theta - (a-\alpha\beta\delta\rho)v)\,\phi'(v) - \frac{1}{2}\alpha(1-\alpha)\beta^2 v\,\phi(v)\,.$$

By direct calculation, we obtain an admissible eigenpair of $\mathcal{L}$, given by

$$(\lambda, \phi(v)) = \left(\theta\kappa, e^{-\kappa v}\right)\,,$$

where

$$\kappa := \frac{1}{\delta^2}\left(\sqrt{(a-\alpha\beta\delta\rho)^2 + \alpha(1-\alpha)\beta^2\delta^2} - a + \alpha\beta\delta\rho\right)\,.$$

Let $\mathbb{Q}$ be the transformed measure with respect to this pair $(\lambda, \phi(v))$. Then, the process v_t satisfies another re-parametrized CIR model

$$dv_t = (\theta - \sqrt{(a-\alpha\beta\delta\rho)^2 + \alpha(1-\alpha)\beta^2\delta^2}\,v_t)\,dt + \delta\sqrt{v_t}\,dW_t\,,$$

where W_t is a $\mathbb{Q}$-Brownian motion. The expected utility can be written as

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2\int_0^t v_s\,ds}]\,e^{\alpha(r+\beta(\mu-r))t} = \mathbb{E}^{\mathbb{Q}}[e^{\kappa v_t}]\,e^{(\alpha r+\alpha\beta(\mu-r)-\theta\kappa)t-\kappa v_0}.$$

Note that $\mathbb{E}^{\mathbb{Q}}[e^{\kappa r_t}]$ converges to a nonzero constant as $t\to\infty$. We refer to Appendix A for more details about this martingale extraction.

Proposition 4.1. *Let $\beta\in\mathbb{R}$. Assume that L is the LETF whose reference price X satisfies the Heston model* (4.1)*. Then, we have*

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] &= \alpha r + \alpha\beta(\mu-r) - \theta\kappa \\ &= \alpha r + \alpha\beta(\mu-r) - \frac{\theta}{\delta^2}\left(\sqrt{(a-\alpha\beta\delta\rho)^2+\alpha(1-\alpha)\beta^2\delta^2} - a + \alpha\beta\delta\rho\right). \end{aligned} \quad (4.2)$$

We now determine the optimal leverage ratio β^* for the risk-averse static investor. To understand the dependence of the long-term limit in (4.2) on β, we define the function

$$\Lambda(\beta) := \left(\frac{\alpha\delta^2(\mu-r)}{\theta} - \alpha\delta\rho\right)\beta - \sqrt{(a-\alpha\beta\delta\rho)^2+\alpha(1-\alpha)\beta^2\delta^2}\,.$$

Let

$$C_1 = \alpha(1-\alpha)\delta^2 + \alpha^2\delta^2\rho^2\,,\ C_2 = -a\alpha\delta\rho\,,\ C_3 = a^2\,,\ D = \frac{\alpha\delta^2(\mu-r)}{\theta} - \alpha\delta\rho\,. \quad (4.3)$$

Then, we rewrite $\Lambda(\beta)$ to highlight the dependence on β as

$$\Lambda(\beta) = D\beta - \sqrt{C_1\beta^2 + 2C_2\beta + C_3}\,. \quad (4.4)$$

In turn, we obtain the derivatives:

$$\Lambda'(\beta) = D - \frac{C_1\beta + C_2}{\sqrt{C_1\beta^2+2C_2\beta+C_3}}\,,\ \Lambda''(\beta) = \frac{C_2^2 - C_1C_3}{(C_1\beta^2+2C_2\beta+C_3)^{3/2}}\,.$$

Since $C_2^2 - C_1C_3 < 0$, we know that $\Lambda(\beta)$ is a strictly concave function of β.

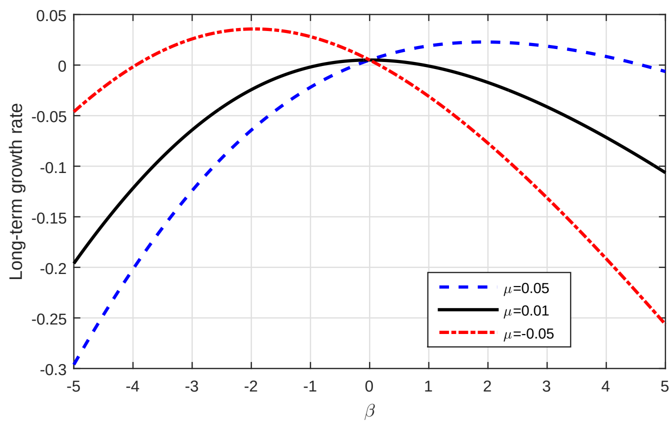

Figure 1: Long-term growth rate of expected utility as a function of the leverage ratio β under the Heston model. Under three different values of the drift $\mu \in \{0.05, 0.01, -0.05\}$, the optimal β^* maximizing the growth rates are $\{1.93, 0, -1.95\}$, respectively.

(i) If $C_1 > D^2$, then $\Lambda'(\beta) = 0$ has a unique solution, which gives the optimal leverage ratio

$$\beta^* = -\frac{C_2}{C_1} + \frac{|D|}{C_1}\sqrt{\frac{C_1C_3 - C_2^2}{C_1 - D^2}}\,.$$

(ii) If $C_1 \leq D^2$, then $\Lambda'(\beta) = 0$ has no solutions. Furthermore, if $D > 0$, $\Lambda'(\beta)$ is positive for all β, thus $\Lambda(\beta)$ is an increasing function. The optimal $\beta^* = 3$ (or the maximum available leverage ratio) in practice. If $D < 0$, $\Lambda'(\beta)$ is negative for all β, thus $\Lambda(\beta)$ is a decreasing function and the most negative leverage ratio is preferred. In practice, the investor would select $\beta^* = -3$. This is intuitive since $D < 0$ means that $\mu < r$ (see (4.3)).

Figure 1 depicts the long-term growth rate in Proposition 4.1 as a function of β. The parameters are: $\alpha = 0.5$, $r = 0.01$, $\theta = 0.16$, $\delta = 0.89$, $a = 3.1$, $\rho = -0.5$, along with $\mu \in \{0.05, 0.01, -0.05\}$. As we can see, when the excess return $(\mu - r)$ is positive, then the optimal leverage ratio is positive ($\beta^* = 1.93$ when $\mu - r = 0.04$). In contrast, when the reference price X_t is trending downward ($\mu = -0.05$), then it is optimal for the investor to select a short LETF (i.e. $\beta = -1.95$).

4.2 The 3/2 volatility model

Under the 3/2 volatility model proposed by Carr and Sun (2007), the reference price X_t follows

$$\begin{aligned} dX_t &= \mu X_t\,dt + \sqrt{v_t}X_t\,dB_t\,, \\ dv_t &= (\theta - av_t)v_t\,dt + \delta v_t^{3/2}\,dZ_t\,, \end{aligned} \tag{4.5}$$

where B_t and Z_t are two standard Brownian motions with instantaneous correlation $\rho \in [-1, 1]$.

As discussed in Section 2, we define the measure $\hat{\mathbb{P}}$ so that the process

$$\hat{B}_t = -\alpha\beta\int_0^t \sqrt{v_s}\,ds + dB_t$$

is a standard Brownian motion under $\hat{\mathbb{P}}$. As a result of (2.8), the expected utility admits the expression

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2 \int_0^t v_s\, ds}]\, e^{\alpha(r+\beta(\mu-r))t} \ .$$

The stochastic volatility process v_t follows a re-parametrized 3/2 model

$$dv_t = (\theta - (a - \alpha\beta\delta\rho)v_t)v_t\, dt + \delta v_t^{3/2}\, d\hat{Z}_t$$

where $\hat{Z}_t$ is a $\hat{\mathbb{P}}$-Brownian motion.

We apply the martingale extraction method by viewing the stochastic volatility process v_t as the process G_t in Section 2.2. The infinitesimal generator $\mathcal{L}$ of the diffusion v_t with killing rate $\frac{1}{2}\alpha(1-\alpha)\beta^2 v_t$ is

$$\mathcal{L}\phi(v) = \frac{1}{2}\delta^2 v^3\, \phi''(v) + (\theta - (a - \alpha\beta\delta\rho)v)v\, \phi'(v) - \frac{1}{2}\alpha(1-\alpha)\beta^2 v\, \phi(v) \ .$$

It can be shown that

$$(\lambda, \phi(v)) := \big(\theta\kappa\, ,\, v^{-\kappa}\big)$$

is an admissible eigenpair of $\mathcal{L}$, where

$$\kappa := \frac{1}{\delta^2}(\sqrt{(a - \alpha\beta\delta\rho + \delta^2/2)^2 + \alpha(1-\alpha)\beta^2\delta^2} - (a - \alpha\beta\delta\rho + \delta^2/2)) \ .$$

Let $\mathbb{Q}$ be the corresponding transformed measure. The process v_t satisfies

$$dv_t = (\theta - (\sqrt{(a - \alpha\beta\delta\rho + \delta^2/2)^2 + \alpha(1-\alpha)\beta^2\delta^2} - \delta^2/2)v_t)v_t\, dt + \delta v_t^{3/2}\, dW_t \ ,$$

where W_t is a $\mathbb{Q}$-Brownian motion. Consequently, we express the expected utility as

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha)\beta^2 \int_0^t v_s\, ds}]\, e^{\alpha(r+\beta(\mu-r))t} = \mathbb{E}^{\mathbb{Q}}[v_t^\kappa]\, e^{(\alpha r + \alpha\beta(\mu-r) - \theta\kappa)t} v_0^{-\kappa} \ .$$

We show that $\mathbb{E}^{\mathbb{Q}}[v_t^\kappa]$ converges to a positive constant by considering the density of the CIR process $1/v_t$ given that

$$\frac{1}{\delta^2}(\sqrt{(a - \alpha\beta\delta\rho + \delta^2/2)^2 + \alpha(1-\alpha)\beta^2\delta^2} + (a - \alpha\beta\delta\rho + \delta^2/2)) + 1 > 0 \ .$$

Otherwise, we have $\mathbb{E}^{\mathbb{Q}}[v_t^\kappa] = \infty$. In conclusion, we have the following proposition.

Proposition 4.2. *Let $\beta \in \mathbb{R}$. Assume that L_t is the LETF with reference process X_t satisfying the 3/2 model* (4.5)*. Then, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \ \alpha r + \alpha\beta(\mu - r) - \theta\kappa \, , \tag{4.6}$$

if

$$\frac{1}{\delta^2}(\sqrt{(a - \alpha\beta\delta\rho + \delta^2/2)^2 + \alpha(1-\alpha)\beta^2\delta^2} + (a - \alpha\beta\delta\rho + \delta^2/2)) + 1 > 0 \ .$$

Otherwise, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \infty \ .$$

The explicit long-term limit allows us to determine conveniently the optimal leverage ratio β^* for the risk-averse static investor. To this end, we define the following function out of (4.6)

$$\Lambda(\beta) := ((\mu - r)\delta/\theta - \rho)\alpha\delta\beta - \sqrt{(a - \alpha\beta\delta\rho + \delta^2/2)^2 + \alpha(1-\alpha)\beta^2\delta^2}\,.$$

We define the constants

$$\begin{aligned}
C_1 &= \alpha(1-\alpha)\delta^2 + \alpha^2\delta^2\rho^2\,,\\
C_2 &= -\alpha\delta\rho(a + \delta^2/2)\,,\\
C_3 &= (a + \delta^2/2)^2\,,\\
D &= ((\mu - r)\delta/\theta - \rho)\alpha\delta\,,
\end{aligned}$$

and to highlight its dependence on β we rewrite $\Lambda(\beta)$ as

$$\Lambda(\beta) = D\beta - \sqrt{C_1\beta^2 + 2C_2\beta + C_3}\,.$$

Given the same structure of $\Lambda(\beta)$, the optimal β^* can be derived by the exactly same way as in Section 4.1. In summary, if $C_1 > D^2$, then the optimal β^* is

$$\beta^* = -\frac{C_2}{C_1} + \frac{|D|}{C_1}\sqrt{\frac{C_1C_3 - C_2^2}{C_1 - D^2}}\,.$$

If $C_1 \leq D^2$, then it is optimal to pick the most positive (resp. most negative) β possible if $D > 0$ (resp. $D < 0$).

5 LETF with stochastic reference and interest rate

In this section, we analyze the long-term growth rate of the expected utility from holding an LETF when both the reference price X_t and short interest rate r_t are stochastic.

5.1 Vasicek interest rate

We first consider the Vasicek interest rate model introduced by Vasicek (1977). The reference price process X_t and the short interest rate r_t satisfy the SDEs

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t\,, \tag{5.1}$$

$$dr_t = (\theta - ar_t)\,dt + \delta\,dZ_t\,, \tag{5.2}$$

for $\mu, \sigma, \theta, a, \delta > 0$, where B_t and Z_t are two Brownian motions such that $\langle Z, W\rangle_t = \rho t$ with $-1 \leq \rho \leq 1$.

Define $\hat{\mathbb{P}}$ as discussed in Section 2, then the process $\hat{B}_t$ given by

$$\hat{B}_t := -\alpha\beta\sigma t + B_t$$

is a $\hat{\mathbb{P}}$-Brownian motion. From equation (2.8), it follows that

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\alpha(\beta-1)\int_0^t r_s\,ds}]\,e^{\alpha\beta\mu t - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 t}\,.$$

The $\hat{\mathbb{P}}$-dynamics of r_t is

$$dr_t = (\theta + \alpha\beta\delta\sigma\rho - ar_t)\,dt + \delta\,d\hat{Z}_t$$

with a $\hat{\mathbb{P}}$-Brownian $\hat{Z}_t$.

We now explore the martingale extraction of $e^{-\alpha(\beta-1)\int_0^t r_s\,ds}$ with the process r_t playing the role of G_t in Section 2.2. Consider the infinitesimal generator $\mathcal{L}$ of the diffusion r_t with killing rate $\alpha(\beta-1)r_t$. We know that the generator $\mathcal{L}$ is

$$\mathcal{L}\phi(r) = \frac{1}{2}\delta^2\phi''(r) + (\theta + \alpha\beta\delta\sigma\rho - ar)\,\phi'(r) - \alpha(\beta-1)r\,\phi(r)\,.$$

It can be shown that

$$(\lambda, \phi(r)) := \left(\frac{1}{2a^2}\alpha(1-\beta)(-\alpha\delta^2(1-\beta) + 2a(\theta+\alpha\beta\delta\sigma\rho)), e^{-\frac{\alpha(1-\beta)r}{a}}\right)$$

is an admissible eigenpair of $\mathcal{L}$. The process r_t satisfies

$$dr_t = (\theta + \alpha\beta\delta\sigma\rho - \alpha\delta^2(1-\beta)/a - ar_t)\,dt + \delta\,dW_t\,,$$

where W_t is a Brownian motion under the corresponding transformed measure $\mathbb{Q}$. It follows that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[L_t^\alpha] &= \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\alpha(\beta-1)\int_0^t r_s\,ds}]\,e^{\alpha\beta\mu t - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 t} \\ &= \mathbb{E}^{\mathbb{Q}}[e^{\kappa r_t}]\,e^{(\alpha\beta\mu - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 + \frac{1}{2a^2}\alpha^2\delta^2(1-\beta)^2 - \frac{1}{a}\alpha(1-\beta)(\theta+\alpha\beta\delta\sigma\rho))t - \kappa r_0}\end{aligned}$$

and we know that $\mathbb{E}^{\mathbb{Q}}[e^{\kappa r_t}]$ converges to a positive constant because r_t is again an OU process under $\mathbb{Q}$. In conclusion, we have the following proposition.

Proposition 5.1. *Let $\beta \in \mathbb{R}$. Suppose that the reference price process X_t and the interest rate r_t satisfy* (5.1) *and* (5.2) *respectively. Then, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \alpha\beta\mu - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 + \frac{1}{2a^2}\alpha^2\delta^2(1-\beta)^2 - \frac{1}{a}\alpha(1-\beta)(\theta+\alpha\beta\delta\sigma\rho). \quad (5.3)$$

We now find the optimal leverage ratio β^* for a static investor. To examine the dependence of the limit (5.3) on β, we define

$$\Lambda(\beta) = C_1\beta^2 + C_2\beta,$$

with the constants

$$C_1 = -\frac{1}{2}\alpha(1-\alpha)\sigma^2 + \frac{\alpha^2\delta^2}{2a^2} + \frac{\alpha^2\delta\sigma\rho}{a}\,,\; C_2 = \alpha\mu - \frac{\alpha^2\delta^2}{a^2} + \frac{\alpha\theta}{a} - \frac{\alpha^2\delta\sigma\rho}{a}\,.$$

Note that $\Lambda(\beta)$ is a quadratic function. If $C_1 < 0$, then $\beta^* = -\frac{C_2}{2C_1}$ is optimal. If $C_1 > 0$, then a more positive (resp. more negative) β is always more favorable when $\frac{C_2}{2C_1} > 0$ (resp. $\frac{C_2}{2C_1} < 0$). In the special case with $C_1 = 0$, then a more positively (resp. more negatively) leveraged ETF is preferred when $C_2 > 0$ (resp. $C_2 < 0$).

In Figure 2, we display the long-term growth rate (5.3) as a function of β for different values of μ. The parameters are: $\alpha = 0.8$, $r = 0.01$, $\theta = 0.16$, $\delta = 0.89$, $a = 3$, $\sigma = 0.3$, and $\rho = -0.5$. We can see that a positive (resp. negative) leverage ratio β^* is optimal in a bull (resp. bear) market with $\mu = 0.05$ (resp. $\mu = -0.05$). A positively leveraged ETF with $\beta^* = 1.52$ is preferred here even when the reference asset offers no excess return (i.e. $\mu - r = 0$). This presents an interesting contrast to the Heston model depicted in Figure 1 and the GBM model whereby the optimal leverage ratio $\beta^* = 0$ whenever $\mu = r$.

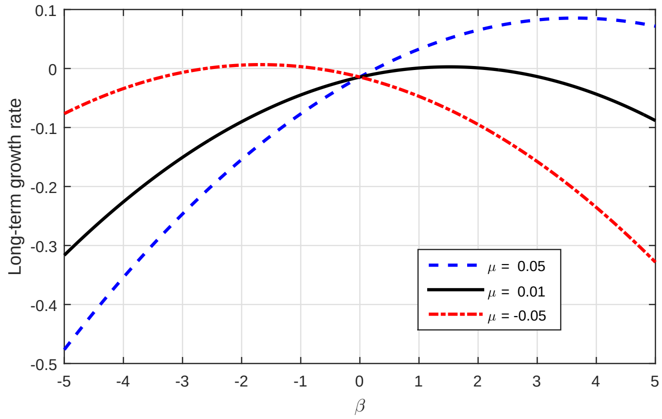

Figure 2: Long-term growth rates of expected utility under the GBM model with Vasicek interest rate corresponding to three values of $\mu \in \{0.05, 0.01, -0.05\}$. The optimal leverage ratios β^* (maximizers of these curves) are $\{3.65, 1.52, -1.68\}$, respectively.

5.2 Inverse GARCH interest rate

Another model for the stochastic short interest rate is the inverse GARCH diffusion model, which was discussed in Section 3.3. Suppose that the reference price X_t and the short interest rate r_t satisfy the SDEs

$$\begin{aligned} dX_t &= \mu X_t\,dt + \sigma X_t\,dB_t\,, \\ dr_t &= (\theta - ar_t)r_t\,dt + \delta r_t\,dZ_t\,, \end{aligned} \tag{5.4}$$

where B_t and Z_t are two Brownian motions such that $\langle Z, W\rangle_t = \rho t$ with $-1 \leq \rho \leq 1$. We assume that $\mu, a, \delta > 0$ and $\theta > \delta^2$.

Following the procedure in Section 2, we define the measure $\hat{\mathbb{P}}$, and express under this measure the expected utility

$$\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\alpha(\beta-1)\int_0^t r_s\,ds}]\,e^{\alpha\beta\mu t - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 t}\,.$$

The stochastic interest rate evolves according to

$$dr_t = (\theta + \alpha\beta\delta\sigma\rho - ar_t)r_t\,dt + \delta r_t\,d\hat{Z}_t\,,$$

where $\hat{Z}_t$ is a $\hat{\mathbb{P}}$-Brownian motion.

We now present the martingale extraction of $e^{-\alpha(\beta-1)\int_0^t r_s\,ds}$. The infinitesimal generator $\mathcal{L}$ of r_t is

$$\mathcal{L}\phi(r) = \frac{1}{2}\delta^2 r^2\phi''(r) + (\theta + \alpha\beta\delta\sigma\rho - ar)r\,\phi'(r) - \alpha(\beta-1)r\,\phi(r)\,.$$

It can be verified that

$$(\lambda, \phi(r)) = \left(-\frac{1}{2a^2}\alpha\delta^2(\beta-1)(\alpha\beta-\alpha+a) + \frac{1}{a}\alpha(\beta-1)(\theta+\alpha\beta\delta\sigma\rho), r^{\alpha(1-\beta)/a}\right)$$

is an admissible eigenpair of $\mathcal{L}$ provided $\theta + \alpha\beta\delta\sigma\rho - \alpha\delta^2(\beta-1)/a > \delta^2$, which explains our condition on the parameters.

After a change to the transformed measure $\mathbb{Q}$, r_t satisfies

$$dr_t = (\theta + \alpha\beta\delta\sigma\rho - \alpha\delta^2(\beta-1)/a - ar_t)r_t\,dt + \delta r_t\,dW_t\ ,$$

where W_t is a Brownian motion under $\mathbb{Q}$. Connecting the measures through the expected utility, we write

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[L_t^\alpha] \\
=&\mathbb{E}^{\hat{\mathbb{P}}}[e^{-\alpha(\beta-1)\int_0^t r_s\,ds}]\,e^{\alpha\beta\mu t-\frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 t} \\
=&\mathbb{E}^{\mathbb{Q}}[r_t^{-\alpha(1-\beta)/a}]\,e^{(\alpha\beta\mu-\frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2+\frac{1}{2a^2}\alpha\delta^2(\beta-1)(\alpha\beta-\alpha+a)-\frac{1}{a}\alpha(\beta-1)(\theta+\alpha\beta\delta\sigma\rho))t}\,r_0^{\alpha(1-\beta)/a}\ .
\end{aligned}\tag{5.5}$$

Inspecting the last line (5.5), we point out that

$$\begin{cases}
\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}[r_t^{-\alpha(1-\beta)/a}] = (\text{positive const}) & \text{if } \alpha(1-\beta)/a + \dfrac{2}{\delta^2}(\theta+\alpha\beta\delta\sigma\rho) - 1 > 0\,, \\
\mathbb{E}^{\mathbb{Q}}[r_t^{-\alpha(1-\beta)/a}] = \infty & \text{otherwise}\ .
\end{cases}\tag{5.6}$$

We refer to Appendix B for the equality in (5.5) and two equalities in (5.6).

Proposition 5.2. *Consider the reference price X_t and the short interest rate r_t that satisfy equation* (5.4). *If $\alpha(1-\beta)/a + \frac{2}{\delta^2}(\theta+\alpha\beta\delta\sigma\rho) - 1 > 0$, then*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \alpha\beta\mu - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 + \frac{1}{2a^2}\alpha^2\delta^2(1-\beta)^2 - \frac{1}{a}\alpha(1-\beta)(\theta+\alpha\beta\delta\sigma\rho).\tag{5.7}$$

Otherwise, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}}[L_t^\alpha] = \infty.$$

Using this result, we can find the optimal leverage ratio β^* for a long-term investor by analyzing the limit (5.7) as a function of β, namely,

$$\Lambda(\beta) := \alpha\beta\mu - \frac{1}{2}\alpha(1-\alpha)\beta^2\sigma^2 + \frac{1}{2a^2}\alpha^2\delta^2(1-\beta)^2 - \frac{1}{a}\alpha(1-\beta)(\theta+\alpha\beta\delta\sigma\rho).$$

The function $\Lambda(\beta)$ is quadratic provided that $\alpha(1-\beta)/a + \frac{2}{\delta^2}(\theta+\alpha\beta\delta\sigma\rho) - 1 > 0$ on $|\beta| \le 3$. The procedure to determine the maximum of $\Lambda(\beta)$ is the same as that presented in Section 5.1, and is thus omitted.

6 Quadratic models

In this section, we consider a quadratic model given by $X_t = e^{|Y_t|^2}$, where Y_t is a d-dimensional Ornstein-Uhlenbeck (OU) process

$$dY_t = (b + BY_t)\,dt + \sigma\,dB_t\ , \qquad Y_0 = 0_d\ .\tag{6.1}$$

Here, b is a d-dimensional column vector, B is a $d\times d$ matrix, and σ is a non-singular $d\times d$ matrix, so that $a = \sigma\sigma^\top$ is strictly positive definite. We refer to Ahn et al. (2002) and Qin and Linetsky

(2015) for more details about this quadratic model. The interest rate r is assumed to be a positive constant. Under the quadratic model, the LETF price L_t can be expressed as

$$L_t = X_t^{\beta} e^{-r(\beta-1)t-2\beta(\beta-1)\int_0^t Y_u^\top aY_u\,du} ,$$

which is derived in Appendix C. From equation (2.8), the expected utility is given by

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[L_t^{\alpha}] &= \mathbb{E}^{\mathbb{P}}[e^{-2\alpha\beta(\beta-1)\int_0^t Y_u^\top aY_u\,du}\, X_t^{\alpha\beta}]\, e^{(r\alpha(1-\beta))t} \\ &= \mathbb{E}^{\mathbb{P}}[e^{-2\alpha\beta(\beta-1)\int_0^t Y_u^\top aY_u\,du}\, e^{\alpha\beta|Y_t|^2}]\, e^{(r\alpha(1-\beta))t} .\end{aligned}$$

We now apply our martingale extraction method developed in Section 2.2. The infinitesimal generator $\mathcal{L}$ corresponding to Y_t in (6.1) is

$$\mathcal{L}\phi(y) = \nabla\phi(y)(b+By) + \frac{1}{2}\sum_{i,j}(H\phi(y))_{ij}\,a_{ij} - 2\alpha\beta(\beta-1)y^\top ay\,\phi(y) ,$$

where $\nabla\phi(y)$ is the gradient row vector and $H\phi(y)$ is the Hessian matrix. We can find an admissible eigenpair of $\mathcal{L}$ by the following way. Let V be the *stabilizing solution* (i.e., V is symmetric, $B-2aV$ is non-singular and the eigenvalues of $B-2aV$ have negative real parts) of

$$2VaV - B^\top V - VB - 2\alpha\beta(\beta-1)a = 0 , \tag{6.2}$$

and define a vector u by

$$u = 2(2a - V^{-1}B^\top)^{-1}b . \tag{6.3}$$

It is known that the stabilizing solution V always exists if $\beta(\beta-1)>0$. If $\beta(\beta-1)\leq 0$, then the existence of the stabilizing solution can be checked case by case. In both cases, an admissible eigenpair of $\mathcal{L}$ is

$$(\lambda,\phi(y)) = (-\frac{1}{2}u^\top au + u^\top b + tr(aV)\, ,\, e^{-u^\top y - y^\top Vy}).$$

This leads to the martingale extraction of

$$e^{-2\alpha\beta(\beta-1)\int_0^t Y_u^\top aY_u\,du} .$$

Refer to Section 6.2.2 in Qin and Linetsky (2015) for more details about this eigenpair. Denote by $\mathbb{Q}$ the transformed measure with respect to (λ,ϕ) (see (2.11)). Under the measure $\mathbb{Q}$, Y_t evolves according to

$$dY_t = (b - au + (B-2aV)Y_t)\,dt + \sigma\,dW_t , \tag{6.4}$$

where W_t is a Brownian motion under $\mathbb{Q}$.

The expected utility from holding the LETF can be expressed as

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[L_t^{\alpha}] &= \mathbb{E}^{\mathbb{P}}[e^{-2\alpha\beta(\beta-1)\int_0^t Y_u^\top aY_u\,du}\, e^{\alpha\beta|Y_t|^2}]\, e^{(r\alpha(1-\beta))t} \\ &= \mathbb{E}^{\mathbb{Q}}[e^{u^\top Y_t + Y_t^\top VY_t + \alpha\beta|Y_t|^2}]\, e^{(r\alpha(1-\beta)+\frac{1}{2}u^\top au - tr(aV) - u^\top b)t} .\end{aligned}$$

For any fixed t, the random variable Y_t has a multivariate normal distribution. Therefore, we compute explicitly the expectation

$$\mathbb{E}^{\mathbb{Q}}[e^{u^\top Y_t + Y_t^\top VY_t + \alpha\beta|Y_t|^2}] = \frac{1}{\sqrt{(2\pi)^d \det\Sigma_t}}\int_{\mathbb{R}^d} e^{u^\top y + y^\top Vy + \alpha\beta|y|^2 - \frac{1}{2}(y-\mu_T)^\top\Sigma_t^{-1}(y-\mu_t)}\,dy , \tag{6.5}$$

where μ_t and Σ_t are the mean vector and the covariance matrix of Y_t under $\mathbb{Q}$, respectively. Under $\mathbb{Q}$, the coefficient of Y_t in the drift term of equation (6.4) is $B - 2aV$, all of whose eigenvalues have negative real parts. Thus, the distribution of Y_t is convergent to an invariant distribution, which is a non-degenerate multivariate normal random variable. Let Σ_∞ be the covariance matrix of the invariant distribution, and define

$$C := V + \alpha\beta I_d - \frac{1}{2}\Sigma_\infty^{-1}\,, \tag{6.6}$$

which is a symmetric matrix. The convergence or divergence of the integral in (6.5) depends on the eigenvalues of C. If at least one eigenvalue of C is positive, then it follows that

$$\mathbb{E}^{\mathbb{Q}}[e^{u^\top Y_t + Y_t^\top V Y_t + \alpha\beta|Y_t|^2}] = \infty$$

for sufficiently large t. If all eigenvalues of C are non-positive, then we can compute directly the limit

$$\ell := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}^{\mathbb{Q}}[e^{u^\top Y_t + Y_t^\top V Y_t + \alpha\beta|Y_t|^2}]\,. \tag{6.7}$$

In particular, if all eigenvalues of C are negative, then $\ell = 0$.

Proposition 6.1. *For $\beta \notin (0,1)$, the long-term growth rate of the LETF under the quadratic model is*

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}^{\mathbb{P}}\left[L_t^\alpha\right] = \begin{cases} r\alpha(1-\beta) + \frac{1}{2}u^\top a u - u^\top b - tr(aV) + \ell & \text{if all eigenvalues of } C \text{ are non-positive,} \\ \infty & \text{otherwise.} \end{cases} \tag{6.8}$$

When $\beta \in [0,1]$, if there exists a stabilizing solution to equation (6.2)*, then the LETF's long-term growth rate is also given by* (6.8).

In the limit (6.8), the matrix V and vector u (see (6.2) and (6.3)) depend on the leverage ratio β, so it is generally difficult to find the optimal β^* analytically. Nevertheless, in the special case when Y is one-dimensional with scalar $b = 0$, we can compute β^*. Recall from (6.1) the scalar parameters B and σ, and assume that $|B| \geq \sigma^2$ so that the stabilizing solution V of equation (6.2) exists for all $\beta \in \mathbb{R}$. In fact, the stabilizing solution is explicitly given by

$$V = \frac{B}{2\sigma^2} + \sqrt{\left(\frac{B}{2\sigma^2}\right)^2 + \alpha\beta(\beta-1)}\,, \tag{6.9}$$

and $u = 0$ according to (6.3).

The optimal value β^* depends on the sign of B. First, consider the case with $B \geq \sigma^2$. Then, the optimal growth rate achieves infinity when $\beta^* = 1$. Indeed, the number

$$C = V + \alpha\beta^* - \frac{1}{2}\Sigma_\infty^{-1} = -V + \alpha\beta^* + \frac{B}{\sigma^2} = \alpha$$

is positive given $\beta^* = 1$, resulting in the second case in (6.8). Next, suppose $B \leq -\sigma^2$. It is well known that the variance of the invariant distribution of the one-dimensional OU process Y in (6.4) is $\Sigma_\infty = \sigma^2/2(2\sigma^2 V - B)$. Putting this and (6.9) into (6.6), we get

$$C = \frac{B}{2\sigma^2} - \alpha\beta - \sqrt{\left(\frac{B}{2\sigma^2}\right)^2 + \alpha\beta(\beta-1)}\,.$$

It can be easily checked that C is negative for all $\beta \in \mathbb{R}$, thus the value ℓ in (6.7) is zero for all $\beta \in \mathbb{R}$. Indeed, the long-term growth rate of the LETF is

$$\begin{aligned}\Lambda(\beta) &= r\alpha(1-\beta) - \sigma^2\sqrt{\Big(\frac{B}{2\sigma^2}\Big)^2 + \alpha\beta(\beta-1)} - \frac{B}{2} \\ &= -r\alpha\beta - \sqrt{c_1\beta^2 + 2c_2\beta + c_3} + r\alpha - \frac{B}{2}\end{aligned}$$

where

$$c_1 = \alpha\sigma\,,\ c_2 = -\frac{\alpha\sigma}{2}\,,\ c_3 = \Big(\frac{B}{2\sigma^2}\Big)^2 .$$

In fact, $\Lambda(\beta)$ is of the same form, but with different coefficients, as that in the Heston model (see (4.4)), and thus, the optimal β^* can be derived in exactly the same way as in Section 4.1.

7 Conclusions

In our study of the long-term growth rate of expected utility of LETF, we have proposed the martingale extraction approach that removes path-dependence in the expectation of utility, which significantly simplifies the analysis and leads to explicit solutions. We have also transformed the original stochastic problem into a deterministic problem by solving the embedded eigenpair (eigenvalue and eigenfunction) problems to determine the long-term growth rate of expected utility (or expected return). In each of the single-factor and multi-factor models studied herein, the eigenpair as well as the limit of the growth rate are derived. The formula for the long-term growth rate allows us to determine the optimal leverage ratio and examine the effects of various model parameters. The results are useful not only for individual or institutional investors, but also ETF providers and regulators as they ought to know the long-term performance for any LETF traded in the market.

There are a number of directions for future research. One direction is to investigate the long-term price behavior of options written on LETFs (see e.g. Leung and Sircar (2015); Leung et al. (2016)). Recent studies on the long-term price behavior of options can be found in Park (2016). Given that the martingale extraction method studied herein works very well for LETFs which are constant proportion portfolios, one interesting and practical extension is to adapt it to other dynamic portfolio strategies.

A The extended CIR model

We evaluate the limit stated in (3.17) under the extended CIR model. We recall the $\mathbb{Q}$-dynamics of X_t in (3.15):

$$dX_t = (\ell - \mu X_t)\,dt + \sigma\sqrt{X_t}\,dW_t\,,$$

with $\ell := \theta + \kappa\sigma^2$.

Proposition A.1. *For $p \in \mathbb{R}$, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{Q}}[X_t^p\, e^{\frac{2\mu}{\sigma^2}X_t}] = \begin{cases}\Big(p + \frac{2\ell}{\sigma^2}\Big)\mu & \text{if } \ p + \frac{2\ell}{\sigma^2} > 0\,, \\ \infty & \text{if } \ p + \frac{2\ell}{\sigma^2} \le 0\,.\end{cases}$$

Before proving this proposition, we define the following notation.

Notation. Let $p(x)$ and $q(x)$ be two positive functions of x. Denote this by

$$p(x) \simeq q(x) \quad \text{at } x = x_0$$

if $\lim_{x \to x_0} p(x)/q(x)$ exists and is a nonzero constant. We denote this by

$$p \lesssim q$$

if there exists a positive constant c such that $p(x) \leq c \cdot q(x)$.

Proof. The density function $g(x;t)$ of X_t at a fixed time t is known to be

$$g(x;t) = h_t \, e^{-u-v} \left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}) \,, \tag{A.1}$$

where I_q is the modified Bessel function of the first kind of order q, and

$$h_t = \frac{2\mu}{\sigma^2(1-e^{-\mu t})} \,, \; q = \frac{2\ell}{\sigma^2} - 1 \,, \; u = h_t e^{-\mu t} \,, \; v = h_t X_0 \,.$$

After rewriting slightly, we find

$$g(x;t) = k_t \, h_t \, e^{-h_t x} x^{q/2} I_q(2h_t e^{-\mu t/2}\sqrt{x}) \,.$$

Here, $k_t = e^{-h_t e^{-\mu t}} e^{\mu q t/2}$. The quantity

$$\lim_{t \to \infty} \frac{1}{t} \log \int_0^\infty x^p e^{\frac{2\mu}{\sigma^2}x} \, g(x;t) \, dx$$

is of interest to us.

By inspection, we obtain

$$\int_0^\infty x^{p+q} e^{-p_t x} \, dx \lesssim \int_0^\infty x^p e^{\frac{2\mu}{\sigma^2}x} \, g(x;t) \, dx \lesssim \int_0^\infty x^{p+q} e^{-p_t x} e^{2h_t e^{-\mu t/2}\sqrt{x}} \, dx \,, \tag{A.2}$$

where $p_t = h_t - \frac{2\mu}{\sigma^2}$. This follows from $z^q \lesssim I_q(z) \lesssim z^q e^z$. We now show that if $p+q+1 > 0$, then

$$\text{(right- and left-hand sides of (A.2))} \simeq e^{(p+q+1)at} \,.$$

For the right-hand side of (A.2), substitute $y = p_t x$, then

$$\int_0^\infty x^{p+q} e^{-p_t x} e^{2h_t e^{-\mu t/2}\sqrt{x}} \, dx = p_t^{-p-q-1} \int_0^\infty y^{p+q} e^{-y} e^{2h_t e^{-\mu t/2} p_t^{-1/2} \sqrt{y}} \, dy \,.$$

As t approaches to infinity, $h_t e^{-\mu t/2} p_t^{-1/2}$ converges to a constant, so the integral term converges to a positive constant. By direct calculation, $p_t^{-p-q-1} \simeq e^{(p+q+1)\mu t}$. This implies that

$$\text{(right-hand side of (A.2))} \simeq e^{(p+q+1)\mu t} \,.$$

The proof is similar for the left-hand side of (A.2) given that $p+q+1 > 0$. On the other hand, if $p+q+1 \leq 0$, then the left-hand side of (A.2) is infinity. This completes the proof. $\square$

B The inverse GARCH model

With refernce to Section 5.2, recall the $\hat{\mathbb{P}}$-dynamics of the stochastic interest rate

$$dr_t = (\theta + \alpha\beta\delta\sigma\rho - ar_t)r_t\,dt + \delta r_t\,d\hat{Z}_t\ .$$

We investigate the long-term growth rate of

$$\mathbb{E}^{\hat{\mathbb{P}}}[e^{-\alpha(\beta-1)\int_0^t r_s\,ds}].$$

This in turn yields the equalities in (5.5) and (5.6). For convenience, in this appendix, we put

$$\hat{\mathbb{P}} \to \mathbb{P}\,,\ \hat{Z}_t \to B_t\,,\ \theta + \alpha\beta\delta\sigma\rho \to \theta\,,\ \delta \to \sigma\,,\ \alpha(\beta-1) \to c\ .$$

With these new parameters, we investigate the long-term growth rate of the expectation

$$\mathbb{E}^{\mathbb{P}}[e^{-c\int_0^t r_s\,ds}]$$

when the process r_t follows the inverse GARCH diffusion model:

$$dr_t = (\theta - ar_t)r_t\,dt + \sigma r_t\,dB_t\,,\ r_0 = 1$$

with $a, \sigma > 0$ and $\theta > \sigma^2$.

Proposition B.1. *Let $\kappa := c/a$ and assume that $\theta > (\kappa+1)\sigma^2$. The long-term growth rate of the above expectation is given by*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[e^{-c\int_0^t r_s\,ds}] = -\theta\kappa + \frac{1}{2}\sigma^2\kappa(\kappa+1)\ .$$

To prove this proposition, we will apply the martingale extraction to $e^{-c\int_0^t r_s\,ds}$. The corresponding infinitesimal generator is

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2\,\phi''(x) + (\theta - ax)x\,\phi'(x) - cx\,\phi(x)\ .$$

With $\kappa = c/a$, it follows that the corresponding eigenpair is

$$(\lambda, \phi(x)) := \left(\theta\kappa - \frac{1}{2}\sigma^2\kappa(\kappa+1)\,,\ x^{-\kappa}\right)$$

is an admissible eigenpair of $\mathcal{L}$ when $\theta > (\kappa+1)\sigma^2$. Let $\mathbb{Q}$ be the corresponding transformed measure. The process r_t follows

$$dr_t = (\theta - \kappa\sigma^2 - ar_t)r_t\,dt + \sigma r_t\,dW_t\ ,$$

where W_t is a $\mathbb{Q}$-Brownian motion. Through the martingale extraction with respect to this eigenpair, the expectation is expressed by

$$\mathbb{E}^{\mathbb{P}}[e^{-c\int_0^t r_s\,ds}] = \mathbb{E}^{\mathbb{Q}}[r_t^{\kappa}]\,e^{(-\theta\kappa+\frac{1}{2}\sigma^2\kappa(\kappa+1))t}\ .$$

We show that

$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}[r_t^{\kappa}] = (\text{positive constant}) \ .$$

Under $\mathbb{Q}$, the process $Y_t = \frac{2a}{\sigma^2} r_t$ converges to the Gamma random variable with parameter $\gamma = \frac{2\theta}{\sigma^2} - 2\kappa - 1$, that is, then density function $p(y;t)$ of Y_t converges to $p(y;\infty) := \frac{1}{\Gamma(\gamma)} y^{\gamma-1} e^{-y}$ as $t \to \infty$ (see Theorem 2.5 in Zhao (2009) and Section 6.5.4 in Linetsky (2004)) By the same argument in equation (3.9), we know

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[r_t^{\kappa}] = \left(\frac{\sigma^2}{2a}\right)^{\kappa} \mathbb{E}^{\mathbb{P}}[Y_t^{\kappa}] &= \left(\frac{\sigma^2}{2a}\right)^{\kappa} \int_0^{\infty} y^{\kappa}\, p(y;t)\, dy \\ &\to \left(\frac{\sigma^2}{2a}\right)^{\kappa} \frac{1}{\Gamma(\gamma)} \int_0^{\infty} y^{\frac{2\theta}{\sigma^2}-\kappa-2} e^{-y}\, dy \ . \end{aligned}$$

when $\theta > (\kappa+1)\sigma^2$. This gives the desired result.

C Quadratic models

We now derive the SDE for X_t by using the fromula $X_t = e^{|Y_t|^2}$, where

$$dY_t = (b + BY_t)\, dt + \sigma\, dW_t \ , \ X_0 = 0_d \ .$$

Define $f(y) = e^{|y|^2}$, then the gradient row vector and Hessian matrix are, respectively,

$$\nabla f(y) = 2f(y) y^{\top} \ , \qquad Hf(y) = 2f(y) \begin{pmatrix} 1+2y_1^2 & 2y_1y_2 & \cdots & 2y_1y_d \\ 2y_2y_1 & 1+2y_2^2 & \cdots & 2y_2y_d \\ \vdots & \vdots & \ddots & \vdots \\ 2y_dy_1 & 2y_dy_2 & \cdots & 1+2y_d^2 \end{pmatrix} . \tag{C.1}$$

By Ito's formula, it follows that

$$dX_t = df(Y_t) = \left(\nabla f(Y_t)(b + BY_t) + \frac{1}{2}\sum_{i,j} (Hf(Y_t))_{ij} (\sigma\sigma^{\top})_{ij} \right) dt + \nabla f(Y_t) \sigma\, dW_t \ . \tag{C.2}$$

Applying (C.1) to (C.2), we get

$$\frac{dX_t}{X_t} = (\cdots)\, dt + 2Y_t^{\top} \sigma\, dW_t \ ,$$

where we have abstracted the drift term since its expression is not needed here. From this and (2.2), the LETF price can be expressed as

$$L_t = X_t^{\beta} e^{-r(\beta-1)t - 2\beta(\beta-1)\int_0^t |\sigma^{\top} Y_u|^2\, du} .$$

From this we observe the path dependence of L_t on the d-dimensional process Y.

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