Credit Risk and Lagged Credit Rating Revision

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Abstract

We develop a dynamic credit-risk model to study the interactions between default risk and lagged credit-rating revision. In the model, credit ratings provide useful information about a firm's default risk, but a credit rating agency may not respond promptly to changes in the credit quality of the firm. Lagged credit rating revision can benefit bondholders by reducing the chance of default, but can also hurt bondholders by exacerbating the adverse selection problem, which lowers the market price of bonds. We show that in general, when the rating revision cost increases, bondholders who hold bonds that are about to be downgraded are better off, while bondholders holding more superior bonds are worse off. We also analyze how the interaction between default risk and rating revision is affected by reputation costs, debt maturity, and asset volatility.

Keywords: credit risk, lagged credit rating revision, information asymmetry

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1 Introduction

Credit rating agencies have received significant criticism for their slow responses to changes in corporate credit risks (Cheng and Neamtiu, 2009; Abinzano et al., 2022). Many critics attribute lagged adjustment of ratings to the lack of competition among rating agencies and to the inability of rating agencies to acquire timely information about credit risks. However, others view the lagged adjustment of ratings as a necessary feature because it enhances the stability of rating assignments and reduces excess bond market fluctuations.[1] Despite these debates on the causes and consequences of lagged rating revisions, there is no theoretical framework that formally analyzes the interaction between lagged rating revisions and firms' default risks.

In this paper, we fill this gap by developing a dynamic credit-risk model that features lagged rating revisions and debt rollover in the presence of information asymmetry. In the model, A firm generates stochastic cashflows and has finite-maturing debts that need to be rolled over. When the firm issues new debt to replace maturing debt, it suffers from information asymmetry as outside investors are uninformed of the firm's credit risk. In this setting, credit ratings provide useful information about the firm's default risks, thereby alleviating informational frictions in the debt issuance market. Yet, the credit rating agency may lag its rating revisions due to rating-revision costs and concerns about rating stability.

We show that the presence of a lag in rating revisions has two competing effects on the firm's default risk and debt value. First, it lowers the probability of a high-rated firm being downgraded, which reduces the firm's default risk. Second, it lowers the informational value of credit ratings, resulting in lower rollover proceeds and thus a higher credit risk for the firm. Our model suggests there is an optimal level of revision lag that maximizes debt value. Further, we find that the sensitivity of debt value to the rating-revision lag is higher for debt with shorter maturity.

Let us describe the model in more details. Time is continuous, and there is a firm whose assets produce stochastic cash flows. The firm has a fixed amount of outstanding bonds that mature with a Poisson intensity. At each point in time, a certain fraction of bonds mature,

[1] See Cheng and Neamtiu (2009), Jiang et al. (2012), Cornaggia and Cornaggia (2013), Goldstein and Huang (2020), and Altman and Rijken (2006) among others.

and the firm commits to issue new bonds so as to keep the total amount of outstanding bonds fixed over time. The firm issues new bonds to a large group of bond investors who are willing to finance as long as they break even.

We assume the firm faces liquidity risks when rolling over its debt. Specifically, when the firm's net operating cashflow is insufficient to cover its financing costs for debt rollover, the equityholders inject new capital to cover the shortfall. However, due to liquidity constraints, the equityholders may fail to provide additional capital, in which case the firm is forced to default. We assume the liquidity-driven default occurs with a Poisson intensity. Besides, equityholders can choose to default voluntarily when injecting capital to keep the firm alive is no longer profitable.

When issuing new bonds to replace maturing bonds, the firm also suffers from information asymmetry: the firm's manager is perfectly informed of the firm's current cashflow level, which will be referred to as the firm's fundamental, while bond investors are uninformed. Because of information asymmetry between the firm manager and bond investors, the firm may not be able to receive the intrinsic value of its bond as rollover proceeds.

There are two potential mechanisms that mitigate the informational friction in the bond market. First, we assume there is a credit rating agency who is perfectly informed of the firm's fundamental and can produce rating signals to inform investors of the firm's credit risk. To simplify the analysis, we assume the rating agency classifies firms into only two rating classes, namely, high (H) and low (L). In the model, the rating agency is concerned about rating accuracy, because we assume wrong assessment may cause reputation losses. Yet it is also concerned about rating stability, because rating changes are assumed to be costly. Therefore, the rating agency dynamically revises the firm's rating in order to minimize the presence value of future rating-revision costs and potential reputation losses.

Second, we assume that the manager can signal its current fundamental in a costly way. Specifically, the firm can intentionally burns some amount of cash to reveal the firm's current fundamental. For instance, we may interpret cash burning as a firm's decision to incerase its marketing expenses to signal its financial strength to investors, as analyzed in Milgrom and Roberts (1986). Bond investors then extract information about the true value of a bond from the amount of cash burned by the firm. We show that a separating equilibrium, in which the firm's current fundamental is truthfully revealed, can arise in this model. In such an equilibrium, due to the presence of the signaling cost, the firm receives the lowest possible rollover proceeds, depending on the current credit rating assigned to the firm.

The model features an interesting interaction between the firm's credit risk and the credit rating. On the one hand, a change in the firm's credit risk changes the credit rating agency's expected reputation losses, inducing the agency to re-optimize its rating-revision decisions. On the other hand, a change in the credit rating agency's rating-revision decisions affects the degree of information asymmetry in the bond market, leading to changes in the firm's rollover proceeds and thus in the firm's (liquidity-driven) default risk.

The interaction between credit risk and credit rating leads to multiple equilibria. In our model, the presence of multiple equilibria is driven by the effect of credit rating on debt rollover. Intuitively, when the credit rating agency downgrades a firm earlier (at a higher cashflow level), the firm's rollover proceeds drop at a higher cashflow level. As such, the firm faces a higher risk of rollover failure and liquidity-driven default in the future, thereby justifying the credit rating agency's initial decision to downgrade the firm earlier. Among all equilibria, we focus on the equilibrium that achieves the highest rating stability. This equilibrium selection is motivated by the fact that credit rating agencies aim to maintain a stable rating system to some extent to avoid causing unnecessary market turbulence (Altman and Rijken, 2006; Cantor and Mann, 2006).

In equilibrium, the rating revisions are indeed lagged due to the costs associated with rating revision. Specifically, the credit rating agency does not downgrade the firm right at the threshold below which the firm's credit risk substantially increases. Instead, the credit rating agency chooses to downgrade the firm at a cashflow level strictly lower than that threshold, which gives the H-rated firm more time for its cashflow level to recover and reduces the probability of the firm being downgraded. Symmetrically, the rating agency also postpones raising the firm's credit rating when the firm's fundamental improves over time.

We show that lagged rating revisions may be beneficial to debt value. In the model, an increase in the rating-revision lag, caused by an increase in the rating-revision cost, has two effects on the firm's default risk and debt value. First, the increase in the revision lag makes it more likely for a H-rated firm to maintain the rating, reducing the probability of being downgraded and thus the probability of liquidity-driven default. Second, a larger lag in credit rating revision also has an indirect effect on debt value through debt rollover. When the firm is downgraded at a lower cashflow threshold, the degree of information asymmetry in the debt market is higher, causing a decrease in rollover proceeds for H-rated firms and a higher probability of liquidity-driven default. Our model suggests that the tradeoff of the two effects leads to an optimal rating-revision lag that maximizes debt value.

Further, we study the effect of debt maturity on the interaction between credit rating and default risk. Debt value is higher when debt maturity is shorter, because shorter-term debts are more likely to be repaid in full and thus less sensitive to default risks. However, we show that the values of shorter-term debts are more sensitive to a change in the rating-revision lag. Intuitively, the effect of rating lag on debt value works through debt rollover, and therefore, the effect is amplified when the firm's debts mature faster and are rolled over more frequently.

This paper contributes to the literature by first developing a credit-risk model with lagged credit-rating revision by credit rating agencies. In the theoretical literature, Boot et al. (2006) study how credit rating can serve as a coordination mechanism in resolving the multiple equilibria problem in the presence of the feedback effects between credit ratings and a default policy. Bolton et al. (2012) show that credit-rating inflation can arise because of behavioral bias of bond investors. Opp et al. (2013) study the variations in credit-rating standards by examining the regulatory impacts of credit ratings on institutional bond investors. Manso (2013) studies the feedback effects between credit ratings and a default policy in a dynamic settup, assuming credit rating agencies can adjust their assessment at any instant, while our model considers the costs associated with the credit-rating revision. Goldstein and Huang (2020) show that credit-rating inflation can arise due to the feedback effects between credit ratings and the investment decision of a firm. In the empirical literature, Altman and Rijken (2006), Cantor and Mann (2006), Cheng and Neamtiu (2009), Alsakka and ap Gwilym (2010), and Abinzano et al. (2022) among others analyze the accuracy of credit ratings examining slow responses of credit rating agencies.

The paper is organized as follows. In Section 2, we develop the model. In Section 3, we solve the model. In Section 4, we discuss the implications of the model. In Section 5, we present an extended model. In Section 6, we conclude.

2 Model

We develop a credit-risk model that incorporates lagged credit rating revision from a credit rating agency in the presence of information frictions. Time is continuous. The market players include the equityholders of a firm, bond investors, and a credit rating agency. All market players have a risk-neutral preference and a constant discount rate of r.

We consider a firm whose asset produces stochastic cash flows of x_t at each time t. The cash-flow level x_t is modeled as a geometric Brownian motion with a constant growth rate μ and volatility σ:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dZ_t,$$

where Z_t is a standard Brownian motion. As in Mella-Barral and Perraudin (1997), we assume that any owners of the asset can liquidate it at a constant scrapping value of A at any time. This assumption simplifies the asset liquidation mechanism in the presence of information frictions. In this setting, as shown in Mella-Barral and Perraudin (1997), an unlevered firm terminates operating the asset when the cash-flow level x_t falls below

$$x_A = \frac{A(r-\mu)\eta}{\eta - 1},$$

where η is a negative root of the equation $\eta(\eta-1)\sigma^2/2 + \eta\mu = r$. Also, the first-best value of this asset is given by

$$V^{FB}(x) = \frac{x}{r-\mu} + \left(A - \frac{x_A}{r-\mu}\right)\left(\frac{x}{x_A}\right)^{\eta}, \quad \forall x \geq x_A.$$

We assume $\mu < r$ to ensure the first-best value is finite.

The cash-flow level x_t is observable to the firm's manager, who acts in the best interest of the equityholders, but not to bond investors including the firm's current bondholders. In fact, we may assume that bond investors can at least observe some noisy or lagged information about the cash flow of a firm. However, in our model, introducing this more realistic assumption does not alter any of our model outcomes, because we later show that a separating equilibrium, in which the cash-flow level is truthfully revealed to bond buyers, arises in the model. In this regard, we do not explicitly adopt this assumption to keep the model exposition to be concise. Moreover, as in Duffie and Lando (2001) and Manso (2013), we assume that equity claims are not traded on a public market. This assumption allows us to avoid a situation that investors can learn the current fundamental value of a firm from the stock price data.

The firm issues bonds to exploit tax benefits. Specifically, the face value of a bond is denoted by F and the coupon rate is denoted by c. As in Leland (1998) and Hackbarth et al. (2006), bonds mature in a stationary manner without a stated maturity date. Specifically, we assume that a randomly chosen fraction λdt of all outstanding bonds mature at each point in time. In this regard, the average time to maturity is equal to $1/\lambda$. In particular, a perpetuity bond has $\lambda = 0$. As mentioned in Leland (1998), this maturity structure is similar to a sinking fund that buys back bonds at par continuously. The corporate tax rate is π and the coupon payment is tax deductible. Throughout, we assume that the debt structure is exogenously given and will discuss the implications for an optimal debt structure by conducting comparative statics analysis.

When a bond has reached its maturity, the firm issues a new bond with the same contract terms regarding the face value, coupon rate, and maturity. But because of information asymmetry between the firm manager and bond investors, the firm may not be able to receive the intrinsic value of a bond as rollover proceeds. As a way of mitigating this information friction, we assume that the manager intentionally burns some amount of cash to reveal her firm's current fundamental. For instance, we can imagine that the firm aggressively increases its marketing expenses to signal its financial strength to investors, as examined by Milgrom and Roberts (1986). Bond investors then extract some information about the true value of a bond from the amount of cash burned by the firm. Throughout, we focus on a separating equilibrium in which the firm's current fundamental is always truthfully revealed through this signaling behavior. We also assume bond investors behave competitively. For clarification, similar to Manso (2013), we assume that the selling price of a bond and the amount of cash burned are not publicized immediately because if so, the current fundamental can be backed out from the past trajectories of the bond prices or the amount of cash burned in this continuous-time setup.

The firm faces liquidity risks when making debt payments. Specifically, note that the net cash flow to equity at time t is equal to $x_t - (1-\pi)cF + \lambda(P_t - F)$, where P_t is the net rollover proceeds that the firm obtains by issuing new bonds at time t, which is endogenously determined. When this net cash flow is negative, the equityholders need to inject new capital to pay back their debts. However, due to liquidity constraints, the equityholders may fail to provide additional capital, in which case the firm is forced to default. Specifically, such a liquidity-constrained firm defaults when a Poisson shock with an arrival intensity of ϕ hits the firm, similar to He and Xiong (2012a). We call this default event liquidity-driven default. The firm's equityholders can also strategically default on their debt if injecting additional capital is no longer profitable. This strategic default occurs when the cash-flow level hits a default threshold x_D, which is endogenously determined.

When the firm defaults at time t, we assume that the equityholders are wiped out and the bondholders take over the asset. Assuming bondholders have no asset management skills, they immediately liquidate the asset at the scrapping value of A. We assume that the scrapping value is less than the face value of debt, so that the recovery rate defined as $\alpha = \frac{A}{F}$ is less than 1. In Section 5, we endogenize the recovery rate of the asset, considering a signaling game between asset sellers and buyers in the presence of information asymmetry.

In this model, a credit rating agency assesses the credit risk of the firm. Following Manso (2013) and Goldstein and Huang (2020), we assume that the credit rating agency precisely knows the current fundamental of the firm. Also, as in Mathis et al. (2009) and Bolton et al. (2012), we assume that the rating agency classifies firms into only two rating classes, namely, high (H) or low (L). The rating agency is concerned about the accuracy of its assessment because wrong assessment may damage the reputation of the rating agency.

Specifically, we assume that if an H-rated firm defaults, the rating agency incurs a reputation loss of B. In other words, B is the size of the reputation loss caused by a late alarm. The rating agency's reputation is also damaged by a false alarm. Specifically, we assume that if the low rating is assigned to a firm even if there is no chance that the firm would default today, the rating agency incurs a reputation loss of bdt, which is a flow cost. In other words, in our context, this second type of reputation loss would occur if the low rating is assigned to a firm when its net cash flow to equity is positive so that the chance that the firm would default today is zero. We can rationalize this assumption as follows. Suppose the firm's fundamental is occasionally publicly revealed at the end of some day, which arrives according to a Poisson shock with intensity θ. When the cash flow is publicly revealed at some day, the rating agency's reputation falls by J if the firm's current rating is low even if the firm is not facing any default risk at the moment. Under this interpretation, the above flow reputation cost b can be regarded as θJ.

We further assume that the credit rating agency faces frictions when trying to revise a firm's credit rating. In practice, credit rating agencies have received significant criticism for their slow response to changes in corporate credit quality, especially when credit quality deteriorates (Cheng and Neamtiu, 2009; Abinzano et al., 2022). Critics attribute lagged adjustment of ratings to the lack of competition among rating agencies, the conflict of interest between issuers and rating agencies, and the stability of assessment that rating agencies intend to maintain.[2] To model these frictions, we assume that the credit rating agency faces a fixed cost of K when downgrading a firm's credit rating from H to L. We assume that the rating agency does not face any costs when raising a firm's credit rating from L to H, although this assumption can be easily relaxed.

Facing the reputation costs and rating revision costs, the credit rating agency's goal is to minimize the present value of those costs. At date 0, we assume that a certain credit rating has been already assigned to the firm in a consistent way with the rating agency's rating revision policy. In other words, we can say that the firm had been already established before date 0, in line with our another assumption regarding the firm's capital structure.

The sequence of the events over each dt-period of time is as follows: the cash-flow level is first realized; the credit rating agency then decides whether to revise the firm's current credit rating; the firm manager then decides whether to default; if the firm has decided not to default, the firm issues new bonds to replace retiring bonds; if the firm still needs to inject additional capital to repay debt, a liquidity shock may hit the firm.

[2] See Cheng and Neamtiu (2009), Jiang et al. (2012), Cornaggia and Cornaggia (2013), Goldstein and Huang (2020), and Altman and Rijken (2006) among others.

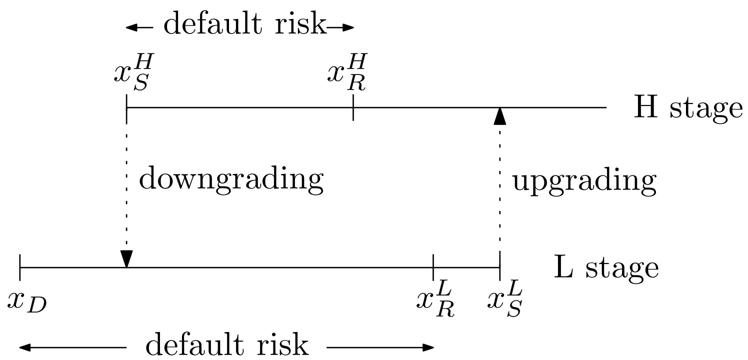

Figure 1: This figure describes the default threshold x_D, the rollover thresholds x_R^H and x_R^L, and the credit-rating revision thresholds x_S^H and x_S^L.

3 Model Solutions

3.1 Equilibrium Conditions

Before solving the model, we first impose a parameter condition that $K < B$. This condition means that the penalty for a late alarm is so large that the rating agency decides to downgrade a firm at least at the last moment when the firm's fundamental hits the default boundary x_D. Although we can consider the other case of $K \geq B$ without any difficulty, the above assumption is considered more realistic and facilitates the exposition of our model analysis.

We now refer to the stage when the firm's current rating is L (H) as the L (H) stage. Then, we postulate that in equilibrium, (i) during the L stage, the net cash flow to equity is negative if $x_t < x_R^L$ and the rating agency raises the firm's credit rating from L to H when the fundamental x_t hits a certain threshold x_S^L from below, where $x_R^L \leq x_S^L$ and (ii) during the H stage, the net cash flows to equity is negative when $x_t < x_R^H$ and the rating agency downgrades the firm when the fundamental x_t hits a certain threshold x_S^H from above, where $x_S^H \leq x_R^H$. Throughout, we focus on this threshold-type equilibrium, which is described in Figure 1.

Under this postulation, the intrinsic value of a bond at each $k \in \{L, H\}$ stage, denoted as $D^k(x)$, must satisfy the following Bellman equations:

$$rD^L(x) = cF + \lambda(F - D^L(x)) + \phi 1_{x<x_R^L}(A - D^L(x)) + \mathcal{L}D^L(x), \ \forall x \in (x_D, x_S^L), \quad (1)$$

$$rD^H(x) = cF + \lambda(F - D^H(x)) + \phi 1_{x<x_R^H}(A - D^H(x)) + \mathcal{L}D^H(x), \ \forall x \in (x_S^H, \infty), \quad (2)$$

subject to

$$D^L(x_D) = A,\ D^L(x_S^L) = D^H(x_S^L),\ D^H(x_S^H) = D^L(x_S^H), \tag{3}$$

where $\mathcal{L}f(x) = \mu x f_x(x) + \frac{\sigma^2}{2}x^2 f_{xx}(x)$ for any value function f considered in this model. In particular, the third terms on the right-hand side in both equations represent the default event caused by a liquidity shock. The second and third boundary conditions indicate the value-matching condition that has to be satisfied between $D^L(x)$ and $D^H(x)$ when the credit rating is revised. Understanding the other terms in the above equations is standard.

To derive the Bellman equation for the equity value at each stage $k \in \{L, H\}$, denoted as $E^k(x)$, we first pin down the amount of net rollover proceeds the firm obtains when issuing new bonds. Suppose that during the L stage, buyers believe the firm with a fundamental value of $x \in [x_D, x_S^L]$ to burn $q^L(x)$ amounts of cash, where the lowest-quality firm among the L-rated firms is believed not to burn any amount of cash. We postulate that a separating equilibrium exists, in which each firm's fundamental is truthfully revealed through the amount of cash burned by that firm. If this equilibrium exists, once the firm chooses to burn $q^L(x)$ amount of cash, buyers will pay $D^L(x)$, that is, the true value of the bond, for the new bond issue. Taking these beliefs and reaction of the market as given, each firm with a fundamental value of x solves the following problem:

$$\max_{x_D \le y \le x_S^L} D^L(y) - q^L(y),$$

where $D^L(y) - q^L(y)$ represents the net rollover proceeds the firm would obtain if the firm mimics the strategy of another firm with a fundamental y. For the separating equilibrium to arise, the term $D^L(y) - q^L(y)$ has to be constant for all $y \in [x_D, x_S^L]$ because otherwise, some of the firms would deviate. Hence, we must have

$$D^L(x) - q^L(x) \equiv D^L(x_D),$$

which means that although the separating equilibrium arises in this economy, the net rollover proceeds to each L-rated firm are reduced to the same level equal to $D^L(x_D)$. This simple outcome obtains because we consider only a pure cash-burning strategy in this model.

Similarly, each H-rated firm with a fundamental value of $x \in [x_S^H, \infty)$ burns $q^H(x) = D^H(x) - D^H(x_S^H)$ amount of cash and earns $D^H(x_S^H)$ as the net rollover proceeds when issuing a new bond. This way, the net rollover proceeds jumps down from $D^H(x_S^H)$ to $D^L(x_D)$ when the firm is downgraded and vice versa.

These results imply that the value functions of equity, $E^L(x)$ and $E^H(x)$, satisfy the following Bellman equations:

$$rE^L(x) = x - (1-\pi)cF + \lambda(D^L(x_D) - F) - \phi 1_{x<x_R^L} E^L(x) + \mathcal{L}E^L(x), \tag{4}$$

$$E^H(x) = x - (1-\pi)cF + \lambda(D^H(x_S^H) - F) - \phi 1_{x<x_R^H} E^H(x) + \mathcal{L}E^H(x), \tag{5}$$

subject to

$$E^L(x_D) = 0,\ E_x^L(x_D) = 0,\ E^L(x_S^L) = E^H(x_S^L),\ E^H(x_S^H) = E^L(x_S^H). \tag{6}$$

In particular, the third terms on the right-hand side in both the equations indicate the total net proceeds the firm obtains when replacing the retiring bonds by new bonds. The second boundary condition is the optimality condition for the default decision. All other terms and conditions can be understood similarly as above.

Now, let $C^k(x)$ denote the present value of the reputation costs and credit-rating revision costs that will be incurred to the credit rating agency in the future, given that the current credit rating assigned to the firm is $k \in \{L, H\}$. Then, we see that $C^L(x)$ and $C^H(x)$ must satisfy the following Bellman equations:

$$rC^L(x) = b1_{x>x_R^L} - \phi 1_{x<x_R^L} C^L(x) + \mathcal{A}C^L(x), \tag{7}$$

$$rC^H(x) = \phi 1_{x<x_R^H}(B - C^H(x)) + \mathcal{A}C^H(x), \tag{8}$$

subject to

$$C^L(x_D) = 0,\ C^L(x_S^L) = C^H(x_S^L),\ C_x^L(x_S^L) = C_x^H(x_S^L),\ C^H(x_S^H) = C^L(x_S^H) + K, \tag{9}$$

$$\begin{cases} C_x^H(x_S^H) = C_x^L(x_S^H), & \text{if } x_D < x_S^H \\ C_x^H(x_S^H) < C_x^L(x_S^H), & \text{if } x_D = x_S^H. \end{cases} \tag{10}$$

The first term on the right-hand side in equation (7) means that the rating agency incurs a reputation loss of b if a false alarm remains uncorrected. The second term indicates that the rating agency does not incur any reputation losses if an L-rated firm defaults. The first term on the right-hand side in equation (8) means that if a H-rated firm defaults, the rating agency incurs a reputation loss of A. The first boundary condition also means that the rating agency does not incur any reputation losses if a L-rated firm defaults. The second boundary condition means that the rating agency can raise a credit rating without facing any costs. The third boundary condition indicates the optimality condition for revising a credit rating from L to H. The fourth boundary condition means that the rating agency incurs fixed costs of K when downgrading a firm. The last boundary condition indicates the optimality condition for revising a credit rating from H to L. In particular, the corner condition described in the second line means that if $C^L(x)$ increases in x faster than $C^H(x)$ at x_D, the rating agency postpones downgrading the firm until the last moment when the firm is expected to default today for sure.

Lastly, the equilibrium conditions that must be satisfied at the rollover thresholds x_R^L and x_R^H are as follows:

$$\begin{cases} x_R^L - (1-\pi)cF + \lambda(D^L(x_D) - F) = 0, & \text{if } x_R^L < x_S^L \\ x_R^L - (1-\pi)cF + \lambda(D^L(x_D) - F) \leq 0, & \text{if } x_R^L = x_S^L \end{cases} \tag{11}$$

and

$$\begin{cases} x_R^H - (1-\pi)cF + \lambda(D^H(x_S^H) - F) = 0, & \text{if } x_S^H < x_R^H \\ x_R^H - (1-\pi)cF + \lambda(D^H(x_S^H) - F) \geq 0, & \text{if } x_S^H = x_R^H \end{cases} \tag{12}$$

Specifically, the interior conditions for both x_R^L and x_R^H are straightforward to understand due to the definition of the net cash flows to equity. The corner case in (11) means that the rating agency raises the credit rating too early so that the low rating is never assigned to a firm that does not face any immediate chances of a liquidity. The corner case in (12) means that the rating agency downgrades the firm too early so that the default event never occurs to a high-rated firm. Also, the above conditions imply that x_R^H must be smaller than or equal to x_R^L in equilibrium because the firm earns higher rollover proceeds when its current rating is high.

In sum, an equilibrium is defined as the set of the value functions and thresholds, $\{D^L(x), D^H(x), E^L(x), E^H(x), C^L(x), C^H(x), x_D, x_R^L, x_S^L, x_R^H, x_S^H\}$, that satisfy the above equations and conditions from (1) to (12).

3.2 Closed-form Solution

In this section, we solve for an equilibrium in closed form up to a system of polynomial equations with a power of real numbers. To begin, note that at the L stage, the value functions are written as

$$D^L(x) = \begin{cases} \frac{(c-\delta+\lambda)F+\phi A}{r+\lambda+\phi} + A_1 x^{\xi_1} + A_2 x^{\xi_2}, & \text{if } x_D < x < x_R^L \\ \frac{(c-\delta+\lambda)F}{r+\lambda} + A_3 x^{\xi_3} + A_4 x^{\xi_4}, & \text{if } x_R^L < x < x_S^L, \end{cases}$$

$$E^L(x) = \begin{cases} \frac{\lambda D^L(x_D)-(1-\pi)cF-\lambda F}{r+\phi} + \frac{x}{r+\phi-\mu} + B_1 x^{\eta_1} + B_2 x^{\eta_2}, & \text{if } x_D < x < x_R^L \\ \frac{\lambda D^L(x_D)-(1-\pi)cF-\lambda F}{r} + \frac{x}{r-\mu} + B_3 x^{\eta_3} + B_4 x^{\eta_4}, & \text{if } x_R^L < x < x_S^L, \end{cases}$$

$$C^L(x) = \begin{cases} C_1 x^{\eta_1} + C_2 x^{\eta_2}, & \text{if } x_D < x < x_R^L \\ \frac{b}{r} + C_3 x^{\eta_3} + C_4 x^{\eta_4}, & \text{if } x_R^L < x < x_S^L, \end{cases}$$

where

$$\eta_1, \eta_2 = \frac{-\mu + \frac{\sigma^2}{2} \pm \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2(r+\phi)}}{\sigma^2},$$

$$\eta_3, \eta_4 = \frac{-\mu + \frac{\sigma^2}{2} \pm \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2},$$

$$\xi_1, \xi_2 = \frac{-\mu + \frac{\sigma^2}{2} \pm \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda+\phi)}}{\sigma^2},$$

$$\xi_3, \xi_4 = \frac{-\mu + \frac{\sigma^2}{2} \pm \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda)}}{\sigma^2}.$$

At the H stage, the value functions are written as

$$D^H(x) = \begin{cases} \frac{(c+\lambda)F+\phi A}{r+\lambda+\phi} + A_5 x^{\xi_1} + A_6 x^{\xi_2}, & \text{if } x_S^H < x < x_R^H \\ \frac{(c+\lambda)F}{r+\lambda} + A_7 x^{\xi_4}, & \text{if } x_R^H < x. \end{cases}$$

$$E^H(x) = \begin{cases} \frac{\lambda D^H(x_S^H)-(1-\pi)cF+\lambda F}{r+\phi} + \frac{x}{r+\phi-\mu} + B_5 x^{\eta_1} + B_6 x^{\eta_2}, & \text{if } x_S^H < x < x_R^H \\ \frac{\lambda D^H(x_S^H)-(1-\pi)cF+\lambda F}{r} + \frac{x}{r-\mu} + B_7 x^{\eta_4}, & \text{if } x_R^H < x, \end{cases}$$

$$C^H(x) = \begin{cases} \frac{\phi B}{r+\phi} + C_5 x^{\eta_1} + C_6 x^{\eta_2}, & \text{if } x_S^H < x < x_R^H \\ C_7 x^{\eta_4}, & \text{if } x_R^H < x. \end{cases}$$

The coefficients A_1, ..., A_7 are pinned down from the following conditions:

$$D^L(x_D) = A,\ D^L(x_R^L-) = D^L(x_R^L+),\ D_x^L(x_R^L-) = D_x^L(x_R^L+),\ D^L(x_S^L) = D^H(x_S^L),$$

$$D^L(x_S^H) = D^H(x_S^H),\ D^H(x_R^H-) = D^H(x_R^H+),\ D_x^H(x_R^H-) = D_x^H(x_R^H+),$$

which lead to a system of linear equations. Similarly, the coefficients B_1, ..., B_7 are pinned down from the following conditions:

$$E^L(x_D) = 0,\ E^L(x_R^L-) = E^L(x_R^L+),\ E_x^L(x_R^L-) = E_x^L(x_R^L+),\ E^L(x_S^L) = E^H(x_S^L),$$

$$E^L(x_S^H) = E^H(x_S^H),\ E^H(x_R^H-) = E^H(x_R^H+),\ E_x^H(x_R^H-) = E_x^H(x_R^H+).$$

The coefficients C_1, ..., C_7 are pinned down from the following conditions:

$$C^L(x_D) = 0,\ C^L(x_R^L-) = C^L(x_R^L+),\ C_x^L(x_R^L-) = C_x^L(x_R^L+),\ C^L(x_S^L) = C^H(x_S^L),$$

$$C^H(x_S^H) = C^L(x_S^H) + K,\ C^H(x_R^H-) = C^H(x_R^H+),\ C_x^H(x_R^H-) = C_x^H(x_R^H+).$$

The equilibrium thresholds, x_D, x_R^L, x_R^H, x_S^L, and x_S^H, are jointly determined from the remaining conditions described in Section 3.1. That is, the default threshold is pinned down from the condition $E_x^L(x_D) = 0$. The rollover thresholds are pinned down from the conditions in (11) and (12). The credit-rating revision thresholds are pinned from the condition $C_x^L(x_S^L) = C_x^H(x_S^L)$ and condition (10). We compute these equilibrium thresholds numerically. We discuss equilibrium multiplicity in the next section.

3.3 Multiple Equilibria

In the literature, a number of papers have shown that a credit-risk model with credit rating agencies may generate multiple equilibria because of the feedback effects between credit ratings and default or investment decisions; for instance, see Bolton et al. (2012), Manso (2013), and Goldstein and Huang (2020). A common argument is that if a credit rating agency assigns a high (low) rating, then bankruptcy will be less (more) likely to occur, thereby justifying the validity of the initial rating assigned by the rating agency. In our model, multiple equilibria may also arise because the credit ratings and the net rollover proceeds to the firm cause positive feedback effects to each other.

We first show that multiple equilibria indeed arise in our model in a special case where the credit-rating revision cost, that is, K, is 0. When the revision cost is non-zero, our numerical results suggest that the model has a unique equilibrium, but we are not aware of the rigorous proof for this result at the moment. When the revision cost is 0, the credit rating agency always assesses the credit risk of a firm accurately to avoid any reputation losses caused by a late alarm or false alarm. Nonetheless, multiple equilibria arise because the credit risk itself depends on the credit rating announced by the rating agency through the rollover channel.

To be more concrete, note that when K is 0, the rollover threshold x_R^k must be glued to the credit-rating revision threshold x_S^k at each stage $k \in \{L, H\}$, that is, $x_R^k = x_S^k$, because otherwise, the rating agency may suffer some reputation loss since $B > 0$ and $b > 0$, which is not the optimal assessment policy. For such an equilibrium to arise, the following two conditions must hold:

$$x_S^L - (1-\pi)cF + \lambda(D^L(x_D) - F) \leq 0 \quad \text{and} \quad x_S^H - (1-\pi)cF + \lambda(D^H(x_S^H) - F) \geq 0, \quad (13)$$

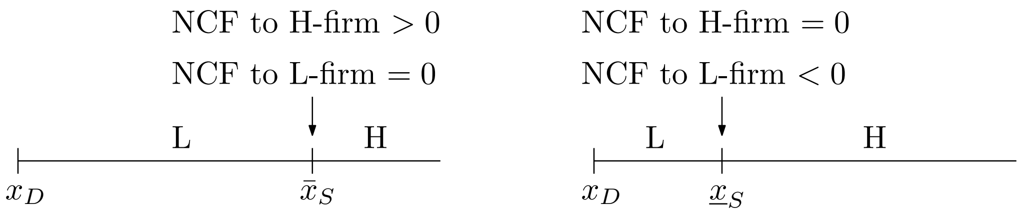

Figure 2: The left figure describes the equilibrium conditions satisfied by the largest equilibrium revision threshold $\bar{x}_S$. The right figure describes the equilibrium conditions satisfied by the smallest equilibrium revision threshold $\underline{x}_S$. In both figures, NCF stands for the net cash flow.

which correspond to the corner conditions in (11) and (12). As mentioned above, the first condition means that the net cash flow to equity of a low-rated firm is always less than or equal to 0 so that the type II error never occurs; the second condition means that a high-rated firm will never default so that the type I error never occurs. Hence, the present value of the future reputation costs and revision costs would be always 0, which must be optimal for the credit rating agency.

We now show that there are actually infinitely many pairs of (x_S^L, x_S^H) satisfying the above conditions. To facilitate our exposition, let us first consider a special case of $x_S^L = x_S^H$, that is, the case where the rating agency uses a single threshold when making a revision decision. We let $x_S := x_S^L = x_S^H$ denote such a single revision threshold. Among all possible equilibrium thresholds x_S satisfying both conditions in (13), the largest threshold, denoted as $\bar{x}_S$, is obtained from the following condition:

$$\bar{x}_S - (1-\pi)cF + \lambda(D^L(x_D) - F) = 0, \tag{14}$$

while the smallest threshold, denoted as $\underline{x}_S$, is obtained from the following condition:

$$\underline{x}_S - (1-\pi)cF + \lambda(D^H(\underline{x}_S) - F) = 0. \tag{15}$$

Here, note that when condition (14) holds, the other condition for a high-rated firm in (13) holds with strict inequality, because the net rollover proceeds to a high-rated firm is larger than the net rollover proceeds to a low-rated firm. For the same reason, when condition (15) holds, the other condition for a low-rated firm in (13) holds with strict inequality. Figure 2 describes the equilibrium conditions satisfied by the largest and smallest equilibrium revision

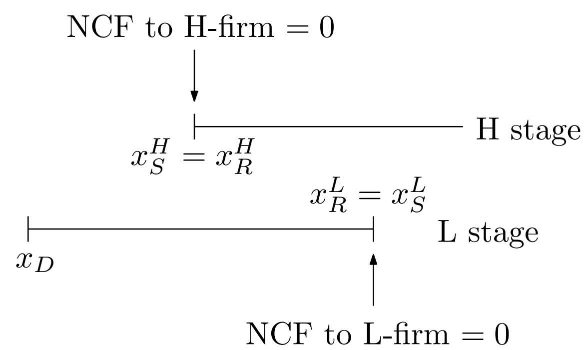

Figure 3: This figure describes a situation in which both conditions in (11) and (12) are satisfied with equality when the revision cost is 0.

thresholds. This observation then further implies that any threshold x_S between $\underline{x}_S$ and $\bar{x}_S$ can be an equilibrium revision threshold because both conditions in (13) hold with strict inequality under this choice of a revision threshold.

Moreover, we can find more pairs of the equilibrium thresholds (x_S^L, x_S^H) because x_S^L and x_S^H do not necessarily need to be the same. Among many, we pay particular attention to a pair of (x_S^L, x_S^H) that satisfies the conditions in (13) with equality for both x_S^L and x_S^H:

$$\begin{cases} x_S^L - (1-\pi)cF + \lambda(D^L(x_D) - F) = 0 \\ x_S^H - (1-\pi)cF + \lambda(D^H(x_S^H) - F) = 0, \end{cases}$$

as described in Figure 3. In fact, in Proposition 3.1, we show that when the revision cost is non-zero, the rollover thresholds (x_R^L, x_R^H) satisfy both conditions in (11) and (12) with equality. As such, when $K = 0$, we naturally select an equilibrium that satisfies both conditions in (11) and (12) with equality, especially when we conduct comparative statics analysis later, because otherwise, the equilibrium would not change continuously when the revision cost increases from 0.

Moreover, note that an equilibrium satisfying both conditions in (11) and (12) with equality corresponds to the most stable equilibrium from the perspective of the rating agency if we define the stability of credit-rating assessment as the average duration required for a credit rating to transition to another rating once it has been revised recently. In other equilibria, the rating agency downgrades or upgrades a credit rating relatively earlier, creating a room for improvement in stability. Given that stability is one of the main concerns for

Table 1: Baseline Parameter Values

Risk-free rate	$r = 3.5\%$
Corporate tax rate	$\tau = 27\%$
Asset growth rate	$\mu = 1.5\%$
Asset volatility	$\sigma = 22\%$
Face value of debt	$F = 100$
Coupon rate	$c = 6$
Average debt maturity	$1/\lambda = 5$
Bond recovery rate	$\alpha = 55\%$
Bond holding costs	$\delta = 0.01$
Liquidity shock intensity	$\phi = 0.3$
Reputation loss due to a late alarm	$B = 2.1$
Reputation loss due to a false alarm	$b = 0.6$
Fixed costs for downgrading	$K = 1$

rating agencies, as pointed out by Altman and Rijken (2006) and Cantor and Mann (2006), we may assert that our equilibrium choice reasonably reflects the concerns of those agencies in stability.

Proposition 3.1. When the credit-rating revision cost K is non-zero, (i) both conditions in (11) and (12) hold with equality and (ii) the rollover threshold x_R^k cannot be the same as the rating revision threshold x_S^k for each rating class $k \in \{H, L\}$.

Proof. See Appendix A. □

4 Model Implications

In Section 4.1, we calibrate the model and analyze the model's main properties. In Section 4.2 and 4.3, we study the model's comparative statics with respect to the credit rating agency's rating-revision costs and reputation losses. Section 4.4 and 4.5 analyze how debt maturity and asset volatility affect the interaction between a firm's default risk and credit rating assignment.

4.1 Baseline Parameter Values

The baseline parameter values that will be used in this model are provided in Table 1. We set the risk-free rate r to 3.5%, close to the numbers widely used in the literature, because the average 10-year Treasury rate over the period between January 2000 and December 2019 was about 3.42%. We set the tax rate τ to 27%, following the arguments by He and Xiong (2012b) and He and Milbradt (2014), which take into account the partial tax exemptions enjoyed by institutional bond investors. We set the asset growth rate μ to 1.5% because the asset growth rate is equal to the risk-free rate minus the asset payout ratio under the risk-neutral measure, where the average asset payout ratio is about 2% according to Zhang et al. (2009) and Huang et al. (2020). We set the asset volatility to 22% because BBB-rated bonds have an average asset volatility of 22% according to Zhang et al. (2009). We normalize the face value of debt to $F = 100$ and set the coupon rate c to 6%. This choice is reasonable because the estimated coupon rate for BBB-rated bonds is around 4% according to He and Milbradt (2014), while the historical coupon rate is about 9% according to Huang and Huang (2012). We set the average debt maturity to $1/\lambda = 5$ because the average time to maturity of corporate bonds is around 5 years according to Barclay and Smith Jr (1995) and Stohs and Mauer (1996). We set the bond recovery rate to 55% because according to Chen (2010) and Glover (2016), the estimated recovery rate ranges from 50% to 60%.

Before we estimate the remaining parameters, note that we can normalize the cost associated with downgrading, K, to 1 because the rating agency's decision is not affected by the units used to measure the reputation costs and the credit-rating revision costs, as long as the same unit is used for these costs. Now, to estimate the parameters ϕ, B, and b by targeting the following four moments as closely as possible: an average annual default rate of speculative grade bonds, the average annual default rate of investment grade bonds, the type I error of credit-rating assessment, and the type II error of credit-rating assessment. According to 2021 Annual Global Corporate Default and Rating Transition Study by S&P Global, over the period from 1981 to 2021, the average annual default rate of investment grade bonds was 0.09% and the average annual default rate of speculative grade bonds was 3.95%. Cheng and Neamtiu (2009) define the type I error as the proportion of defaulted firms that held credit ratings above a specific cutoff level one year prior to default and define the type II error as the proportion of firms that held credit ratings below that cutoff level over the past one year among all firms that stayed alive at a specific point in time. Following this definition, Cheng and Neamtiu (2009) report that the type I error is about 21.4% and the

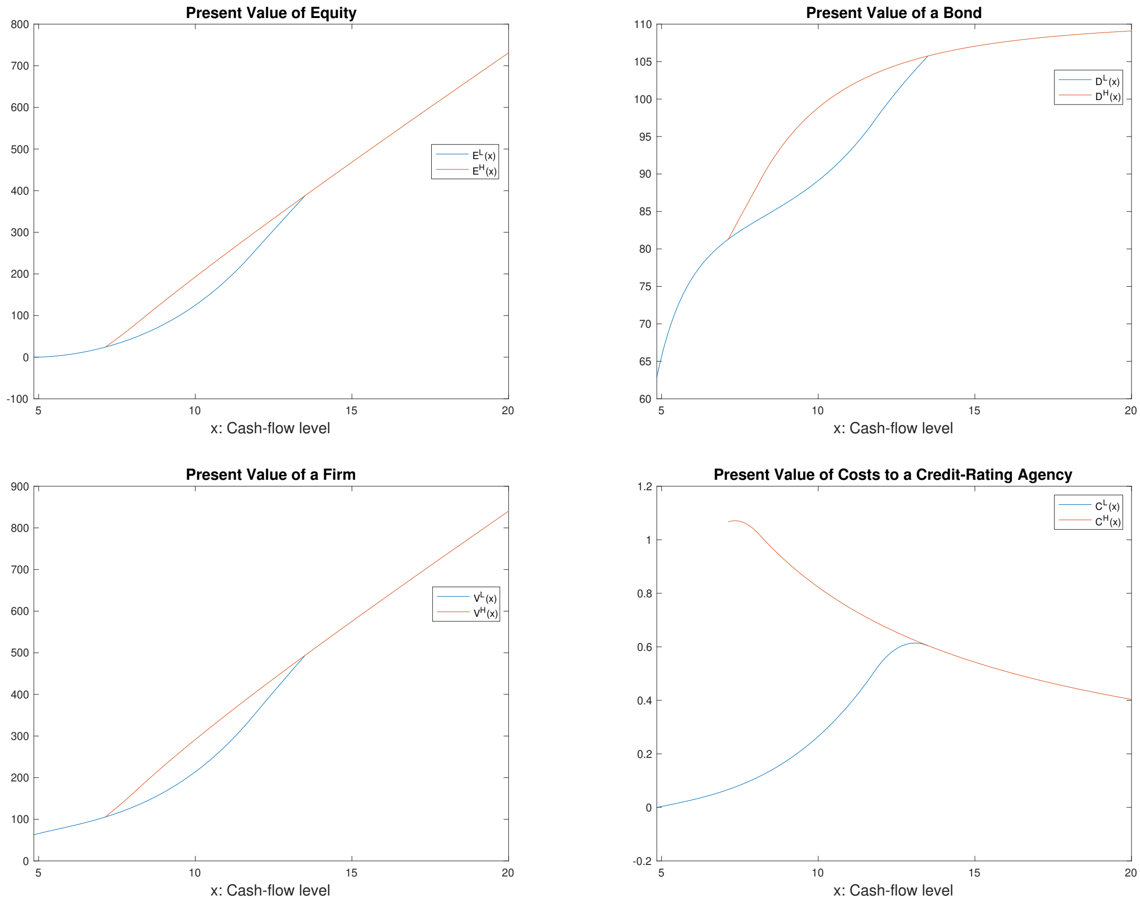

Figure 4: The left-top, right-top, left-bottom, and right-bottom panels plot the present value of equity, the present value of debt, the present value of a firm, and the present value of costs to the rating agency, respectively, for each rating class.

type II error is about 10.3%, when the BB rating is used as the cutoff level. To match these target moments, we run a simulation by setting the initial fundamental value of a firm, x_0, to a certain level that makes the yield spread of the firm equal to the average yield spread of investment-grade bonds. The parameter values estimated from this simulation are $\phi = 0.3$, $B = 2.1$, and $b = 0.6$. Under this parameter choice, the annual default rate is 2.7%, the type I error is 17.59%, and the type II error is 8.71%, which are reasonably close to the target moments.

Under the above baseline parameter values, Figure 4 plots the value functions $E^k(x)$, $D^k(x)$, $V^k(x)$, and $C^k(x)$ for each rating class $k \in \{L, H\}$, where $V^k(x)$ is defined as the value of a firm, that is, $E^k(x) + D^k(x)$. The resulting equilibrium thresholds are $x_D = 5.23$, $x_S^H = 8.33$, $x_R^H = 9.56$, $x_R^L = 13.38$, and $x_S^L = 15.39$. Moreover, according to Ross (1985),

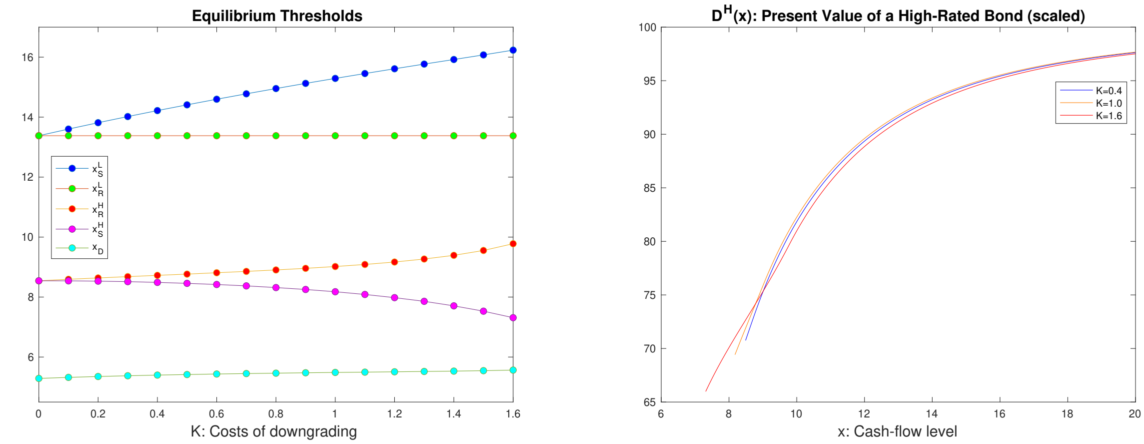

Figure 5: This graph presents the effects of the credit-rating revision cost K. The left panel plots the effects of K on the equilibrium thresholds, x_D, x_S^H, x_R^H, x_R^L, and x_S^L. The right panel plots the effects of K on the debt value of a high-rated firm, by scaling the debt value by a factor $\frac{r+\lambda}{c+\lambda}$ so that the maximum possible debt value is set to F.

the average yield spreads of BBB-rated bonds and B-rated bonds with medium maturity are 235 bps and 510 bps, respectively. In our model, under the above baseline parameter choice, the cash-flow levels that generate these two yield spreads are $x_{BBB} = 12.9$ and $x_B = 10.3$, respectively.

4.2 Effects of the Rating-Revision Cost

We first study the effects of the credit-rating revision cost K on the firm's default risk and its debt value. An increase in K makes it more costly for the credit rating agency to change the rating, therefore, we expect a larger lag length between a substantial change in the firm's credit risk and a credit-rating revision. In the left panel of Figure 5, we indeed find that following an increase in the rating-revision cost K, the downgrading lag $(x_R^H - x_S^H)$ and the upgrading lag $(x_R^L - x_S^L)$, measured by the distance between the firm's liquidity-driven default threshold and the corresponding rating-revision threshold, become larger.

The increase in rating-revision cost causes more delay in credit rating revisions, which affects the firm's credit risk and debt value. To simplify the discussion, we focus on a H-rated firm in our calibrated model, which corresponds to investment-grade bonds in reality. For this H-rated firm, a larger lag in credit rating revision has two effects on its default risk and debt value.

First, a larger lag in credit rating revision has a direct effect on the H-rated firm's probability of maintaining its rating in the future. As the firm would now be downgraded at a lower cashflow threshold, its expected probability to remain H-rated becomes higher. A higher probability of staying H-rated leads to lower default risk and higher debt value. This is because, when compared with L-rated firms, H-rated firms have a higher rollover proceed and a lower probability of liquidity-driven default in the future.

However, a larger lag in credit rating revision also has an indirect effect on default risk and debt value through debt rollover. When the firm is downgraded at a lower cashflow threshold, the lowest-possible cashflow value of a H-rated firm becomes smaller, and thus the degree of information asymmetry in the debt issuance market is higher. In other words, the informational value of credit rating becomes smaller.[3] Consequently, the H-rated firm's rollover proceeds shrink, so it is more likely to suffer from liquidity-driven default when staying H-rated. This leads to higher default risk and lower debt value.

In sum, an increase in K causes a larger lag in credit rating revision, yet its effects on the firm's default risk and debt value are ambiguous, as the direct and the indirect effects are opposite in sign. The right panel of Figure 5 shows that the tradeoff between the two effects leads to interesting predictions about the effect of rating-revision cost on firms' debt values.

In the right panel of Figure 5, we plot the debt value for H-rated firms at different cashflow levels for different rating-revision cost. There are two interesting results from the graph. First, we find that for H-rated firms with relatively low cashflow levels, their debt values are higher when the rating-revision cost is larger. Note that these firms are close to being downgraded. For these firms, the higher lag in rating revision can substantially reduce the probability of being downgraded in the future, and therefore, the direct effect of a higher lag in rating revision dominates. Consequently, a larger lag in rating revision can increase the debt value for these H-rated firms with relatively weak fundamentals.

Second, for H-rated firms with relatively high cashflow levels, we find that there exists a positive rating-revision cost that leads to the highest debt value. In the graph, we find

[3] In the extreme case where $K \to \infty$, the credit rating would never be revised, and the informational value of credit rating goes to zero.

that all else being equal, the debt value when $K = 1$ is larger than that when $K = 0.4$ or when $K = 1.6$. This result is driven by the tradeoff of the two effects discussed above. On the one hand, when the rating-revision cost is too small, the direct effect dominates, so that an increase in rating-revision cost can enhance debt value by reducing the likelihood of downgrading. On the other hand, when the rating-revision cost is too large, the indirect effect dominates, so that an increase in rating-revision cost can reduce debt value by increasing the likelihood of liquidity-driven default when the debt stays H-rated.

Credit rating agencies are often criticized because their slow response to changes in credit quality of corporate bonds. Yet, our model suggests that revising credit ratings fast according changes in credit risks, assuming that credit rating agency has the ability to do so, may not be optimal for debt value. A certain amount of lag in rating revision can enhance debt value because it reduces the probability of being downgraded and grants the firm more time for its fundamental to recover. Yet, when the lag is too large, a further increase in the lag can hurt debt value because it reduces the informational value of credit rating and thus makes H-rated firms more prone to liquidity-driven default. We find that the debt value is pretty high in the calibrated model. This suggests that the current level of lag in credit rating revision is close to the optimal level that maximizes corporate bond values, at least for firms with strong fundamentals.

4.3 Effects of Reputation Losses

Now we analyze the effects of the reputation losses caused by late alarm (B) or false alarm (b). Here B is the reputation loss to the credit rating agency when a H-rated firm defaults on its debt, so it is associated with the credit rating agency's failure to alarm the firm's default risk by downgrading. b is the flow reputation loss to the credit rating agency when a L-rated firm keeps rolling over its debt, so it is associated with the credit rating agency generating false alarm about the firm's credit risk.

Figure 6 plots the comparative statics with respect to B. An increase in B makes it more costly for the credit rating agency to postpone its downgrading, which results in a higher downgrading threshold x_S^H. As such, the increase in B increases the probability of a

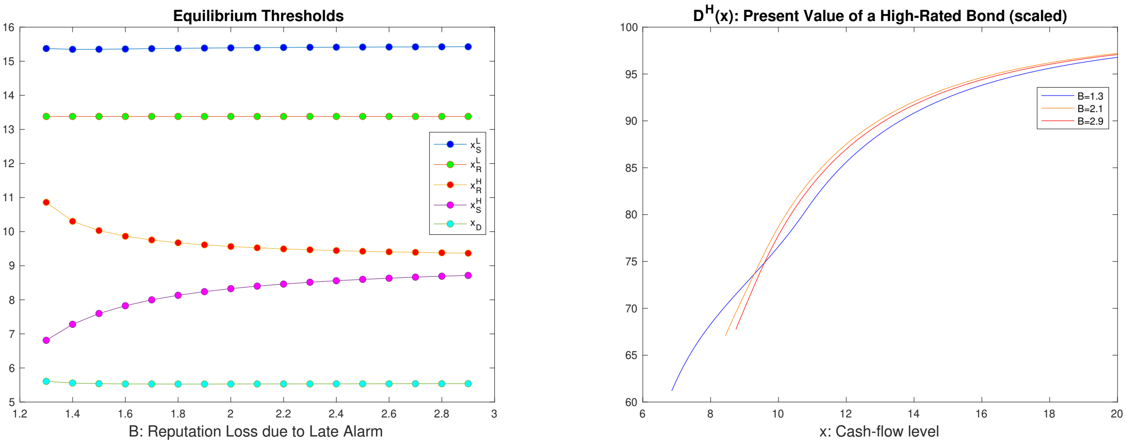

Figure 6: This graph presents the effects of the reputation loss, B, caused by a late alarm. The left panel plots the effects of B on the equilibrium thresholds, x_D, x_S^H, x_R^H, x_R^L, and x_S^L. The right panel plots the effects of B on the debt value of a high-rated firm, by scaling the debt value by a factor $\frac{r+\lambda}{c+\lambda}$ so that the maximum possible debt value is set to F.

H-rated firm being downgraded in the future, which is described as the direct effect discussed in Section 3.2. Nevertheless, the higher downgrading threshold alleviates information asymmetry in the bond market, resulting in higher rollover proceeds for H-rated firms and thus a lower rollover threshold x_R^H. As such, the increase in B reduces the H-rated firm's defaultable region $(x_R^H - x_S^H)$, and therefore reduces the firm's conditional default risk (conditional on the firm staying at the H rating), which is the indirect effect discussed in Section 3.2.

Similarly as in Section 3.2, the increase in B has an ambiguous effect on the firm's debt value. For a H-rated firm whose cashflow level is close to the downgrading threshold, its debt value is larger when B becomes smaller, because a smaller B leads to a larger lag in the credit rating agency's downgrading decision, giving the firm more time for its cashflow level to recover. In contrasts, for a H-rated firm whose cashflow level is relatively high, there seems to be an intermediate value of B that leads to the highest debt value. The non-monotonic effect of B on debt value is again driven by the tradeoff between the two effects discussed in Section 3.2.

Figure 7 plots the comparative statics with respect to b. An increase in b makes it more costly for the credit rating agency to postpone its upgrading, which results in a lower upgrading threshold x_S^L. As the credit rating agency concerns more about lag in upgrading rather than about lag in downgrading, it re-optimizes its rating-revision decision by setting

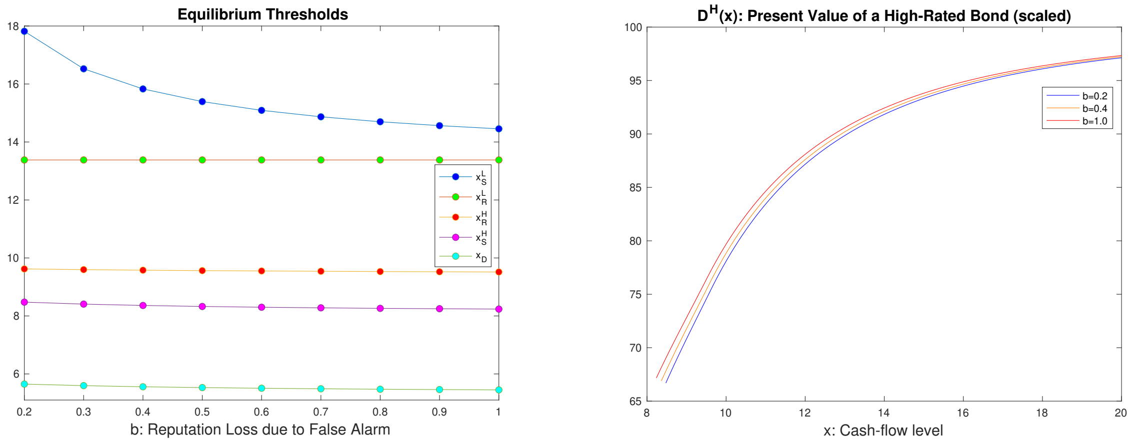

Figure 7: This graph presents the effects of the reputation loss, b, caused by a false alarm. The left panel plots the effects of b on the equilibrium thresholds, x_D, x_S^H, x_R^H, x_R^L, and x_S^L. The right panel plots the effects of b on the debt value of a high-rated firm, by scaling the debt value by a factor $\frac{r+\lambda}{c+\lambda}$ so that the maximum possible debt value is set to F.

a lower downgrading threshold x_S^H and thus increasing the lag in downgrading. As such, the increase in b decreases the probability of a H-rated firm being downgraded in the future, which is described as the direct effect discussed in Section 3.2. Nevertheless, the higher downgrading threshold exacerbates information asymmetry in the bond market, resulting in lower rollover proceeds for H-rated firms and thus a higher rollover threshold x_R^H. As such, the increase in b expands the H-rated firm's defaultable region $(x_R^H - x_S^H)$, and therefore heightens the firm's conditional default risk (conditional on the firm staying at the H rating), which is the indirect effect discussed in Section 3.2.

Again the increase in b has an ambiguous effect on the firm's debt value. For a H-rated firm whose cashflow level is close to the downgrading threshold, its debt value is larger when b becomes larger, because a larger b leads to a larger lag in the credit rating agency's downgrading decision, giving the firm more time for its cashflow level to recover. In contrasts, for a H-rated firm whose cashflow level is relatively high, there seems to be an intermediate value of b that leads to the highest debt value. The non-monotonic effect of b on debt value is again driven by the tradeoff between the two effects discussed in Section 3.2.

In sum, our model suggests that the effect of penalizing the credit agency's rating inaccuracy depends crucially on whether the additional penalty is imposed on Type I or Type II error, that is, whether penalizing the credit rating agency because of false alarm or because of

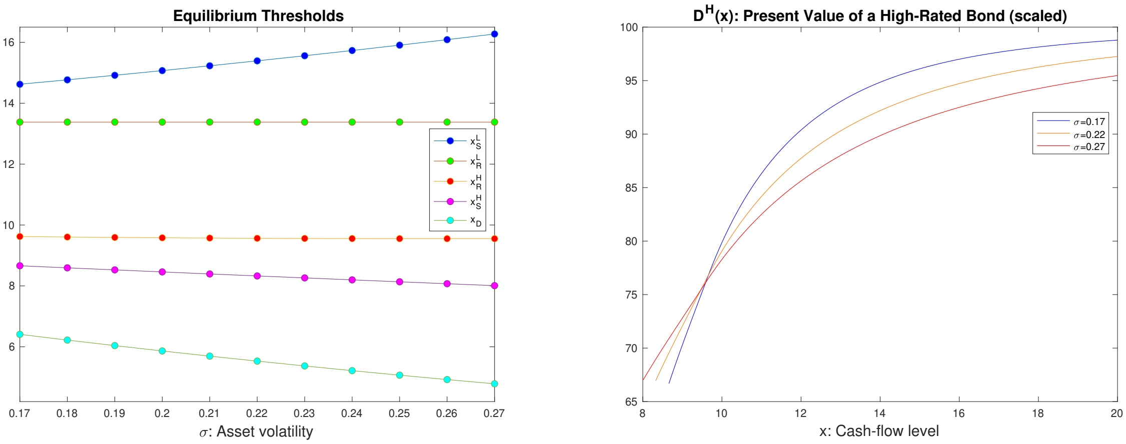

Figure 8: This graph presents the effects of the asset volatility σ. The left panel plots the effects of σ on the equilibrium thresholds, x_D, x_S^H, x_R^H, x_R^L, and x_S^L. The right panel plots the effects of σ on the debt value of a high-rated firm, by scaling the debt value by a factor $\frac{r+\lambda}{c+\lambda}$ so that the maximum possible debt value is set to F.

late alarm. Increasing the agency's reputation loss when a H-rated firm defaults (late alarm) can reduce the lag in downgrading, while increasing the agency's reputation loss when a L-rated firm stays alive (false alarm) can increase the lag in downgrading. We show that the overall effects of changes in reputation losses on the firm's default risk and debt value are ambiguous, because a higher level of downgrading delay has both a positive direct effect that reduces the probability of downgrading and a negative indirect effect through exacerbating information asymmetry in the debt issuance market.

4.4 Effects of Asset Volatility

Now we study how asset volatility affects the firm's default risk, its rating revisions, and its debt value. Figure 8 plots the comparative static result with respect to the firm's cashflow volatility σ.

When the firm's cashflow level is more volatile, the firm is more likely to enter into low cashflow regions, resulting in a higher probability of default in the future. Therefore, an increase in asset volatility leads to a decrease in the debt value, which is shown in the right panel of Figure 8. Consequently, an increase in asset volatility causes the rollover thresholds x_R^H and x_R^L to be higher, which further increases the probability of the firm entering into a liquidity-driven default event. For rating revisions, we find that both the lag in downgrading

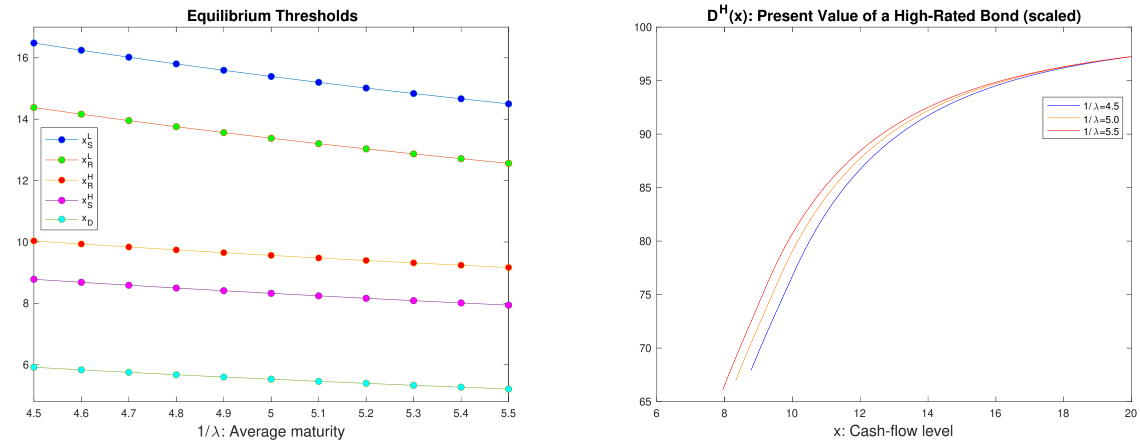

Figure 9: This graph presents the effects of the average debt maturity, which is equal to $1/\lambda$ in this model. The left panel plots the effects of debt maturity on the equilibrium thresholds, x_D, x_S^H, x_R^H, x_R^L, and x_S^L. The right panel plots the effects of debt maturity on the debt value of a high-rated firm, by scaling the debt value by a factor $\frac{r+\lambda}{c+\lambda}$ so that the maximum possible debt value is set to F.

$(x_R^H - x_S^H)$ and the lag in upgrading $(x_S^L - x_R^L)$ are higher when the asset volatility becomes larger. Intuitively, when the firm's cashflow level is more volatile, the credit rating agency is more likely to incur reputation losses, and therefore, she optimally sets large laps for her rating-revision decisions.

4.5 Effects of Debt Maturity

In this section, we focus on debt maturity. Specifically, we are interested at how the interaction between default risk and lagged rating differs across bonds with different maturities. Figure 9 plots the effect of debt maturity on default risk, lagged ratings, and debt value.

When the firm's debt has shorter maturity, the firm needs to roll over more frequently, and therefore, the firm suffers a higher amount of rollover loss when its cashflow level is relatively low. As such, the endogenous default boundary x_D, the rollover threshold in the L stage x_R^L, and the rollover threshold in the H stage x_R^H are higher when the debt maturity is shorter. Nevertheless short-term debt matures faster and is more likely to get paid in full. In our calibrated model, the debt value is higher when the debt maturity is shorter, which is shown in the right panel of Figure 9. For rating revisions, we find that the distance between the upgrading and the downgrading thresholds $x_S^L - x_S^H$ is larger for debts with

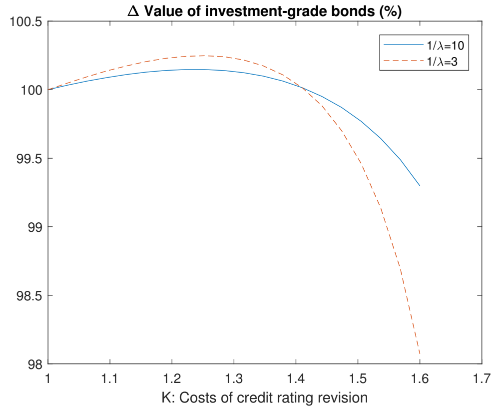

Figure 10: This graph compares the effects of the credit-rating revision cost on the value of debt at the H stage by changing the average maturity of debt. The solid curve represents the effect in the case where the average maturity is 10 years. The dashed curve represents the case where the average maturity is 3 years. When calculating the value of debt, we set $x = x_{BBB}$ to match the average yield spread of BBB-rated bonds.

shorter maturity. In other words, rating stability is higher for shorter-term debt. This is because when compared to long-term debts, short-term debts are more likely to be paid in full and less subject to default risks.

In addition, Figure 10 shows how the effect of an increase in the rating-revision cost on debt value varies across bonds with different maturities. First, notice that, debt value is non-monotonic in the rating-revision cost for a given debt maturity. This result echoes our analysis in Section 3.2. More importantly, we show that the value of short-term debt is more sensitive to a change in rating-revision cost. Intuitively, an increase in rating-revision cost induces a larger time lag in the credit rating agency's rating revision decision, which leads a direct effect that benefits debt value by reducing the probability of H-rated firms getting downgraded and an indirect effect that hurts debt value by reducing the rollover proceeds for H-rated firms. Notice that both effects work through debt rollover. As short-term debts are rolled over more frequently, the value of short-term debt should be more sensitive to a change in rating-revision cost.

5 Extensions

5.1 Endogenous Recovery Rate

In this section, we endogenize the recovery rate of failed assets, considering a signaling game in the liquidation market. To begin, we first assume that upon default, the firm's manager provides the precise information about the firm's current fundamental to the existing creditors of the firm. Or, we can equivalently assume that the firm's manager liquidates the assets on behalf of the creditors who acquire the assets after default. Potential buyers of the assets are not informed of the firm's fundamental.

To mitigate this informational asymmetry, the creditors, who are now the asset owners, choose to retain some portion of the assets, similar to Leland and Pyle (1977). If the creditors continue to operate any portion of the assets as a going concern, the future cash-flow level will drop to βx_t at each point in time, where $\beta < 1$. If potential asset buyers operate the assets, the cash-flow level will drop to only γx_t, where $\beta < \gamma \leq 1$, which means that potential asset buyers may not be the first-best asset users, but they are at least more skillful than the current creditors who typically do not have any industry-specific skills. This assumption is consistent with Shleifer and Vishny (1992).

In this setting, the market believes that the creditors of a firm with a fundamental value of $x \in [x_D, x_R^L)$ will retain a fraction $f^L(x)$ of the assets during the L stage and the creditors of a firm with a fundamental value of $x \in [x_S^H, x_R^H)$ will retain a fraction $f^H(x)$ of the assets during the H stage. During each stage, the creditors of a firm with the lowest fundamental are believed to sell the entire fraction of their assets. As in the bond market, we focus on a separating equilibrium in which each firm's fundamental is truthfully revealed through the asset-retention strategy.

Under this market belief, during the L stage, the creditors of a firm with a fundamental value of $x \in [x_D, x_R^L)$ solve the following problem:

$$\max_{y \in [x_D, x_R^L)} \underbrace{f^L(y)\beta V(x)}_{\text{from retained assets}} + \underbrace{(1 - f^L(y))\gamma V(y)}_{\text{from liquidation}},$$

where the first term denotes the present value of the cash flows that will be obtained from the retained portion of the assets and the second term represents the liquidation proceeds that will be immediately obtained from selling the remaining portion of the assets, if the above creditors mimic the creditors of another firm with a fundamental value of y. The first-order condition of this problem is written as

$$f_x^L(y)\beta V(x) - f_x^L(y)\gamma V(y) + (1 - f(y))\gamma V_x(y) = 0.$$

For the separating equilibrium to arise, this differential equation must hold with $y = x$. Using the boundary condition, $f(x_D) = 0$, the solution for $f^L(x)$ is given by

$$f^L(x) = 1 - \left(\frac{x}{x_D}\right)^{-\frac{\gamma}{\gamma-\beta}}, \quad \forall x \in [x_D, x_R^L).$$

As a result, the total recovery value of the assets is given by

$$[f^L(x)\beta + (1 - f^L(x))\gamma]V(x),$$

which means the endogenous recovery rate is given by $\alpha^L(x) := f^L(x)\beta + (1 - f^L(x))\gamma$.

Similarly, during the H stage, the creditors of a firm with a fundamental value of $x \in [x_S^H, x_R^H)$ retain a fraction $f^H(x)$ of the assets, where

$$f^H(x) = 1 - \left(\frac{x}{x_S^H}\right)^{-\frac{\gamma}{\gamma-\beta}}, \quad \forall x \in [x_S^H, x_R^H).$$

The recovery rate of the assets is then given by $\alpha^H(x) := f^H(x)\beta + (1 - f^H(x))\gamma$.

Now, note that we can still characterize an equilibrium in closed form because the recovery rate $\alpha^k(x)$ is a polynomial function with a power of real numbers. The closed-form solution is provided in Section B. Moreover, since the total recovery value $\alpha^k(x)V(x)$ is also increasing in x at each stage $k \in \{L, H\}$, we see that most of the main qualitative properties of the model must remain the same even if we use the endogenous recovery rate.

6 Conclusion

We study the interaction between the firm's default risk and the credit rating agency's rating decision in a dynamic credit-risk model with debt rollover. While credit rating provides useful information about debt value and alleviates information asymmetry in the debt issuance market, the credit rating agency has an incentive to lag its rating revisions so as to minimize future rating-revision costs and to enhance rating stability. We show that such a lag in rating revisions may actually lead to higher debt values, because lagged downgrading decisions give high-rated firms more time for their fundamentals to recover, thereby reducing the risk of getting downgraded and the occurrence of liquidity-driven default in the future. Our model also yields interesting predictions about how this interaction between the firm default risk and the rating revision lag is affected by firm characteristics such as debt maturity and asset volatility.

A Proof of Proposition 3.1

Proof. To prove this proposition, we first note that $D^L(x)$ strictly increases in x over the interval $[x_D, x_R^L]$ because $D^L(x_D) = 0$ and the rating agency does not incur any reputation losses when the firm's fundamental lies between x_D and x_R^L. Similarly, we show that $D^H(x)$ strictly decreases in x over the interval $[x_R^H, \infty)$ because $\lim_{x \to \infty} D^H(x) = 0$ and the rating agency does not incur any reputation losses when the firm's fundamental lies above x_R^H. Therefore, given that x_R^H is at least larger than x_D, if $x_S^H = x_R^H$, the smooth-pasting condition for x_S^H, that is, $C_x^H(x_S^H) = C_x^L(x_S^H)$, cannot hold. Also, the other smooth-pasting for regarding x_S^L, that is, $C_x^L(x_S^L) = C_x^H(x_S^L)$, cannot hold if $x_S^L = x_R^L$. □

B Closed-form Solution for the Extended Model

In this section, we provide a closed-form solution for the equilibrium of the model with an endogenous recovery rate. We first note that for any given thresholds, (x_D, x_R^L, x_S^L, x_S^H, x_R^H), the recovery value of the failed assets affects only the debt values, we only need to solve for the debt values in closed form. To this aim, using the fact that the total recovery value of the failed assets is equal to

$$R^L(x) = \beta f^L(x)V(x) + \gamma(1 - f^L(x))V(x) = \beta V(x) + (\gamma - \beta)(x/x_D)^\zeta V(x)$$

and

$$R^H(x) = \beta f^H(x)V(x) + \gamma(1 - f^H(x))V(x) = \beta V(x) + (\gamma - \beta)(x/x_S^H)^\zeta V(x)$$

at the L and H stages, respectively, where $\zeta = -\gamma/(\gamma - \beta)$, the value functions of debt are described as

$$D^L(x) = \begin{cases} \frac{(c+\lambda)F}{r+\lambda+\phi} + \frac{\phi\beta V(x)}{r+\lambda+\phi-\mu} + \frac{\phi(\gamma-\beta)(x/x_D)^\zeta V(x)}{r+\lambda+\phi-\mu(1+\zeta)-\sigma^2(1+\zeta)\zeta/2} + A_1 x^{\xi_1} + A_2 x^{\xi_2}, & \text{if } x_D < x < x_R^L \\ \frac{(c+\lambda)F}{r+\lambda} + A_3 x^{\xi_3} + A_4 x^{\xi_4}, & \text{if } x_R^L < x < x_S^L, \end{cases}$$

$$D^H(x) = \begin{cases} \frac{(c+\lambda)F}{r+\lambda+\phi} + \frac{\phi\beta V(x)}{r+\lambda+\phi-\mu} + \frac{\phi(\gamma-\beta)(x/x_S^H)^\zeta V(x)}{r+\lambda+\phi-\mu(1+\zeta)-\sigma^2(1+\zeta)\zeta/2} + A_5 x^{\xi_1} + A_6 x^{\xi_2}, & \text{if } x_S^H < x < x_R^H \\ \frac{(c+\lambda)F}{r+\lambda} + A_7 x^{\xi_4}, & \text{if } x_R^H < x, \end{cases}$$

where ξ_1 and ξ_2 are the same as those numbers given in Section 3.2. The coefficients A_1, ..., A_7 are pinned down from the same boundary conditions used in Section 3.2. The equilibrium thresholds are also constructed by the same method used in the main model.

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