Optimal Consumption and Investment with Welfare Constraints *

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Abstract

In this study, we investigate an optimal consumption and investment problem of an economic agent who faces a *welfare constraint*; the agent does not accept her expected utility (continuation value) falls below a certain fixed level regardless of the time and state. This optimization problem involves an infinite number of constraints. Using a duality approach, we transform infinitely many constraints into a single constraint and define the dual problem, which becomes a two-dimensional singular control problem. The dual problem provides its associated Hamilton-Jacobi-Bellman (HJB) equation with a gradient constraint. Under a general class of utility functions, we obtain an explicit solution to the HJB equation and provide optimal strategies by establishing a duality theorem. As an example, we consider hyperbolic absolute risk aversion (HARA) utility, which may incorporate a government subsidy or a basic support, and provide the solution and its implications.

Keywords : consumption and investment, welfare constraints, general utility, singular control problem, duality approach, dynamic constraints

1 Introduction

This study investigates the optimal consumption and investment problems of an economic agent with a general utility function in the presence of welfare constraints. Under welfare constraints, it is assumed that an agent's expected utility (continuation value) should always be greater than or equal to a certain minimum welfare level. An economic agent can determine the minimum welfare level based on various rationales. One example is the reservation utility of an economic agent in a limited commitment framework, as in Choi et al. (2021). In Choi et al. (2021) with limited commitment, the economic agent may default and not repay the debt, and consequently, the agent is not able to participate in the financial market and is allowed to consume a fixed proportion of income afterwards as a default penalty. In this case, the economic agent repays the debt only when repayment is incentive-compatible, and the corresponding reservation utility is the lifetime utility the economic agent would have by choosing immediate default and consuming a fixed proportion of his/her labor income without participating in the financial market.[1]

Recently, Campbell and Martin (2022) studied the optimal consumption and investment problem with constant relative risk aversion (CRRA) utility under the sustainability constraint that expected utility, which is a function of current wealth and thus a random variable, should not be expected to decrease over time. More precisely, the drift of the expected utility process should be nonnegative under the sustainability constraint. By contrast, we impose a constraint on the level of expected utility, not its dynamics, and consider a more general class of utility functions that include hyperbolic absolute risk aversion (HARA) class utility functions.

To investigate our optimization problem, we employ the duality approach developed by Karatzas et al. (1987) and Cox and Huang (1989). As the welfare constraint should be satisfied regardless of the time and state of the world, our optimization problem faces an infinite number of constraints. To overcome this difficulty, we transformed infinitely many constraints for the primal problem into one constraint for the dual problem, as in He and Pagés (1993). Because the dual problem involves the choice of a non-decreasing shadow price process, which is the cumulative Lagrange multiplier process arising from dynamic welfare constraints, the dual problem can be formulated as a two-dimensional singular control problem, and we can obtain an associated Hamilton-Jacobi-Bellman (HJB) equation with a gradient constraint. For a general class of utility functions, we derive an explicit solution to the HJB equation and provide a verification theorem that guarantees that the solution to the HJB equation is indeed the solution to the dual-value function. Previous studies that incorporated similar singular control problems established the verification theorem by utilizing the standard argument in the literature that puts some growth conditions on the dual conjugate function of the utility function. However, without imposing a growth condition on the dual conjugate function of the utility function, it is difficult to apply standard arguments to our verification theorem. To resolve this difficulty, based on many technical and non-standard arguments, we directly show the identification between the dual value function and the solution to the HJB equation using Fubini's theorem and expressing the solution to the HJB equation as an integral of optimal stopping problems. This is a main technical contribution of this study. Finally, we prove the duality theorem and obtain the explicit solutions to the optimal consumption and investment strategies.

Based on the explicit solutions for the general class of utility functions, we can show that there exists

[1] More details will be provided in Remark 3 in Section 2.

a minimum wealth level endogenously determined by the welfare constraint. In other words, an economic agent's wealth process under welfare constraints should always be greater than or equal to a certain endogenous minimum wealth level. In contrast to the model with the borrowing constraint, the endogenous minimum wealth level in our model can have a positive value depending on the minimum welfare level imposed by the welfare constraint. As expected, as the minimum welfare level increases, the endogenous minimum wealth level increases. On the other hand, the optimal consumption and investment (provided the risky asset provides positive excess return) for a given wealth level decrease as the minimum welfare level increases, which is an intuitive result. We can also show that the optimal investment becomes zero when the agent's wealth reaches the endogenous minimum wealth level, to avoid the risk of violating the welfare constraint.

For example, we consider a HARA class utility function that may incorporate a government subsidy or basic support that cannot be stored, and should be consumed immediately. It is obvious that it becomes easier to meet the welfare constraint as basic support increases, which implies that the solutions may have different properties depending on the relationship between the minimum welfare level and basic support. This is confirmed and discussed in Section 6.

Our study is closely related to Choi et al. (2021), who investigate the optimal consumption and investment problem of a household with limited commitment to debt repayment, since the agent's limited commitment in Choi et al. (2021) can be regarded as time varying welfare constraints. Our study is also related to the literature on optimal consumption and portfolio problems with non-negative wealth constraints (see He and Pagés (1993), El Karoui and Jeanblanc-Picqué (1998)) in that the welfare constraints in our problem generate minimum wealth constraints. However, the wealth constraints generated by welfare constraints are determined endogenously according to the minimum welfare level. Although the mathematical structure of the two problems seems to be similar, our dual problem is formulated into a two-dimensional singular control, whereas the dual problem in He and Pagés (1993) and El Karoui and Jeanblanc-Picqué (1998) is formulated as a one-dimensional singular control. Moreover, He and Pagés (1993), El Karoui and Jeanblanc-Picqué (1998) and Choi et al. (2021) provide explicit solutions only for the case of the CRRA utility function. As our dual problem involves the choice of a non-decreasing shadow price process, the mathematical structure is similar to the incremental irreversible investment problems studied by Pindyck (1988), Dixit and Pindyck (1994) and Riedel and Su (2011). There have been many studies on the continuous-time portfolio selection problem with the feature of a singular control (see Davis and Norman (1990), He and Pagés (1993), Dybvig (1995), El Karoui and Jeanblanc-Picqué (1998), Deng et al. (2022), and references therein). We add to the literature by investigating the welfare constraint on the agent's consumption and investment problem using the general utility function.

We introduce our model and primal optimization in Section 2, and the corresponding dual problem and its associated HJB equation are derived in Section 3. Section 4 derives a solution to the HJB equation and provides a verification theorem. The duality theorem and optimal strategies are presented in Section 5. Section 6 provides an example of the HARA class utility function that can incorporate basic support and discusses the implications of the solution. Finally, Section 7 concludes the paper. All proofs are provided in the Appendix.

2 Model

We consider a simple, standard continuous-time financial market.

Preference: The agent's objective is to maximize the following expected utility from intertemporal consumption c_t:

$$U \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right], \tag{1}$$

where $\beta > 0$ denotes the subjective discount rate.

The agent's expected utility (continuation value) at time $t \geq 0$ is given by

$$\mathcal{W}_t^a \equiv \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)} u(c_s)ds\right], \tag{2}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t]$ denotes the conditional expectation at time t on the filtration $\mathcal{F}_t$.

We assume that the agent wants to maintain the expected utility (continuation value) above a certain level P. In other words, the agent's consumption process $\{c_t\}_{t=0}^\infty$ satisfies the following *welfare constraints*: for all $t \geq 0$

$$\mathcal{W}_t^a \geq P. \tag{3}$$

Financial Market: The financial market consists of two assets: a risk-free asset and a risky asset (or a market index). We assume that the risk-free rate$r > 0$ is constant. The price S_t of the risky asset evolves as follows:

$$dS_t/S_t = \mu dt + \sigma dB_t,$$

where μ, σ are constants, $\mu > r$, and B_t is a Brownian motion on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Brownian motion B_t.

The Budget Constraint and Admissible Strategies: We assume that the agent receives the labor wage at the constant rate $\epsilon > 0$. The agent's wealth process $(X_t^{c,\pi})_{t=0}^\infty$ corresponding to strategy (c, π) evolves according to the following dynamics:

$$dX_t^{c,\pi} = [rX_t^{c,\pi} + \pi_t(\mu - r) - c_t + \epsilon]dt + \sigma\pi_t dB_t, \quad X_0^{c,\pi} = x > -\frac{\epsilon}{r}, \tag{4}$$

where $c_t \geq 0$ and π_t are the consumption rate and the dollar amount invested in the risky asset, respectively, at time t.

Throughout this paper, (c, π) belongs to the admissible class $\mathcal{A}(x)$ if they are $\mathcal{F}_t$-progressively measurable processes that satisfy the following conditions:

(a) c_t and π_t satisfy

$$\int_0^t c_s ds < \infty, \text{ a.s. and } \int_0^t \pi_s^2 ds < \infty, \text{ a.s., } \forall\ t \geq 0, \tag{5}$$

(b) $(c_t)_{t=0}^\infty$ satisfies the dynamic welfare constraints in (3),

(c) The wealth $(X_t^{c,\pi})_{t=0}^\infty$ corresponding to $(c_t, \pi_t)_{t=0}^\infty$ in (4) satisfies

$$X_t^{c,\pi} > -\frac{\epsilon}{r} \text{ a.s. } \forall\, t \geq 0.$$

Utility Functions: We make the following assumptions on the felicity function to guarantee the existence of a solution to the agent's optimization problem:

Assumption 1. *Felicity function* $u : [0, \infty) \to \mathbb{R}$ *is strictly increasing, strictly concave and continuously differentiable, and* $\lim_{c \to +\infty} u'(c) = 0$.

The strictly decreasing and continuous function $u' : (0, \infty) \overset{\text{onto}}{\longrightarrow} (0, u'(0))$ has strictly decreasing continuous inverse $I : (0, u'(0)) \overset{\text{onto}}{\longrightarrow} (0, \infty)$. We extend I by setting $I(y) = 0$ for $y \geq u'(0)$. Then, we have

$$u'(I(y)) = \begin{cases} y, & 0 < y < u'(0), \\ u'(0), & y \geq u'(0), \end{cases}$$

and $I(u'(c)) = c$ when $0 < c < \infty$. We note that $\lim_{y \to \infty} I(y) = 0$.

Remark 1. *The conditions in Assumption 1 are commonly employed in the consumption-investment choice problem. The increasing property and strict concavity of the utility function in Assumption 1 imply that the agent has monotone preference and exhibits risk aversion. The last limiting condition is the Inada condition, which states that the marginal utility of consumption approaches zero if one consumes larger amounts of consumption goods. This assumption is standard in the literature (see Merton (1969, 1971)).*

Assumption 2. *For any* $y > 0$,

$$\int_0^y \xi^{-n_2} I(\xi) d\xi < \infty.$$

where $n_1 > 0$ *and* $n_2 < 0$ *are the two roots of the quadratic equation:*

$$q(n) := \frac{\theta^2}{2} n^2 + \left(\beta - r - \frac{\theta^2}{2} \right) n - \beta = 0. \tag{6}$$

Remark 2. *Assumption 2 is a necessary and sufficient condition for the existence of optimal policies for standard consumption-investment choice problem, that is, the agent's problem without the welfare constraints (see Chapter 3.9 in Karatzas and Shreve (1998)). Quantity* $I(y)$ *is the consumption corresponding to marginal utility* y*. Hence, the assumption implies that the improper integral near-zero consumption weighted by the positive power* $-n_2$ *of the marginal utility is finite. Accordingly, it implies that consumption does not explode too quickly as marginal utility becomes very small.*

For the welfare constraint in (3), we make the following assumptions.

Assumption 3. P *satisfies the following condition.*

$$\lim_{c \to 0+} u(c) < \beta P < \lim_{c \to \infty} u(c).$$

Remark 3. *As described in the Introduction, one example of the minimum welfare level* P *is the reservation utility of an agent with limited commitment. Suppose that the agent can choose to default and not repay the debt, and the consequential default penalty is that the agent is not allowed to participate in the financial market and is only able to consume a fixed proportion* δ_I *of income afterwards. In this case we have*

$$P = \int_0^\infty e^{-\beta t} u(\delta_I \epsilon) dt.$$

We now describe the optimization problem for the agent.

Problem 1 (Primal Problem)**.**
Given $x > -\frac{\epsilon}{r}$, we consider the following optimization problem:

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right]. \tag{7}$$

Because Problem 1 involves the choice of consumption and portfolio as well as dynamic constraints, our strategy for solving the problem consists of the following steps:

(1) By using the duality approach, we derive the budget constraint in static form for Problem 1. To transform the infinitely many constraints in (3) into one constraint, we employ the method proposed by He and Pagés (1993) with a modification to fit our purpose. To do this, we introduce a non-decreasing process, which can be thought of as the integral of infinitesimal Lagrange multipliers for the constraint (3). Then, we can set the Lagrangian for Problem 1.

(2) By maximizing the Lagrangian over consumption, we derive the candidates of optimal consumption. Putting this in the Lagrangian, we define the dual value function, which takes the form of a two-dimensional singular control problem.

(3) Applying the dynamic programming principle to the dual problem, we derive its associated Hamilton-Jacobi-Bellman(HJB) equation with a gradient constraint. We provide an explicit-form solution to the HJB equation and verification that it is the equivalent to the dual value function.

(4) Finally, we prove the duality theorem and provide the optimal strategies in the explicit-forms.

3 A Dual Optimization Problem

First, we transform the agent's dynamic wealth process in (4) into a static-form constraint using the linearization method developed by Cox and Huang (1989) and Karatzas et al. (1987). Let us define

$$\theta \equiv \frac{\mu - r}{\sigma}, \quad \mathcal{H}_t \equiv e^{-(r+\frac{\theta^2}{2})t-\theta B_t}, \quad \mathcal{H}_s^t \equiv \frac{\mathcal{H}_s}{\mathcal{H}_t} \quad \text{for } 0 \le t \le s,$$

where $\mathcal{H}_t$ is the *stochastic discount factor* or *state price density.* The agent's wealth process can then be transformed into the following static budget constraint:

$$\mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t - \epsilon)dt\right] \le x. \tag{8}$$

Proposition 4.1 in Knudsen et al. (1998) is useful for investigating the optimization problem. Thus, we briefly introduce a proposition based on our notation. For an arbitrary measurable function $\phi(y)$ defined on $\mathbb{R}_+$, let us define operator Γ as

$$\Gamma_\phi(y) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\phi(\mathcal{Y}_t^y)dt\right], \tag{9}$$

where $\mathcal{Y}_t^y \equiv ye^{\beta t}\mathcal{H}_t$. Note that

$$\frac{d\mathcal{Y}_t^y}{\mathcal{Y}_t^y} = (\beta - r)dt - \theta dB_t \ \text{ with } \ \mathcal{Y}_0^y = y. \tag{10}$$

The following proposition provides some properties of the operator Γ:

Proposition 1 (Proposition 4.1 in Knudsen et al. (1998)). *Let $\phi(y)$ be an arbitrary measurable function defined on $\mathbb{R}^+$. Then, the following conditions are equivalent:*

(i) for every $y > 0$

$$\Gamma_{|\phi|}(y) < \infty,$$

(ii) for every $y > 0$

$$\int_0^y \xi^{-n_2-1}|\phi(\xi)|d\xi + \int_y^\infty \xi^{-n_1-1}|\phi(\xi)|d\xi < \infty.$$

Under condition (i) or (ii), the following statements are true:

(a) $\liminf_{y\downarrow 0} y^{-n_2}|\phi(y)| = \liminf_{y\uparrow\infty} y^{-n_1}|\phi(y)| = 0,$

(b) Γ_ϕ *has the following form:*

$$\Gamma_\phi(y) = \frac{2}{\theta^2(n_1 - n_2)}\left[y^{n_2}\int_0^y \xi^{-n_2-1}\phi(\xi)d\xi + y^{n_1}\int_y^\infty \xi^{-n_1-1}\phi(\xi)d\xi\right],$$

(c) $\Gamma_\phi(y)$ *is twice differentiable and satisfies*

$$\frac{\theta^2}{2}y^2\Gamma_\phi''(y) + (\beta - r)y\Gamma_\phi'(y) - \beta\Gamma_\phi(y) + \phi(y) = 0,$$

(d) there exists a positive constant C such that

$$|\Gamma_\phi'(y)| \le C(y^{n_1-1} + y^{n_2-1}) \quad \textit{for all} \;\; y > 0,$$

(e) $\lim_{t\to\infty} e^{-\beta t}\mathbb{E}\left[|\Gamma_\phi(\mathcal{Y}_t^y)|\right] = 0.$

We now derive a dual problem arising from Problem 1. For this purpose, we first provide an informal heuristic derivation of the Lagrangian for the dual problem. The key is to write a part of the Lagrangian corresponding to the infinitely many constraints (2). By utilizing the method developed by He and Pagés (1993), we transform the constraint (2) into the following condition:

$$\mathbb{E}\left[\int_0^\infty \eta_t\left(\mathcal{W}_t - P\right)dt\right] = \mathbb{E}\left[\int_0^\infty \eta_t\left(\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right)dt - P\int_0^\infty \eta_t dt\right] \ge 0, \tag{11}$$

where $\eta_t \ge 0$ denotes the Lagrangian multiplier of constraint (2) at each time $t \ge 0$.

We now write the Lagrangian $\mathfrak{L}$ for Problem 1 with constraints (2) and (8) as follows:

$$\begin{aligned}\mathfrak{L} \equiv &\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] + y\left(x - \mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t - \epsilon)dt\right]\right) + \mathbb{E}\left[\int_0^\infty e^{\beta t}\eta_t\left(\int_t^\infty e^{-\beta s}u(c_s)ds\right)dt\right.\\ &\left.- P\int_0^\infty \eta_t dt\right],\end{aligned} \tag{12}$$

where $y > 0$ denotes the Lagrangian multiplier for the static budget constraint (8). By ignoring the technical conditions to derive the dual problem, we deduce the following.

$$\begin{aligned}\mathfrak{L} =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] + y\left(x - \mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t - \epsilon)dt\right]\right) + \mathbb{E}\left[\int_0^\infty e^{\beta t}\eta_t\left(\int_t^\infty e^{-\beta s}u(c_s)ds\right)dt - P\int_0^\infty \eta_t dt\right]\\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] + y\left(x - \mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t - \epsilon)dt\right]\right) + \mathbb{E}\left[\int_0^\infty \left(\int_0^t e^{\beta s}\eta_s ds\right)e^{-\beta t}u(c_t)dt - P\int_0^\infty \eta_t dt\right]\\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\left(1 + \int_0^t e^{\beta s}\eta_s ds\right)u(c_t) - \mathcal{Y}_t^y c_t + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}e^{\beta t}\eta_t dt\right],\end{aligned} \tag{13}$$

where $\mathcal{Y}_t^y = ye^{\beta t}\mathcal{H}_t$ and we use integration by parts in the second equality.

We define process $\mathcal{Z}$ by

$$\mathcal{Z}_t = \int_0^t e^{\beta s}\eta_s ds \;\text{ for }\; t \geq 0. \tag{14}$$

Note that the process $\mathcal{Z}$ is non-decreasing in $t \geq 0$ with $d\mathcal{Z}_t = e^{\beta t}\eta_t dt$. Then, we have

$$\mathfrak{L} = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\,u(c_t) - \mathcal{Y}_t^y c_t + \mathcal{Y}_t^y \epsilon\right)dt - P\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right]. \tag{15}$$

Since the dual conjugate function $\tilde{u}$ of u is defined as

$$\tilde{u}(y) = \sup_{c\geq 0}\,(u(c) - yc) = u(I(y)) - yI(y) \quad \text{with} \quad I(y) = (u')^{-1}(y), \tag{16}$$

we deduce that the candidate for the optimal consumption $\hat{c}(\mathcal{Y}_t^y)$ for $y > 0$ is given by

$$\hat{c}(\mathcal{Y}_t^y) := I\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right) \;\text{ for }\; t \geq 0. \tag{17}$$

Optimizing over c_t in (15) yields

$$\mathfrak{L}(y, \mathcal{Z}) := \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right) + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right] + yx. \tag{18}$$

Motivated by the discussion above, we formally state the dual problem as follows.

Problem 2 (Dual Problem)**.**

$$J(y) = \inf_{\mathcal{Z}\in\Pi(0)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right) + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right], \tag{19}$$

where $\Pi(z)$ is the set of all positive non-decreasing, right-continuous processes $\mathcal{Z}$ with left-limits, and $\mathcal{Z}_{0-} = z \geq 0$ such that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left|u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)\right| d\nu dt\right] < \infty, \tag{20}$$

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right] < \infty. \tag{21}$$

We call $J(y)$ and the process $\mathcal{Z}$ the dual value function and shadow price, respectively.

Remark 4. *One might think that integral condition (20) is somewhat unusual. However, the key idea for the verification of the dual problem is to apply Fubini's theorem. As can be seen in the proof of Theorem 1, the integral condition allows us to utilize Fubini's theorem.*

Lemma 1. *Let $z \geq 0$ be given. For any $(\mathcal{Z}_t)_{t=0}^\infty \in \Pi(z)$, the integrability condition (20) implies that:*

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right| dt\right] < \infty. \tag{22}$$

Thus, the integrability conditions (20) and (21) guarantee that the dual value function $J(y)$ is finite.

Proof. See Appendix A. □

The next proposition states that the Lagrangian $\mathfrak{L}(y, \mathcal{Z})$ in (18) is greater than or equal to the value function $V(x)$ defined in Problem 1.

Proposition 2. *For any $y > 0$ and $\mathcal{Z} \in \Pi(0)$, the following inequality holds.*

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right] \le \mathfrak{L}(y, \mathcal{Z}).$$

The equality holds if and only if, for all $t \ge 0$

$$\begin{aligned} c_t =& I\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right), \quad x = \mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t - \epsilon)dt\right], \\ 0 =& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right] - P\right)d\mathcal{Z}_t\right]. \end{aligned}$$

Proof. See Appendix B. □

From Proposition 2, we have

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] \le \inf_{y>0} \inf_{\mathcal{Z}\in\Pi(0)} \mathfrak{L}(y, \mathcal{Z}) \le \inf_{y>0} \left(J(y) + yx\right). \tag{23}$$

Thus, we derive the following weak-duality:

$$V(x) \le \inf_{y>0} \left(J(y) + yx\right). \tag{24}$$

We demonstrate that the above weak-duality holds with equality in Theorem 2.

To apply the dynamic programming principle to the dual function in Problem 2, we consider the following two-dimensional singular control problem of $\mathfrak{J}(y, z)$ given by

$$\mathfrak{J}(y, z) = \inf_{\mathcal{Z}\in\Pi(z)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right) + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right]. \tag{25}$$

Note that

$$J(y) = \mathfrak{J}(y, 0). \tag{26}$$

Remark 5. *As mentioned in the Introduction, the structure of the singular control problem in (25) is similar to that of the irreversible investment problem with the capacity expansion decision studied by Pindyck (1988), Bertola (1998), Riedel and Su (2011) and references therein.*

From the standard theory of singular control (see Harrison (1985) and Stokey (2009)), we consider the following two-dimensional HJB equation with a gradient constraint arising from the problem (25):

$$\min\left\{\mathcal{L}\mathcal{Q}(y, z) + (1+z)\tilde{u}\left(\frac{y}{1+z}\right) + y\epsilon,\ \partial_z\mathcal{Q}(y, z) - P\right\} = 0, \tag{27}$$

where the differential operator $\mathcal{L}$ is given by

$$\mathcal{L} \equiv \frac{\theta^2}{2}y^2\frac{d^2}{dy^2} + (\beta - r)y\frac{d}{dy} - \beta.$$

Usually, many studies on singular control problems show the sufficient conditions[2], under which the value function $\mathfrak{J}(y, z)$ of the control problem (25) identifies with a solution to $\mathcal{Q}(y, z)$ by imposing proper growth condition on $\tilde{u}(\cdot)$. However, under Assumptions 1-2 without the growth condition on $\tilde{u}$, it is not easy to show the sufficient conditions (28) and (29) in our optimization problem.

To overcome this difficulty, in the next section, we directly show the identification between $\mathfrak{J}(y, z)$ and $\mathcal{Q}(y, z)$ instead of imposing the growth condition on $\tilde{u}$ to show the sufficient conditions. The main idea of the proof is to utilize Fubini's theorem and express $\mathcal{Q}(y, z)$ as the integral of the optimal stopping problems (see Proposition 5 and the proof of Theorem 1). This is one of the technical contributions of this study.

4 Solution Analysis

Consider the following substitution:

$$\mathcal{Q}(y, z) = (1 + z)\mathcal{M}(\nu) \quad \text{with} \quad \nu = \frac{y}{1 + z}. \tag{30}$$

It follows from (27) that:

$$\min \left\{\mathcal{L}\mathcal{M}(\nu) + \tilde{u}\left(\nu\right) + \nu\epsilon,\ \mathcal{M}(\nu) - \nu\mathcal{M}'(\nu) - P\right\} = 0 \quad \text{for} \ \ \nu \in \mathbb{R}_+. \tag{31}$$

To derive a solution to HJB (31), we consider the following free boundary problem (FBP) arising from HJB (31): for $\nu \in \mathbb{R}_+$

$$\begin{cases} \mathcal{L}\mathcal{M}(\nu) + \tilde{u}(\nu) + \nu\epsilon = 0 & \text{for} \ \ 0 < \nu < \bar{\nu}, \\ \mathcal{M}(\nu) - \nu\mathcal{M}'(\nu) = P & \text{for} \ \ \nu \geq \bar{\nu}, \\ (\mathcal{M} - \nu\mathcal{M}')'(\bar{\nu}) = -\bar{\nu}\mathcal{M}''(\bar{\nu}) = 0 & \text{(smooth-pasting)}. \end{cases} \tag{32}$$

We will find a twice continuously differentiable solution to FBP (32).

Lemma 2. *The following inequality holds for any $\nu > 0$:*

$$\int_0^\nu \xi^{-n_2-1}|u(I(\xi))|d\xi + \int_\nu^\infty \xi^{-n_1-1}|u(I(\xi))|d\xi < \infty$$

and

$$\int_0^\nu \xi^{-n_2-1}|\tilde{u}(\xi)|d\xi + \int_\nu^\infty \xi^{-n_1-1}|\tilde{u}(\xi)|d\xi < \infty.$$

Proof. See Appendix C. □

[2] The sufficient conditions are twice continuously differentiability of $\mathcal{Q}(y, z)$, martingale condition

$$\int_0^t e^{-\beta t}\mathcal{Y}_t^y \partial_y \mathcal{Q}(\mathcal{Y}_t^y, \mathcal{Z}_t)dB_t \ \text{ is } \mathcal{F}_t\text{-martingale}, \tag{28}$$

and transversality condition

$$\lim_{t\to\infty} e^{-\beta t}\mathbb{E}\left[\mathcal{Q}(\mathcal{Y}_t^y, \mathcal{Z}_t)\right] = 0 \tag{29}$$

for any $\mathcal{Z} \in \Pi(z)$

In the region $0 < \nu < \bar{\nu}$, a solution to FBP (32) can be expressed as the sum of a general solution to the homogeneous equation and a particular solution:

$$\mathcal{M}(\nu) = D_1\nu^{n_1} + D_2\nu^{n_2} + \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right].$$

From Proposition 1 and Lemma 2, we deduce that

$$\begin{aligned}&\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right]\\ =&\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}\tilde{u}(\xi)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}\tilde{u}(\xi)d\xi\right] + \frac{\epsilon}{r}\nu\end{aligned}$$

is well-defined. As $\mathcal{Q}(y,z)$ should satisfy the transversality condition $\lim_{t\to\infty}\mathbb{E}\left[e^{-\beta t}\mathcal{Q}(\mathcal{Y}_t^y,\mathcal{Z}_t)\right]=0$, we set $D_2 = 0$. Thus, we can rewrite $\mathcal{M}(\nu)$ as

$$\mathcal{M}(\nu) = D_1\nu^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}\tilde{u}(\xi)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}\tilde{u}(\xi)d\xi\right] + \frac{\epsilon}{r}\nu.$$

From Lemma 2 and Proposition 1 (a), we have

$$\liminf_{y\downarrow 0} y^{-n_2+1}I(y) = 0.$$

It follows that

$$\begin{aligned}\nu\mathcal{M}'(\nu) &= n_1D_1\nu^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[n_2\nu^{n_2}\int_0^\nu \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + n_1\nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right]\\ &= n_1D_1\nu^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}(-\xi I(\xi)+\xi\epsilon)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}(-\xi I(\xi)+\xi\epsilon)d\xi\right],\end{aligned}$$

where we used integration by parts for the Riemann-Stieltjes integrals in the second equality. Because $\tilde{u}(\xi) = u(I(\xi)) - \xi I(\xi)$, we obtain that for $0 < \nu \le \bar{\nu}$

$$\begin{aligned}\mathcal{M} - \nu\mathcal{M}'(\nu) =&(1-n_1)D_1\nu^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}(\tilde{u}(\xi)+\xi I(\xi))d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi I(\xi))d\xi\right]\\ =&(1-n_1)D_1\nu^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}u(I(\xi))d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}u(I(\xi))d\xi\right].\end{aligned} \tag{33}$$

By the smooth-pasting condition (32), we have

$$\begin{aligned}D_1 =&\frac{2}{\theta^2(n_1-1)(n_1-n_2)}\int_{\bar{\nu}}^\infty \xi^{-n_1-1}u(I(\xi))d\xi - \frac{-n_2P}{(n_1-1)(n_1-n_2)}\bar{\nu}^{-n_1}\\ =&\frac{2}{\theta^2(n_1-1)(n_1-n_2)}\int_{\bar{\nu}}^\infty \xi^{-n_1-1}\left(u(I(\xi))-\beta P\right)d\xi\end{aligned}$$

and

$$0 = \int_0^{\bar{\nu}} \xi^{-n_2-1}u(I(\xi))d\xi - \frac{\theta^2}{2}n_1\bar{\nu}^{-n_2}P = \int_0^{\bar{\nu}} \xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi.$$

Lemma 3. *There exists a unique free boundary $\bar{\nu} > 0$ such that*

$$\Psi(\bar{\nu}) := \int_0^{\bar{\nu}} \xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi = 0.$$

Then,

$$u(I(\bar{\nu})) - \beta P < 0.$$

Proof. See Appendix D. ☐

Because $\mathcal{M}(\nu) - \nu\mathcal{M}'(\nu) - P = 0$ for all $\nu \geq \bar{\nu}$, it follows that:

$$\mathcal{M}''(\nu) = 0 \quad \text{for} \quad \nu \geq \bar{\nu}.$$

This implies that $\mathcal{M}'(\nu)$ is constant on the domain $\nu \geq \bar{\nu}$, that is,

$$\mathcal{M}'(\nu) = \chi \quad \text{for} \quad \nu \geq \bar{\nu}.$$

for some constant χ. Under the assumption that $\mathcal{M}$ is C^2, we can deduce that

$$\mathcal{L}\mathcal{M}(\bar{\nu}) + \tilde{u}(\bar{\nu}) + \bar{\nu}\epsilon = 0.$$

Since $\mathcal{M}''(\bar{\nu}) = 0$ and $\mathcal{M}(\bar{\nu}) - \bar{\nu}\mathcal{M}'(\bar{\nu}) = P$, we have

$$\chi = \frac{1}{r\bar{\nu}} \left(\tilde{u}(\bar{\nu}) - \beta P + \bar{\nu}\epsilon \right).$$

This leads to

$$\mathcal{M}(\nu) = \nu\mathcal{M}'(\nu) + P = \left(\tilde{u}(\bar{\nu}) - \beta P + \bar{\nu}\epsilon \right) \frac{\nu}{r\bar{\nu}} + P \quad \text{for} \quad \nu \geq \bar{\nu}.$$

In summary, we obtain the following explicit-form of $\mathcal{M}(\nu)$ satisfying FBP (32):

$$\mathcal{M}(\nu) = \begin{cases} D_1 \nu^{n_1} + \dfrac{2}{\theta^2(n_1 - n_2)} \left[\nu^{n_2} \displaystyle\int_0^{\nu} \xi^{-n_2-1} \tilde{u}(\xi) d\xi + \nu^{n_1} \int_{\nu}^{\infty} \xi^{-n_1-1} \tilde{u}(\xi) d\xi \right] + \dfrac{\epsilon}{r}\nu & \text{for} \quad 0 < \nu < \bar{\nu}, \\ \left(\tilde{u}(\bar{\nu}) - \beta P + \bar{\nu}\epsilon \right) \dfrac{\nu}{r\bar{\nu}} + P & \text{for} \quad \nu \geq \bar{\nu}, \end{cases} \tag{34}$$

where

$$D_1 = \frac{2}{\theta^2(n_1 - 1)(n_1 - n_2)} \int_{\bar{\nu}}^{\infty} \xi^{-n_1-1} \left(u(I(\xi)) - \beta P \right) d\xi, \tag{35}$$

$$0 = \int_0^{\bar{\nu}} \xi^{-n_2-1} \left(u(I(\xi)) - \beta P \right) d\xi. \tag{36}$$

Proposition 3. *$\mathcal{M}(\nu)$ given in (34) is twice continuously differentiable, and satisfies the HJB equation (31).*

Proof. See Appendix E. ☐

Let us denote $\hat{z}(y)$ as

$$\hat{z}(y) = \frac{y}{\bar{\nu}} - 1.$$

From substitution (30), we deduce that $\mathcal{Q}(y, z)$ is given by:
(i) for $\hat{z}(y) \leq z$

$$\begin{aligned} \mathcal{Q}(y, z) =& (1 + z) \left\{ D_1 \left(\frac{y}{1+z} \right)^{n_1} + \frac{2}{\theta^2(n_1 - n_2)} \left[\left(\frac{y}{1+z} \right)^{n_2} \int_0^{\frac{y}{1+z}} \xi^{-n_2-1} \tilde{u}(\xi) d\xi \right.\right. \\ & \left.\left. + \left(\frac{y}{1+z} \right)^{n_1} \int_{\frac{y}{1+z}}^{\infty} \xi^{-n_1-1} \tilde{u}(\xi) d\xi \right] + \frac{\epsilon}{r} \left(\frac{y}{1+z} \right) \right\}, \end{aligned} \tag{37}$$

(ii) for $0 \leq z < \hat{z}(y)$

$$\mathcal{Q}(y, z) = \left(\tilde{u}(\bar{\nu}) - \beta P + \bar{\nu}\epsilon \right) \frac{y}{r\bar{\nu}} + P(1 + z) = \mathcal{Q}\left(y, \hat{z}(y)\right) + P\left(z - \hat{z}(y)\right). \tag{38}$$

Corollary 1. *$\mathcal{Q}(y,z)$ given in (37) and (38) is twice continuously differentiable, and satisfies HJB equation (27).*

For a given $y>0$, we define a stopping time $\hat{\tau}(y)$ as the first hitting time of $\mathcal{Y}_t^y$ to reach the boundary $\bar{\nu}$:

$$\hat{\tau}(y) = \inf\{t \geq 0 \mid \mathcal{Y}_t^y \geq \bar{\nu}\}. \tag{39}$$

In the following proposition, we can characterize that $\hat{\tau}(y)$ is the solution to a certain optimal stopping problem, which is useful for the verification theorem for the dual value function:

Proposition 4. *$\hat{\tau}(y)$ is the solution to the following optimal stopping problem.*

$$\varphi(y) \equiv \inf_{\tau\in\mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right] = \inf_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_\tau^\infty e^{-\beta t}\left(u(I(\mathcal{Y}_t^y)) - \beta P\right)dt\right] \tag{40}$$

$$= \mathbb{E}\left[e^{-\beta\hat{\tau}(y)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\hat{\tau}(y)}^y)\right], \tag{41}$$

where $\mathcal{S}$ is the set of all $\mathcal{F}$-stopping times, taking values in $[0,\infty]$.

Moreover, $\varphi(y)$ is given in the following explicit-form.

$$\varphi(y) = \begin{cases} \Gamma_{(u\circ I-\beta P)}(y) & \text{for } y \geq \bar{\nu}, \\ \Gamma_{(u\circ I-\beta P)}(\bar{\nu})\left(\dfrac{y}{\bar{\nu}}\right)^{n_1} & \text{for } 0 < y < \bar{\nu}. \end{cases} \tag{42}$$

Proof. See Appendix F. □

For a given $y>0$ and $z\geq 0$, let us consider the barrier strategy $\widehat{\mathcal{Z}}^z(y)$ defined by

$$\widehat{\mathcal{Z}}_t^z(y) := \max\left\{1+z, \sup_{0\leq s\leq t}\frac{\mathcal{Y}_t^y}{\bar{\nu}}\right\} - 1 \quad \text{for} \quad t\geq 0, \tag{43}$$

with $\widehat{\mathcal{Z}}_{0-}^z(y) = z$. Here, $\bar{\nu}$ denotes the free boundary given in Lemma 3.

Proposition 5. *Let $y>0$ and $z\geq 0$ be given.*

(a) $(\widehat{\mathcal{Z}}_t^z(y))_{t=0}^\infty \in \Pi(z)$.

(b) For given $y>0$ and $z\geq 0$, the following relationship holds:

$$\mathcal{Q}(y,z) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right) + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right]$$

$$= \int_z^\infty \varphi\left(\frac{y}{1+\nu}\right)d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y$$

where $\varphi(y)$ is defined in Proposition 4.

Proof. See Appendix G. □

Theorem 1.

(a) For given $y>0$ and $z\geq 0$, $\mathfrak{J}(y,z) = \mathcal{Q}(y,z)$.

(b) The dual value function $J(y)$ has the following explicit-form:

$$J(y) = \begin{cases} D_1 y^{n_1} + \dfrac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\displaystyle\int_0^y \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + y^{n_1}\int_y^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right] & \text{for } 0<y<\bar{\nu}, \\ (\tilde{u}(\bar{\nu})-\beta P+\bar{\nu}\epsilon)\dfrac{y}{r\bar{\nu}}+P & \text{for } y\geq\bar{\nu}, \end{cases} \tag{44}$$

where D_1 and $\bar{\nu}$ are given in (35)*.*

Proof. See Appendix H. □

Note that

$$\mathcal{Q}(y,z) = (1+z)\mathcal{M}\left(\frac{y}{1+z}\right).$$

It follow that

$$\mathcal{Q}_z(y,z) = \mathcal{M}\left(\frac{y}{1+z}\right) - \frac{y}{1+z}\mathcal{M}'\left(\frac{y}{1+z}\right).$$

Since $J(y) = \mathfrak{J}(y,0) = \mathcal{Q}(y,0) = \mathcal{M}(y)$, we have

$$\mathfrak{J}_y(y,0) = J'(y) \ \text{ and } \ \mathfrak{J}_z(y,0) = \mathcal{M}(y) - y\mathcal{M}'(y) = J(y) - yJ'(y). \tag{45}$$

The following lemma provides the meaning of the relation in (45).

Lemma 4. *For a given $y>0$, the candidate for optimal consumption $\hat{c}$ in* (17) *satisfies*

$$\mathbb{E}\left[\int_0^\infty \mathcal{H}_t\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)-\epsilon\right)dt\right] = -J'(y) \tag{46}$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}u\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)\right)dt\right] = J(y) - yJ'(y). \tag{47}$$

Proof. See Appendix I. □

Let us define $\mathcal{W}(y)$ as

$$\mathcal{W}(y) := \mathbb{E}\left[\int_0^\infty e^{-\beta t}u\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)\right)dt\right]. \tag{48}$$

From Proposition 5 (b), Theorem 1, and Lemma 4, we have

$$\begin{aligned}\mathcal{W}(y) = \mathfrak{J}_z(y,0) &= -\varphi(y) + \Gamma_{u\circ I}(y) \\ &= -\varphi(y) + \Gamma_{(u\circ I-\beta P)}(y) + P,\end{aligned}$$

where we used the fact that:

$$\Gamma_{(u\circ I-\beta P)}(y) = \Gamma_{u\circ I}(y) + P.$$

Recall that in Proposition 4,

$$\varphi(y) = \inf_{\tau\in\mathcal{S}}\mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right] \leq \Gamma_{(u\circ I-\beta P)}(y),$$

where the equality holds if $y \geq \bar{\nu}$. Overall, we deduce that

$$\mathcal{W}(y) = -\varphi(y) + \Gamma_{(u \circ I - \beta P)}(y) + P \geq P. \tag{49}$$

By the Markov property and the definition of $\widehat{\mathcal{Z}}_t^0(y)$ in (43), we can easily derive that for any $t \geq 0$

$$\mathcal{W}(\mathcal{Y}_t^y) = \mathbb{E}_t \left[\int_t^\infty e^{-\beta(s-t)} u \left(\hat{c} \left(\frac{\mathcal{Y}_s^y}{1 + \widehat{\mathcal{Z}}_s^0(y)} \right) \right) ds \right] \geq P, \tag{50}$$

where the equality in the inequality above holds only when $d\widehat{\mathcal{Z}}_s^0(y) \neq 0$. Thus, we can directly obtain the following corollary.

Corollary 2. *For a given $y > 0$, we have*

$$\int_0^t e^{-\beta s} \left(\mathcal{W}(\mathcal{Y}_s^y) - P \right) d\widehat{\mathcal{Z}}_s^0(y) = 0 \;\; \forall\, t \geq 0. \tag{51}$$

5 Duality Theorem and Optimal Strategies

Let us define $\bar{x}$ as

$$\bar{x} := -J(\bar{\nu}) = -\left(\frac{\tilde{u}(\bar{\nu}) - \beta P}{\bar{\nu}} + \epsilon \right) \frac{1}{r}. \tag{52}$$

Since $\tilde{u}(y) = u(I(y)) - yI(y)$ and $u(I(\bar{\nu})) - \beta P < 0$ (see Lemma 3), we deduce that:

$$\bar{x} = -\left(\frac{u(I(\bar{\nu})) - \beta P - \bar{\nu} I(\bar{\nu})}{\bar{\nu}} + \epsilon \right) \frac{1}{r} > -\frac{\epsilon}{r}.$$

We now state our main theorem.

Theorem 2. *Let $x \geq \bar{x}$ be given.*

(a) The value function $V(x)$ in Problem 1 and the dual value function $J(y)$ in Problem 2 satisfy the following duality relationship:

$$V(x) = \inf_{y>0} \left[J(y) + yx \right]. \tag{53}$$

There exist a unique solution $y^* \in (0, \bar{\nu}]$ to the minimization problem (53) such that

$$x = -J'(y^*). \tag{54}$$

(b) The optimal consumption c_t^* and portfolio π_t^* at time $t \geq 0$ are given by

$$c_t^* = \hat{c} \left(\frac{\mathcal{Y}_t^*}{1 + \mathcal{Z}_t^*} \right) \tag{55}$$

and

$$\pi_t^* = \frac{\theta}{\sigma} \frac{\mathcal{Y}_t^*}{1 + \mathcal{Z}_t^*} J'' \left(\frac{\mathcal{Y}_t^*}{1 + \mathcal{Z}_t^*} \right), \tag{56}$$

where

$$\mathcal{Y}_t^* := \mathcal{Y}_t^{y^*} = y^* e^{\beta t} \mathcal{H}_t \;\; \text{and} \;\; \mathcal{Z}_t^* = \max \left\{ 0, \sup_{0 \leq s \leq t} \frac{\mathcal{Y}_t^*}{\bar{\nu}} - 1 \right\}.$$

(c) The agent's wealth $X_t^{c^*,\pi^*}$ *corresponding to* (c^*,π^*) *at time* $t \geq 0$ *is given by*

$$X_t^* = -J'\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right) \tag{57}$$

and satisfies

$$X_t^* \geq \bar{x} \ \ \textit{for all} \ \ t \geq 0. \tag{58}$$

Proof. See Appendix J. □

Remark 6. *As shown in part (c) of Theorem 2, the agent's wealth should always be greater than or equal to* $\bar{x}$*. If* $\bar{x} \leq 0$*, then* $-\bar{x}$ *can be interpreted as the endogenous credit limit (or borrowing limit) generated by the welfare constraint. Note that, if we impose a credit limit (or borrowing limit) on the agent's wealth as* $X_t \geq \bar{x}$*, it is natural to assume that* $\bar{x} \leq 0$*. In contrast, if we impose a welfare constraint as in our model,* $\bar{x}$ *can also be positive, depending on the level of* P*.*

From Theorem 1, we deduce that for $y < \bar{\nu}$

$$\begin{aligned}
J'(y) =& n_1 D_1 y^{n_1-1} + \frac{2}{\theta^2(n_1-n_2)}\left[n_2 y^{n_2-1}\int_0^y \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + n_1 y^{n_1-1}\int_y^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right] \\
=& n_1 D_1 y^{n_1-1} + \frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2-1}\int_0^y \xi^{-n_2-1}(\epsilon - I(\xi))d\xi + y^{n_1-1}\int_y^\infty \xi^{-n_1-1}(\epsilon - I(\xi))d\xi\right],
\end{aligned}$$

where we used integration by parts in the second equality and

$$\liminf_{y\downarrow 0} y^{-n_2}|\tilde{u}(y)+y\epsilon| = \liminf_{y\uparrow\infty} y^{-n_1}|\tilde{u}(y)+y\epsilon| = 0. \quad \text{(see Proposition 1)}.$$

Since

$$\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*} \leq \bar{\nu} \ \ \text{for all} \ \ t \geq 0,$$

we can directly obtain the following corollary.

Corollary 3. *For* $t \geq 0$*, the agent's wealth* $X_t^{c^*,\pi^*}$ *corresponding to* (c^*,π^*)*, and the optimal portfolio* π_t^* *have the following explicit-forms:*

$$X_t^{c^*,\pi^*} = -n_1 D_1 (\Upsilon_t^*)^{n_1-1} + \frac{2}{\theta^2(n_1-n_2)}\left[(\Upsilon_t^*)^{n_2-1}\int_0^{\Upsilon_t^*} \xi^{-n_2-1}(I(\xi)-\epsilon)d\xi + (\Upsilon_t^*)^{n_1-1}\int_{\Upsilon_t^*}^\infty \xi^{-n_1-1}(I(\xi)-\epsilon)d\xi\right]$$

and

$$\begin{aligned}
\pi_t^* =& \frac{\theta}{\sigma}\Bigg\{n_1(n_1-1)D_1(\Upsilon_t^*)^{n_1-1} - \frac{2}{\theta^2(n_1-n_2)}\Bigg[(n_2-1)(\Upsilon_t^*)^{n_2-1}\int_0^{\Upsilon_t^*}\xi^{-n_2-1}(I(\xi)-\epsilon)d\xi \\
&+ (n_1-1)(\Upsilon_t^*)^{n_1-1}\int_{\Upsilon_t^*}^\infty \xi^{-n_1-1}(I(\xi)-\epsilon)d\xi\Bigg]\Bigg\},
\end{aligned}$$

where

$$\Upsilon_t^* = \frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}.$$

In particular, $X_t^{c^*,\pi^*} = \bar{x}$ *and* $\pi_t^* = 0$ *when* Υ_t^* *reaches the boundary* $\bar{\nu}$*.*

Proposition 6 (Comparative Statics)**.** *Let* $x \geq \bar{x}$ *be given.*

(a) *The minimum level of wealth $\bar{x}$ increases in P.*

(b) *The consumption c_t^* decreases in P.*

(c) *The optimal portfolio π_t^* decreases (resp. increases) in P if $\theta > 0$ (resp. $\theta < 0$).*

Proof. See Appendix K. □

6 Examples and Implications

In this section, we consider the following hyperbolic absolute risk aversion (HARA) class utility function:

$$u(c) = \frac{(c+a)^{1-\gamma}}{1-\gamma} \quad (0 < \gamma \neq 1,\ a \geq 0). \tag{59}$$

It is obvious that the above HARA utility function satisfies Assumption 1, whereas the following condition is required to satisfy Assumption 2:

$$K := -q(1 - \frac{1}{\gamma}) > 0,$$

where $q(\cdot)$ is the quadratic function defined in (6). Here, K is the Merton constant and this condition is equivalent to $n_2 < 1 - \frac{1}{\gamma} < n_1$.

In this example, a can be interpreted as government subsidy or basic support, as in Bae et al. (2020) and Park et al. (2021), which cannot be stored and should be consumed immediately. If $a > 0$, c can be interpreted as additional consumption by the agent on top of a. If $a = 0$, this example becomes constant relative risk aversion (CRRA) utility with the coefficient of relative risk aversion $0 < \gamma \neq 1$.[3]

We can easily check that $I(y)$ and $\tilde{u}(y)$ have different formulas depending on the relationship between y and $a^{-\gamma}$ as follows:

$$I(y) = \begin{cases} y^{-\frac{1}{\gamma}} - a & \text{for } 0 < y < a^{-\gamma}, \\ 0 & \text{for } y \geq a^{-\gamma}, \end{cases} \quad \text{and} \quad \tilde{u}(y) = \begin{cases} \frac{\gamma}{1-\gamma} y^{1-\frac{1}{\gamma}} + ay & \text{for } 0 < y < a^{-\gamma}, \\ \frac{1}{1-\gamma} a^{1-\gamma} & \text{for } y \geq a^{-\gamma}. \end{cases} \tag{60}$$

Because $J(y)$ has a different formula depending on the relationship between y and $\bar{\nu}$, this implies that the solution will differ depending on the relationship between $\bar{\nu}$ and $a^{-\gamma}$, which is equivalent to the relationship between P and $\hat{P}(a) := \frac{-n_2}{\beta(1-n_2-\frac{1}{\gamma})} \frac{a^{1-\gamma}}{1-\gamma}$, an increasing function of a. In other words, depending on the relationship between the minimum welfare P and basic support a, we obtain different solutions as follows:

- If $P > \hat{P}(a)$, by substituting (60) into the solution in Theorem 1, we can obtain

$$\bar{\nu} = \left\{ \frac{(1 - n_2 - \frac{1}{\gamma})\beta(1-\gamma)P}{-n_2} \right\}^{\frac{\gamma}{\gamma-1}} < a^{-\gamma}, \tag{61}$$

  and consequently,

$$J(y) = \begin{cases} -\frac{(1-\gamma)P\bar{\nu}^{-n_1}}{(n_1-1)(n_1-1+\frac{1}{\gamma})\gamma} y^{n_1} + \frac{\gamma}{(1-\gamma)K} y^{1-\frac{1}{\gamma}} + \frac{\epsilon + a}{r} y & \text{for } 0 < y < \bar{\nu}, \\ \left(\frac{\gamma}{1-\gamma} \bar{\nu}^{1-\frac{1}{\gamma}} - \beta P \right) \frac{y}{r\bar{\nu}} + \frac{\epsilon + a}{r} y + P & \text{for } y \geq \bar{\nu}. \end{cases} \tag{62}$$

[3] We can also consider log utility such as $u(c) = \ln(c + a)$ $(a \geq 0)$ that incorporates the basic support a. As it does not provide new properties or implications, we focus on the example in (59) to avoid redundancy.

Note that $\bar{\nu}$ for this case is irrelevant to a, and the role of basic support a in $J(y)$ is simply an additional income added to labor income ϵ. By substituting (60) into the solution in Theorem 2, we have

$$V(x) = \begin{cases} \frac{(1-\gamma)P\bar{\nu}^{-n_1}}{(n_1-1+\frac{1}{\gamma})\gamma} y^{*n_1} + \frac{1}{(1-\gamma)K} y^{*1-\frac{1}{\gamma}} & \text{for } x > \bar{x}, \\ P & \text{for } x = \bar{x}, \end{cases} \tag{63}$$

$$c^* = \begin{cases} y^{*-\frac{1}{\gamma}} - a & \text{for } x > \bar{x}, \\ \bar{\nu}^{-\frac{1}{\gamma}} - a & \text{for } x = \bar{x}, \end{cases} \tag{64}$$

$$\pi^* = \begin{cases} \frac{\theta}{\sigma\gamma}\left(x + \frac{\epsilon + a}{r}\right) - \frac{\theta}{\sigma}\frac{n_1(1-\gamma)P\bar{\nu}^{-n_1}}{(n_1-1)\gamma} y^{*n_1-1} & \text{for } x > \bar{x}, \\ 0 & \text{for } x = \bar{x}, \end{cases} \tag{65}$$

where $y^* \in (0, \bar{\nu})$ is the unique solution to the following equation for given $x > \bar{x}$:

$$x = \frac{(1-\gamma)P\bar{\nu}^{-n_1}}{(n_1-1)(n_1-1+\frac{1}{\gamma})\gamma} n_1 y^{*n_1-1} + \frac{1}{K} y^{*-\frac{1}{\gamma}} - \frac{\epsilon + a}{r},$$

and

$$\bar{x} = -\left(\frac{\gamma}{1-\gamma}\bar{\nu}^{1-\frac{1}{\gamma}} - \beta P\right)\frac{1}{r\bar{\nu}} - \frac{\epsilon + a}{r}.$$

As we can see in the equation (65) and the right panel of Figure 1, the optimal investment becomes zero at the minimum wealth level $\bar{x}$, at which the welfare constrain is binding, to prevent the wealth level from being less than the minimum wealth level $\bar{x}$. Note that the agent may keep the additional consumption c stay at 0 for a while in the presence of the basic support $a > 0$. However, when $P > \hat{P}(a)$, the minimum welfare P imposed by the welfare constraint is high enough relative to the basic support a so that the minimum level of optimal additional consumption c^* (attained when the welfare constraint is binding at $x = \bar{x}$) is positive ($\bar{\nu}^{-\frac{1}{\gamma}} - a > 0$) because $\bar{\nu} < a^{-\gamma}$ when $P > \hat{P}(a)$. This can be observed in the left panel In Figure 1.

In contrast, when $P \leq \hat{P}(a)$, we have a quite different result as follows.[4]

- If $P \leq \hat{P}(a)$, we have a different formula for $\bar{\nu}$ that depends on the basic support a as follows:

$$\bar{\nu} = \left\{\gamma(1 - n_2 - \frac{1}{\gamma})\left(\beta P - \frac{1}{1-\gamma}a^{1-\gamma}\right)a^{\gamma(1-n_2-\frac{1}{\gamma})}\right\}^{\frac{1}{n_2}} \geq a^{-\gamma} \tag{66}$$

and we have

$$J(y) = \begin{cases} \left(D_1 - \frac{2a^{\gamma(n_1-1+\frac{1}{\gamma})}}{\theta^2(n_1-n_2)n_1(n_1-1)(n_1-1+\frac{1}{\gamma})\gamma}\right) y^{n_1} + \frac{\gamma}{(1-\gamma)K} y^{1-\frac{1}{\gamma}} + \frac{\epsilon + a}{r} y & \text{for } 0 < y < a^{-\gamma}, \\ D_1 y^{n_1} + \frac{2a^{-\gamma(1-n_2-\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(-n_2)(1-n_2)(1-n_2-\frac{1}{\gamma})\gamma} y^{n_2} + \frac{1}{(1-\gamma)\beta} a^{1-\gamma} + \frac{\epsilon}{r} y & \text{for } a^{-\gamma} \leq y < \bar{\nu}, \\ \left(\frac{\gamma}{1-\gamma} a^{1-\frac{1}{\gamma}} - \beta P\right)\frac{y}{r\bar{\nu}} + \frac{\epsilon}{r} y + P & \text{for } y \geq \bar{\nu}, \end{cases} \tag{67}$$

[4] Note that we only have the case $P > \hat{P}(a)$ when there is no basic support because $P > \hat{P}(0+) := \lim_{a\to 0+} \hat{P}(a)$ regardless of γ.

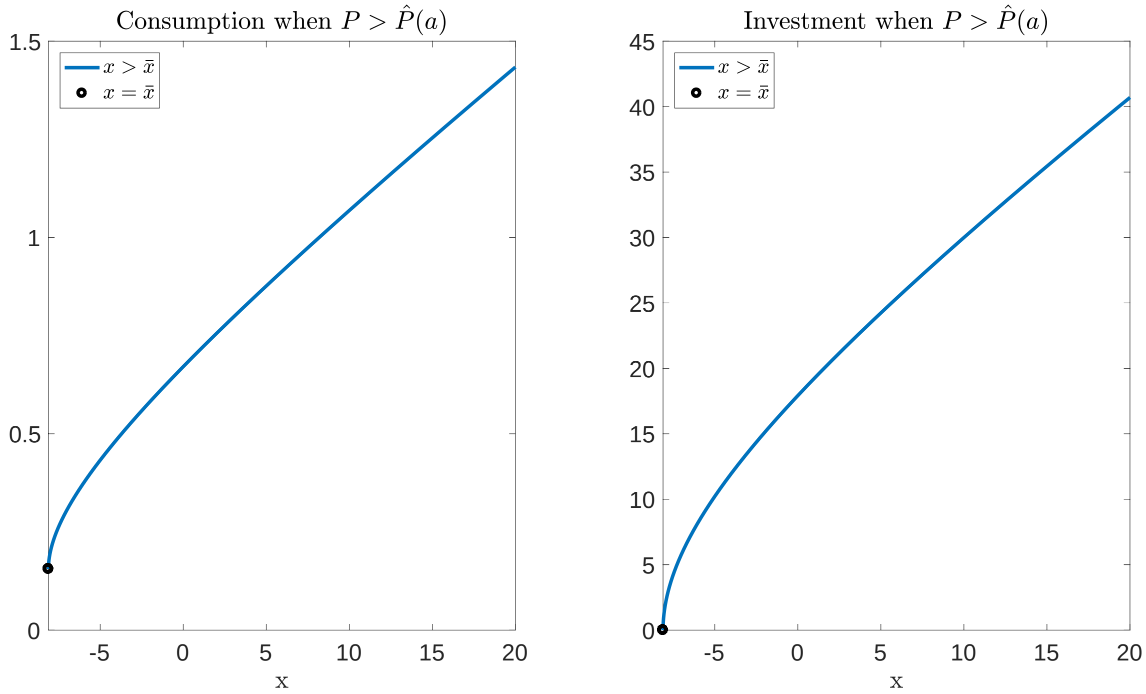

Figure 1: Consumption and Investment when $P > \hat{P}(a)$.
Baseline parameters: $r = 0.03,\ \beta = 0.04,\ \mu = 0.07,\ \sigma = 0.3,\ \gamma = 0.5,\ P = 55,\ \epsilon = 1.$
By setting $a = 0.2$, we have $\hat{P}(a) = 41.2346$ that is less than P.

where

$$D_1 = -\frac{2}{\theta^2(n_1 - n_2)n_1(n_1 - 1)}\left(\beta P - \frac{1}{1-\gamma}a^{1-\gamma}\right)\bar{\nu}^{-n_1} < 0. \tag{68}$$

In contrast to the previous case with $P > \hat{P}(a)$, we have three regions of y that gives different form of $J(y)$: $(0, a^{-\gamma})$, $[a^{-\gamma}, \bar{\nu})$, $[\bar{\nu}, \infty)$ when $P \leq \hat{P}(a)$. Let $\hat{x}$ and $\bar{x}$ be the values of primal variable x that correspond to $y = a^{-\gamma}$ and $y = \bar{\nu}$, respectively, then we can verify that

$$\hat{x} = -D_1 n_1 a^{-\gamma(n_1-1)} + \frac{2a^{-\gamma(1-n_2-\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(1-n_2)(1-n_2-\frac{1}{\gamma})\gamma}a^{\gamma(1-n_2)} - \frac{\epsilon}{r}, \tag{69}$$

$$\bar{x} = -\left(\frac{1}{1-\gamma}a^{1-\gamma} - \beta P\right)\frac{1}{r\bar{\nu}} - \frac{\epsilon}{r}. \tag{70}$$

Moreover, it follows from Theorem 2 that

$$V(x)=\begin{cases}\left(D_1-\dfrac{2a^{\gamma(n_1-1+\frac{1}{\gamma})}}{\theta^2(n_1-n_2)n_1(n_1-1)(n_1-1+\frac{1}{\gamma})\gamma}\right)(1-n_1)y^{n_1}+\dfrac{1}{(1-\gamma)K}y^{1-\frac{1}{\gamma}} & \text{for } x>\hat{x},\\ D_1(1-n_1)y^{n_1}+\dfrac{2a^{-\gamma(1-n_2-\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(-n_2)(1-n_2-\frac{1}{\gamma})\gamma}y^{n_2}+\dfrac{1}{(1-\gamma)\beta}a^{1-\gamma} & \text{for } \bar{x}<x\le\hat{x},\\ P & \text{for } x=\bar{x},\end{cases} \tag{71}$$

$$c^*=\begin{cases}y^{*-\frac{1}{\gamma}}-a & \text{for } x>\hat{x},\\ 0 & \text{for } \bar{x}<x\le\hat{x},\\ 0 & \text{for } x=\bar{x},\end{cases} \tag{72}$$

$$\pi^*=\begin{cases}\dfrac{\theta}{\sigma\gamma}\left(x+\dfrac{\epsilon+a}{r}\right)+\dfrac{\theta}{\sigma}\left[D_1n_1(n_1-1+\dfrac{1}{\gamma})-\dfrac{2a^{\gamma(n_1-1+\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(n_1-1)\gamma}\right]y^{n_1-1} & \text{for } x>\hat{x},\\ \dfrac{\theta}{\sigma}\left[D_1n_1(n_1-1)y^{n_1-1}+\dfrac{2a^{-\gamma(1-n_2-\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(1-n_2-\frac{1}{\gamma})\gamma}y^{n_2-1}\right] & \text{for } \bar{x}<x\le\hat{x},\\ 0 & \text{for } x=\bar{x},\end{cases} \tag{73}$$

where $y^*\in(0,\bar{\nu})$ is the unique solution to the following equation for given $x>\bar{x}$:

$$x=\begin{cases}\left(-D_1n_1+\dfrac{2a^{\gamma(n_1-1+\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(n_1-1)(n_1-1+\frac{1}{\gamma})\gamma}\right)y^{n_1-1}+\dfrac{1}{K}y^{-\frac{1}{\gamma}}-\dfrac{\epsilon+a}{r} & \text{for } x>\hat{x},\\ -D_1n_1y^{n_1-1}+\dfrac{2a^{-\gamma(1-n_2-\frac{1}{\gamma})}}{\theta^2(n_1-n_2)(1-n_2)(1-n_2-\frac{1}{\gamma})\gamma}y^{n_2-1}-\dfrac{\epsilon}{r} & \text{for } \bar{x}<x\le\hat{x}.\end{cases} \tag{74}$$

Recall that c, the additional consumption on top of the basic support a can stay at 0 for a while if there is a basic support $a>0$. When $P\le\hat{P}(a)$, the minimum welfare P required by the welfare constraint P is not large enough. Therefore, when the wealth level x goes below $\hat{x}$ but still above $\bar{x}$ ($x\in(\bar{x},\hat{x}]$), it is optimal for the agent to keep the additional consumption c as zero, while the optimal investment is not zero yet. As time goes, and the wealth level goes up above $\hat{x}$, the agent will have positive additional consumption again. On the other hand, if the wealth level drops and reaches $\bar{x}$, then it is optimal to make both the additional consumption c and the investment π be zero. This result can be observed in Figure 2.

7 Concluding Remarks

We have studied the consumption and investment problem of an agent under the welfare constraints with a general class of utility functions. The welfare constraint requires that the expected utility of the agent should not be less than a fixed minimum welfare level, which is an optimization problem with an infinite number

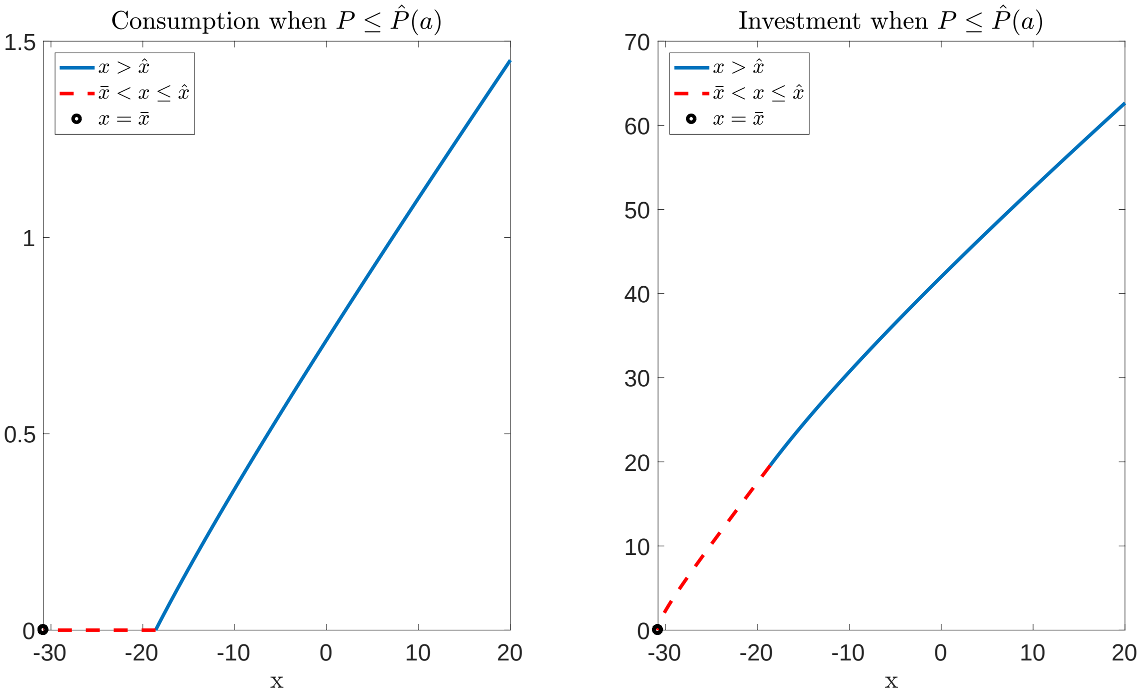

Figure 2: Consumption and Investment when $P \leq \hat{P}(a)$.
Baseline parameters: $r = 0.03,\ \beta = 0.04,\ \mu = 0.07,\ \sigma = 0.3,\ \gamma = 0.5,\ P = 55,\ \epsilon = 1$.
By setting $a = 1$, we have $\hat{P}(a) = 92.2033$ that is greater than P.

of constraints. We have derived the corresponding dual problem, which becomes a two-dimensional singular control problem, and its associated HJB equation with a gradient constraint. The explicit solutions for the general utility functions are obtained, and the optimal consumption and investment strategies are provided.

The welfare constraint endogenously generates a minimum wealth level at which the optimal investment becomes zero to satisfy the welfare constraint without violating it. Moreover, we show that the minimum wealth level increases, while the optimal consumption and investment decrease as the minimum welfare level increases. These results consistently hold for the general class of utility functions.

Using a HARA class utility function, we introduce an example that includes a government subsidy or basic support that can only be consumed as soon as the agent receives it and cannot be stored. In this case, c_t in our model can be interpreted as additional consumption in excess of basic support. Depending on the relationship between the minimum welfare level and basic support, solutions have different properties. In particular, when basic support is large enough relative to the minimum welfare level, there exists an range of wealth levels in which the optimal additional consumption is zero, and this phenomenon does not appear when basic support is not large enough.

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Appendix

A Proof of Lemma 1

Note that
$$\frac{d}{dz}\left((1+z)\tilde{u}\left(\frac{y}{1+z}\right)\right)=\tilde{u}\left(\frac{y}{1+z}\right)+\frac{y}{1+z}I\left(\frac{y}{1+z}\right)=u\left(I\left(\frac{y}{1+z}\right)\right).$$
It follows that
$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right|dt \mid \mathcal{Z}_{0-}=z\right]\\
=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|\int_z^{\mathcal{Z}_t} u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)d\nu+(1+z)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+z}\right)\right|dt\right]\\
\le&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|\int_z^{\mathcal{Z}_t} u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)d\nu\right|dt\right]+\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|(1+z)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+z}\right)\right|dt\right]\\
\le&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left|u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)\right|d\nu dt\right]+(1+z)\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+z}\right)\right|dt\right]\\
=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t}\left|u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)\right|d\nu dt\right]+(1+z)\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|\tilde{u}\left(\mathcal{Y}_t^{\frac{y}{(1+z)}}\right)\right|dt\right]\\
=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t}\left|u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)\right|d\nu dt\right]+(1+z)\Gamma_{|\tilde{u}|}(y)\\
<&\infty
\end{aligned}$$

Therefore, for any $\mathcal{Z}\in\Pi(0)$,
$$|J(y)|\le\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\left|(1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right|+\mathcal{Y}_t^y\epsilon\right)dt+|P|\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right]<\infty.$$

B Proof of Proposition 2

For given $y > 0$, any admissible strategy $(c, \pi) \in \mathcal{A}(x)$, and shadow price process $\mathcal{Z} \in \Pi(0)$, we have

$$(1+\mathcal{Z}_t)u(c_t) - \mathcal{Y}_t^y c_t \le (1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right). \tag{75}$$

This implies that

$$(1+\mathcal{Z}_t)u(c_t))^{\pm} \le (1+\mathcal{Z}_t)|(u(c_t))| \le \mathcal{Y}_t^y c_t + (1+\mathcal{Z}_t)\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right|,$$

where $(u(c_t))^+$ and $(u(c_t))^-$ are the positive and negative parts of $u(c_t)$, respectively.

It follows from Lemma 1 that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)|u(c_t)|dt\right] \le& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Y}_t^y c_t dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right|dt\right] \\ \le& y\mathbb{E}\left[\int_0^\infty \mathcal{H}_t c_t dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)\left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)\right|dt\right] \\ <& \infty.\end{aligned}$$

Hence,

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}|u(c_t)|dt\right] < \mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)|u(c_t)|dt\right] < \infty \tag{76}$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)(u(c_t))^{\pm}dt\right] < \infty. \tag{77}$$

Then, integration by parts yields that for a fixed $T > 0$,

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\beta t}(1+\mathcal{Z}_t)(u(c_t))^+dt\right] =& -\mathbb{E}\left[\left(\int_t^T e^{-\beta s}(u(c_s))^+ds\right)(1+\mathcal{Z}_t)\right]\Bigg|_{t=0}^T \\ &+\mathbb{E}\left[\int_0^T e^{-\beta t}\mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}(u(c_s))^+ds\right]d\mathcal{Z}_t\right] \\ =& \mathbb{E}\left[\int_0^T e^{-\beta s}(u(c_s))^+ds\right] + \mathbb{E}\left[\int_0^T e^{-\beta t}\mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}(u(c_s))^+ds\right]d\mathcal{Z}_t\right].\end{aligned}$$

Letting $T \to +\infty$, the monotone convergence theorem implies that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}(u(c_t))^+dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^+ds\right]d\mathcal{Z}_t\right] = \mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)(u(c_t))^+dt\right] < \infty.$$

Thus,

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^+ds\right]d\mathcal{Z}_t\right] < \infty. \tag{78}$$

Note that for any $t \ge 0$

$$P \le \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^+ds\right] - \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^-ds\right].$$

It follows from (78) that

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^- ds\right]d\mathcal{Z}_t\right]+\mathbb{E}\left[\int_0^\infty e^{-\beta t}Pd\mathcal{Z}_t\right]\\ \le&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^+ds\right]d\mathcal{Z}_t\right]<\infty.\end{aligned}$$

Since $\mathbb{E}\left[\int_0^\infty e^{-\beta t}Pd\mathcal{Z}_t\right]<\infty$, we have

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}(u(c_s))^- ds\right]d\mathcal{Z}_t\right]<\infty. \tag{79}$$

From (78) and (79),

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}|u(c_s)|ds\right]d\mathcal{Z}_t\right]<\infty. \tag{80}$$

Thus, we deduce that for a fixed $T>0$

$$\begin{aligned}\mathbb{E}\left[\int_0^T(1+\mathcal{Z}_t)u(c_t)dt\right]=&-\mathbb{E}\left[(1+\mathcal{Z}_t)\int_t^T e^{-\beta s}u(c_s)ds\right]\Big|_{t=0}^T+\mathbb{E}\left[\int_0^T e^{-\beta t}\mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}u(c_s)ds\right]d\mathcal{Z}_t\right]\\ =&\mathbb{E}\left[\int_0^T e^{-\beta t}u(c_t)dt\right]+\mathbb{E}\left[\int_0^T e^{-\beta t}\mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}u(c_s)ds\right]d\mathcal{Z}_t\right].\end{aligned}$$

From (76) and (80), the dominated convergence theorem implies that

$$\mathbb{E}\left[\int_0^\infty(1+\mathcal{Z}_t)u(c_t)dt\right]=\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right]+\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right]d\mathcal{Z}_t\right]. \tag{81}$$

Thus, we derive that

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right]\\ \le&\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right]+y\left(x-\mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t-\epsilon)dt\right]\right)+\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right]-P\right)d\mathcal{Z}_t\right]\\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right]+y\left(x-\mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t-\epsilon)dt\right]\right)+\mathbb{E}\left[\int_0^\infty \mathcal{Z}_t u(c_t)dt\right]-\mathbb{E}\left[\int_0^\infty e^{-\beta t}Pd\mathcal{Z}_t\right]\\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1+\mathcal{Z}_t)\left(u(c_t)-c_t\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}-\epsilon\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)dt-\int_0^\infty e^{-\beta t}Pd\mathcal{Z}_t\right]+yx\\ \le&\mathfrak{L}(y,\mathcal{Z}).\end{aligned}$$

The equalities in the above inequalities hold if and only if for all $t\ge 0$

$$\begin{aligned}c_t=&I\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right),\quad x=\mathbb{E}\left[\int_0^\infty \mathcal{H}_t(c_t-\epsilon)dt\right],\\ 0=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}u(c_s)ds\right]-P\right)d\mathcal{Z}_t\right].\end{aligned}$$

C Proof of Lemma 2

Note that for any $\nu > 0$

$$u(I(\nu)) - \nu I(\nu) = \tilde{u}(\nu) \text{ and } \tilde{u}'(\nu) = -I(\nu),$$

and thus

$$u(I(\xi)) = u(I(\nu)) - \nu I(\nu) + \xi I(\xi) + \int_{\xi}^{\nu} I(\zeta) d\zeta.$$

It follows that

$$\begin{aligned}
&\int_0^{\nu} \xi^{-n_2-1}|u(I(\xi))|d\xi + \int_{\nu}^{\infty} \xi^{-n_1-1}|u(I(\xi))|d\xi \\
\leq& -\frac{1}{n_2}\nu^{-n_2}|u(I(\nu)) - \nu I(\nu)| + \int_0^{\nu} \xi^{-n_2} I(\xi) d\xi + \int_0^{\nu}\int_{\xi}^{\nu} \xi^{-n_2-1} I(\zeta) d\zeta d\xi \\
&+\frac{1}{n_1}\nu^{-n_1}|u(I(\nu)) - \nu I(\nu)| + \int_{\nu}^{\infty} \xi^{-n_1} I(\xi) d\xi + \int_{\nu}^{\infty}\int_{\nu}^{\xi} \xi^{-n_1-1} I(\zeta) d\zeta d\xi \\
=& -\frac{1}{n_2}\nu^{-n_2}|u(I(\nu)) - \nu I(\nu)| + \frac{1}{n_1}\nu^{-n_1}|u(I(\nu)) - \nu I(\nu)| \\
&+ \left(1 + \frac{1}{n_1}\right)\int_{\nu}^{\infty} \xi^{-n_1} I(\xi) d\xi + \left(1 - \frac{1}{n_2}\right)\int_0^{\nu} \xi^{-n_2} I(\xi) d\xi < \infty.
\end{aligned} \tag{82}$$

where we have used Fubini's theorem in last equality. From Assumptions 1 and 2, it is easy to check that

$$\int_0^{\nu} \xi^{-n_2} I(\xi) d\xi + \int_{\nu}^{\infty} \xi^{-n_1} I(\xi) d\xi < \infty.$$

Therefore, it follows from $|\tilde{u}(\xi)| \leq |u(I(\xi))| + I(\xi)$ that

$$\int_0^{\nu} \xi^{-n_2-1}|\tilde{u}(\xi)| d\xi + \int_{\nu}^{\infty} \xi^{-n_1-1}|\tilde{u}(\xi)| d\xi < \infty.$$

D Proof of Lemma 3

Lemma 2 implies that $\Psi(\nu)$ is well-defined for any $\nu > 0$. By Assumption 3, there exists a unique $\hat{\nu} > 0$ such that

$$\begin{aligned}
&u(I(\hat{\nu})) - \beta P = 0, \\
&u(I(\nu)) - \beta P > 0, \quad \text{for } \nu < \hat{\nu}, \\
&u(I(\nu)) - \beta P < 0, \quad \text{for } \nu > \hat{\nu}.
\end{aligned}$$

Then, it is easy to confirm that $\Psi(\nu)$ increases in $\nu \in (0, \hat{\nu})$ and decreases in $(\hat{\nu}, \infty)$. Moreover, we have that

$$\Psi(\nu) > 0 \quad \text{for} \quad \nu \in (0, \hat{\nu}].$$

For sufficiently large $M > 0$, there exists a constant $\delta > 0$ such that

$$u(I(\nu)) - \beta P < -\delta \quad \text{for} \quad \nu \geq M.$$

Since

$$\int_M^{\infty} \xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right) d\xi \leq -\delta \int_M^{\infty} \xi^{-n_2-1} d\xi = -\delta\left[-\frac{\xi^{-n_2}}{n_2}\right]_{\xi=M}^{\infty} = -\infty,$$

we have

$$
\begin{aligned}
&\lim_{\nu\to+\infty}\int_0^{\nu}\xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi\\
=&\int_0^{M}\xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi+\lim_{\nu\to+\infty}\int_M^{\nu}\xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi=-\infty.
\end{aligned}
$$

Therefore, we can conclude that there exist a unique $\bar{\nu}>\hat{\nu}$ such that $\Psi(\bar{\nu})=0$. Clearly, $u(I(\bar{\nu}))-\beta P<0$.

E Proof of Proposition 3

By the construction of $\mathcal{M}(\nu)$ in (34), we can easily show that $\mathcal{M}(\nu)$ is twice continuously differentiable. Since $\mathcal{M}(\nu)=(\tilde{u}(\bar{\nu})-\beta P+\bar{\nu}\epsilon)\dfrac{\nu}{r\bar{\nu}}+P$ for $\nu\ge\bar{\nu}$, it is clear that $\mathcal{M}(\nu)-\nu\mathcal{M}'(\nu)=P$ for $\nu\ge\bar{\nu}$. Moreover, for $\nu\ge\bar{\nu}$,

$$
\begin{aligned}
\mathcal{L}\mathcal{M}(\nu)+\tilde{u}(\nu)+\nu\epsilon=&-\left(\tilde{u}(\bar{\nu})-\beta P+\bar{\nu}\epsilon\right)\frac{\nu}{\bar{\nu}}-\beta P+\tilde{u}(\nu)+\epsilon\nu\\
=&-\left(\tilde{u}(\bar{\nu})-\beta P\right)\frac{\nu}{\bar{\nu}}-\beta P+\tilde{u}(\nu).
\end{aligned}
$$

Let us temporarily denote $l(\nu)$ as

$$
l(\nu):=-\left(\tilde{u}(\bar{\nu})-\beta P\right)\frac{\nu}{\bar{\nu}}-\beta P+\tilde{u}(\nu).
$$

Then, we deduce that

$$
l'(\nu)=-\left(\tilde{u}(\bar{\nu})-\beta P\right)\frac{1}{\bar{\nu}}-I(\nu)=-(u(I(\bar{\nu}))-\beta P)\frac{1}{\bar{\nu}}+(I(\bar{\nu})-I(\nu))>0\quad\text{for}\ \ \nu\ge\bar{\nu},
$$

where we have used the fact that $u(I(\bar{\nu}))-\beta P<0$ (Lemma 3). That is, $l(\nu)$ is increasing in $\nu\ge\bar{\nu}$ and thus

$$
\mathcal{L}\mathcal{M}(\nu)+\tilde{u}(\nu)+\nu\epsilon=l(\nu)\ge l(\bar{\nu})=0\quad\text{for}\ \ \nu\ge\bar{\nu}.
$$

Since $\mathcal{M}(\nu)$ in (34) is the solution of the FBP (32), it is clear that

$$
\mathcal{L}\mathcal{M}(\nu)+\tilde{u}(\nu)+\epsilon\nu=0\quad\text{for}\ \ 0<\nu<\bar{\nu}.
$$

For $0<\nu<\bar{\nu}$, it follows from (33), (34), and (35) that

$$
\begin{aligned}
&\mathcal{M}(\nu)-\nu\mathcal{M}'(\nu)\\
=&(1-n_1)D_1\nu^{n_1}+\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^{\nu}\xi^{-n_2-1}u(I(\xi))d\xi+\nu^{n_1}\int_{\nu}^{\infty}\xi^{-n_1-1}u(I(\xi))d\xi\right]\\
=&-\frac{2}{\theta^2(n_1-n_2)}\int_{\bar{\nu}}^{\infty}\xi^{-n_1-1}u(I(\xi))d\xi+\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^{\nu}\xi^{-n_2-1}u(I(\xi))d\xi+\nu^{n_1}\int_{\nu}^{\infty}\xi^{-n_1-1}u(I(\xi))d\xi\right]\\
=&\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^{\nu}\xi^{-n_2-1}u(I(\xi))d\xi+\nu^{n_1}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}u(I(\xi))d\xi\right].
\end{aligned}\tag{83}
$$

Note that

$$
P=\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^{\nu}\xi^{-n_2-1}\beta Pd\xi+\nu^{n_1}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}\beta Pd\xi\right].\tag{84}
$$

From (83) and (84), we have that for $0 < \nu < \bar{\nu}$.

$$
\begin{aligned}
&\mathcal{M}(\nu) - \nu\mathcal{M}'(\nu) - P \\
=&\frac{2}{\theta^2(n_1 - n_2)}\left[\nu^{n_2}\int_0^{\nu}\xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right)d\xi + \nu^{n_1}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}\left(u(I(\xi)) - \beta P\right)d\xi\right].
\end{aligned}
\tag{85}
$$

Let us temporarily denote $g(\nu)$ as

$$
\begin{aligned}
g(\nu) :=&\frac{1}{(n_1 - n_2)}\left[n_2\int_0^{\nu}\xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right)d\xi + n_1\nu^{n_1-n_2}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}\left(u(I(\xi)) - \beta P\right)d\xi\right] \\
=&\frac{\theta^2}{2}\nu^{-n_2}(\mathcal{M}(\nu) - \nu\mathcal{M}'(\nu) - P)'.
\end{aligned}
\tag{86}
$$

Then, we have

$$
\begin{aligned}
g'(\nu) =& -\nu^{-n_2-1}(u(I(\nu)) - \beta P) + n_1\nu^{n_1-n_2}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi \\
=& -\bar{\nu}^{-n_2-1}(u(I(\bar{\nu})) - \beta P) + \nu^{n_1-n_2}\int_{\nu}^{\bar{\nu}}\xi^{-n_1}d(u(I(\xi)) - \beta P),
\end{aligned}
\tag{87}
$$

where the above equality follows from integration by parts. From (87), $g'(\nu)$ is strictly increasing function in $\nu \in (0, \bar{\nu})$.

By Lemma 3,

$$
g'(\bar{\nu}) = -\bar{\nu}^{-n_2-1}(u(I(\bar{\nu})) - \beta P) > 0.
$$

On the other hand, it follows from $u(I(\hat{\nu})) - \beta P = 0$ that

$$
g'(\hat{\nu}) = n_1\hat{\nu}^{n_1-n_2}\int_{\hat{\nu}}^{\bar{\nu}}\xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi < 0.
$$

Thus, there exists a unique $\nu_1 \in (\hat{\nu}, \bar{\nu})$ such that $g'(\nu_1) = 0$. Moreover,

$$
g'(\nu) \begin{cases} < 0 & \text{if } 0 < \nu < \nu_1, \\ = 0 & \text{if } \nu = \nu_1, \\ > 0 & \text{if } \nu > \nu_1. \end{cases}
\tag{88}
$$

This implies that $g(\nu)$ is strictly decreasing in $\nu \in (0, \nu_1)$ and strictly increasing in $\nu \in (\nu_1, \bar{\nu})$.

It is clear that

$$
g(\bar{\nu}) = \frac{n_2}{(n_1 - n_2)}\int_0^{\bar{\nu}}\xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right)d\xi = 0.
$$

Since

$$
\lim_{\nu\to 0+}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}\left(u(I(\xi)) - \beta P\right)d\xi = +\infty,
$$

it follows from L'Hôspitals rule that

$$
\begin{aligned}
g(0+) =& \lim_{\nu\to 0+}\frac{n_1}{(n_1 - n_2)}\left(\nu^{n_1-n_2}\int_{\nu}^{\bar{\nu}}\xi^{-n_1-1}\left(u(I(\xi)) - \beta P\right)d\xi\right) \\
=&\frac{n_1}{(n_1 - n_2)}\lim_{\nu\to 0+}\frac{\nu^{-n_1-1}\left(u(I(\nu)) - \beta P\right)}{(n_1 - n_2)\nu^{n_2-n_1-1}} \\
=&\frac{n_1}{(n_1 - n_2)^2}\lim_{\nu\to 0+}\nu^{n_1-n_2}\nu^{-n_1}\left(u(I(\nu)) - \beta P\right) = 0,
\end{aligned}
$$

where we used the fact that

$$\lim_{\nu \to 0+} \nu^{-n_1} \left(u(I(\nu)) - \beta P \right) = 0 \quad \text{(see (a) in Proposition 1 and Lemma 2).}$$

Hence, we deduce that

$$g(\nu) < 0 \;\; \text{for} \;\; \nu \in (0, \bar{\nu}) \tag{89}$$

or

$$(\mathcal{M}(\nu) - \nu \mathcal{M}'(\nu) - P)' < 0 \;\; \text{for} \;\; \nu \in (0, \bar{\nu}). \tag{90}$$

Since $\mathcal{M}(\bar{\nu}) - \bar{\nu}\mathcal{M}'(\bar{\nu}) - P = 0$, we conclude that

$$\mathcal{M}(\nu) - \nu \mathcal{M}'(\nu) - P > 0 \;\; \text{for} \;\; \nu \in (0, \bar{\nu}).$$

In conclusion, $\mathcal{M}(z)$ satisfies the following HJB equation with mixed boundary condition:

$$\min \left\{ \mathcal{L}\mathcal{M}(\nu) + \tilde{u}\,(\nu) + \nu\epsilon, \; \mathcal{M}(\nu) - \nu \mathcal{M}'(\nu) - P \right\} = 0 \quad \text{for} \;\; \nu \in \mathbb{R}_+.$$

F Proof of Proposition 4

We will prove this proposition in the following steps.

Step 1: $\varphi(y)$ given in (42) is continuously differentiable in $y > 0$ and satisfies the following HJB equation:

$$\min \left\{ \mathcal{L}\varphi(y), \; \Gamma_{(u \circ I - \beta P)}(y) - \varphi(y) \right\} = 0. \tag{91}$$

Moreover, the two regions (the continuation region **CR** and the stopping region **SR**) defined as

$$\mathbf{CR} := \{ y > 0 \mid \Gamma_{(u \circ I - \beta P)}(y) > \varphi(y) \} \;\; \text{and} \;\; \mathbf{SR} := \{ y > 0 \mid \Gamma_{(u \circ I - \beta P)}(y) = \varphi(y) \}$$

can be rewritten as

$$\mathbf{CR} = \{ y > 0 \mid 0 < y < \bar{\nu} \} \;\; \text{and} \;\; \mathbf{SR} = \{ y > 0 \mid y \geq \bar{\nu} \}.$$

Proof of Step 1:

Since

$$0 = \int_0^{\bar{\nu}} \xi^{-n_2 - 1} \left(u(I(\xi)) - \beta P \right) d\xi,$$

we deduce that

$$\begin{aligned} \Gamma_{(u \circ I - \beta P)}(\bar{\nu}) =& \frac{2}{\theta^2 (n_1 - n_2)} \left[\bar{\nu}^{n_2} \int_0^{\bar{\nu}} \xi^{-n_2 - 1} (u(I(\xi)) - \beta P) d\xi + \bar{\nu}^{n_1} \int_{\bar{\nu}}^{\infty} \xi^{-n_1 - 1} (u(I(\xi)) - \beta P) d\xi \right] \\ =& \frac{2}{\theta^2 (n_1 - n_2)} \bar{\nu}^{n_1} \int_{\bar{\nu}}^{\infty} \xi^{-n_1 - 1} (u(I(\xi)) - \beta P) d\xi. \end{aligned}$$

Thus, it is easy to see that $\varphi(y)$ and $\varphi'(y)$ is continuous at $y = \bar{\nu}$. That is, $\varphi(y)$ is continuously differentiable in $y > 0$.

Note that $u(I(\bar{\nu})) - \beta P < 0$ (see Lemma 3). By (c) in Proposition 1, we have

$$\mathcal{L}\Gamma_{(u\circ I-\beta P)}(y) = -\left(u(I(y)) - \beta P\right) \geq -\left(u(I(\bar{\nu})) - \beta P\right) > 0.$$

On the other hand, it follow from the inequality in (85) that for $0 < y < \bar{\nu}$

$$\begin{aligned}\Gamma_{(u\circ I-\beta P)}(y) - \varphi(y) = &\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \xi^{-n_2-1}(u(I(\xi)) - \beta P)d\xi + y^{n_1}\int_y^{\bar{\nu}} \xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi\right]\\ =&\mathcal{M}(y) - y\mathcal{M}'(y) - P > 0.\end{aligned}$$

Clearly, $\mathcal{L}\varphi(y) = 0$ for $0 < y < \bar{\nu}$. This completes the proof of Step 1.

Step 2: The following inequality holds:

$$\inf_{\tau\in\mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right] \geq \varphi(y).$$

Proof of Step 2: Even though $\varphi(y)$ is C^1 on $(0,\infty)$ and C^2 on $(0,\infty)\backslash\{\bar{\nu}\}$, one can still apply Itô's lemma (see Exercise 6.24 in Karatzas and Shreve (1991)), and it follows that

$$d\left(e^{-\beta t}\varphi(\mathcal{Y}_t^y)\right) = e^{-\beta t}\left((\beta - r)\mathcal{Y}_t^y\varphi'(\mathcal{Y}_t^y) + \frac{\theta^2}{2}(\mathcal{Y}_t^y)^2\varphi''(\mathcal{Y}_t^y) - \beta\varphi(\mathcal{Y}_t^y)\right)dt - \theta\mathcal{Y}_t^y\varphi'(\mathcal{Y}_t^y)dB_t.$$

Then, for any $\tau\in\mathcal{S}$, we have

$$e^{-\beta(\tau\wedge t)}\varphi(\mathcal{Y}_{\tau\wedge t}^y) - \varphi(y) = \int_0^{\tau\wedge t} e^{-\beta s}\mathcal{L}\varphi(\mathcal{Y}_t^y)ds + \int_0^{\tau\wedge t}(-\theta)\mathcal{Y}_t^y\varphi'(\mathcal{Y}_t^y)dB_s. \tag{92}$$

By Proposition 1 (d), there exists a constant $C_1 > 0$ such that

$$|\Gamma'_{(u\circ I-\beta P)}(y)| \leq C_1\left(y^{n_1-1} + y^{n_2-1}\right).$$

It follows that

$$|\varphi'(y)| \leq C_2\left(y^{n_1-1} + y^{n_2-1}\right).$$

for some constant $C_2 > 0$. Thus, we can easily show that for every constant $T > 0$, the process

$$\int_0^{T\wedge t} e^{-\beta s}(-\theta)\mathcal{Y}_t^y\varphi'(\mathcal{Y}_t^y)dB_t$$

is a martingale (for detail, see Lemma 3.4 in Knudsen et al. (1998)).

By taking the expectation on both sides of the equation (92), we deduce that

$$\varphi(y) + \mathbb{E}\left[\int_0^{\tau\wedge t} e^{-\beta s}\mathcal{L}\varphi(\mathcal{Y}_t^y)ds\right] = \mathbb{E}\left[e^{-\beta(\tau\wedge t)}\varphi(\mathcal{Y}_{\tau\wedge t}^y)\right]. \tag{93}$$

Since $\varphi(y)$ satisfies the HJB equation (91), it follows that

$$\varphi(y) \leq \mathbb{E}\left[e^{-\beta(\tau\wedge t)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\tau\wedge t}^y)\right].$$

The dominated convergence theorem implies that

$$\varphi(y) \leq \lim_{t\to\infty}\mathbb{E}\left[e^{-\beta(\tau\wedge t)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\tau\wedge t}^y)\right] = \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right]$$

for any $\tau \in \mathcal{S}$. Thus, we have

$$\inf_{\tau \in \mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right] \geq \varphi(y).$$

Step 3: The stopping time $\hat{\tau}(y)$ defined as

$$\hat{\tau}(y) = \inf\left\{t \geq 0 \mid \mathcal{Y}_t^y \geq \bar{\nu}\right\}.$$

is the solution to the optimal stopping problem in (40).

Proof of Step 3: Replacing τ by $\hat{\tau}(y)$ in (93), it follows from the results in **Step 1** that

$$\varphi(y) = \mathbb{E}\left[e^{-\beta(\hat{\tau}(y)\wedge t)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\hat{\tau}(y)\wedge t}^y)\right].$$

By the dominated convergence theorem, we have

$$\varphi(y) = \mathbb{E}\left[e^{-\beta\hat{\tau}(y)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\hat{\tau}(y)}^y)\right].$$

Since $\hat{\tau}(y) \in \mathcal{S}$,

$$\inf_{\tau \in \mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right] \geq \varphi(y) = \mathbb{E}\left[e^{-\beta\hat{\tau}(y)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\hat{\tau}(y)}^y)\right] \geq \inf_{\tau \in \mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_\tau^y)\right].$$

This complete the proof.

G Proof of Proposition 5

To proceed the proof, we first show the following lemma:

Lemma 5. *For any* $(\mathcal{Z}_t)_{t=0}^{\infty} \in \Pi(z)$,

$$\lim_{t\to\infty} e^{-\beta t}\mathbb{E}\left[\mathcal{Z}_t\right] = 0 \quad and \quad z + \mathbb{E}\left[\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right] = \beta\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right]. \tag{94}$$

Proof. For given $T > 0$, integration by parts implies

$$\mathbb{E}\left[\int_0^T e^{-\beta t} d\mathcal{Z}_t\right] = \mathbb{E}\left[e^{-\beta T}\mathcal{Z}_T\right] - z + \beta\mathbb{E}\left[\int_0^T e^{-\beta t}\mathcal{Z}_t dt\right].$$

By the monotone convergence theorem, we have

$$\begin{aligned} z + \mathbb{E}\left[\int_0^\infty e^{-\beta t} d\mathcal{Z}_t\right] &= \lim_{T\to\infty}\left(\mathbb{E}\left[e^{-\beta T}\mathcal{Z}_T\right] + \beta\mathbb{E}\left[\int_0^T e^{-\beta t}\mathcal{Z}_t dt\right]\right) \\ &= \lim_{T\to\infty}\mathbb{E}\left[e^{-\beta T}\mathcal{Z}_T\right] + \beta\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right] < \infty. \end{aligned}$$

Thus, we have

$$\lim_{T\to\infty}\mathbb{E}\left[e^{-\beta T}\mathcal{Z}_T\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right] < \infty.$$

If $\lim_{T\to\infty} e^{-\beta T}\mathbb{E}[\mathcal{Z}_T] > 0$, then there exits positive constants $\widehat{C}$ and $\widehat{T}$ such that

$$e^{-\beta t}\mathcal{Z}_t \geq \widehat{C} \;\text{ for all }\; T \geq \widehat{T}.$$

It follows that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right] \geq \int_{\widehat{T}}^\infty \widehat{C}dt = \infty. \tag{95}$$

This is contradiction to

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right] < \infty.$$

Hence, we deduce that

$$\lim_{t\to\infty} e^{-\beta t}\mathbb{E}\left[\mathcal{Z}_t\right] = 0 \ \text{ and } \ z + \mathbb{E}\left[\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right] = \beta\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Z}_t dt\right].$$

□

(a): It is sufficient to show that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right] < \infty$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|(1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)\right|dt\right] < \infty.$$

For given $T > 0$, it follows from integration by parts that

$$\mathbb{E}\left[\int_0^T e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right] = \mathbb{E}\left[e^{-\beta T}(1+\widehat{\mathcal{Z}}_T^z)\right] - (1+z) + \beta\mathbb{E}\left[\int_0^T e^{-\beta t}(1+\widehat{\mathcal{Z}}_t^z(y))dt\right].$$

Note that for any $t \geq 0$

$$1 + \widehat{\mathcal{Z}}_t^z(y) \leq 1 + z + \sup_{0\leq s\leq t}\frac{\mathcal{Y}_t^y}{\bar{\nu}} = 1 + z + \frac{1}{\bar{\nu}}\overline{\mathcal{Y}}_t^y.$$

For given $\lambda \in (0, n_1)$, Lemma 1 in Merhi and Zervos (2007) implies that there exists $\epsilon_1,\ \epsilon_2 > 0$ such that

$$\mathbb{E}\left[e^{-\beta t}(\overline{\mathcal{Y}}_t^y)^\lambda\right] \leq \frac{\theta^2\lambda^2+\epsilon_2}{\epsilon_2}y^\lambda e^{-\epsilon_1 t} \quad \text{and} \quad \mathbb{E}\left[\sup_{t\geq 0} e^{-\beta t}(\overline{\mathcal{Y}}_t^y)^\lambda\right] \leq \frac{\theta^2\lambda^2+\epsilon_2}{\epsilon_2} + y^\lambda,$$

where $\overline{\mathcal{Y}}_t^y = \sup_{0\leq s\leq t}\mathcal{Y}_t^y$ with $\mathcal{Y}_t^y = ye^{\beta t}\mathcal{H}_t$.

Since $0 < 1 < n_1$, it follows that

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right] &\leq \frac{1}{\bar{\nu}}\left(\mathbb{E}\left[e^{-\beta T}\overline{\mathcal{Y}}_T^y\right] + \beta\int_0^T \mathbb{E}\left[e^{-\beta t}\overline{\mathcal{Y}}_t^y\right]dt\right)\\ &\leq \frac{\beta}{\bar{\nu}}\frac{\theta^2+\epsilon_2}{\epsilon_1\epsilon_2}y\\ &<\infty.\end{aligned}$$

For any measurable function ϕ defined on $\mathbb{R}_+$, recall that the operator Γ in (9) is given by

$$\Gamma_\phi(y) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\phi(\mathcal{Y}_t^y)dt\right].$$

Let us temporarily denote $\Phi(y,z)$ as

$$\Phi(y,z) \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)}\left|u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)\right|d\nu dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)}\left|u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)\right|d\nu dt\right].$$

Then, Fubini-Tonelli theorem implies that

$$\begin{aligned}\Phi(y,z) =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)} \left|u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)\right| d\nu dt\right]\\ =&\int_z^\infty \mathbb{E}\left[\int_{\hat{\tau}(\frac{y}{1+\nu})}^\infty e^{-\beta t}\left|u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)\right| dt\right] d\nu. \end{aligned} \tag{96}$$

Let us denote $\Lambda(y)$ by

$$\begin{aligned}\Lambda(y) \equiv \mathbb{E}\left[\int_{\hat{\tau}(y)}^\infty e^{-\beta t}\left|u\left(I\left(\mathcal{Y}_t^y\right)\right)\right| dt\right] =&\mathbb{E}\left[e^{-\beta\hat{\tau}(y)}\mathbb{E}_{\hat{\tau}(y)}\left[\int_{\hat{\tau}(y)}^\infty e^{-\beta(t-\hat{\tau}(y))}\left|u\left(I\left(\mathcal{Y}_t^y\right)\right)\right| dt\right]\right]\\ =&\mathbb{E}\left[e^{-\beta\hat{\tau}(y)}\Gamma_{|u\circ I|}(\mathcal{Y}_{\hat{\tau}(y)}^y)\right].\end{aligned}$$

Note that the strong Markov property implies that

$$\Gamma_{|u\circ I|}(\mathcal{Y}_{\hat{\tau}(y)}^y) = \mathbb{E}_{\hat{\tau}(y)}\left[\int_{\hat{\tau}(y)}^\infty e^{-\beta(t-\hat{\tau}(y))}\left|u\left(I\left(\mathcal{Y}_t^y\right)\right)\right| dt\right].$$

Moreover, $\Gamma_{|u\circ I|}(y)$ is twice differentiable (see Proposition 1 (c)). For $y \geq \bar{\nu}$, it is clear that

$$\Lambda(y) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left|u\left(I\left(\mathcal{Y}_t^y\right)\right)\right| dt\right] = \Gamma_{|u\circ I|}(y). \tag{97}$$

For $0 < y < \bar{\nu}$, the dynamic programming principle implies that $\Lambda(y)$ satisfies the following ordinary differential equation(ODE):

$$\begin{cases}\mathcal{L}\Lambda(y) = 0, \quad \text{for} \ \ 0 < y < \bar{\nu},\\ \Lambda(\bar{\nu}) = \Gamma_{|u\circ I|}(\bar{\nu}).\end{cases} \tag{98}$$

Moreover, the following transversality condition holds:

$$\lim_{t\to\infty} e^{-\beta t}\mathbb{E}\left[\Lambda(\mathcal{Y}_t^y)\right] = 0. \tag{99}$$

By solving the ODE (98) with the condition (99), we can easily confirm that

$$\Lambda(y) = \Gamma_{|u\circ I|}(\bar{\nu})\left(\frac{y}{\bar{\nu}}\right)^{n_1} \quad \text{for} \ \ 0 < y < \bar{\nu}. \tag{100}$$

From (97) and (100), we have

$$\Lambda(y) = \begin{cases}\Gamma_{|u\circ I|}(y) & \text{for} \ \ y \geq \bar{\nu},\\ \Gamma_{|u\circ I|}(\bar{\nu})\left(\frac{y}{\bar{\nu}}\right)^{n_1} & \text{for} \ \ 0 < y < \bar{\nu}.\end{cases}$$

From (96), we obtain

$$\begin{aligned}\Phi(y,z) =& \int_z^\infty \Lambda\left(\frac{y}{1+\nu}\right) d\nu\\ =& \int_z^{z\vee\hat{z}(y)} \Gamma_{|u\circ I|}\left(\frac{y}{1+\nu}\right) d\nu + \int_{z\vee\hat{z}(y)}^\infty \Gamma_{|u\circ I|}(\bar{\nu})\left(\frac{y}{(1+\nu)\bar{\nu}}\right)^{n_1} d\nu\\ =& \int_z^{z\vee\hat{z}(y)} \Gamma_{|u\circ I|}\left(\frac{y}{1+\nu}\right) d\nu + \frac{1}{n_1-1}\left((z\vee\hat{z}(y))+1\right)^{1-n_1}\Gamma_{|u\circ I|}(\bar{\nu})\left(\frac{y}{\bar{\nu}}\right)^{n_1} < \infty.\end{aligned} \tag{101}$$

(b): The proof of this part is almost similar to that of (a) in this proposition.

Let us denote $\widehat{\mathcal{Q}}(y,z)$ by

$$\widehat{\mathcal{Q}}(y,z) \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right].$$

By Lemma 5, we have

$$\begin{aligned}
\widehat{\mathcal{Q}}(y,z) &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^z(y)\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)+\mathcal{Y}_t^y\epsilon-\beta P\widehat{\mathcal{Z}}_t^z(y)\right)dt\right]+Pz \qquad (102)\\
&= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)+\mathcal{Y}_t^y\epsilon-\beta P(1+\widehat{\mathcal{Z}}_t^z(y))\right)dt\right]+P(1+z)\\
&= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)-\beta P(1+\widehat{\mathcal{Z}}_t^z(y))\right)dt\right]+\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Y}_t^y\epsilon dt\right]+P(1+z).
\end{aligned}$$

As similar to the proof of part (a), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)-\beta P(1+\widehat{\mathcal{Z}}_t^z(y))\right)dt\right] \qquad (103)\\
=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)}\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)d\nu dt\right]+\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+z)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+z}\right)-\beta P(1+z)\right)dt\right]
\end{aligned}$$

By Lemma 2 (a), for any $y>0$,

$$\begin{aligned}
&\int_0^\nu \xi^{-n_2-1}|\tilde{u}(\xi)+\xi\epsilon|d\xi+\int_\nu^\infty \xi^{-n_1-1}|\tilde{u}(\xi)+\xi\epsilon|d\xi\\
\leq&\int_0^\nu \xi^{-n_2-1}|\tilde{u}(\xi)|d\xi+\int_\nu^\infty \xi^{-n_1-1}|\tilde{u}(\xi)|d\xi+\epsilon\int_0^\nu \xi^{-n_2}d\xi+\epsilon\int_\nu^\infty \xi^{-n_1}d\xi<\infty.
\end{aligned}$$

By applying Proposition 1, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\tilde{u}\left(\mathcal{Y}_t^{\frac{y}{1+z}}\right)+\mathcal{Y}_t^{\frac{y}{1+z}}\epsilon\right)dt\right] \qquad (104)\\
=&\frac{2}{\theta^2(n_1-n_2)}\left[\left(\frac{y}{1+z}\right)^{n_2}\int_0^{\frac{y}{1+z}}\xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi+\left(\frac{y}{1+z}\right)^{n_1}\int_{\frac{y}{1+z}}^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right]\\
=&\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right)+\frac{\epsilon}{r}\left(\frac{y}{1+z}\right).
\end{aligned}$$

It follows from (102), (103), and (104) that

$$\begin{aligned}
\widehat{\mathcal{Q}}(y,z) =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)}\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)d\nu dt\right]+(1+z)\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\tilde{u}\left(\mathcal{Y}_t^{\frac{y}{1+z}}\right)+\mathcal{Y}_t^{\frac{y}{1+z}}\epsilon\right)dt\right]\\
=&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)d\nu dt\right]+(1+z)\left(\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right)+\frac{\epsilon}{r}\left(\frac{y}{1+z}\right)\right).
\end{aligned}$$

Let us temporarily denote $\widehat{\Phi}(y,z)$ as

$$\widehat{\Phi}(y,z) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)} \left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right) - \beta P\right) d\nu dt\right].$$

Since $\widehat{\mathcal{Z}} \in \Pi(z)$, Fubini's theorem implies that

$$\begin{aligned}
\widehat{\Phi}(y,z) =& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\widehat{\mathcal{Z}}_t^z(y)} \left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right) - \beta P\right) d\nu dt\right] \\
=& \int_z^\infty \mathbb{E}\left[\int_{\hat{\tau}(\frac{y}{1+\nu})}^\infty e^{-\beta t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right) - \beta P\right) dt\right] d\nu \\
=& \int_z^\infty \mathbb{E}\left[e^{-\beta\hat{\tau}(\frac{y}{1+\nu})}\mathbb{E}_{\hat{\tau}(\frac{y}{1+\nu})}\left[\int_{\hat{\tau}(\frac{y}{1+\nu})}^\infty e^{-\beta(t-\hat{\tau}(\frac{y}{1+\nu}))}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right) - \beta P\right) dt\right]\right] d\nu,
\end{aligned} \tag{105}$$

where $\hat{\tau}(y) = \inf\{t \geq 0 \mid \mathcal{Y}_t^y \geq \bar{\nu}\}$. By Proposition 4,

$$\varphi(\nu) = \mathbb{E}\left[e^{-\beta\hat{\tau}(\nu)}\mathbb{E}_{\hat{\tau}(\nu)}\left[\int_{\hat{\tau}(\nu)}^\infty e^{-\beta(t-\hat{\tau}(\nu))}\left(u\left(I\left(\mathcal{Y}_t^\nu\right)\right) - \beta P\right) dt\right]\right] = \mathbb{E}\left[e^{-\beta\hat{\tau}(\nu)}\Gamma_{(u\circ I-\beta P)}(\mathcal{Y}_{\hat{\tau}(\nu)}^\nu)\right],$$

where

$$\begin{aligned}
\Gamma_{(u\circ I-\beta P)}(y) &= \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\left(u\left(I\left(\mathcal{Y}_t^y\right)\right) - \beta P\right) ds\right] \\
&= \frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \xi^{-n_2-1}(u(I(\xi)) - \beta P)d\xi + y^{n_1}\int_y^\infty \xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi\right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
\varphi\left(\frac{y}{1+\nu}\right) =& \mathbb{E}\left[e^{-\beta\hat{\tau}(\frac{y}{1+\nu})}\mathbb{E}_{\hat{\tau}(\frac{y}{1+\nu})}\left[\int_{\hat{\tau}(\frac{y}{1+\nu})}^\infty e^{-\beta(t-\hat{\tau}(\frac{y}{1+\nu}))}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right) - \beta P\right) dt\right]\right] \\
=& \begin{cases} \Gamma_{(u\circ I-\beta P)}\left(\dfrac{y}{1+\nu}\right) & \text{for } \dfrac{y}{1+\nu} \geq \bar{\nu}, \\ \Gamma_{(u\circ I-\beta P)}(\bar{\nu})\left(\dfrac{y}{\bar{\nu}(1+\nu)}\right)^{n_1} & \text{for } 0 < \dfrac{y}{1+\nu} \leq \bar{\nu}. \end{cases}
\end{aligned} \tag{106}$$

Since

$$0 = \int_0^{\bar{\nu}} \xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right) d\xi,$$

we deduce that

$$\begin{aligned}
\Gamma_{(u\circ I-\beta P)}(\bar{\nu}) =& \frac{2}{\theta^2(n_1-n_2)}\left[\bar{\nu}^{n_2}\int_0^{\bar{\nu}} \xi^{-n_2-1}(u(I(\xi)) - \beta P)d\xi + \bar{\nu}^{n_1}\int_{\bar{\nu}}^\infty \xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi\right] \\
=& \frac{2}{\theta^2(n_1-n_2)}\bar{\nu}^{n_1}\int_{\bar{\nu}}^\infty \xi^{-n_1-1}(u(I(\xi)) - \beta P)d\xi \\
=& (n_1-1)\bar{\nu}^{n_1}D_1.
\end{aligned} \tag{107}$$

It follows from (102), (105), (106), and (107) that

$$\widehat{Q}(y,z) = \int_z^\infty \varphi\left(\frac{y}{1+\nu}\right) d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y.$$

When $\hat{z}(y) \leq z$,

$$\begin{aligned}\widehat{Q}(y,z) =& \int_z^\infty \varphi\left(\frac{y}{1+\nu}\right) d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y \\ =& \int_z^\infty \Gamma_{(u\circ I-\beta P)}(\bar{\nu})\left(\frac{y}{\bar{\nu}(1+\nu)}\right)^{n_1} d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y \\ =& D_1 y^{n_1}(1+z)^{1-n_1} + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y \\ =& (1+z)\left\{D_1\left(\frac{y}{1+z}\right)^{n_1} + \frac{2}{\theta^2(n_1-n_2)}\left[\left(\frac{y}{1+z}\right)^{n_2}\int_0^{\frac{y}{1+z}} \xi^{-n_2-1}\tilde{u}(\xi)d\xi \right.\right. \\ & \left.\left. + \left(\frac{y}{1+z}\right)^{n_1}\int_{\frac{y}{1+z}}^\infty \xi^{-n_1-1}\tilde{u}(\xi)d\xi\right] + \frac{\epsilon}{r}\left(\frac{y}{1+z}\right)\right\}.\end{aligned} \tag{108}$$

When $0 \leq z < \hat{z}(y)$,

$$\begin{aligned}\widehat{Q}(y,z) =& \int_z^\infty \varphi\left(\frac{y}{1+\nu}\right) d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y \\ =& \int_z^{\hat{z}(y)} \Gamma_{(u\circ I-\beta P)}\left(\frac{y}{1+\nu}\right) d\nu + \int_{\hat{z}(y)}^\infty \Gamma_{(u\circ I-\beta P)}(\bar{\nu})\left(\frac{y}{\bar{\nu}(1+\nu)}\right)^{n_1} d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}y.\end{aligned} \tag{109}$$

Note that

$$\begin{aligned}\frac{d}{dz}\left((1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right)\right) &= \Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) - \frac{y}{1+z}\Gamma'_{\tilde{u}}\left(\frac{y}{1+z}\right) \\ &= \Gamma_{u\circ I}\left(\frac{y}{1+z}\right) \\ &= \Gamma_{(u\circ I-\beta P)}\left(\frac{y}{1+z}\right) + P,\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\Gamma'_{\tilde{u}}(\nu) =& \frac{2}{\theta^2(n_1-n_2)}\left[n_2\nu^{n_2-1}\int_0^\nu \xi^{-n_2-1}\tilde{u}(\xi)d\xi + \nu^{n_1-1}\int_\nu^\infty \xi^{-n_1-1}\tilde{u}(\xi)d\xi\right] \\ =& -\frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2-1}\int_0^\nu \xi^{-n_2}I(\xi)d\xi + \nu^{n_1-1}\int_\nu^\infty \xi^{-n_1}I(\xi)d\xi\right]\end{aligned}$$

and

$$\begin{aligned}\Gamma_{(u\circ I-\beta P)}(\nu) =& \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}\left(u(I(\xi))-\beta P\right)d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}\left(u(I(\xi))-\beta P\right)d\xi\right] \\ =& \frac{2}{\theta^2(n_1-n_2)}\left[\nu^{n_2}\int_0^\nu \xi^{-n_2-1}u(I(\xi))d\xi + \nu^{n_1}\int_\nu^\infty \xi^{-n_1-1}u(I(\xi))d\xi\right] - P \\ =& \Gamma_{u\circ I}(\nu) - P.\end{aligned}$$

This implies that

$$\int_z^{\hat{z}(y)} \Gamma_{(u\circ I-\beta P)}\left(\frac{y}{1+\nu}\right) d\nu = (1+\hat{z}(y))\Gamma_{\tilde{u}}\left(\frac{y}{1+\hat{z}(y)}\right) - (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + P(z-\hat{z}(y)). \tag{110}$$

It follows from (109) and (110) that

$$\begin{aligned}\widehat{\mathcal{Q}}(y,z) &= \int_{\hat{z}(y)}^{\infty} \Gamma_{(u\circ I-\beta P)}(\bar{\nu})\left(\frac{y}{\bar{\nu}(1+\nu)}\right)^{n_1} d\nu + (1+\hat{z}(y))\Gamma_{\tilde{u}}\left(\frac{y}{1+\hat{z}(y)}\right) + \frac{\epsilon}{r}y + P(z-\hat{z}(y)) \\ &= \widehat{\mathcal{Q}}(y,\hat{z}(y)) + P(z-\hat{z}(y)) \quad \text{for } 0 \le z < \hat{z}(y).\end{aligned} \tag{111}$$

By comparing $\mathcal{Q}(y,z)$ in (37), (38) with $\widehat{\mathcal{Q}}(y,z)$ in (108), (111), we conclude that

$$\mathcal{Q}(y,z) = \widehat{\mathcal{Q}}(y,z).$$

H Proof of Theorem 1

(a): For any $\mathcal{Z}_t \in \Pi(z)$, it follows from Lemma 5, (102) and (103) that

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right) + \mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right] \\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)d\nu dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+z)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+z}\right)+\epsilon\mathcal{Y}_t^y\right)dt\right] \\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)d\nu dt\right] + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right) + \frac{\epsilon}{r}.\end{aligned} \tag{112}$$

Since

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{Z_t}\left|u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)\right|d\nu dt\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right] < \infty,$$

we can easily deduce that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left|\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)\right|d\nu dt\right] < \infty.$$

Fubini's theorem implies that

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left(u\left(I\left(\frac{\mathcal{Y}_t^y}{1+\nu}\right)\right)-\beta P\right)d\nu dt\right] \\ =&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\int_z^{\mathcal{Z}_t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)d\nu dt\right] \\ =&\int_z^\infty \mathbb{E}\left[\int_{\kappa(\nu)}^\infty e^{-\beta t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)dt\right]d\nu,\end{aligned} \tag{113}$$

where the stopping time $\kappa(\nu)$ for $\nu \ge z$ is defined as

$$\kappa(\nu) = \inf\{t \ge 0 \mid \mathcal{Z}_t \ge \nu\}. \tag{114}$$

Moreover, by Proposition 4, we deduce that for any $\nu \ge z$

$$\begin{aligned}\mathbb{E}\left[\int_{\kappa(\nu)}^\infty e^{-\beta t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)dt\right] \ge& \inf_{\tau\in\mathcal{S}}\left[\int_\tau^\infty e^{-\beta t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)dt\right] \\ =&\varphi\left(\frac{y}{1+\nu}\right).\end{aligned} \tag{115}$$

It follows from (112), (113), (115), and Proposition 5 that for any $\mathcal{Z}_t \in \Pi(z)$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right]\\
=&\int_z^\infty \mathbb{E}\left[\int_{\kappa(\nu)}^\infty e^{-\beta t}\left(u\left(I\left(\mathcal{Y}_t^{\frac{y}{1+\nu}}\right)\right)-\beta P\right)dt\right]d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right)+\frac{\epsilon}{r} \qquad (116)\\
\geq&\int_z^\infty \varphi\left(\frac{y}{1+\nu}\right)d\nu + (1+z)\Gamma_{\tilde{u}}\left(\frac{y}{1+z}\right)+\frac{\epsilon}{r}\\
=&\mathcal{Q}(y,z).
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{Q}(y,z) \leq &\inf_{\mathcal{Z}_t\in\Pi(z)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right] \qquad (117)\\
\leq&\mathcal{Q}(y,z),
\end{aligned}$$

we conclude that

$$\mathfrak{J}(y,z) = \inf_{\mathcal{Z}_t\in\Pi(z)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right] = \mathcal{Q}(y,z).$$

(b): Since $J(y) = \mathfrak{J}(y,0) = \mathcal{Q}(y,z)$, it is clear that

$$\begin{aligned}
J(y) =&\mathcal{Q}(y,0)\\
=&\begin{cases} D_1 y^{n_1} + \dfrac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\displaystyle\int_0^y \xi^{-n_2-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi + y^{n_1}\int_y^\infty \xi^{-n_1-1}(\tilde{u}(\xi)+\xi\epsilon)d\xi\right] & \text{for } 0<y<\bar{\nu}\\
(\tilde{u}(\bar{\nu})-\beta P+\bar{\nu}\epsilon)\dfrac{y}{r\bar{\nu}}+P \quad \text{for } y\geq\bar{\nu} & \text{for } y\geq\bar{\nu}.\end{cases}
\end{aligned}$$

I Proof of Lemma 4

For sufficiently small $\delta>0$, it follows from Theorem 1 (a) that

$$\begin{aligned}
\mathfrak{J}(y\pm\delta,0) =& \inf_{\mathcal{Z}\in\Pi(0)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^{y\pm\delta}}{1+\mathcal{Z}_t}\right)+\mathcal{Y}_t^{y\pm\delta}\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right]\\
\leq&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^0(y))\tilde{u}\left(\frac{\mathcal{Y}_t^{y\pm\delta}}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)+\mathcal{Y}_t^{y\pm\delta}\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^0(y)\right].
\end{aligned}$$

It follows that

$$\mathfrak{J}(y\pm\delta,0)-\mathfrak{J}(y,0) \leq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^0)\tilde{u}\left(\frac{\mathcal{Y}_t^{y\pm\delta}}{1+\widehat{\mathcal{Z}}_t^0}\right)-(1+\widehat{\mathcal{Z}}_t^0)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0}\right)\pm\delta e^{\beta t}\mathcal{H}_t\epsilon\right)dt\right].$$

By the dominated convergence theorem, we have

$$J'(y) = \lim_{\delta\to0+}\frac{\mathfrak{J}(y+\delta,0)-\mathfrak{J}(y,0)}{\delta} \leq \mathbb{E}\left[\int_0^\infty \mathcal{H}_t\left(-\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)+\epsilon\right)dt\right]$$

and

$$J'(y) = \lim_{\delta\to 0+} \frac{\mathfrak{J}(y,0)-\mathfrak{J}(y-\delta,0)}{-\delta} \geq \mathbb{E}\left[\int_0^\infty \mathcal{H}_t\left(-\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)+\epsilon\right)dt\right].$$

Hence,

$$\mathbb{E}\left[\int_0^\infty \mathcal{H}_t\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^0(y)}\right)-\epsilon\right)dt\right] = -J'(y).$$

Assume that $z>0$ and recall that for given $y>0$

$$\widehat{\mathcal{Z}}_t^z(y) = \max\left\{z, \sup_{0\leq s\leq t}\frac{\mathcal{Y}_t^y}{\bar{\nu}}\right\}-1.$$

For sufficiently small $\delta>0$, let us denote $\widehat{\mathcal{Z}}_t^{z,\pm\delta}(y)$ by

$$\widehat{\mathcal{Z}}_t^{z,\pm\delta}(y) = \widehat{\mathcal{Z}}_t^z(y) \pm \delta. \tag{118}$$

As similar to the proof of Proposition 5 (a), we can easily deduce that

$$\widehat{\mathcal{Z}}_t^{z,\pm\delta}(y) \in \Pi(z\pm\delta).$$

Since $(1+z)\tilde{u}(y/(1+z))$ is convex in $z\geq 0$ for given $y>0$, we deduce that for any $\rho\in(0,\delta)$

$$\begin{aligned}
&\frac{(1+\widehat{\mathcal{Z}}_t^{z,-\delta}(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,-\delta}(y)}\right)-(1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)}{-\delta}\\
\leq&\frac{(1+\widehat{\mathcal{Z}}_t^{z,\pm\rho})\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,\pm\rho}(y)}\right)-(1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)}{\pm\rho} &(119)\\
\leq&\frac{(1+\widehat{\mathcal{Z}}_t^{z,+\delta}(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,+\delta}(y)}\right)-(1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^z(y)}\right)}{\delta}. &(120)
\end{aligned}$$

By Theorem 1 (a),

$$\begin{aligned}
\mathfrak{J}(y,z\pm\rho) =& \inf_{\mathcal{Z}\in\Pi(z\pm\rho)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\mathcal{Z}_t)\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\mathcal{Z}_t}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\mathcal{Z}_t\right]\\
\leq&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left((1+\widehat{\mathcal{Z}}_t^{z,\pm\delta})\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,\pm\delta}}\right)+\mathcal{Y}_t^y\epsilon\right)dt - P\int_0^\infty e^{-\beta t}d\widehat{\mathcal{Z}}_t^{z,\pm\delta}\right].
\end{aligned}$$

It follows that

$$\frac{\mathfrak{J}(y,z+\rho)-\mathfrak{J}(y,z)}{\rho} \leq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{(1+\widehat{\mathcal{Z}}_t^{z,+\rho}(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,+\rho}(y)}\right)-(1+\widehat{\mathcal{Z}}_t^z(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,+\rho}(y)}\right)}{+\rho}dt\right]$$

and

$$\frac{\mathfrak{J}(y,z-\rho)-\mathfrak{J}(y,z)}{-\rho} \geq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{(1+\widehat{\mathcal{Z}}_t^{z,-\rho}(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,-\rho}(y)}\right)-(1+\widehat{\mathcal{Z}}_t^{z,-\rho}(y))\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1+\widehat{\mathcal{Z}}_t^{z,-\rho}(y)}\right)}{-\rho}dt\right].$$

Note that $\widehat{\mathcal{Z}}_t^{z,\pm\delta}(y) \in \Pi(z \pm \delta)$ implies

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}(1 + \widehat{\mathcal{Z}}_t^{z,\pm\delta}(y)) \left|\tilde{u}\left(\frac{\mathcal{Y}_t^y}{1 + \widehat{\mathcal{Z}}_t^{z,\pm\delta}(y)}\right)\right| dt\right] < \infty \quad \text{(see Lemma 1)}.$$

From (119), the dominated convergence theorem yields

$$\partial_z \mathfrak{J}(y,z) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} u\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1 + \widehat{\mathcal{Z}}_t^z(y)}\right)\right) dt\right].$$

The monotone convergence theorem implies that

$$J(y) = \mathfrak{J}(y,0) = \lim_{z \to 0+} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1 + \widehat{\mathcal{Z}}_t^z(y)}\right)\right) dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\beta t} u\left(\hat{c}\left(\frac{\mathcal{Y}_t^y}{1 + \widehat{\mathcal{Z}}_t^0(y)}\right)\right) dt\right].$$

J Proof of Theorem 2

From the relations in (45), we deduce that for $y \in (0, \bar{\nu})$

$$(\mathcal{M}(y) - y\mathcal{M}'(y) - P)' = (J(y) - yJ'(y) - P)' = -yJ''(y). \tag{121}$$

It follows from (90) in Proposition 3 that

$$-yJ''(y) = (\mathcal{M}(y) - y\mathcal{M}'(y) - P)' = (J(y) - yJ'(y) - P)' < 0 \;\text{ for }\; y \in (0, \bar{\nu}). \tag{122}$$

That is, $J(y)$ is strictly convex in $y \in (0, \bar{\nu})$.

Since $J'(y) = \mathcal{M}'(y)$, it is clear that

$$\lim_{y \to \bar{\nu}} J'(y) = J'(\bar{\nu}) = \mathcal{M}'(\bar{\nu}) = -\bar{x}. \tag{123}$$

By Theorem 1, the explicit-form of $J(y)$ for $y \in (0, \bar{\nu})$ is given by

$$J(y) = D_1 y^{n_1} + \frac{2}{\theta^2(n_1 - n_2)}\left[y^{n_2}\int_0^y \xi^{-n_2-1}(\tilde{u}(\xi) + \xi\epsilon)d\xi + y^{n_1}\int_y^\infty \xi^{-n_1-1}(\tilde{u}(\xi) + \xi\epsilon)d\xi\right]. \tag{124}$$

Thus,

$$\begin{aligned}
J'(y) =& n_1 D_1 y^{n_1-1} + \frac{2}{\theta^2(n_1 - n_2)}\left[n_2 y^{n_2-1}\int_0^y \xi^{-n_2-1}(\tilde{u}(\xi) + \xi\epsilon)d\xi + n_1 y^{n_1-1}\int_y^\infty \xi^{-n_1-1}(\tilde{u}(\xi) + \xi\epsilon)d\xi\right] \\
=& n_1 D_1 y^{n_1-1} + \frac{2}{\theta^2(n_1 - n_2)}\left[y^{n_2-1}\int_0^y \xi^{-n_2}(\epsilon - I(\xi))d\xi + y^{n_1-1}\int_y^\infty \xi^{-n_1}(\epsilon - I(\xi))d\xi\right],
\end{aligned}$$

where we used integration by parts in the second equality. Since

$$\int_0^y \xi^{-n_2} I(\xi)d\xi + \int_y^\infty \xi^{-n_1} I(\xi)d\xi < \infty \;\text{ for } y > 0,$$

it follows from Proposition 1 that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Y}_t^y(\epsilon - I(\mathcal{Y}_t^y))dt\right] = \frac{2}{\theta^2(n_1 - n_2)}\left[y^{n_2}\int_0^y \xi^{-n_2}(\epsilon - I(\xi))d\xi + y^{n_1}\int_y^\infty \xi^{-n_1}(\epsilon - I(\xi))d\xi\right].$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty \mathcal{H}_t(\epsilon - I(\mathcal{Y}_t^y))dt\right] =& \frac{1}{y}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\mathcal{Y}_t^y(\epsilon - I(\mathcal{Y}_t^y))dt\right] \\ =& \frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2-1}\int_0^y \xi^{-n_2}(\epsilon - I(\xi))d\xi + y^{n_1-1}\int_y^\infty \xi^{-n_1}(\epsilon - I(\xi))d\xi\right].\end{aligned}$$

By the monotone convergence theorem,

$$\lim_{y\to 0+}\mathbb{E}\left[\int_0^\infty \mathcal{H}_t I(\mathcal{Y}_t^y)dt\right] = +\infty.$$

Therefore,

$$\lim_{y\to 0+}\left\{\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2-1}\int_0^y \xi^{-n_2}(\epsilon - I(\xi))d\xi + y^{n_1-1}\int_y^\infty \xi^{-n_1}(\epsilon - I(\xi))d\xi\right]\right\} = -\infty.$$

and thus

$$\lim_{y\to 0+} J'(y) = -\infty. \tag{125}$$

By the limits (123) and (125), the strict convexity of $J(y)$ implies that for given $x \geq \bar{x}$ there exist a unique $y^* \in (0,\bar{\nu})$ such that

$$x = -J'(y^*).$$

Let us denote c^*, $\mathcal{Y}_t^*$, and $\mathcal{Z}_t^*$ as

$$c_t^* = \hat{c}\left(\frac{\mathcal{Y}_t^{y^*}}{1+\widehat{\mathcal{Z}}_t^0(y^*)}\right),\quad \mathcal{Y}_t^* = \mathcal{Y}_t^{y^*},\quad \text{and}\quad \mathcal{Z}_t^* = \mathcal{Z}_t^0(y^*),$$

respectively. By Proposition 2 and Lemma 4, we deduce that

$$\begin{aligned}V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] \leq& \inf_{y>0}\left(\inf_{\mathcal{Z}\in\Pi(0)}\mathfrak{L}(y,\mathcal{Z})\right) \\ =& \inf_{y>0}(J(y)+yx) \\ \leq& J(y^*) - y^*J(y^*) \\ =& \mathbb{E}\left[\int_0^\infty e^{-\beta t}u\left(c_t^*\right)dt\right] \\ \leq& \sup_{(c,\pi)\in\mathcal{A}(x)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right] = V(x).\end{aligned}$$

Therefore,

$$V(x) = \inf_{y>0}(J(y)+yx) = J(y^*) + y^*x \tag{126}$$

and thus (c^*,π^*) is optimal.

It follows from Lemma 4 that

$$x = -J'(y^*) = \mathbb{E}\left[\int_0^\infty \mathcal{H}_t\left(c_t^* - \epsilon\right)dt\right]. \tag{127}$$

By slightly modifying Theorem 9.4 in Chapter 3.9 of Karatzas and Shreve (1998), we easily show that there exists a portfolio π^* such that $(c^*,\pi^*)\in\mathcal{A}(x)$. Moreover, the wealth $X_t^{c^*,\pi^*}$ corresponding to (c^*,π^*) is

$$X_t^{c^*,\pi^*} = \mathbb{E}_t\left[\int_t^\infty \frac{\mathcal{H}_s}{\mathcal{H}_t}\left(c_t^* - \epsilon\right)dt\right]. \tag{128}$$

and

$$dX_t^{c^*,\pi^*} = [rX_t^{c^*,\pi^*} + (\mu - r)\pi_t^* - c_t^* + \epsilon]dt + \sigma\pi_t^* dB_t. \tag{129}$$

The strong Markov property implies that

$$X_t^{c^*,\pi^*} = -J'\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right). \tag{130}$$

By the generalized Itô's lemma (see Harrison (1985), Stokey (2009)), we have

$$\begin{aligned} dX_t^{c^*,\pi^*} =& d\left(-J'\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)\right) \\ =& -J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)d\mathcal{Y}_t^* - \frac{1}{2}J'''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)(d\mathcal{Y}_t^*)^2 + \frac{\mathcal{Y}_t^*}{(1+\mathcal{Z}_t^*)^2}J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)d\mathcal{Z}_t^*. \end{aligned} \tag{131}$$

Note that for $0 < y \le \bar{\nu}$, $J(y)$ satisfy

$$\frac{\theta^2}{2}y^2 J''(y) + (\beta - r)yJ'(y) - \beta J(y) + \tilde{u}(y) + y\epsilon = 0. \tag{132}$$

By differentiating the above equation with respect to y, we have

$$\frac{\theta^2}{2}y^2 J'''(y) + (\beta - r + \theta^2)yJ''(y) - rJ(y) + \epsilon - I(y) = 0 \text{ for } 0 < y \le \bar{\nu}. \tag{133}$$

Moreover,

$$d\mathcal{Z}_t^* = 0 \text{ for } \mathcal{Y}_t^*/(1+\mathcal{Z}_t^*) < \bar{\nu} \tag{134}$$

and

$$d\mathcal{Z}_t^* > 0 \text{ for } \mathcal{Y}_t^*/(1+\mathcal{Z}_t^*) = \bar{\nu}. \tag{135}$$

Since $J''(\bar{\nu}) = 0$, we have

$$J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)d\mathcal{Z}_t^* = 0 \text{ for all } t \ge 0. \tag{136}$$

Thus, it follows from (131), (133), and (136) that

$$\begin{aligned} dX_t^{c^*,\pi^*} =& \left[-rJ'\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right) + \theta^2\mathcal{Y}_t^* J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right) - I\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right) + \epsilon\right]dt + \theta\mathcal{Y}_t^* J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)dB_t \\ & \left[rX_t^{c^*,\pi^*} + \theta^2\mathcal{Y}_t^* J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right) - c_t^* + \epsilon\right]dt + \theta\mathcal{Y}_t^* J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right)dB_t. \end{aligned} \tag{137}$$

By comparing the wealth process $X_t^{c^*,\pi^*}$ in (129) and (137), we have

$$\pi_t^* = \frac{\theta}{\sigma}\mathcal{Y}_t^* J''\left(\frac{\mathcal{Y}_t^*}{1+\mathcal{Z}_t^*}\right). \tag{138}$$

K Proof of Proposition 6

(a): From Lemma 3,

$$0 = \int_0^{\bar{\nu}} \xi^{-n_2-1}\left(u(I(\xi)) - \beta P\right)d\xi = \int_0^{\bar{\nu}} \xi^{-n_2-1}u(I(\xi))d\xi + \beta P\frac{\bar{\nu}^{-n_2}}{n_2}.$$

By differentiating the above equation with respect to P, we have

$$0 = \bar{\nu}^{-n_2-1}u(I(\bar{\nu}))\frac{d\bar{\nu}}{dP} - \beta P\bar{\nu}^{-n_2-1}\frac{d\bar{\nu}}{dP} + \beta\frac{\bar{\nu}^{-n_2}}{n_2} = \bar{\nu}^{-n_2-1}\left(u(I(\bar{\nu})) - \beta P\right)\frac{d\bar{\nu}}{dP} + \beta\frac{\bar{\nu}^{-n_2}}{n_2}.$$

It follows that

$$\frac{d\bar{\nu}}{dP} = -\beta\frac{\bar{\nu}^{-n_2}}{n_2}\frac{1}{\bar{\nu}^{-n_2-1}\left(u(I(\bar{\nu})) - \beta P\right)} < 0,$$

where we have used the fact that $u(I(\bar{\nu})) - \beta P < 0$. Since

$$\bar{x} = -\left(\frac{\tilde{u}(\bar{\nu}) - \beta P}{\bar{\nu}} + \epsilon\right)\frac{1}{r},$$

we deduce that

$$\begin{aligned}\frac{d\bar{x}}{dP} =& \frac{\beta}{r\bar{\nu}} - \left(\frac{-I(\bar{\nu})\bar{\nu} - \tilde{u}(\bar{\nu}) + \beta P}{r\bar{\nu}^2}\right)\frac{d\bar{\nu}}{dP}\\ =& \frac{\beta}{r\bar{\nu}} + \left(\frac{u(I(\bar{\nu})) - \beta P}{r\bar{\nu}^2}\right)\frac{d\bar{\nu}}{dP} > 0.\end{aligned}$$

(b): Since $\dfrac{d\bar{\nu}}{dP} < 0$,

$$\frac{dD_1}{dP} = -\,\bar{\nu}^{-n_1-1}\left(u(I(\bar{\nu})) - \beta P\right)\frac{d\bar{\nu}}{dP} - \beta\frac{\bar{\nu}^{-n_1}}{n_1} < 0.$$

Since $x = -J'(y^*)$, we have

$$0 = -J''(y^*)\frac{dy^*}{dP} - n_1(y^*)^{n_1-1}\frac{dD_1}{dP}.$$

It follows from $J''(y^*) > 0$ and $dD_1/dP < 0$ that

$$\frac{dy^*}{dP} > 0.$$

This leads that

$$\frac{dc_t^*}{dP} < 0.$$

(c): Since $\pi_t^* = 0$ when $x = \bar{x}$, we focus on the case $x > \bar{x}$. For given $x > \bar{x}$, by using the relationship $x = -J'(y^*)$, we can derive

$$\begin{aligned}\pi_t^* &= \frac{\theta}{\sigma}y^*J''(y^*) = \frac{\theta}{\sigma}\left[y^*J''(y^*) + x + J'(y^*)\right]\\ &= \frac{\theta}{\sigma}\left[x + \frac{\epsilon}{r} + D_1 n_1^2 {y^*}^{n_1-1} + \frac{2}{\theta^2(n_1-n_2)}\left\{n_2^2{y^*}^{n_2-1}\int_0^{y^*} d\xi + n_1^2{y^*}^{n_1-1}\int_{y^*}^{\infty} d\xi\right\}\right] - \frac{2}{\theta^2}\frac{\tilde{u}(y^*)}{y^*}.\end{aligned}$$

Taking into account that D_1 depends on P, and differentiating both sides with respect to P, we have

$$\begin{aligned}\frac{d\pi_t^*}{dP} &= \frac{\theta}{\sigma}\left[\frac{dD_1}{dP}n_1^2{y^*}^{n_1-1} + (y^*J'''(y^*) + 2J''(y^*)\frac{dy^*}{dP}\right]\\ &= \frac{\theta}{\sigma}\frac{-y^*J'''(y^*) + (n_1-2)J''(y^*)}{J''(y^*)}n_1{y^*}^{n_1-1}\frac{dD_1}{dP}\end{aligned}$$

because

$$\frac{dy^*}{dP} = -\frac{n_1 {y^*}^{n_1 - 1}}{J''(y^*)} \frac{dD_1}{dP}.$$

We can show that

$$-y^* J'''(y^*) + (n_1 - 2) J''(y^*) = -\frac{2}{\theta^2} \int_0^{y^*} \xi^{1-n_2} DI(\xi) d\xi > 0$$

because the weak derivative $DI(\xi) > 0$ almost everywhere, and we also know that $J''(y^*) > 0$ and $dD_1/dP < 0$. Thus, it follows that $\frac{d\pi_t^*}{dP}$ and θ always have opposite signs, which completes the proof.