# Personal Bankruptcy and Post-bankruptcy Liquidity Constraint

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**Abstract**

Once a debtor files for bankruptcy under Chapter 7, all or some of the unsecured debts are discharged and the debtor is endowed with a financial fresh start. However, a post-bankruptcy consumer faces restrictions on borrowing against future income and is likely to be liquidity constrained. This paper intends to provide quantitative analysis regarding to the effects of limits for borrowing against future income imposed on a post-bankruptcy consumer. We obtain explicit expressions for the optimal consumption, investment in the risky asset, and discretionary bankruptcy decision through a duality approach when there exists a liquidity constraint after bankruptcy. The quantitative results show that a post-bankruptcy constraint has significant impacts on a debtor's consumption, investment, and bankruptcy wealth level. The effects of an opportunity to file for bankruptcy compete with those of post-bankruptcy liquidity constraint. We also provide implications on the expected time to bankruptcy.

*Keywords:* Bankruptcy, consumption and investment, liquidity constraint, duality approach
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## 1. Introduction

Consumer bankruptcy is a form of social insurance which provides overburdened debtors with debt relief. The United States law offers two alternatives to debtors for filing bankruptcy. Under Chapter 7, debtors' unsecured debts (e.g. credit cards debt, medical bills, etc) are discharged at the expense of nonexempt assets. Chapter 7 debtors are allowed to preserve future incomes, but are obliged to turn nonexempt assets over to a bankruptcy trustee, who will liquidate them in order to repay creditors. Debtors who file for bankruptcy under Chapter 13 get to keep all of their assets, while all or a portion of their debts should be repaid using future income in accordance with a repayment plan. In short, filing for bankruptcy under chapter 7 means a 'Fresh Start' while Chapter 13 bankruptcy is classified as a reorganization of debts.

Numerous studies regarding personal bankruptcy in the U.S. have reported substantial increases in the number of people filing for bankruptcy, and have investigated reasons for such rise in bankruptcy filing rates. Bukley and Brinig (1998) attributed the dramatic rise of filing for bankruptcy in the U.S. during $1984-1991$ to changes in social norms. Fay et al. (2002) estimated a model to access the impact of financial benefit on personal bankruptcy decision. Zhu (2011) argued that consumption patterns are significant determinants of filing for bankruptcy. According to White (2007), there was a nearly five-fold increase in the number of the U.S. personal bankruptcy filings during $1980-2004$ and the major reason for the dramatic increase in personal bankruptcy was the expansion of credit card borrowing. Literature on the effects of bankruptcy protection on consumers include the following studies. Dobbie and Song (2015) found that Chapter 13 protection may increase earnings and employment. Mahoney (2015) investigated the role of bankruptcy as an implicit health insurance. A justification for establishing a fresh start bankruptcy system is to provide work incentive, which was quantified as a 12.3% average increase in the labor supply of Chapter 7 filers, following the evaluation of Chen and Zhao (2017).

Once a debtor files for bankruptcy under Chapter 7 and preserve future earnings (e.g. labor income), the filer would finance consumption with future earnings to smooth out consumption. However, someone who has filed for bankruptcy has a limited access to the credit market (See Livshits et al. (2007)). Filer and Fisher (2007) also confirmed empirically that filers have less access to credit and investigated filers' consumption sensitivity. The empirical results of Han and Li (2011), which are based on the Survey of Consumer Finances (SCF), imply that bankruptcy filers have restricted access to unsecured credit and have more access to secured debt. As in Filer and Fisher (2007), this restriction on borrowing against future income is a non-pecuniary cost of filing for bankruptcy, while the value of nonexempt assets is a financial cost. For the interdependence between bankruptcy rate and credit market, Gross et al. (2021) shows that the rate of bankruptcy declaration is positively correlated with the endogenous credit interest rate of the debtor.

In this paper, we quantify the impact of post-bankruptcy restrictions on borrowing against future income on consumption, investment, and the bankruptcy decision of a consumer. A consumer enjoys utility from consumption, invests in the financial market, receives an income stream, and repays unsecured debt until bankruptcy. We assume that the consumer is eligible for Chapter 7 bankruptcy, and will face constraints on borrowing against future income if the consumer files for bankruptcy. So, after being the filer, she wants to maximize the expected utility from consumption, but is liquidity constrained. This study is in line with Jeanblanc et al. (2004), which investigated a debtor's decision to file for bankruptcy and the consumption/investment choice problem. However, in Jeanblanc et al. (2004), the debtor is not a wage earner and is not supposed to be faced with liquidity constraints after bankruptcy. We focus the effect of the post-bankruptcy liquidity constraints, and do not impose any constraint on the debtor except for the cost of bankruptcy.

Our debtor's problem as well as that of Jeanblanc et al. (2004) can be cast into a mixed optimal stopping and control problem, in which there exists a wealth jump at bankruptcy due to the cost of bankruptcy. In Jeanblanc et al. (2004), however, the value function at bankruptcy has an explicit form in terms of wealth level and they apply the dynamic programming method to derive the solution. In contrast, in our problem, the value function at bankruptcy is given in an implicit form due to the liquidity constraint, and we apply a duality approach to obtain the closed-form solutions. We first determine the post-bankruptcy value function in the presence of a liquidity constraint. We then solve the optimal stopping time problem of the debtor with a jump in wealth level at the time of stopping. This is quite different from the problems including both borrowing limits and voluntary retirement, such as Choi and Shim (2006), Farhi and Panageas (2007), Dybvig and Liu (2011), Lim and Shin (2011) and others. In those studies, the value function after stopping time is an explicit function of wealth and there is no wealth jump at the time of stopping.

Our explicit solutions show that the post-bankruptcy liquidity constraint significantly impacts on a debtor's decisions on bankruptcy, consumption and investment. First of all, for stricter post-bankruptcy liquidity constraint, the debtor files for bankruptcy at a higher wealth level. With a tightener liquidity constraint, the debtor who wants to smooth consumption before and after bankruptcy declaration should increase the wealth level right after filing for bankruptcy (or the initial wealth of the filer). On the other hand, a strong post-bankruptcy liquidity constraint compels the debtor to become a conservative investor so that she reduces consumption and investment in the risky asset.

Second, the effects of bankruptcy opportunity compete with those of post-bankruptcy liquidity constraint. With a sufficiently strong post-bankruptcy liquidity constraint, the effects of the post-bankruptcy liquidity constraint dominates those of bankruptcy opportunity, and the debtor reduces the consumption and risky investment. In addition, the marginal effects becomes more sensitive for the tighter liquidity constraint. In particular, a sufficiently strong post-bankruptcy liquidity constraint forces the debtor to reduce consumption and investment in the risky asset more dramatically as the wealth level approaches to the bankruptcy wealth level from above.

Third, the expected time to bankruptcy decreases when the post-bankruptcy liquidity constraint is more tightened. The debtor's bankruptcy option is a discretionary decision, and the intertemporal consumption smoothing motive makes the debtor reduce the probability of binding the liquidity constraints by increasing the wealth level right after filing for bankruptcy. If the bankruptcy wealth level is negative, the bankruptcy benefit gives the incentive to accumulate more debt. However, the post-bankruptcy liquidity constraint restricts this accumulation, and it leads to the earlier bankruptcy declaration as the post-bankruptcy liquidity constraint becomes stronger. Finally, we extend the model to the problem where the debt maturity is finite, and the effective period of a liquidity constraint after bankruptcy is also finite. We confirm that the qualitative results remain unchanged.

The rest of this paper proceeds as follows. Section 2 introduces the continuous-time financial market and the bankruptcy procedure. Section 3 describes the debtor's optimization problem in the presence of the post-bankruptcy liquidity constraint and formulates a mixed optimal stopping and control problem. The explicit solutions are obtained in Section 4 and we provide the implications on the optimal policies in Section 5. Section 6 provides further discussions and Section 7 concludes.

## 2. The Model

*Financial market*

In a continuous-time financial market, we assume that there exist two assets. One is a riskless asset and the other is a risky asset, whose prices at time $t$ are denoted by $S_t^0$ and $S_t$, respectively. The riskless asset has a constant rate of return $r > 0$ so it has dynamics as $dS_t^0 = rS_t^0 dt$. The risky asset follows a geometric Brownian motion with constant coefficients, which evolves as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the $\mathbb{P}$-augmentation of the natural filtration generated by the standard Brownian motion $(B_t)_{t\geq 0}$ by $(\mathcal{F}_t)_{t\geq 0}$. We consider a consumer who receives an income stream at a constant rate of $\bar{I}$ (for example, labor income). In addition, we assume that there are no transaction/information costs, or other market frictions. We denote by $\mathbf{c} \equiv (c_t)_{t\geq 0}$ and $\boldsymbol{\pi} \equiv (\pi_t)_{t\geq 0}$ the consumption rate process and portfolio process (amount of money invested in the risky asset), respectively. We assume that $c_t \geq 0$ and $\pi_t$ are $\mathcal{F}_t$-progressively measurable and satisfy the following technical conditions

$$\int_0^t c_s ds < \infty, \text{ for all } t \geq 0 \ a.s., \qquad \int_0^t \pi_s^2 ds < \infty \text{ for all } t \geq 0 \ a.s. \tag{2.1}$$

We denote $S_{[0,T]}$ by the set of all $\mathcal{F}_T$-stopping times for a fixed $T$ and define the set of stopping times when $T \to \infty$ by $S$.

*Bankruptcy and liquidity constraint*

We suppose that the consumer is a debtor and suffers from a continuous debt repayment at a rate of $\delta$. The continuous debt repayment can be discharged by filing for bankruptcy under Chapter 7 without income garnishment. After bankruptcy, the debt repayment is forgiven, and the debtor is supposed to receive the same income stream $\bar{I}$ as she did previously. If we denote the time for bankruptcy by $\tau$, the wealth level process for $t \geq 0$ evolves according to the following equation

$$dX_t = \begin{cases} \left(rX_t + \pi_t(\mu - r) - c_t - \delta + \bar{I}\right) dt + \sigma\pi_t dB_t, & 0 \leq t \leq \tau, \\ \left(rX_t + \pi_t(\mu - r) - c_t + \bar{I}\right) dt + \sigma\pi_t dB_t, & t > \tau, \end{cases}$$

with the initial wealth level $X_0 = x$. We assume that $\tau$ is an $\mathcal{F}_t$-stopping time. A static budget constraint for the debtor is given as follows

$$\mathbb{E}\left[\int_0^\tau H_t\left(c_t+\delta\right)dt + H_\tau\left(X_\tau+\frac{\bar{I}}{r}\right)\right] \le x + \frac{\bar{I}}{r}, \tag{2.2}$$

for any $\tau \in S$. The pricing kernel $H_t$ is given by $H_t \equiv e^{-rt}e^{-\frac{1}{2}\theta^2 t-\theta B_t}$, where $\theta = (\mu - r)/\sigma$. Similarly, the static budget constraint of a liquidity constrained filer with an initial wealth $X_0 = x$ is as follows

$$\mathbb{E}\left[\int_0^\infty H_t(c_t-\bar{I})dt\right] \le x. \tag{2.3}$$

At the time of bankruptcy, the debtor is obliged to pay a fixed cost $F$ (for example, a legal service fee) and we assume that the fee for filing for bankruptcy $F$ should not exceed the present value of the total debt $\delta/r$, i.e., $\delta/r > F$. Moreover, the debtor is supposed to retain the rate $\alpha$ of the wealth level $X_\tau$ less the cost $F$. Thus, the wealth level immediately following bankruptcy is given by

$$X_{\tau_+} = \alpha(X_\tau - F), \quad 0 < \alpha < 1.$$

Given $(X_\tau - F) > 0$, $\alpha$ can be considered as the exemption rate. When $(X_\tau - F) < 0$, however, $(1-\alpha)$ stands for the rate of the value of discharged asset to the lump sum debt $|(X_\tau - F)|$ (apart from the debt repayment stream $\delta$).[1]

We impose the liquidity constraint with which the filer will be faced. Due to the bankruptcy flag, the consumer who has previously filed for bankruptcy has a limited access to the credit market. Therefore, the filer experiences difficulty in borrowing against future income stream. We assume that the filer can borrow only up to a fraction $w \in [0,1]$ of the present value of the total income stream, that is

$$X_t \ge -\frac{\bar{I}}{r}w, \quad \text{for } t > \tau. \tag{2.4}$$

Note that if there is no liquidity constraint ($w = 1$), the filer's borrowing limit is $\bar{I}/r$. Whenever $w < 1$, the liquidity constraint in (2.4) can be rewritten as $X_t \ge -\bar{I}/(r/w)$. This

[1] In practice, a bankruptcy debtor can exempt a certain amount of his/her property after filing for bankruptcy. Specifically, the necessities of real life such as motor vehicles, clothing, household goods, and furnishings up to specific values are exempt. In addition, a portion of unpaid but earned wages, pension, and public benefits, including public assistance, social security, and accumulated in a bank account, are exempt. If the exempt assets have less correlated with the debtor's financial wealth, we can regard that $\alpha$ is low enough. Since $\alpha \in (0,1)$ in our model, however, our model includes the case when the exempt assets have a low correlation with the financial wealth.

means that the filer's effective interest rate (the interest rate in the credit market) becomes larger than $r$. Since the liquidity constraint can be regarded as the higher interest rates in the credit market, the permanent liquidity constraint of the filer can be thought as the higher effective interest rate in the credit market $(r/w)$ without liquidity constraint.[2]

## 3. The optimization problem

We assume that the debtor has a CRRA utility function of consumption, which is defined by

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1, \ \gamma > 0,$$

both before and after bankruptcy. Then if $\beta > 0$ is the time preference rate, the optimization problem of the debtor is to maximize the following expected discounted lifetime utility by choosing optimal consumption, portfolio, and bankruptcy time[3]

$$\mathbb{E}\left[\int_0^\tau e^{-\beta t} u(c_t)dt + e^{-\beta\tau} V_f(\alpha(X_\tau - F))\right],$$

subject to the budget constraint (2.2) and liquidity constraint (2.4). The function $V_f(\cdot)$ is the liquidity-constrained filer's value function which will be obtained below. We assume that the filer's lifetime is infinite and optimally chooses consumption and portfolio. The debtor's optimal bankruptcy time is characterized through a wealth threshold, we call the *bankruptcy wealth level*. If the debtor files for bankruptcy, the debtor becomes a filer who faces a liquidity constraint.

The tradeoff between debt repayment and liquidity constraint with an initial wealth $\alpha(X_\tau - F)$ makes the debtor choose the bankruptcy option or not. In other words, for a debtor before bankruptcy declaration, repaying the debt is less painful than facing the liquidity constraint with a wealth adjustment $(\alpha(X_\tau - F))$.

Our debtor's problem can be solved by two consecutive procedures. First, we determine the post-bankruptcy value function, $V_f(\cdot)$. Second, we consider the optimal stopping time

[2] In practice the filer faces a liquidity constraint within a limited time periods. We will provide the model with a limited effectiveness of a post-bankruptcy liquidity constraint in Subsection 5.1.

[3] If there is no option to file for bankruptcy, the present value of the remaining debt is defined by $\mathbb{E}\left[\int_0^\infty H_t \delta dt\right] = \delta/r$. With a bankruptcy option, however, the economic value of the remaining debt is less than $\delta/r$. This implies that the full repayment $(\delta/r)$ before bankruptcy is sub-optimal. Thus, the debtor never repays the debt in full even though the debtor's wealth is high enough, and the full repayment option does not affect our qualitative result.

problem, where $V_f(\cdot)$ is the function at optimal stopping time. Each step is well studied in the literature but the combined procedure is nontrivial in the sense that the value function at stopping time is an implicit function of wealth. Moreover, there is a wealth level jump at the time of stopping. This is quite different from the problems including both borrowing limits and voluntary retirement, such as Farhi and Panageas (2007), Choi et al. (2008), Dybvig and Liu (2011), and Lim and Shin (2011). In these studies, the value function after stopping time is an explicit function of wealth and there is no wealth jump at the time of stopping.

### *3.1. Filer's optimization problem*

We first consider the filer's optimization problem with the initial wealth level $X_0 = x$. Note that the remaining debt is vanished if the debtor chooses to file for bankruptcy. Instead, the filer is supposed to face a liquidity constraint defined in (2.4). For defining the filer's optimization problem, we define admissible policy pair as follows.

**Definition 3.1.** *We call* $(\mathbf{c}, \boldsymbol{\pi})$ *an admissible policy pair at* $x$ *if*

*(a)* $\mathbf{c}$ *and* $\boldsymbol{\pi}$ *satisfy* (2.1),

*(b)* $X_t \geq -\frac{\bar{I}}{r}w,$ *for* $t \geq 0$,

*(c)* $\mathbb{E}\displaystyle\int_0^\infty e^{-\beta t}|u(c_t)|dt < \infty.$

Denote by $\mathcal{A}_f(x)$ the set of all admissible policy pairs at $x$. We define the value function

$$V_f(x) \equiv \sup_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}_f(x)} J_{f,(\mathbf{c},\boldsymbol{\pi})}(x), \tag{3.1}$$

where

$$J_{f,(\mathbf{c},\boldsymbol{\pi})}(x) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}u(c_t)dt\right],$$

subject to the budget constraint (2.3). We give an assumption that makes $V_f(x)$ attainable[4]. The function $V_f(x)$ in (3.1) is the liquidity-constrained filer's value function which will be obtained analytically. We assume that the filer's lifetime is infinite and optimally chooses consumption and portfolio. The debtor's optimal bankruptcy time is characterized through a wealth threshold, we call the bankruptcy wealth level. If the debtor files for bankruptcy, the debtor becomes a filer who faces a liquidity constraint.

[4] $V_f(x)$ is said to be *attainable* if there exists an admissible policy pair $(\tilde{\mathbf{c}}, \tilde{\boldsymbol{\pi}})$ such that $V_f(x) = J_{f,(\tilde{\mathbf{c}},\tilde{\boldsymbol{\pi}})}(x)$.

**Assumption 3.1.**

$$K \equiv r + \frac{\beta - r}{\gamma} + \frac{(\gamma - 1)}{2\gamma^2}\theta^2 > 0. \tag{3.2}$$

Assumption 3.1 guarantees nonnegative consumption and there is not much difference from reality, especially for the case where $\gamma$ is fairly large. Under the condition (3.2), we can find the solution to the optimization problem (3.1). Let $c_t^f = C^f(X_t)$ and $\pi_t^f = \Pi^f(X_t)$ be the optimal consumption rate and investment in the risky asset for the value function $V_f$, respectively. Then the solutions to the filer's problem is given in Lemma Appendix C.1. The optimal consumption and investment in the risky asset of the filer can be rewritten as follows

$$C^f(X_t) = K\left(X_t + \frac{\bar{I}}{r}\right) + n_+ K A(\mathcal{Y}_0(X_t))^{n_+ - 1}, \tag{3.3}$$

$$\Pi^f(X_t) = \frac{\theta}{\gamma\sigma}\left(X_t + \frac{\bar{I}}{r}\right) + \frac{\theta}{\gamma\sigma}(1 + \gamma n_+ - \gamma) A(\mathcal{Y}_0(X_t))^{n_+ - 1}, \tag{3.4}$$

where $A$ and $\mathcal{Y}_0(\cdot)$ are given in (C.2) and (C.6), respectively. The first terms of the optimal consumption (3.3) and investment in the risk asset (3.4) are exactly the same as those in Merton (1969). On the other hand, the last terms present a penalty due to the liquidity constraint. Since the liquidity constraint makes the investment opportunity set shrunken, it forces reduce consumption. Actually, the coefficient $A$ is negative, so these penalty terms are always negative. In words, the liquidity constraint always leads the filer to reduce consumption and investment in the risky asset. In Section 4, we will show that the liquidity constraint after bankruptcy also negatively impacts on the debtor's decisions for consumption and investment in the risky asset.

*3.2. Debtor's optimization problem*

After bankruptcy the filer will adopt the optimal strategies $C^f(X_t)$ and $\Pi^f(X_t)$ to maximize expected lifetime utility from consumption, i.e., to obtain the value function $V_f(X_t)$. This implies that we only need to determine the optimal bankruptcy time $\tau^*$, the optimal consumption rate $c_t^*$, and the optimal investment in the risky asset $\pi_t^*$ before bankruptcy. Therefore, the debtor's optimization problem is to find the following value function

$$V(x) \equiv \sup_{(\mathbf{c},\boldsymbol{\pi},\tau)\in\mathcal{A}(x)} J_{(\mathbf{c},\boldsymbol{\pi},\tau)}(x), \tag{3.5}$$

subject to the budget constraint (2.2) with the initial wealth level $X_0 = x$, where

$$J_{(\mathbf{c},\boldsymbol{\pi},\tau)}(x) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} u_1(c_t)dt + e^{-\beta\tau} u_2(X_\tau)\right],$$

$$u_1(x) = u(x), \quad u_2(x) = V_f\left(\alpha\left(x - F\right)\right),$$

and $\mathcal{A}(x)$ is the set of all admissible policy triples at $x$, which is defined below. The debtor is not liquidity constrained, so he can finance up to the present value of the total income stream $\bar{I}/r$. Filing for bankruptcy is discretionary so it is assumed that the debtor can afford to pay the present value of the debt stream $\delta/r$ as well as the lump sum cost of filing for bankruptcy $F$. Therefore, we impose the following inequality

$$X_t - \frac{\delta}{r} - F \geq -\frac{\bar{I}}{r}, \; 0 \leq t \leq \tau. \tag{3.6}$$

If the debtor files for bankruptcy at time $t$, $X_{t+} = \alpha(X_t - F)$ should satisfy the inequality of Definition 3.1 (b), so we have

$$\alpha(X_t - F) \geq -\frac{\bar{I}}{r}w, \; 0 \leq t \leq \tau. \tag{3.7}$$

Thus we require that (3.6) and (3.7) should hold for defining admissible policy triples.

**Definition 3.2.** *We call* $(\mathbf{c}, \boldsymbol{\pi}, \tau)$ *an admissible policy triple at* $x$ *if*

1. $\mathbf{c}$ *and* $\boldsymbol{\pi}$ *satisfy* (2.1),
2. $\tau \in S$,
3. $X_t \geq \hat{x} \equiv \max\left(-\frac{\bar{I} - \delta}{r} + F, \; -\frac{\bar{I}}{\alpha r}w + F\right), \quad 0 \leq t \leq \tau,$
4. $\mathbb{E}\int_0^\tau e^{-\beta t}|u(c_t)| \, dt < \infty.$

As is shown in Karatzas and Wang (2000), the optimization problem (3.5) can be cast into a pure optimal stopping problem. For any fixed $\tau \in S$, we define the set $\mathcal{B}_\tau(x)$ of consumption and portfolio pairs $(\mathbf{c}, \boldsymbol{\pi})$ which satisfies $(\mathbf{c}, \boldsymbol{\pi}, \tau) \in \mathcal{A}(x)$ and

$$V_\tau(x) \equiv \sup_{(\mathbf{c}, \boldsymbol{\pi}) \in \mathcal{B}_\tau(x)} J_{(\mathbf{c}, \boldsymbol{\pi}, \tau)}(x), \tag{3.8}$$

then we have

$$V(x) = \sup_{\tau \in S} V_\tau(x) = \inf_{y > 0}\left[\widetilde{V}(y) + y\left(x + \frac{\bar{I}}{r}\right)\right], \tag{3.9}$$

where $\widetilde{V}(y)$ is defined in (D.5).

By applying a duality approach, we can determine the debtor's optimal policies in the following theorem. The detailed derivation is provided in Appendix D.

**Theorem 3.1.** *Let us denote the bankruptcy wealth level by $\underline{x}$. For a given initial wealth level $X_0 = x > \underline{x}$, the debtor's value function $V(x)$ of* (3.5) *is given by*

$$V(x) = \widetilde{V}(y^*) + y^* \left( x + \frac{\bar{I}}{r} \right),$$

*where $y^*$ is implicitly determined from the equation*

$$x = h\left(y^*\right), \tag{3.10}$$

*where $\underline{x}$ and $h(\cdot)$ are given* (D.14) *and* (D.15)*, respectively. The optimal bankruptcy time $\tau^*$ is given by*

$$\tau^* = \inf\{t > 0 | y_t^* \geq \bar{y}\} = \inf\{t > 0 \mid X_t \leq \underline{x}\}, \tag{3.11}$$

*where $y_t^* = y^* e^{\beta t} H_t$ and $\bar{y}$ is given in* (D.9)*. For a given debtor's wealth level $X_t$, the optimal consumption rate $c_t^*$ and investment in the risky asset $\pi_t^*$ in feedback forms are given by*

$$c_t^* = K \left( X_t + \frac{\bar{I} - \delta}{r} \right) + n_+ K B \left(\mathcal{Y}(X_t)\right)^{n_+ - 1}, \tag{3.12}$$

$$\pi_t^* = \frac{\theta}{\gamma\sigma} \left( X_t + \frac{\bar{I} - \delta}{r} \right) + \frac{\theta}{\gamma\sigma}(1 + \gamma n_+ - \gamma) n_+ B \left(\mathcal{Y}(X_t)\right)^{n_+ - 1}, \tag{3.13}$$

*where $n_+$ is defined in* (C.1)*, $B$ is given by*

$$B = \frac{A}{\alpha^{n_+}} + \frac{\frac{\bar{I}}{r}\left(\frac{1}{\alpha} - 1\right) + \frac{\delta}{r} - F}{1 - \gamma + n_+\gamma} \bar{y}^{1 - n_+}, \tag{3.14}$$

*and $\mathcal{Y}(X_t)$ satisfies*

$$X_t = h\left(\mathcal{Y}(X_t)\right). \tag{3.15}$$

From (3.11) and (3.15), we see that

$$\tau^* = \inf\{t > 0 \mid c_t^* \leq \bar{y}^{-\frac{1}{\gamma}}\},$$

which implies that the debtor files for bankruptcy at the first time when the consumption level reaches a certain level $\bar{y}^{-\frac{1}{\gamma}}$ from above. Note that $\bar{y}$ is not dependent on $w$. Therefore, the debtor files for bankruptcy at the same level of consumption rate regardless of how strong the post-bankruptcy liquidity constraint is.

From now on, to emphasize the dependencies, let us denote $\underline{x}$, $A$ and $B$ by $\underline{x}(w)$, $A(w)$ and $B(w)$, respectively. Similar to the filer's problem, the first terms of the optimal consumption and investment in the risky asset in (3.12) and (3.13) are exactly the same as those of Merton (1969). The last terms which involve the coefficient $B(w)$ reflect the combined effects of discretionary bankruptcy and post-bankruptcy liquidity constraint. More precisely, the effects of bankruptcy opportunity compete with those of the post-bankruptcy liquidity constraint. The first term of $B(w)$ in (3.14), $A(w)/\alpha^{n_+}$, reflects the effect of the post-bankruptcy liquidity constraint and is always negative. Thus, we can say that the existence of a liquidity constraint after filing for bankruptcy lowers a debtor's consumption and investment in the risky asset. In contrast, the second term of $B(w)$, $\frac{\frac{\bar{I}}{r}(\frac{1}{\alpha}-1)+\frac{\delta}{r}-F}{1-\gamma+n_+\gamma}\bar{y}^{1-n_+} > 0$, reflects the effects due to bankruptcy opportunity and is positive. Thus, we can also say that the opportunity to file for bankruptcy leads the debtor to increase consumption and investment in the risky asset.

For a sufficiently low $w$, i.e., in the case of a strong post-bankruptcy liquidity constraint, $B(w)$ can be negative. While, under a sufficiently high $w$, $B(w)$ can be positive and the effect of bankruptcy opportunity on consumption and investment dominates that of the post-bankruptcy liquidity constraint. The following proposition provides such a condition.

**Proposition 3.1.** *Under the following condition, the aggregate effects of bankruptcy opportunity and post-bankruptcy liquidity constraint on optimal consumption and investment in the risky asset are positive, i.e.,* $B(w) > 0$.

$$1 - \frac{1}{\gamma}\left\{\frac{1}{\alpha} - 1 + \frac{r}{\bar{I}}\left(\frac{\delta}{r} - F\right)\right\}\left[n_+\gamma\alpha^{n_+}\left\{\frac{1-\gamma}{1-\frac{1}{\alpha}^{1-1/\gamma}}\right\}^{\gamma n_+ - \gamma}\right]^{\frac{1}{1+\gamma n_+ - \gamma}} < w. \tag{3.16}$$

Figure 1 shows a debtor's consumption and investment in the risky asset for different values of $w$. With the given parameters, $w = 0.5$ satisfies the condition (3.16) and $w = 0.1$ does not. This implies that when $w = 0.5$ (or $w = 0.1$), the effect of the bankruptcy option (or post-bankruptcy liquidity constraint) dominates so the combined effect is positive (or negative). Note that the condition (3.16) is independent of the wealth level, so the dominant effect is unchanged for any wealth level before bankruptcy.

## 4. Implications

### 4.1. Comparative statistics

In this subsection, we provide analytic results on comparative statics of the optimal polices with respect to $w$. Due to the fact that $w$ is involved in the coefficient $A$ in (C.2) but not in $\bar{y}$, the dependencies of the bankruptcy wealth level, optimal consumption and investment on post-bankruptcy liquidity constraint in Theorem 3.1 are described only by the coefficient $B$ in (3.14).

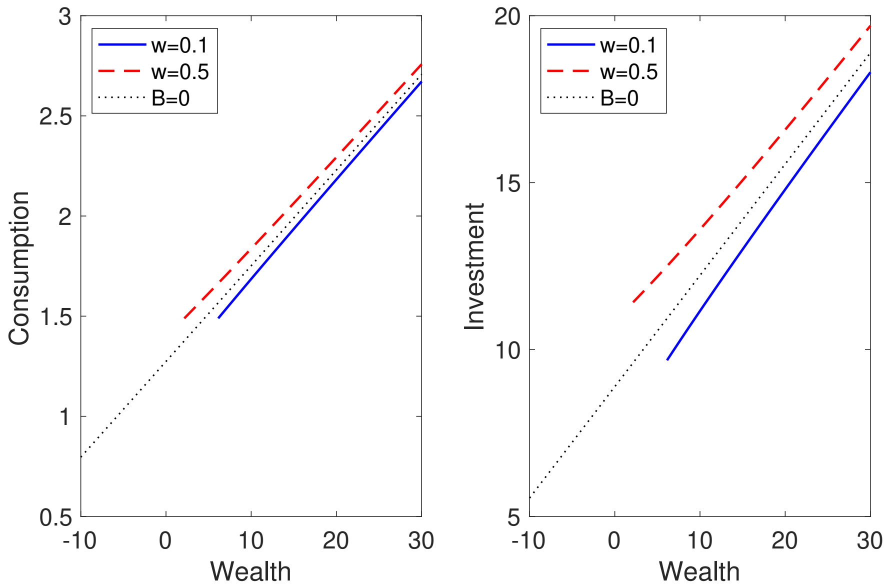
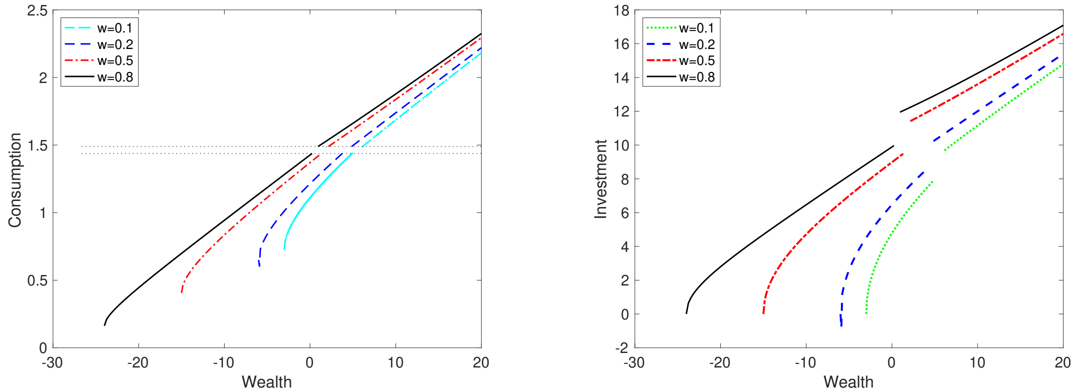



Figure 1: The combined effects of a bankruptcy option and a post-bankruptcy liquidity constraint. ($\gamma = 3, \mu = 0.07, \sigma = 0.2, \beta = 0.07, r = 0.03, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, \alpha = 0.9$).

**Proposition 4.1.** *The strategic bankruptcy wealth level $\underline{x}(w)$ decreases with $w$, whereas the optimal consumption $c_t^*$ and the investment in the risky asset $\pi_t^*$ both increase with $w$ for a given wealth level $X_t$.*

Proposition 4.1 implies that a more stringent liquidity constraint leads to lower consumption, less investment in the risky asset, and higher bankruptcy wealth level. This is consistent with an intuition that a debtor who is faced with a stronger liquidity constraint after filing bankruptcy tends to increase saving and allocate more into the risk-free asset. By doing that, the debtor who wants to smooth consumption before and after filing for bankruptcy tends to increase bankruptcy wealth level so as to reduce the probability of binding the post-bankruptcy liquidity constraint. Figure 2 illustrates the effects of post-bankruptcy liquidity constraint for different values of $w$ on the optimal consumption and investment in the risky asset.

There are consumption drops at bankruptcy, and the size of reduction is independent of $w$. This is obvious because $w$ has no effect on the free boundary value $\bar{y}$ and the optimal consumption rate is given by $\bar{y}^{-1/\gamma}$ as we can see in Theorem 3.1. The horizontal dotted lines in the left panel of Figure 2 show the constant size of consumption reduction as well as the




(a) Consumption

(b) Portfolio

Figure 2: Consumption and investment in the risky asset for $-w\bar{I}/r < $ Filer's wealth level $\leq \alpha(\underline{x}(w) - F)$ & Debtor's wealth level $\geq \underline{x}(w)$. The market parameters are given by $\gamma = 3, \mu = 0.07, \sigma = 0.2, \beta = 0.07, r = 0.03, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, \alpha = 0.9$.

constant consumption rate at bankruptcy.[5]

*4.2. Consumption near bankruptcy*

We take a closer look at the consumption and investment in the risky asset when a debtor's wealth level approaches to the bankruptcy wealth level through investigating the sensitivity of consumption and that of investment in the risky asset to wealth, i.e., marginal propensity to consume (MPC) and marginal propensity to invest (MPI) out of wealth. We have shown that the combined effect of a bankruptcy opportunity and post-bankruptcy liquidity constraint, which is reflected in the last hedging terms in (3.12) and (3.13) can be either positive or negative. As explained, the main factor which determines the combined effect is how strong the post-bankruptcy liquidity constraint is. If the hedging term is positive (negative), the effect of bankruptcy opportunity (post-bankruptcy liquidity constraint) dominates that of the post-bankruptcy liquidity constraint (bankruptcy opportunity). Hence when a debtor's wealth approaches the bankruptcy wealth level from above, the debtor will try to reduce consumption, but the marginal behavior depends on the degree of the post-bankruptcy liquidity constraint. For a more detailed description, we provide the explicit results on the marginal propensity to consume (MPC) as in the following proposition.

[5]This result does not hold anymore if the time to maturity of the debt is finite. Please see Subsection 5.1 for more details.

**Proposition 4.2.** *The marginal propensity to consume (MPC) out of wealth denoted by* $M_c(X_t)$ *is given by*

$$M_c(X_t) = \frac{1}{n_+(n_+-1)\gamma B(w)(\mathcal{Y}(X_t))^{n_+ + \frac{1}{\gamma}-1} + 1/K},$$

*where* $\mathcal{Y}(X_t)$ *is given in* (3.15)*. Moreover,* $M_c(X_t)$ *increases (decreases) with wealth level if and only if* $B(w) > 0 \quad (B(w) < 0)$*.*

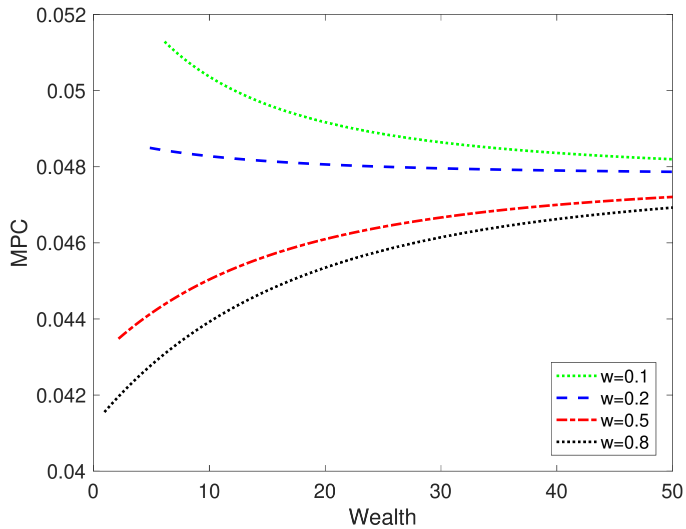



Figure 3: Debtors' MPC out of wealth for Wealth level $\geq \underline{x}(w)$
($\gamma = 3, \mu = 0.07, \sigma = 0.2, \beta = 0.07, r = 0.03, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, \alpha = 0.9$).

Figure 3 shows the MPCs out of wealth for different values of $w$. If $w$ is sufficiently low (for example, $w = 0.1,\ 0.2$ in Figure 3), we can confirm that MPC out of wealth increases as the wealth level declines to the bankruptcy wealth level. In this case, the debtor reduces consumption more dramatically as the wealth level approaches to the bankruptcy wealth level from above. On the other hand, if $w$ is sufficiently high (for example, $w = 0.5\ , 0.8$ in Figure 3), MPC out of wealth decreases as the wealth decreases to the bankruptcy wealth level and the debtor reduces consumption more moderately as the wealth level decreases to the bankruptcy wealth level. In addition, it is easy to check that for any given wealth level $X_t$, MPC out of wealth decreases with $w$. Thus, for a fixed wealth level, the debtor who will be faced with a stronger liquidity constraint after bankruptcy reduces consumption more rapidly for a given reduction in wealth although the consumption level itself of this debtor is lower than that of a debtor with a weaker post-bankruptcy liquidity constraint (See also Figure 2(a)).

*4.3. Investment near bankruptcy*

In this subsection, we present the optimal investment in the risky asset when a debtor's wealth level is very near to the bankruptcy wealth level. For more detailed analysis, we provide the explicit result on the marginal propensity to invest (MPI) out of wealth as in the following proposition.

**Proposition 4.3.** *The marginal propensity to invest (MPI) out of wealth denoted by $M_\pi(X_t)$ is given by*

$$M_\pi(X_t) = \frac{\frac{\theta}{\sigma}\left(-n_+(n_+-1)^2 B(w)(\mathcal{Y}(X_t))^{n_+ + \frac{1}{\gamma}-1} + \frac{1}{\gamma^2 K}\right)}{n_+(n_+-1)B(w)(\mathcal{Y}(X_t))^{n_+ + \frac{1}{\gamma}-1} + \frac{1}{\gamma K}}.$$

*Moreover, $M_\pi(X_t)$ increases (decreases) with wealth level if and only if $B(w) > 0$ $(B(w) < 0)$.*

Proposition 4.3 implies that MPI out of wealth shows similar patterns to MPC out of wealth. Figure 4 describes MPI out of wealth for different values of $w$. If the post-bankruptcy liquidity constraint is sufficiently strong (for example, $w = 0.1,\ 0.2$ in Figure 4), the debtor reduces investment in the risky asset more dramatically as wealth approaches the bankruptcy wealth level from above. However, with a relaxed post-bankruptcy liquidity constraint (for example, $w = 0.5,\ 0.8$ in Figure 4), the debtor reduces investment in the risky asset more moderately as the wealth level decreases to the bankruptcy wealth level. It can be also easily checked that for any given wealth level $X_t$, MPI out of wealth decreases with $w$. Therefore, for a fixed wealth level, a debtor with a strong post-bankruptcy liquidity constraint cuts investment in the risky asset more sharply for a given reduction in wealth than a debtor with a weak post-bankruptcy liquidity constraint does, although the amount of money invested in the risky asset itself for the former is lower (See also Figure 2(b)).

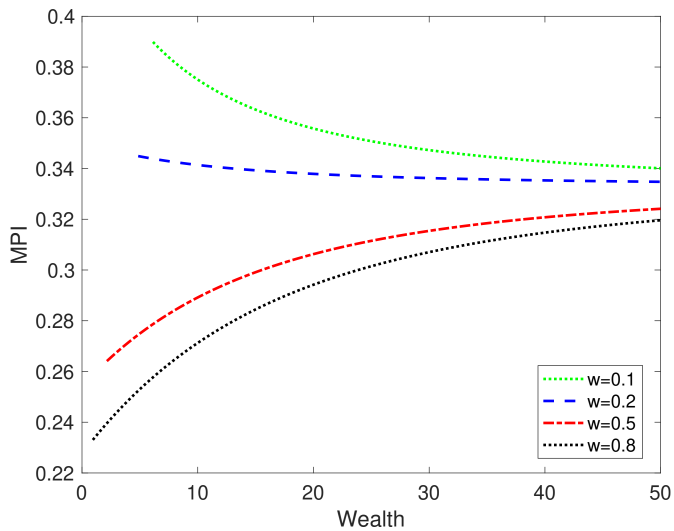


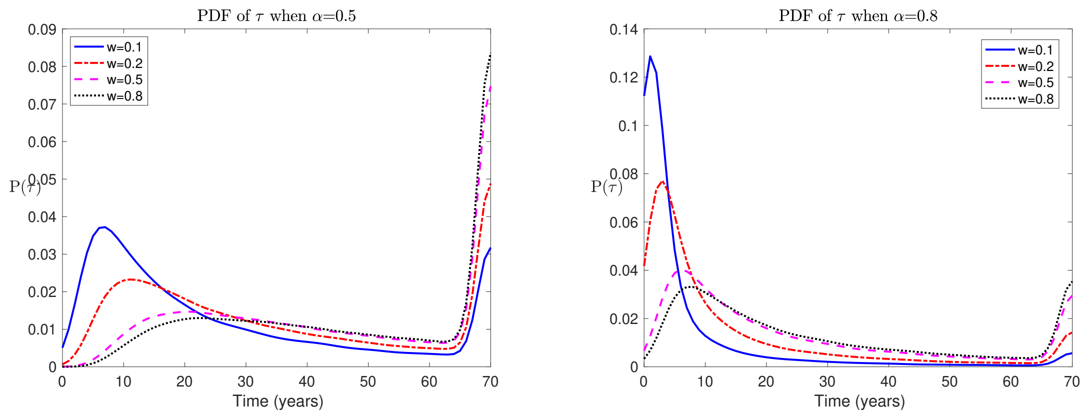

Figure 4: Debtors' MPI out of wealth for Wealth level $\geq \underline{x}(w)$
$(\gamma = 3, \mu = 0.07, \sigma = 0.2, \beta = 0.07, r = 0.03, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, \alpha = 0.9)$.

In sum, when the post-liquidity constraint is tightened, the debtor becomes more sensitive as the debtor's wealth approaches the bankruptcy wealth, and reacts in a conservative way. In particular, the consumption and investment in the risky asset decrease more rapidly compared to those of the debtor who has a less severe post-bankruptcy liquidity constraint. In terms of investment, the debtor invests in the risky asset by using borrowing money from money market account, so the decrease in investment implies the reduced borrowing. Thus, if the bankruptcy wealth level is negative, the stronger post-bankruptcy liquidity constraint makes the debtor file for bankruptcy with a smaller borrowing. We relate the post-bankruptcy liquidity constraint to the expected bankruptcy time in the next subsection.

*4.4. Expected time to bankruptcy*

In this subsection, we focus on the time to bankruptcy. We perform Monte Carlo simulation 100,000 times according to $w$. For experimental purpose, we set 70 years as the end of the time period, due to that it is not possible to simulate for an infinite time horizon. The left panel in Figure 5 shows the probability density function of bankruptcy time when $\alpha = 0.5$, and the right panel shows the probability density function of bankruptcy time when $\alpha = 0.8$. It is shown that the peak of the density function is higher as $w$ decreases and tilted to the left. This implies that the debtor who has a lower $w$ intends to bankrupt more earlier.

On the other hand, we confirm that regardless of the degree of liquidity constraints after




(a) $\alpha = 0.5$ (b) $\alpha = 0.8$

Figure 5: Probability density functions of optimal bankruptcy time $\tau^*$. The parameter set is given by $\mu = 0.7, \sigma = 0.2, r = 0.03, \beta = 0.07, \gamma = 3, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2$, and $X_0 = 10$. We simulate 100,000 sample paths and fit the probability density function by using kernel density estimation with Gaussian kernel.

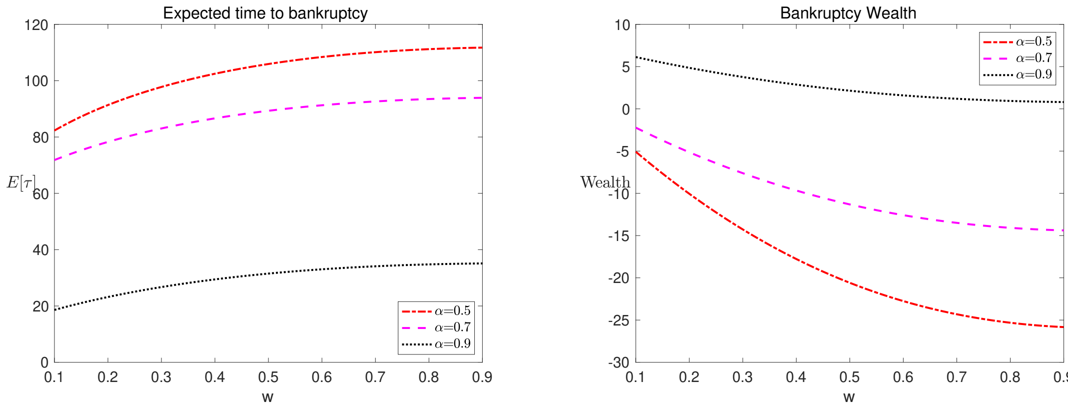
bankruptcy, there exist samples whose bankruptcy time is higher than given time period (70 years). These samples can be seen as the paths of the debtors who do not choose to file for bankruptcy until the end of the time period. With the higher liquidity constraint after bankruptcy, the less samples of the non-default debtors are observed.

For the better understanding of the bankruptcy time, we provide the expected time to bankruptcy analytically. From the optimal stopping time $\tau^*$ in Theorem 3.1, we can derive the expected value of the optimal stopping time as follows.

$$\mathbb{E}\left[\tau^*|X_0 = x\right] = \begin{cases} \frac{\ln y - \ln \bar{y}}{\frac{1}{2}\theta^2 - \beta + r}, & \text{if } \frac{1}{2}\theta^2 - \beta + r < 0, \\ \infty, & \text{otherwise,} \end{cases}$$

where $y$ is determined by the algebraic equation $x = -n_+ B(w) y^{n_+ - 1} + \frac{1}{K} y^{-\frac{1}{\gamma}} - \frac{\bar{I} - \delta}{r}$. We depict the expected time to bankruptcy and the corresponding bankruptcy wealth level in Figure 6. It can be shown that the expected time to bankruptcy increases with $w$ but the bankruptcy wealth level decreases with $w$.

The more rapid decreases in consumption and investment in the risky asset near bankruptcy leads to the earlier bankruptcy. More specifically, when $(X_{\tau^*} - F) > 0$, $\alpha$ is an exemption rate and it is a bankruptcy cost rather than benefit. The debtor has a less incentive to accumulate her wealth before default. However, the post-liquidity constraint makes the debtor accumulate more wealth, and this phenomenon becomes stronger as $w$ decreases. In Figure 6, it is shown that when $\alpha = 0.9$ the bankruptcy wealth level is positive and it increases as $w$ decreases. This




(a) Expected time to bankruptcy

(b) Bankruptcy wealth level

Figure 6: Expected time to bankruptcy ($\mathbb{E}[\tau^*]$) and bankruptcy wealth threshold($\underline{x}$). The parameter set is given by $\mu = 0.7, \sigma = 0.2, r = 0.03, \beta = 0.07, \gamma = 3, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, X_0 = 10$.

leads to the earlier bankruptcy.

On the other hand, when $(X_{\tau^*} - F) < 0$, $\alpha$ would be the bankruptcy benefit rather than cost, so the debtor has an incentive to accumulate more debt before filing for bankruptcy. The post-liquidity constraint restricts this accumulation and this makes the debtor file for bankruptcy earlier as the post-bankruptcy liquidity constraint is more tightened. In Figure 6, when $\alpha = 0.5$ or 0.7, $(X_{\tau^*} - F)$ is negative, and the smaller $w$ leads to the earlier bankruptcy. In summary, as $w$ decreases, the bankruptcy wealth threshold increases and the expected time to bankruptcy decreases regardless of the sign of $(X_{\tau^*} - F)$.

## 5. Further Discussion

### *5.1. Finite debt maturity and limited effective post-bankruptcy liquidity constraints*

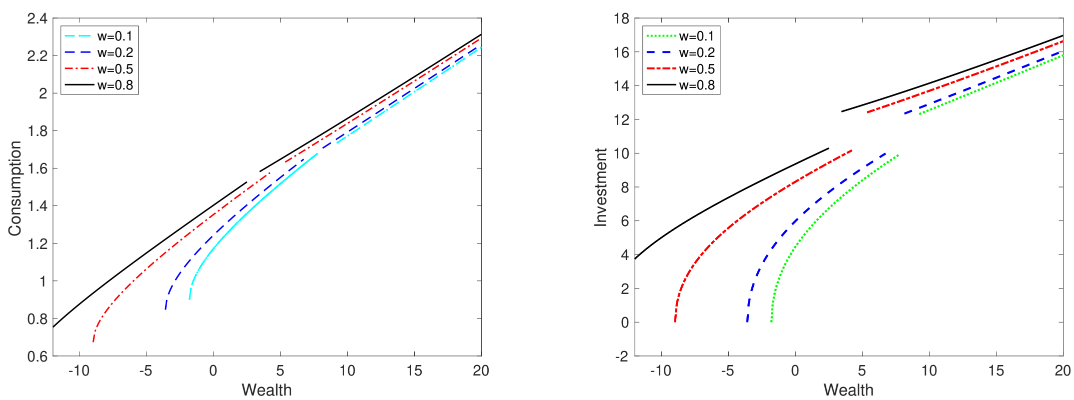
We extend the our model into the problem with a finite debt maturity and a limited effective post-bankruptcy liquidity constraint. Similar to Liu and Loewenstein (2002)[6], we adopt independent random times which are exponentially distributed. In that case, our qualitative

[6]Liu and Loewenstein (2002) study an optimal portfolio selection in the presence of transaction costs. Instead of a fixed time horizon, they suppose an uncertain time, and show that a sequence of the solutions with random times converges to the solution with a deterministic finite horizon. In this paper, we want to show whether our qualitative results hold even with a finite horizon or not. So, it is enough to show one solution with random times.

results still hold because the problem can be solved by interchanging the market parameters which do not affect the comparative analysis of the post-bankruptcy liquidity constraint ($w$).

We consider the maturity time $T_1$ as an independent random time which is exponentially distributed with an intensity $\eta_1 > 0$. Then the expected time for random time $T_1$ is given by $1/\eta_1$, and the survival probability until time $t$ is defined by $F(t) = e^{-\eta_1 t}$. Similarly, for a limited time effectiveness for a post-bankruptcy liquidity constraint, we consider another independent random time $T_2$ which is exponentially distributed with an intensity $\eta_2 > 0$. Then compared to the original problem, we can solve the random time problem by interchanging the discount factor ($\beta$) and interest rate ($r$) into ($\beta + \eta_i$) and ($r + \eta_i$) for $i = 1, 2$. Thus, they do not affect our qualitative results. The detailed model and derivations are given in Appendix J.

In Figure 7, we show the comparative static results of consumption and investment with respect to $w$ when $\eta_1 = 0.003$ and $\eta_2 = 0.02$.[7] As we expected, the bankruptcy wealth level increases as $w$ decreases, so we can confirm that qualitative results in the infinite horizon model still survive.




(a) Consumption (b) Portfolio

Figure 7: Expected time to bankruptcy ($\mathbb{E}[\tau^*]$) and bankruptcy wealth threshold. The parameter set is given by $\mu = 0.7, \sigma = 0.2, r = 0.03, \beta = 0.07, \gamma = 3, \bar{I} = 0.9, \delta = 0.1, F = \delta/r \times 0.2, \eta_1 = 0.003, \eta_2 = 0.02$.

Notice that differently from the infinite horizon model, when a debt maturity is finite, the consumption at the moment of bankruptcy depends on $w$. In this case, there exist additional

[7] We can assume the larger intensities so that the average times of $T_1$ and $T_2$ are smaller. For example, when $\eta_1 = 0.03$ and $\eta_2 = 0.2$, the average debt maturity is 33 years and the limited effective time for post-bankruptcy liquidity constraint is 5 years. In that case, we can confirm the qualitative results still hold. To provide better graphical results, however, we suppose the intensities as $\eta_1 = 0.003$ and $\eta_2 = 0.02$.

tradeoffs between the remaining time to maturity for the debt and the limited time for the effective post-bankruptcy liquidity constraint. The time effects depend on $w$, so that the consumption drops differ according to $w$.

*5.2. Income uncertainty*

Our study is restricted on the consideration of the financial market risk so bad shocks in the financial market is a major factor of consumer's bankruptcy filing. However, income shocks (eg, job loss) are also major factors of consumer bankruptcy. Considering the effects of labor market risk as well as those of financial market risk on decision for consumer bankruptcy is worth to study.

As did in Ahn et al. (2019), if we consider the income risk which is perfectly correlated with the financial market risk, we can obtain the similar results. Specifically, we can suppose the wage income as follows.

$$\frac{dI_t}{I_t} = \mu_I dt + \sigma_I dB_t,$$

where $\mu_I$ and $\sigma_I$ are constant coefficients and $B_t$ is the same Brownian motion defined in Section 2. In this case, the trigger for filing for bankruptcy is a wealth to income ratio rather than a wealth level itself. The optimal bankruptcy time is the first hitting time when the wealth to income ratio reaches a certain critical level of a wealth to income ratio. Similar to the previous results, the bankruptcy wealth to income thresholds would increase as the post-bankruptcy liquidity constraint becomes stronger ($w$ decreases).

On the other hand, if the labor income and financial market is partially correlated,[8] the income risk is impossible to be hedged. The market incompleteness makes it almost impossible to find analytic and numerical solutions. However, we can guess a partial result based on the previous literature. Koo (1998) shows that for a given financial wealth, the consumption and risky investment in the incomplete market are smaller than those in the complete market. Accordingly, we can conjecture that where the market is incomplete, the bankruptcy wealth level would increase due to the debtor's more conservative decisions.

[8] As did in Koo (1998), we can suppose the wage income evolves $dI_t/I_t = \mu_I dt + \sigma_I dZ_t$, where $Z_t$ is a standard Brownian motion which is correlated with $B_t$.

## 6. Conclusion

We investigate the effects of post-bankruptcy liquidity constraint on a debtor's discretionary decision of bankruptcy, consumption, and investment. The Debtor's optimization problem can be cast into a mixed optimal stopping and control problem, but the value function at stopping time is an implicit function of wealth and there is a wealth level jump at the time of stopping. A dualtiy approach is applied to obtain explicit expressions for the optimal bankruptcy, consumption, and investment.

A stringent post-bankruptcy liquidity constraint implies a high bankruptcy wealth level along with reductions in consumption as well as investment in the risky asset. An interesting finding is that the effects of bankruptcy opportunity compete with those of the post-bankruptcy liquidity constraint, and we find the criterion for the latter to dominate the former. As the wealth level approaches the bankruptcy wealth level from above, all debtors reduce consumption and investment in the risky asset. However, a debtor with a stringent (weak) post-bankruptcy liquidity constraint reduces consumption and investment in the risky asset more dramatically (moderately) as the wealth level approaches to the bankruptcy wealth level from above. In addition, at a fixed wealth level, the debtor who will be faced with a stronger liquidity constraint after bankruptcy reduces consumption and investment in the risky asset more rapidly for a given reduction in wealth.

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## Appendix A. Static budget constraints

Let us define the state price density

$$H_t \equiv e^{-rt}\xi_t, \tag{A.1}$$

where $\xi_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t}$, and introduce an equivalent martingale probability measure $\widetilde{\mathbb{P}}(A) \equiv \mathbb{E}[\xi_T \mathbf{1}_A],\ A \in \mathcal{F}_T$, for any fixed $T$. We rewrite wealth level process $X_t$ before bankruptcy under the new measure $\widetilde{\mathbb{P}}$ as follows

$$dX_t = \left(rX_t - c_t - \delta + \bar{I}\right)dt + \sigma\pi_t d\widetilde{B}_t, \quad 0 \le t \le \tau,$$

where $\widetilde{B}_t \equiv B_t + \theta t,\ 0 \le t \le T$, is a standard Brownian motion under $\widetilde{\mathbb{P}}$. By applying Itô's lemma to the product of $e^{-rt}$ and $X_t$, we have the following equation

$$\int_0^t e^{-rs}\left(c_s + \delta - \bar{I}\right)ds + e^{-rt}X_t = x + \int_0^t e^{-rs}\sigma\pi_s d\widetilde{B}_s. \tag{A.2}$$

If $X_t,\ 0 \le t \le \tau$, is lower bounded, the right hand side of (A.2) is a $\widetilde{\mathbb{P}}$ - local martingale which is lower bounded, hence a supermartingale. In this case, from the optional sampling theorem we have

$$\mathbb{E}\left[\int_0^\tau H_t(c_t + \delta - \bar{I})dt + H_\tau X_\tau\right] \le x,$$

and equivalently,

$$\mathbb{E}\left[\int_0^\tau H_t\left(c_t + \delta\right)dt + H_\tau\left(X_\tau + \frac{\bar{I}}{r}\right)\right] \le x + \frac{\bar{I}}{r},$$

for any $\tau \in S$.

## Appendix B. Convex dual functions

**Definition Appendix B.1.** *We define the convex dual function $\widetilde{u}_i$ of concave function $u_i$ as follows*

$$\widetilde{u}_i(z) \equiv \sup_x \left[u_i(x) - zx\right],\ i = 1, 2. \tag{B.1}$$

## Appendix C. Solution to the filer's optimization problem

**Lemma Appendix C.1.** *Let $n_+ > 1$ be the positive real root of the following quadratic equation*

$$\frac{1}{2}\theta^2 n^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n - \beta = 0. \tag{C.1}$$

*Then for given $x \geq -\frac{\bar{I}}{r}w$, the filer's value function is given by*

$$V_f(x) = A\lambda^{*n_+} + \frac{\gamma}{K(1-\gamma)}\lambda^{*1-\frac{1}{\gamma}} + \lambda^*\left(x + \frac{\bar{I}}{r}\right),$$

*where the constant $A$ is given by*

$$A = -\frac{1}{n_+(n_+-1)\gamma K}\left(\frac{\gamma K\bar{I}(1-w)(n_+-1)}{r(1+\gamma n_+-\gamma)}\right)^{1+\gamma n_+-\gamma}, \tag{C.2}$$

*and $\lambda^*$ is implicitly determined from the equation*

$$x = g\left(\lambda^*\right), \tag{C.3}$$

*where*

$$g(\lambda) = -n_+A\lambda^{n_+-1} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}} - \frac{\bar{I}}{r}.$$

*The optimal consumption rate $C^f(X_t)$ and investment in the risky asset $\Pi^f(X_t)$ for given wealth level $X_t$ at time $t$ are given by*

$$c_t^f = C^f(X_t) = \left(\mathcal{Y}_0(X_t)\right)^{-\frac{1}{\gamma}}, \tag{C.4}$$

$$\pi_t^f = \Pi^f(X_t) = \frac{\theta}{\sigma}\left\{n_+(n_+-1)A\left(\mathcal{Y}_0(X_t)\right)^{n_+-1} + \frac{1}{\gamma K}\left(\mathcal{Y}_0(X_t)\right)^{-\frac{1}{\gamma}}\right\}, \tag{C.5}$$

*where $\mathcal{Y}_0(X_t)$ satisfies*

$$X_t = g\left(\mathcal{Y}_0(X_t)\right). \tag{C.6}$$

*Proof.* We provide a sketch of the proof, and all technical details and verification for the optimality can be found in Ahn et al. (2019) and Dybvig and Liu (2011). From the definition of convex dual function, we have

$$\widetilde{u}(z) \equiv \sup_x\left[u(x) - zx\right] = \frac{\gamma}{1-\gamma}z^{\frac{\gamma-1}{\gamma}}. \tag{C.7}$$

Then, for any given $(\mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}_f(x)$ and $\lambda > 0$, from the budget constraint (2.3) we have

$$\begin{aligned} J_{f,(\mathbf{c},\boldsymbol{\pi})}(x) &\leq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\widetilde{u}(\lambda_t) + \lambda_t c_t\right\}dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\widetilde{u}(\lambda_t)dt\right] + \lambda\mathbb{E}\left[\int_0^\infty H_t c_t dt\right] \\ &\leq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\{\widetilde{u}(\lambda_t) + \bar{I}\lambda_t\}dt\right] + \lambda x, \end{aligned} \tag{C.8}$$

where $\lambda_t = \lambda e^{\beta t}H_t$, hence $\lambda_0 = \lambda$. The dual value function $v_f$, defined by

$$v_f(\lambda) \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\widetilde{u}(\lambda_t) + \bar{I}\lambda_t\right\}dt\right], \tag{C.9}$$

solves the following ordinary differential equation (ODE)

$$\frac{1}{2}\theta^2\lambda^2\frac{\partial^2 v_f}{\partial\lambda^2}+(\beta-r)\lambda\frac{\partial v_f}{\partial\lambda}-\beta v_f+\widetilde{u}(\lambda)+\bar{I}\lambda=0, \tag{C.10}$$

by Feynman-Kac formula. We have the relations between the value function $V_f$ and its dual value function $v_f$ defined in (C.9) as follows

$$v_f(\lambda)=\sup_x\left[V_f(x)-\lambda x\right],\quad (V_f')^{-1}(\lambda)=-v_f'(\lambda). \tag{C.11}$$

Due to the liquidity constraint $X_t\geq -\frac{\bar{I}}{r}w,\ t>0$, we conjecture there exists a free boundary $\hat{\lambda}>0$ that corresponds to the wealth level $-\frac{\bar{I}}{r}w$. Therefore, the solution to (C.10) is of the form

$$v_f(\lambda)=A\lambda^{n_+}+\frac{\gamma}{K(1-\gamma)}\lambda^{1-\frac{1}{\gamma}}+\frac{\bar{I}}{r}\lambda,\quad 0<\lambda\leq\hat{\lambda}, \tag{C.12}$$

where the coefficient $A$ and the free boundary $\hat{\lambda}$ can be determined by the following conditions

$$v_f'(\hat{\lambda})=\frac{\bar{I}}{r}w,\quad v_f''(\hat{\lambda})=0.$$

The value function $V_f(x)$ can be obtained through the duality relation as follows

$$V_f(x)=\inf_{\lambda>0}\left[v_f(\lambda)+\lambda x\right]. \tag{C.13}$$

We can verify that the minimizing $\lambda^*$ of (C.13) satisfies (C.3) and the optimal consumption rate (C.4) makes the first inequality in (C.8) hold as an equality. By theorem 3.8.8 in Karatzas and Shreve (1998), the optimal investment in the risky asset is given by

$$\Pi^f(X_t)=-\frac{\theta}{\sigma}\mathcal{Y}_0(X_t)g'(\mathcal{Y}_0(X_t))=\frac{\theta}{\sigma}\left\{n_+(n_+-1)A\left(\mathcal{Y}_0(X_t)\right)^{n_+-1}+\frac{1}{\gamma K}\left(\mathcal{Y}_0(X_t)\right)^{-\frac{1}{\gamma}}\right\}.$$

Let us define $g_0(\lambda)\equiv n_+(n_+-1)A\lambda^{\frac{\gamma n_+-\gamma+1}{\gamma}}+\frac{1}{\gamma K}$, which is strictly decreasing and $g_0(\hat{\lambda})=0$. Therefore $v_f''(\lambda)=\lambda^{-\frac{1}{\gamma}-1}g_0(\lambda)>0$ for $0<\lambda\leq\hat{\lambda}$, i.e., $v_f(\lambda)$ is a strictly convex function for $0<\lambda\leq\hat{\lambda}$. From the duality relation (C.13), $V_f(x)$ is a strictly concave function for $x\geq-\frac{\bar{I}}{r}w$. □

## Appendix D. Solution to the debtor's optimization problem

Since $V_f$ as well as $u$ are strictly concave and strictly increasing functions we can define (B.1) to exploit duality approach. Let us denote by $I_i(\cdot),\ i=1,2$, the inverse function of $u_i'(\cdot),\ i=1,2$. It is easy to see that $I_i(\cdot),\ i=1,2$, are strictly decreasing functions and

$$\widetilde{u}_i'(\cdot)=-I_i(\cdot),\ i=1,2.$$

From (C.7)

$$\widetilde{u}_1(z) = \widetilde{u}(z) = \frac{\gamma}{1-\gamma} z^{\frac{\gamma-1}{\gamma}}.$$

If we denote by $X^*$ the maximizer of $\max_X [V_f(\alpha(X-F)) - zX]$, we have

$$V_f'(\alpha(X^* - F)) = \frac{z}{\alpha}. \tag{D.1}$$

Taking $(V_f')^{-1}(\cdot) = -v_f'(\cdot)$, given in (C.11), to both side of (D.1) we obtain

$$\alpha(X^* - F) = -n_+ A \left(\frac{z}{\alpha}\right)^{n_+ - 1} + \frac{1}{K}\left(\frac{z}{\alpha}\right)^{-\frac{1}{\gamma}} - \frac{\bar{I}}{r} = g\left(\frac{z}{\alpha}\right).$$

Therefore,

$$\begin{aligned}
\widetilde{u}_2(z) &= V_f(\alpha(X^* - F)) - zX^* \\
&= A\left(\frac{z}{\alpha}\right)^{n_+} + \frac{\gamma}{K(1-\gamma)}\left(\frac{z}{\alpha}\right)^{1-\frac{1}{\gamma}} + \frac{\bar{I}}{r}\left(\frac{z}{\alpha}\right) - Fz.
\end{aligned}$$

For any policy triple $(\mathbf{c}, \boldsymbol{\pi}, \tau) \in \mathcal{A}(x)$ and $y > 0$, it follows from (2.2) and (B.1) that

$$\begin{aligned}
J_{(\mathbf{c},\boldsymbol{\pi},\tau)}(x) &\leq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\{\widetilde{u}_1(y_t) + y_t c_t\}\, dt\right] + \mathbb{E}\Big[e^{-\beta\tau}\{\widetilde{u}_2(y_\tau) + y_\tau X_\tau\}\Big] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\widetilde{u}_1(y_t)dt + e^{-\beta\tau}\widetilde{u}_2(y_\tau)\right] + y\mathbb{E}\left[\int_0^\tau H_t c_t dt + H_\tau X_\tau\right] \\
&\leq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\widetilde{u}_1(y_t)dt + e^{-\beta\tau}\widetilde{u}_2(y_\tau)\right] \\
&+ y\left(x + \frac{\bar{I}}{r}\right) - y\mathbb{E}\left[\int_0^\tau H_t \delta dt + H_\tau \frac{\bar{I}}{r}\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta t}\{\widetilde{u}_1(y_t) - \delta y_t\}\, dt + e^{-\beta\tau}\left\{\widetilde{u}_2(y_\tau) - \frac{\bar{I}}{r}y_\tau\right\}\right] \\
&+ y\left(x + \frac{\bar{I}}{r}\right),
\end{aligned} \tag{D.2}$$

where $y_t = ye^{\beta t}H_t$ and $dy_t = (\beta - r)y_t dt - \theta y_t dB_t,\ y_0 = y$. Let us define

$$\widetilde{J}_\tau(y) \equiv \mathbb{E}\left[\int_0^\tau e^{-\beta t}\{\widetilde{u}_1(y_t) - \delta y_t\}\, dt + e^{-\beta\tau}\left\{\widetilde{u}_2(y_\tau) - \frac{\bar{I}}{r}y_\tau\right\}\right]. \tag{D.3}$$

It follows from (D.2) that

$$\begin{aligned}
V(x) = \sup_{\tau\in S} V_\tau(x) &\leq \sup_{\tau\in S}\inf_{y>0}\left[\widetilde{J}_\tau(y) + y\left(x + \frac{\bar{I}}{r}\right)\right] \\
&\leq \inf_{y>0}\sup_{\tau\in S}\left[\widetilde{J}_\tau(y) + y\left(x + \frac{\bar{I}}{r}\right)\right] \\
&= \inf_{y>0}\left[\widetilde{V}(y) + y\left(x + \frac{\bar{I}}{r}\right)\right],
\end{aligned} \tag{D.4}$$

where $\widetilde{V}(y)$ is defined by

$$\widetilde{V}(y) \equiv \sup_{\tau \in S} \widetilde{J}_\tau(y) = \sup_{\tau \in S} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{\widetilde{u}_1(y_t) - \delta y_t\right\} dt + e^{-\beta\tau}\left\{\widetilde{u}_2(y_\tau) - \frac{\bar{I}}{r} y_\tau\right\}\right]. \tag{D.5}$$

Firstly, we solve the pure optimal stopping time problem (D.5). If we consider the case that $\tau = 0$, we see that

$$\widetilde{V}(y) \geq \widetilde{u}_2(y) - \frac{\bar{I}}{r} y. \tag{D.6}$$

For any $\tau > 0$, dynamic programming principle implies

$$\widetilde{V}(y) \geq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{\widetilde{u}_1(y_t) - \delta y_t\right\} dt + e^{-\beta\tau}\widetilde{V}(y_\tau)\right].$$

If $\widetilde{V}(y)$ is a $C^2$ function, we can apply the Itô's lemma to $e^{-\beta t}\widetilde{V}(y_t)$ to obtain

$$\frac{1}{2}\theta^2 y^2 \frac{\partial^2 \widetilde{V}}{\partial y^2} + (\beta - r) y \frac{\partial \widetilde{V}}{\partial y} - \beta\widetilde{V} + \widetilde{u}_1(y) - \delta y \leq 0. \tag{D.7}$$

The optimality arises when equality holds either in (D.6) or in (D.7). Therefore, we formulate the following obstacle problem

$$\max\left\{\mathcal{L}\widetilde{V}(y) - \beta\widetilde{V}(y) + \widetilde{u}_1(y) - \delta y,\ \widetilde{u}_2(y) - \frac{\bar{I}}{r} y - \widetilde{V}(y)\right\} = 0, \tag{D.8}$$

where

$$\mathcal{L} = \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2} + (\beta - r) y \frac{\partial}{\partial y}.$$

**Assumption Appendix D.1.** *We assume that*

$$\bar{y} < \tilde{y},$$

*where $\tilde{y}$ solves*

$$\hat{x} = -\tilde{u}_2'(\tilde{y}) = -\frac{n_+ A}{\alpha}\left(\frac{\tilde{y}}{\alpha}\right)^{n_+ - 1} + \frac{1}{\alpha K}\left(\frac{\tilde{y}}{\alpha}\right)^{-\frac{1}{\gamma}} - \frac{\bar{I}}{\alpha r} + F.$$

Assumption Appendix D.1 is needed to the bankruptcy wealth level satisfies the Inequality Definition 3.1 and Definition 3.2, and so as to have admissible policy. We cast the obstacle problem (D.8) as a variational inequality as follows.

**Variational Inequality Appendix D.1.** *Find a free boundary value $\bar{y}$ and a function $v(\cdot) \in C^1(0, \tilde{y}] \cap C^2((0, \tilde{y}] \setminus \{\bar{y}\})$ satisfying*

*v.1* $\mathcal{L}v(y) - \beta v(y) + \widetilde{u}_1(y) - \delta y = 0,\ 0 < y \leq \bar{y}$

*v.2* $\mathcal{L}v(y) - \beta v(y) + \widetilde{u}_1(y) - \delta y < 0,\ \bar{y} < y \le \tilde{y}$

*v.3* $v(y) > \widetilde{u}_2(y) - \frac{\bar{I}}{r}y,\ 0 < y \le \bar{y}$

*v.4* $v(y) = \widetilde{u}_2(y) - \frac{\bar{I}}{r}y,\ \bar{y} < y \le \tilde{y}.$

We have $\widetilde{V}(y) = v(y)$.

**Proposition Appendix D.1.** *The value function* $\widetilde{V}(y)$ *of the optimization problem* (D.5) *is given by*

$$\widetilde{V}(y) = \begin{cases} By^{n_+} + \dfrac{\gamma}{K(1-\gamma)}y^{1-\frac{1}{\gamma}} - \dfrac{\delta}{r}y, & 0 < y \le \bar{y}, \\ \widetilde{u}_2(y) - \dfrac{\bar{I}}{r}y, & \bar{y} < y \le \tilde{y}, \end{cases}$$

*where the free boundary value* $\bar{y}$ *and constant* $B$ *are given by*

$$\bar{y} = \left[\frac{(n_+ - 1)K(1-\gamma)\left\{\frac{\bar{I}}{r}\left(\frac{1}{\alpha} - 1\right) + \frac{\delta}{r} - F\right\}}{(1-\gamma+n_+\gamma)\left(1 - \frac{1}{\alpha}^{1-\frac{1}{\gamma}}\right)}\right]^{-\gamma}, \tag{D.9}$$

*and*

$$B = \frac{A}{\alpha^{n_+}} + \frac{\frac{\bar{I}}{r}\left(\frac{1}{\alpha} - 1\right) + \frac{\delta}{r} - F}{1-\gamma+n_+\gamma}\bar{y}^{1-n_+}.$$

*The optimal stopping time* $\hat{\tau}_y$ *to the problem* (D.5) *is given by*

$$\hat{\tau}_y = \inf\{t > 0 | y_t \ge \bar{y}\}. \tag{D.10}$$

*Proof.* The general solution to the linear equation of v.1 in Variational inequality Appendix D.1 is of the form

$$v(y) = By^{n_+} + \frac{\gamma}{K(1-\gamma)}y^{1-\frac{1}{\gamma}} - \frac{\delta}{r}y.$$

The condition that $v(y)$ and $v'(y)$ are continuous at $y = \bar{y}$ (smooth pasting condition) enables us to find $\bar{y}$ and $B$ as in (D.9) and (3.14), respectively. By a direct calculation, we have

$$\mathcal{L}v(y) - \beta v(y) + \widetilde{u}_1(y) - \delta y = \left[\frac{\gamma}{1-\gamma}\left(1 - \frac{1}{\alpha}^{1-\frac{1}{\gamma}}\right)y^{-\frac{1}{\gamma}} + (rF - \delta) + \left(1 - \frac{1}{\alpha}\right)\bar{I}\right]y,\ \bar{y} < y \le \tilde{y}. \tag{D.11}$$

By Remark 3.3 of Shim and Shin (2014), we can derive the following inequality

$$\frac{(n_+ - 1)K}{1-\gamma+n_+\gamma} < \frac{r}{\gamma},$$

from which we obtain

$$\bar{y} > \left[\frac{r}{\gamma}(1-\gamma)\frac{\frac{\bar{I}}{r}\left(\frac{1}{\alpha}-1\right)+\frac{\delta}{r}-F}{\left(1-\frac{1}{\alpha}^{1-\frac{1}{\gamma}}\right)}\right]^{-\gamma}. \tag{D.12}$$

From (D.11) and (D.12), we have

$$\mathcal{L}v(y)-\beta v(y)+\widetilde{u}_1(y)-\delta y<0,\ \bar{y}<y\leq\tilde{y},$$

hence the inequality v.2 in Variational inequality Appendix D.1 holds.

To prove that the inequality v.3 of Variational inequality Appendix D.1 holds, we define

$$\begin{aligned}G(y) &\triangleq By^{n_+}+\frac{\gamma}{K(1-\gamma)}y^{1-\frac{1}{\gamma}}-\frac{\delta}{r}y-\left(\widetilde{u}_2(y)-\frac{\bar{I}}{r}y\right)\\ &=\left(B-\frac{A}{\alpha^{n_+}}\right)y^{n_+}+\frac{\gamma}{K(1-\gamma)}\left(1-\frac{1}{\alpha}^{1-\frac{1}{\gamma}}\right)y^{1-\frac{1}{\gamma}}+\left(F-\frac{\delta}{r}+\frac{\bar{I}}{r}-\frac{\bar{I}}{\alpha r}\right)y.\end{aligned}$$

Note that $G(\bar{y})=G'(\bar{y})=0$, by construction. For $0<\gamma<1$, $G''(y)>0$ for any $y>0$ so $G'(y)\leq 0$ for $0<y\leq\bar{y}$. Since $G(\bar{y})=0$, we see that $G(y)\geq 0$ for $0<y\leq\bar{y}$. For the case of $\gamma>1$,

$$G''(\ddot{y})=0,\ \ddot{y}=\left\{\frac{\frac{1}{\gamma K}\left(\frac{1}{\alpha}^{1-\frac{1}{\gamma}}-1\right)}{n_+(n_+-1)\left(B-\frac{A}{\alpha^{n_+}}\right)}\right\}^{\frac{\gamma}{\gamma n_+-\gamma+1}}.$$

Then we have $G''(y)<0$, for $0<y<\ddot{y}$ and $G''(y)\geq 0$, for $\ddot{y}\leq y$. From $\lim_{y\downarrow 0}G(y)=0$ and $\lim_{y\downarrow 0}G'(y)=+\infty$, we obtain that $G(y)\geq 0$ for $0<y\leq\bar{y}$. Therefore, it is verified that v.3 holds. □

If there is no liquidity constraint in post-bankruptcy, $A$ is zero and $B$ becomes positive. Since $A$ is negative, the constant $B$ can has both signs. Moreover, if the value function $\widetilde{V}(y)$ is strictly convex on $(0,\tilde{y})$, we can show the one-to-one correspondence between the variable $y$ and the wealth level $x$. Under the following assumption, $\widetilde{V}(y)$ is indeed strictly convex on $(0,\tilde{y}]$.

**Assumption Appendix D.2.** *If $B<0$,*

$$\widehat{y}\equiv(-\gamma Kn_+(n_+-1)B)^{-\frac{\gamma}{\gamma n_+-\gamma+1}}>\bar{y}.$$

**Lemma Appendix D.1.** *$\widetilde{V}(y)$ is a strictly convex function on $(0,\tilde{y}]$.*

*Proof.* Since

$$\widetilde{V}''(y)=v''(y)=\frac{n_+(n_+-1)By^{n_+-1+\frac{1}{\gamma}}+\frac{1}{\gamma K}}{y^{\frac{1}{\gamma}+1}},$$

$\widetilde{V}''(y) > 0$ on $(0, \tilde{y}]$ if $B \geq 0$. On the other hand, $\widetilde{V}''(y) = 0$ has the unique solution $\widehat{y}$ with $B < 0$. Therefore, $\widetilde{V}''(y) > 0$ for $y \in (0, \widehat{y})$ and consequently $\widetilde{V}''(y) > 0$ for $y \in (0, \bar{y}]$ when $B < 0$ from Assumption Appendix D.2.

For $\bar{y} < y \leq \tilde{y}$, $\widetilde{V}(y) = \widetilde{u}_2(y) - \frac{\bar{I}}{r} y$, is strictly convex with $y$. Therefore, $\widetilde{V}(y)$ is a strictly convex function on $(0, \tilde{y}]$. □

From Lemma Appendix D.1, $\tilde{V}'(y)$ is strictly increasing $(0, \tilde{y}]$ and, as stated in the next lemma, we can also show that $\widetilde{V}'(y)$ is represented by the expected one of the values at optimal stopping time $\hat{\tau}_y$. Since its proof goes along similar lines to the proof of Corollary 8.3 in Karatzas and Wang (2000), we omit the proof.

**Lemma Appendix D.2.** *For given $y > 0$ and the optimal stopping time $\hat{\tau}_y$ determined in Proposition Appendix D.1, we have*

$$\widetilde{V}'(y) = -\mathbb{E}\left[\int_0^{\hat{\tau}_y} H_t \left\{I_1(y_t) + \delta\right\} dt + H_{\hat{\tau}_y} \left\{I_2(y_{\hat{\tau}_y}) + \frac{\bar{I}}{r}\right\}\right]. \tag{D.13}$$

Before we proceed to the main theorem, let us define

$$h(y) \equiv -n_+ B y^{n_+ - 1} + \frac{1}{K} y^{-\frac{1}{\gamma}} - \frac{\bar{I} - \delta}{r}, \tag{D.14}$$

and define the bankruptcy wealth level

$$\underline{x} \equiv -\tilde{u}_2'(\bar{y}). \tag{D.15}$$

Since $\tilde{u}_2(\cdot)$ is a strictly convex function, Assumption Appendix D.1 implies

$$\underline{x} > \hat{x},$$

which guarantees that the dynamic constraint Definition 3.2 3 never binds, and we can tackle the debtor's optimization problem without considering it (see Definition 2.2 (v) and Theorem 3.1 in Jeanblanc et al. (2004) for similar arguments).

## Appendix E. Proof of Theorem 3.1

*Proof.* Let $y^*$ be such that satisfies (3.10), then we have

$$\widetilde{V}(y^*) + y^* \left(x + \frac{\bar{I}}{r}\right) = \inf_{y>0} \left[\widetilde{V}(y) + y \left(x + \frac{\bar{I}}{r}\right)\right],$$

and hence

$$x + \frac{\bar{I}}{r} = -\widetilde{V}'(y^*). \tag{E.1}$$

As in the proof of Corollary 8.3 of Karatzas and Wang (2000), we have

$$\widetilde{V}'(y^*) = -\mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} H_t \left\{I_1(y_t^*) + \delta\right\} dt + H_{\hat{\tau}_{y^*}} \left\{I_2(y_{\hat{\tau}_{y^*}}) + \frac{\bar{I}}{r}\right\}\right], \tag{E.2}$$

where $y_t^* = y^* e^{\beta t} H_t$. In the course of proof of Proposition Appendix D.1, we use smooth pasting condition of $\widetilde{V}(y)$ at $y = \bar{y}$, equivalently

$$n_+ B \bar{y}^{n_+ - 1} - \frac{1}{K}\bar{y}^{-\frac{1}{\gamma}} - \frac{\delta}{r} = \widetilde{u}_2'(\bar{y}) - \frac{\bar{I}}{r}, \tag{E.3}$$

which yields

$$I_2(y_{\hat{\tau}_{y^*}}) = I_2(\bar{y}) = -\widetilde{u}_2'(\bar{y}) = \underline{x}. \tag{E.4}$$

Therefore we have, from (E.1) and (E.2),

$$x + \frac{\bar{I}}{r} = \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} H_t \Big\{I_1(y_t^*) + \delta\Big\} dt + H_{\hat{\tau}_{y^*}} \left\{I_2(y_{\hat{\tau}_{y^*}}) + \frac{\bar{I}}{r}\right\}\right].$$

There exists a portfolio process that can finance consumption level $I_1(y_t^*)$ and bankruptcy wealth level $I_2(y_{\hat{\tau}_{y^*}})$. Observe that

$$\begin{aligned}
V(x) &\geq \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} e^{-\beta t} u_1\Big(I_1(y_t^*)\Big) dt + e^{-\beta \hat{\tau}_{y^*}} u_2\left(I_2(y_{\hat{\tau}_{y^*}})\right)\right] \\
&= \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} e^{-\beta t} \Big\{\widetilde{u}_1(y_t^*) + y_t^* I_1(y_t^*)\Big\} dt + e^{-\beta \hat{\tau}_{y^*}} \left\{\widetilde{u}_2\left(y_{\hat{\tau}_{y^*}}\right) + y_{\hat{\tau}_{y^*}} I_2(y_{\hat{\tau}_{y^*}})\right\}\right] \\
&= \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} e^{-\beta t} \widetilde{u}_1(y_t^*)\, dt + e^{-\beta \hat{\tau}_{y^*}} \widetilde{u}_2\left(y_{\hat{\tau}_{y^*}}\right)\right] + y^* \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} H_t I_1(y_t^*) dt + H_{\hat{\tau}_{y^*}}\ I_2(y_{\hat{\tau}_{y^*}})\right] \\
&= \mathbb{E}\left[\int_0^{\hat{\tau}_{y^*}} e^{-\beta t} \Big\{\widetilde{u}_1(y_t^*) - \delta y_t^*\Big\} dt + e^{-\beta \hat{\tau}_{y^*}} \left\{\widetilde{u}_2\left(y_{\hat{\tau}_{y^*}}\right) - \frac{\bar{I}}{r} y_{\hat{\tau}_{y^*}}\right\}\right] + y^*\left(x + \frac{\bar{I}}{r}\right) \\
&= \widetilde{V}(y^*) + y^*\left(x + \frac{\bar{I}}{r}\right) = \inf_{y>0}\left[\widetilde{V}(y) + y\left(x + \frac{\bar{I}}{r}\right)\right].
\end{aligned} \tag{E.5}$$

Combining (D.4) and (E.5) results in

$$V(x) = \inf_{y>0}\left[\widetilde{V}(y) + y\left(x + \frac{\bar{I}}{r}\right)\right].$$

Therefore, the debtor's optimal strategies are given by

$$\begin{cases}
c_t^* = I_1(y_t^*) = (y_t^*)^{-\frac{1}{\gamma}}, \text{ for } t < \hat{\tau}_{y^*}, \\
X_{\hat{\tau}_{y^*}} = I_2(y_{\hat{\tau}_{y^*}}) = \underline{x}, \\
\hat{\tau}_{y^*} = \inf\{t > 0 | y_t^* \geq \bar{y}\}.
\end{cases}$$

In particular, $c_0^* = (y^*)^{-\frac{1}{\gamma}}$. Therefore, we obtain the feedback form (3.12) of the optimal consumption rate at any $t < \hat{\tau}_{y^*}$. By theorem 3.8.8 in Karatzas and Shreve (1998), the feedback form of the optimal investment in the risky asset is given by

$$\pi_t^*(X_t) = -\frac{\theta}{\sigma}\mathcal{Y}(X_t)h'(\mathcal{Y}(X_t)),$$

which yields (3.13). □

### Appendix F. Proof of Proposition 4.1

From (C.2) and (3.14), we see that $B(w)$ increases with $w$. It is straightforward to see that $w$ reduces $\underline{x}(w)$. Consider two debtors with the same wealth level $X_t$ but have different levels of post-bankruptcy liquidity constraint; $w_1$ and $w_2$ with $w_1 < w_2$. Then it should hold that

$$\begin{aligned} X_t &= -n_+ B(w_1)\left(\mathcal{Y}_{1,t}\right)^{n_+-1} + \frac{1}{K}\left(\mathcal{Y}_{1,t}\right)^{-\frac{1}{\gamma}} - \frac{\bar{I}-\delta}{r} \equiv X_{1,t}(\mathcal{Y}_{1,t}), \\ &= -n_+ B(w_2)\left(\mathcal{Y}_{2,t}\right)^{n_+-1} + \frac{1}{K}\left(\mathcal{Y}_{2,t}\right)^{-\frac{1}{\gamma}} - \frac{\bar{I}-\delta}{r} \equiv X_{2,t}(\mathcal{Y}_{2,t}), \end{aligned} \tag{F.1}$$

for some $\mathcal{Y}_{1,t},\ \mathcal{Y}_{2,t} > 0$. For a given $t$, suppose that $\mathcal{Y}_{1,t} = \mathcal{Y}_{2,t} = u > 0$. Then we have $X_{1,t}(u) > X_{2,t}(u)$, since $B(w_1) < B(w_2)$. Note that both $X_{1,t}(\cdot)$ and $X_{2,t}(\cdot)$ are strictly decreasing functions. Therefore we must have $\mathcal{Y}_{1,t} > u > \mathcal{Y}_{2,t}$ in order to make $X_{1,t}(\mathcal{Y}_{1,t}) = X_{2,t}(\mathcal{Y}_{2,t})$. $X_{1,t}(\mathcal{Y}_{1,t})$ is the optimal wealth level with the optimal consumption $c_{1,t}^*$ and the optimal portfolio $\pi_{1,t}^*$ as follows:

$$c_{1,t}^* = \left(\mathcal{Y}_{1,t}\right)^{-\frac{1}{\gamma}}, \tag{F.2}$$

$$\pi_{1,t}^* = \frac{\theta}{\sigma}\left\{n_+(n_+-1)B(w_1)(\mathcal{Y}_{1,t})^{n_+-1} + \frac{1}{\gamma K}(\mathcal{Y}_{1,t})^{-\frac{1}{\gamma}}\right\}. \tag{F.3}$$

Similarly,

$$c_{2,t}^* = \left(\mathcal{Y}_{2,t}\right)^{-\frac{1}{\gamma}}, \tag{F.4}$$

$$\pi_{2,t}^* = \frac{\theta}{\sigma}\left\{n_+(n_+-1)B(w_2)(\mathcal{Y}_{2,t})^{n_+-1} + \frac{1}{\gamma K}(\mathcal{Y}_{2,t})^{-\frac{1}{\gamma}}\right\}, \tag{F.5}$$

are the optimal consumption and the optimal portfolio for the optimal wealth level $X_{2,t}(\mathcal{Y}_{1,t})$. It is easily seen from (F.2) and (F.4) that $c_{1,t}^* < c_{2,t}^*$. By multiplying both sides of (F.1) by $(n_+ - 1)$, subtracting one from another, we have

$$\frac{\sigma}{\theta}\left\{\pi_{1,t}^* - \pi_{2,t}^*\right\} = \frac{1+\gamma n_+ - \gamma}{\gamma K}\left\{(\mathcal{Y}_{1,t})^{-\frac{1}{\gamma}} - (\mathcal{Y}_{2,t})^{-\frac{1}{\gamma}}\right\},$$

from which we conclude that $\pi_{1,t}^* < \pi_{2,t}^*$.

## Appendix G. Proof of Lemma ??

From (E.3) and (E.4), it follows that

$$X_\tau = \underline{x} = -n_+ B(w)\bar{y}^{n_+-1} + \frac{1}{K}\bar{y}^{-\frac{1}{\gamma}} - \frac{\bar{I}-\delta}{r} \tag{G.1}$$

$$X_{\tau_+} = \alpha(X_\tau - F) = \alpha(\underline{x} - F) = -n_+ A(w)\bar{\lambda}^{n_+-1} + \frac{1}{K}\bar{\lambda}^{-\frac{1}{\gamma}} - \frac{\bar{I}}{r}, \tag{G.2}$$

where $\bar{\lambda} = \bar{y}/\alpha$. From (D.9), (G.1) and (G.2) we obtain

$$\tilde{c}^*_\tau = \bar{y}^{-\frac{1}{\gamma}},\ \tilde{c}^*_{\tau_+} = \bar{\lambda}^{-\frac{1}{\gamma}},$$

so we have

$$\Delta c = \left(\alpha^{\frac{1}{\gamma}} - 1\right)\bar{y}^{-\frac{1}{\gamma}}.$$

Similarly,

$$\begin{aligned}
\tilde{\pi}^*_\tau &= \frac{\theta}{\sigma}\left\{n_+(n_+-1)B(w)\bar{y}^{n_+-1} + \frac{1}{\gamma K}\bar{y}^{-\frac{1}{\gamma}}\right\},\\
\tilde{\pi}^*_{\tau_+} &= \frac{\theta}{\sigma}\left\{n_+(n_+-1)A(w)\bar{\lambda}^{n_+-1} + \frac{1}{\gamma K}\bar{\lambda}^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

yields

$$\Delta\pi = \frac{\theta}{\sigma}n_+(n_+-1)\left\{\left(\alpha^{1-n_+} - \alpha^{-n_+}\right)A(w)\bar{y}^{n_+-1} - \frac{\frac{\bar{I}}{r}\left(\frac{1}{\alpha}-1\right) + \frac{\delta}{r} - F}{1-\gamma+n_+\gamma}\right\} + \frac{\theta}{\sigma\gamma K}\Delta c.$$

## Appendix H. Proof of Proposition 4.2

The marginal propensity to consume out of wealth is defined by

$$M_c(X_t) \triangleq \frac{dc^*_t}{dX_t},$$

thus, we have

$$\begin{aligned}
M_c(X_t) &= \frac{d\left((\mathcal{Y}(X_t))^{-\frac{1}{\gamma}}\right)}{dX_t} = \frac{d\left((\mathcal{Y}(X_t))^{-\frac{1}{\gamma}}\right)}{d\mathcal{Y}(X_t)}\cdot\frac{d\mathcal{Y}(X_t)}{dX_t}\\
&= -\frac{1}{\gamma}(\mathcal{Y}(X_t))^{-\frac{1}{\gamma}-1}\cdot\frac{1}{-n_+(n_+-1)B(\mathcal{Y}(X_t))^{n_+-2} - \frac{1}{\gamma K}(\mathcal{Y}(X_t))^{-\frac{1}{\gamma}-1}}.
\end{aligned}$$

The elasticity of consumption with respect to wealth is defined by

$$\epsilon_M(X_t) \triangleq \frac{dc^*_t}{dX_t}\frac{X_t}{c^*_t} = M_c\cdot\frac{X_t}{c^*_t}.$$

Thus, by substituting $M_c(X_t)$ we obtain the explicit form of $\epsilon_M(X_t)$. $dM_c(X_t)/dX_t$ can be obtained from a direct calculation.

## Appendix I. Proof of Proposition 4.3

From the definition of MPI out of wealth, we have

$$
\begin{aligned}
M_\pi(X_t) &\equiv \frac{\partial \pi_t^*}{\partial X_t} = \frac{\partial \pi_t^*}{\partial \mathcal{Y}(X_t)} \cdot \frac{d\mathcal{Y}(X_t)}{dX_t} \\
&= \frac{\partial}{\partial \mathcal{Y}(X_t)} \left( (-n_+(n_+-1)B(\mathcal{Y}(X_t))^{n_+-1} - \frac{1}{\gamma K}(\mathcal{Y}(X_t))^{-\frac{1}{\gamma}}) \cdot \left(-\frac{\theta}{\sigma}\right)\right) \times \frac{d\mathcal{Y}(X_t)}{dX_t} \\
&= \frac{-\frac{\theta}{\sigma}\left(n_+(n_+-1)^2 B(\mathcal{Y}(X_t))^{n_+-2} - \frac{1}{\gamma^2 K}(\mathcal{Y}(X_t))^{-\frac{1}{\gamma}-1}\right)}{n_+(n_+-1)B(\mathcal{Y}(X_t))^{n_+-2} + \frac{1}{\gamma K}(\mathcal{Y}(X_t))^{-\frac{1}{\gamma}-1}}.
\end{aligned}
$$

We can obtain the result by multiplying $(\mathcal{Y}(X_t))^{\frac{1}{\gamma}+1}$ on both denominator and numerator. $dM_\pi(X_t)/dX_t$ can be obtained from a direct calculation.

## Appendix J. A Finite debt maturity and a finite effective liquidity constraint

In this appendix, we characterize the model where a maturity of the debt and a effective post-bankruptcy liquidity constraint are finite. In particular, if we denote the time to maturity for the debt by $T_1$, the value function can be written as follows. For an admissible policy $(c, \pi, \tau)$[9] and an initial wealth $X_0 = x > 0$,

$$
V(x) = \sup_{c,\pi,\tau} \mathbb{E}\left[\int_0^{\tau\wedge T_1} e^{-\beta t}u(c_t)dt + e^{-\beta\tau}V_f(\alpha(X_\tau - F)) \cdot \mathbf{1}_{\{\tau \le T_1\}} + e^{-\beta T_1}V_M\left(X_{T_1} - \frac{\delta}{r}\right)\cdot \mathbf{1}_{\{\tau > T_1\}}\right],
$$

subject to the static budget constraint

$$
\mathbb{E}\left[\int_0^{\tau\wedge T_1} H_t(c_t - \bar{I} + \delta)dt + H_{\tau\wedge T_1}X_{\tau\wedge T_1}\right] \le x.
$$

The function $V_F(\alpha(X_\tau - F))$ is the filer's value function and $V_M(X_{T_1} + \frac{\bar{I}-\delta}{r})$ is the value function of Merton (1969) problem defined by $\frac{1}{K^\gamma(1-\gamma)}\left(X_{T_1} + \frac{\bar{I}-\delta}{r}\right)^{1-\gamma}$. As did in Liu and Loewenstein (2002), we consider the maturity time $T_1$ as an independent random time which is exponentially distributed with an intensity $\eta_1 > 0$. Then the expected time for random time $T_1$ is given by $1/\eta_1$, and the survival probability until time $t$ is defined by $F_1(t) = e^{-\eta_1 t}$. The value function can be rewritten as

$$
V(x) = \sup_{c,\pi,\tau} \mathbb{E}\left[\int_0^{\tau} F(t)e^{-\beta t}\left(u(c_t)dt + \eta_1 V_M(X_t - \frac{\delta}{r})\right)dt + F_1(\tau)e^{-\beta\tau}V_F(\alpha(X_\tau - F))\right]. \tag{J.1}
$$

[9]We can define the admissible policy $(c, \pi, \tau)$ by slightly modifying Definition 3.1 and 3.2.

In addition, we also suppose a limited time effectiveness for post-bankruptcy liquidity constraint. Similar to $T_1$, we consider another random time $T_2$ which is exponentially distributed with an intensity $\eta_2 > 0$. Then, the expected time for $T_2$ is given by $1/\eta_2$, and the survival probability until time $t$ is given by $F_2(t) = e^{-\eta_2 t}$. The value function of the filer with liquidity constraint within a limited time is defined by

$$V_F(x) = \sup_{c,\pi} \mathbb{E}\left[\int_0^{T_2} e^{-\beta t} u(c_t)dt + e^{-\beta T_2} U(X_{T_2})\right],$$

subject to *(i) the static budget constraint*

$$\mathbb{E}\left[\int_0^{T_2} H_t(c_t - \bar{I})dt + H_{T_2} X_{T_2}\right] \le x,$$

and *(ii) the liquidity constraint* $X_t \ge -\frac{\bar{I}}{r}w, w \in [0,1]$ for $t \le T_2$. Note that $U(X_T)$ is the Merton (1969)'s value function without any debt repayment and constraint, so it is given by $U(X_T) = \frac{1}{K^\gamma}\left(X_T + \frac{\bar{I}}{r}\right)^{1-\gamma}$.

With a survival probability $F_2(t)$, the filer's problem can be rewritten as

$$V_F(x) = \sup_{\{c,\pi\}} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\eta_2)t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \eta_2 \frac{(X_t + \frac{\bar{I}}{r})^{1-\gamma}}{K^\gamma(1-\gamma)}\right)dt\right],$$

subject to the static budget

$$\mathbb{E}\left[\int_0^\infty H_{2,t}(c_t - \bar{I} + \eta_2 X_t)dt\right] \le x,$$

where $H_{2,t} = e^{-(r+\eta_2-\frac{1}{2}\theta^2)t-\theta B_t}$. We summarize the value function of a liquidity-constrained filer as follow.

**Lemma Appendix J.1.** *Let $m_+ > 1$ be the positive real root of the following algebraic quadratic equation*

$$\frac{1}{2}\theta^2 m^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)m - \beta - \eta_2 = 0.$$

*When the liquidity constraint is effective only until a time $T_2$, for given $x \ge -\frac{\bar{I}}{r}w$, the filer's value function is obtained from*

$$V_f(x) = \tilde{A}\tilde{\lambda}^{*^{m_+}} + \frac{\gamma}{K(1-\gamma)}\tilde{\lambda}^{*^{1-\frac{1}{\gamma}}} + \tilde{\lambda}^*\left(x + \frac{\bar{I}}{r}\right),$$

*where the constants $A_{T_2}$ and $\tilde{\lambda}^*$ are given by*

$$\tilde{A} = -\frac{1+\frac{\eta_2}{K}}{m_+(m_+-1)\gamma(K+\eta_2)}\left(-\frac{\gamma(K+\eta_2)(m_+-1)(w-1-\frac{\eta_2}{r})\bar{I}}{(1+\frac{\eta_2}{K})(1-\gamma+\gamma m_+)(r+\eta_2)}\right)^{1-\frac{1}{\gamma}-m_+},$$

*and*

$$x = -m_+\tilde{A}\tilde{\lambda}^{*^{m_+-1}} + \frac{1+\frac{\eta_2}{K}}{K+\eta_2}\tilde{\lambda}^{*^{-\frac{1}{\gamma}}} - \frac{\bar{I}(1+\frac{\eta_2}{r})}{r+\eta_2}.$$

Now we go back the debtor's problem when the maturity of the debt is given by $T_1$. Similar to the method developed in the previous section, we can define the Lagrangian as follows.

$$
\begin{aligned}
\tilde{\mathcal{L}} \triangleq \max_{c,\pi,\tau} \mathbb{E}\left[\int_0^\tau e^{-(\beta+\eta_1)t}\left(\frac{c_t^{1-\gamma}}{1-\gamma}+\eta_1\frac{(X_t+\frac{\bar{I}-\delta}{r})^{1-\gamma}}{K^\gamma(1-\gamma)}\right)dt + e^{-(\beta+\eta_1)\tau}V_F(\alpha(X_\tau-F))\right] \\
-\lambda\left(\mathbb{E}\left[\int_0^\tau H_{1,t}(c_t-\bar{I}+\delta+\eta_1 X_t)dt + H_{1,t}X_\tau\right]-x\right) \\
= \mathbb{E}\left[\int_0^\tau e^{-(\beta+\eta_1)t}\left(\frac{\gamma}{1-\gamma}z_t^{1-\frac{1}{\gamma}}+\eta_1\left(\frac{\gamma}{K(1-\gamma)}z_t^{1-\frac{1}{\gamma}}+\frac{\bar{I}-\delta}{r}z_t\right)+(\bar{I}-\delta)z_t\right)dt \right.\\
\left. + e^{-(\beta+\eta_1)\tau}\left(\tilde{\phi}\left(\frac{z_\tau}{\alpha}\right)-z_\tau F\right)\right]+\lambda x,
\end{aligned}
$$

where $z_t$ is the same process in (3.19) with $z_0 = \lambda$, and $\tilde{\phi}(\cdot)$ is the convex conjugate function of $V_F(x)$ defined by

$$V_F(x) = \min_\lambda\left\{\tilde{\phi}(\lambda)+\lambda x\right\}.$$

Then, the debtor's value function in (J.1) can be obtained from

$$V(x) = \min_\lambda\{\varphi(\lambda)+\lambda x\},$$

where $\varphi(z)$ satisfies the following variational inequalities:

**Variational Inequality Appendix J.1.** *Find a free boundary value $\hat{z}$ and a function $\varphi(\cdot) \in C^1(0,\tilde{z}] \cap C^2((0,\tilde{z}]\setminus\{\hat{z}\})$ satisfying*

1. $\mathcal{L}\varphi(z)-(\beta+\eta_1)\varphi(z)+\frac{\gamma}{1-\gamma}\left(1+\frac{\eta_1}{K}\right)z^{1-\frac{1}{\gamma}}+(\bar{I}-\delta)\left(1+\frac{\eta_1}{r}\right)z = 0,\ 0<z\le\hat{z}$
2. $\mathcal{L}\varphi(z)-(\beta+\eta_1)\varphi(z)+\frac{\gamma}{1-\gamma}\left(1+\frac{\eta_1}{K}\right)z^{1-\frac{1}{\gamma}}+(\bar{I}-\delta)\left(1+\frac{\eta_1}{r}\right)z < 0,\ \hat{z}<z\le\tilde{z}$
3. $\varphi(z) > \tilde{\varphi}(z)-Fz,\ 0<z\le\hat{z}$
4. $v(z) = \tilde{\varphi}(z)-Fz,\ \hat{z}<z\le\tilde{z}.$

*where* $\mathcal{L} = \frac{1}{2}\theta^2 z^2\frac{\partial^2}{\partial z^2}+(\beta-r)z\frac{\partial}{\partial z}$, $\tilde{z} = \left(\frac{\gamma(m_+-1)(K+\eta_2)(1+\frac{\eta_2}{r}-w)\bar{I}}{(r+\eta_2)(1+\frac{\eta_2}{K})(1-\gamma+m_+\gamma)}\right)^{-\gamma}$, *and*

$$\tilde{\varphi}(z) = \tilde{A}\left(\frac{z}{\alpha}\right)^{m_+}\frac{\gamma(1+\frac{\eta_2}{K})}{(K+\eta_2)(1-\gamma)}\left(\frac{z}{\alpha}\right)^{1-\frac{1}{\gamma}}+\frac{\bar{I}(1+\frac{\eta_2}{r})}{r+\eta_2}\left(\frac{z}{\alpha}\right).$$

We can apply the similar methods in Section 4, and value function $\varphi(y)$ can be derived by

$$\varphi(z) = Bz^{\tilde{n}_+}+\frac{\gamma\left(1+\frac{\eta_1}{K}\right)}{(K+\eta_1)(1-\gamma)}z^{1-\frac{1}{\gamma}}+(\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1}z,$$

where $\tilde{n}_+$ is the positive real root of the equation

$$\frac{1}{2}\theta^2\tilde{n}^2+(\beta-r-\frac{1}{2}\theta^2)\tilde{n}-(\beta+\eta_1) = 0,$$

and $\tilde{B}$ is given by

$$\tilde{B} = \left( \frac{1+\frac{\eta_2}{K}}{K+\eta_2} \left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}} - \frac{1+\frac{\eta_1}{K}}{K+\eta_1} \right) \frac{\gamma}{1-\gamma} \hat{z}^{1-\frac{1}{\gamma}-\tilde{n}_+} + \tilde{A}\left(\frac{1}{\alpha}\right)^{m_+} \hat{z}^{m_+-\tilde{n}_+}$$
$$+ \left( \frac{\bar{I}(1+\frac{\eta_2}{r})}{r+\eta_2}\left(\frac{1}{\alpha}\right) + (\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1} - F \right) \hat{z}^{1-\tilde{n}_+},$$

where $\hat{z}$ is the solution to the following algebraic equation

$$\frac{1-\gamma+\tilde{n}_+\gamma}{1-\gamma} \left( \frac{1+\frac{\eta_1}{K}}{K+\eta_1} - \frac{1+\frac{\eta_2}{K}}{K+\eta_2}\left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}} \right) \hat{z}^{-\frac{1}{\gamma}} - \tilde{A}(\tilde{n}_+ - m_+)\left(\frac{1}{\alpha}\right)^{m_+} \hat{z}^{m_+-1}$$
$$+ (m_+ - 1)\left( (\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1} - \frac{\bar{I}(1+\frac{\eta_2}{r})}{\alpha(r+\eta_2)} + F \right) = 0$$

Then, we summarize the optimal consumption, investment, and bankruptcy as follows.

**Theorem Appendix J.1.** *For a given initial wealth $X_0 = x$, the debtor's problem in* (J.1) *is given by*

$$V(x) = \varphi(z^*) + z^* x,$$

*where $z^*$ is implicitly determined from the equation*

$$x = -n_+ \tilde{B} {z^*}^{\tilde{n}_+ - 1} + \frac{1+\frac{\eta_1}{K}}{K+\eta_1} {z^*}^{-\frac{1}{\gamma}} - (\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1}.$$

*When $z_t^* = z^* e^{\beta t} H_t$, the optimal bankruptcy time $\tau^{**}$ is given by*

$$\tau^{**} = \inf\{t > 0 | z_t^* \geq \hat{z}\},$$

*and the optimal wealth $X_t^{**}$, the optimal consumption rate $c_t^{**}$ and the investment in the risky asset $\pi_t^{**}$ in feedback forms are given by*

$$X_t^{**} = -n_+ \tilde{B} {z_t^*}^{\tilde{n}_+ - 1} + \frac{1+\frac{\eta_1}{K}}{K+\eta_1} {z_t^*}^{-\frac{1}{\gamma}} - \frac{(\bar{I}-\delta)(1+\frac{\eta_1}{r})}{r+\eta_1},$$
$$c_t^{**} = \frac{K+\eta_1}{1+\frac{\eta_1}{K}} \left( X_t^{**} + (\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1} \right) + \frac{\tilde{n}_+(K+\eta_1)B}{1+\frac{\eta_1}{K}} {z_t^*}^{\tilde{n}_+ - 1},$$
$$\pi_t^{**} = \frac{\theta}{\gamma\sigma} \left( X_t^{**} + (\bar{I}-\delta)\frac{1+\frac{\eta_1}{r}}{r+\eta_1} \right) + \frac{\theta}{\gamma\sigma}(1-\gamma+\tilde{n}_+\gamma)\tilde{n}_+ B {z_t^*}^{\tilde{n}_+ - 1}$$

Obviously, it is easily checked that the problem with a finite horizon is reduced to the debtor's problem with an infinite horizon when $\eta_1 = \eta_2 = 0$.