Volatility Disagreement and Asset Prices

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Abstract

We study a dynamic equilibrium model in which investors disagree on future volatility and trade volatility derivatives to hedge stock positions and speculate. On average, volatility disagreement makes the variance risk premium more negative. However, volatility trading enables a risk transfer among investors that turns the variance risk premium positive when the market underestimates future volatility. Under higher volatility, investors trade fewer volatility derivatives as these become too risky. These economic mechanisms shed light on empirical regularities during market turmoil. Volatility disagreement also lowers the stock market valuation, increases market volatility, and generates time-variation in the leverage effect.

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1 Introduction

The volatility derivatives market has experienced a rapid growth in the last two decades, allowing more investors to hedge volatility risk and to speculate on future volatility.[1] A recent survey of senior executives of trading firms (Acuiti (2023)) finds that 64% of survey respondents were trading volatility products, and suggests that volatility derivatives have moved from being a niche hedging instrument to a core asset class for many investors. The growth of the volatility derivatives market has spurred a vast literature documenting various empirical regularities in this market that, when taken together, are hard to reconcile within existing theoretical studies. In particular, two interesting puzzles have emerged: the variance risk premium changes sign during market turmoil, and trading volume in volatility derivatives dries up during periods of heightened volatility (e.g., Cheng (2019, 2020)). These findings are surprising as they imply that during market turmoil insurance providers against volatility risk are expected to lose money, and that investors rely less on volatility trading precisely when there is arguably more volatility risk to be hedged.

In this paper, we develop a tractable dynamic asset pricing model that can explain this evidence and delivers a rich set of novel predictions for the volatility derivatives market, as well as the stock market. The key ingredient in our model is that investors disagree about future volatility. Despite growing survey evidence documenting significant disagreement in investors' volatility expectations (e.g., Graham and Harvey (2001), Amromin and Sharpe (2014), Kaplanski et al. (2016)), this form of disagreement and its implications for financial markets have been largely unexplored. To the best of our knowledge, ours is the first theory work that incorporates volatility disagreement in a dynamic equilibrium setting.

We consider an economy in which a stock, representing the aggregate stock market, is a claim to a risky payoff determined by a fundamental process with a stochastic variance. To hedge the volatility risk in their stock holdings and to speculate on their beliefs, investors can also trade variance swaps. Investors disagree on volatility, as they have different expectations about the future variance of the fundamental process. Specifically, we consider two risk-averse investor types: *high-fear* investors, who overestimate the future variance, and *low-fear* investors, who underestimate it.[2] The valuation of the stock market, the variance swap rate (i.e., the price that volatility buyers pay to volatility sellers), and investors' security holdings are determined in equilibrium.

[1]The primary financial instruments in the volatility derivatives market are variance swaps, VIX futures, and VIX options. While the average daily volume of VIX futures was around 450 contracts in 2004 when first introduced, it grew to around 215,000 in 2023. Similarly, Moran and Liu (2020) reports the total annual volume of VIX options to be 127 million contracts in 2019. See Carr and Lee (2009) for a brief survey on the history and the workings of the volatility derivatives market.

[2]Our terminology for investor types is motivated by the financial press and industry commonly referring to the CBOE's volatility index, VIX, as the "fear index."

We first characterize the equilibrium in this economy and show that in the presence of volatility disagreement, the relative wealth distribution across investors arises as an additional endogenous state variable. In accordance with their volatility expectations, investors take different positions in the stock and volatility derivatives, leading to wealth transfers in our dynamic economy. Investors whose beliefs align with realized shocks get relatively wealthier and have a stronger impact on asset prices. In particular, high-fear investors become volatility buyers and low-fear investors volatility sellers in the derivatives market. Thus, following positive (negative) variance shocks, high-fear (low-fear) investors get wealthier and more dominant. Although wealth transfer effects are generally present in heterogeneous-agents models, the novel feature of wealth transfers in our setting is that they arise because of the volatility disagreement and are driven by shocks to the second moment of asset returns.

Investigating the equilibrium behavior of the stock market, as novel predictions, we find that on average higher volatility disagreement leads to lower stock market valuation but higher and more volatile stock return variance. An increase in volatility disagreement makes high-fear investors to discount future stock payoffs more and low-fear investors to discount them less. However, the higher and more persistent variance expectations of high-fear investors make them more sensitive to risk than low-fear investors. So, the impact of their discounting is stronger in equilibrium, inducing an overall downward pressure on the valuation of the stock market. By making the stock more sensitive to variance shocks and amplifying wealth transfers, higher volatility disagreement also increases the stock return variance, and makes it more volatile.

Moving to the volatility derivatives market, we first demonstrate that the ability to trade volatility derivatives induces a *risk transfer* from high-fear to low-fear investors in equilibrium. Such risk transfer occurs because of the opposite positions they take in the variance swaps, making low-fear investors more exposed to variance risk and high-fear investors less exposed to it. Moreover, the risk transfer is exacerbated by the misvaluation of the volatility derivatives that each investor type perceives as induced by the other type. Indeed, the overestimation of future volatility by high-fear investors makes the variance swaps overvalued from the perspective of low-fear investors. Similarly, the underestimation of future volatility by low-fear investors makes the variance swaps undervalued from the perspective of high-fear investors. In equilibrium, these perceived misvaluations create a positive wedge between the subjective variance risk premium of high-fear investors and that of low-fear investors.

Characterizing the objective variance risk premium, i.e., the variance risk premium that an unbiased econometrician would measure in this economy, we show that, while being negative on average, it can actually turn positive when the economy tend to underestimate future volatility, which is when a large fraction of wealth is held by low-fear investors. To fully appreciate this novel result, we decompose the variance risk premium into two economically distinct components: a hedging component and a speculative component. The hedging component reflects the investors' desire to hedge fluctuations in the stock price that are driven by variance shocks. The ensuing demand for variance swaps, enabling the hedging, creates price pressure that leads to a negative variance risk premium on average in equilibrium. The speculative component, which instead reflects the investors' contrasting beliefs and the opposite positions that they take in the variance swaps, can become positive or negative depending on which investor type has a stronger price impact. As low-fear investors become more dominant in the economy, the supply of variance swaps increases while its demand decreases. Hence, the expected return of providing volatility insurance must go down for the derivatives market to clear, resulting in a positive variance risk premium.

We also find that higher volatility disagreement leads to a more negative variance risk premium and a higher variance swap rate on average. These results arise in our model because as disagreement increases, the stock price becomes more volatile, thus intensifying investors' desire to hedge against stock market fluctuations. For the derivatives market to clear, the variance swap increases and the variance risk premium becomes more negative. Taken together, the economics around the variance risk premium in our model significantly differ from those in the existing literature, which are predominantly based on representative agent settings and thus unable to generate our wealth and risk transfer effects (e.g., Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), Eraker and Wu (2017), Atmaz (2022), Lochstoer and Muir (2022)).

A key advantage of our setting is that it allows us to study the equilibrium holdings and trades in volatility derivatives. We find a notable, and perhaps unexpected, hump-shaped relation between volatility disagreement and the variance swap open interest (or trades). This non-monotonic relation occurs because of two opposing effects of disagreement. A higher disagreement increases both high- and low-fear investors' expected profits from trading variance swap contracts. Everything else equal, this effect leads to larger variance swap holdings in equilibrium. However, a higher disagreement also increases the riskiness of these swap contracts, leading to smaller variance swap holdings. For low levels of disagreement, the former effect dominates, inducing a positive relation between volatility disagreement and the variance swap open interest. However, for high levels of disagreement, the relation inverts because trading variance swaps becomes too risky for investors. The fact that trading volatility can become excessively risky for investors is also the reason why in our model we find that the variance swap open interest decreases with the fundamental variance. So, even when investors should arguably be more willing to trade volatility derivatives to speculate (when disagreement increases) or hedge their risky positions (when fundamental risk increases), they find it optimal to reduce their volatility trading. It is the general equilibrium nature of asset prices, increasing the riskiness of volatility derivatives, that is key to these results, which would otherwise not emerge in partial equilibrium settings.

Our theory also provides novel predictions for the so-called "leverage effect," which in our model is characterized by the comovement between the stock and variance swap returns. We find that the volatility disagreement is a key determinant and a source of time-variation in the leverage effect. In our model, the leverage effect arises in equilibrium because positive variance shocks lead to lower stock returns but higher stock return variance, translating to higher returns for the variance swaps. Notably, the leverage effect gets stronger under more volatility disagreement and consistent with its documented behavior in the data (e.g., Bandi and Renò (2012), Andersen, Bondarenko, and Gonzalez-Perez (2015)) its magnitude increases in more volatile periods. These results arise because asset prices become more sensitive to variance shocks under higher disagreement and uncertainty since in these cases there is more room for wealth transfers.

We conclude our analysis by considering an extension of our model that allows us to investigate the implications of an aggregate volatility bias. In equilibrium, by distorting the investors' subjective expectations in the same direction, a higher aggregate volatility bias lowers the stock market valuation, the variance risk premium, while increasing the stock return variance, the magnitude of the leverage effect, and the variance swap rate. Counter to our expectations, we find that a higher aggregate volatility bias always reduces volatility trading in equilibrium. Differently from the effect of volatility disagreement, a higher bias increases only the riskiness of variance swaps, without affecting their expected returns.

Overall, our paper makes several contributions. First, to our best knowledge, ours is the first theory work to study the implications of volatility disagreement for the stock and volatility derivatives market in a dynamic equilibrium setting. Second, all our results for the effects of volatility disagreement are new and are not obtained in the existing literature. Third, to the best of our knowledge, we are the first to reconcile the puzzling empirical evidence that investors on average tend to hold and trade fewer volatility derivatives in periods of high volatility (e.g., Cheng (2019)). Indeed, Cheng finds that during market turmoils, the variance risk premium is largely positive, its magnitude rises in volatility, and investors trade fewer volatility derivatives. Our model predicts that low-fear investors must become sufficiently dominant for the variance risk premium to become positive, which means that the economy must have experienced a sufficiently long period of low volatility. This additional prediction is in line with prolonged "calm" periods that tend to precede market turmoil, such as the 2008 financial crises and the 2020 Covid-19 pandemic. The sudden surge in volatility associated with the outset of these episodes has led the positive variance risk premium to spike (Cheng (2019, 2020)), as our theory predicts.

Related Literature. This paper is related to several strands of literature. Our paper contributes to the theoretical literature studying the effects of belief disagreement on asset prices. In this literature, the vast majority of works focus on first-moment disagreement (e.g., Detemple and Murthy (1994), Zapatero (1998), Basak (2000, 2005), Gallmeyer and Hollifield (2008), Yan (2008), Dumas, Kurshev, and Uppal (2009), Banerjee (2011), Bhamra and Uppal (2014), Buraschi, Trojani, and Vedolin (2014), Atmaz and Basak (2018), Ehling, Graniero, and Heyerdahl-Larsen (2018), Andrei, Carlin, and Hasler (2019a), Panageas (2020), Heyerdahl-Larsen and Walden (2021), Xiouros and Zapatero (2024)). In these works, investors typically disagree on the expected growth rate of aggregate consumption or dividend. Thus, wealth transfers in these models are driven by shocks to the first moment of asset returns, resulting in asset price dynamics that are different than ours. Moreover, these works typically study the effects of such disagreement only on the stock market and do not consider volatility derivatives market as we do.

In contrast, there are only a few works studying the asset pricing effects of second-moment disagreement (Detemple and Selden (1991), Duchin and Levy (2010), Bakshi, Madan, and Panayotov (2015)). Our methodology and modeling of volatility disagreement, and hence our results, differ considerably from these works. For instance, all these works employ static mean-variance frameworks, thus abstracting from the dynamic trading and wealth transfer mechanisms, which are key to our main results. Particularly, Detemple and Selden (1991) and Duchin and Levy (2010) focus only on the stock market and find that a higher volatility disagreement leads to a higher stock price and lower risk premium, the opposite of what we find. This difference in results arises because in these works, investors disagree on the *current* variance. Since the mean-variance security demand is convex function of current variance, due to Jensen's inequality, a cross-investor uncertainty about the current variance (disagreement), increases the demand and hence the stock price. Whereas in our model, investors disagree on the *future* variance while agreeing on the current variance; thus, there is no such Jensen's convexity effect. Bakshi, Madan, and Panayotov (2015), on the other hand, focuses on the implied shape of the pricing kernel and does not have our implications for the volatility disagreement and volatility derivatives market.

Our paper also relates to the recent theoretical literature studying the equilibrium effects of investors' volatility expectations. In this literature, Atmaz (2022) and Lochstoer and Muir (2022) consider representative agent dynamic extrapolative expectations frameworks to study the effects of biased volatility expectations on asset prices. Whereas Ghaderi, Kilic, and Seo (2023) considers an incomplete information setting in which a representative agent rationally learns about a hidden state of the economy. Due to incomplete information and learning, they find that variance risk premium can become positive in some states, similar to ours. Since all these works employ single-agent economies, they cannot provide our predictions about prices *and* quantities in the volatility derivatives market or the effects of volatility disagreement.

Finally, we contribute to the literature studying the effects of trading in nonredundant derivatives (e.g., Brennan and Cao (1996), Franke, Stapleton, and Subrahmanyam (1998), Cao and Ou-Yang

(2008), Garleanu, Pedersen, and Poteshman (2008), Bhamra and Uppal (2009), Banerjee and Graveline (2014), Smith (2019), Chabakauri, Yuan, and Zachariadis (2022)). The derivatives considered in these works are typically equity options rather than volatility derivatives, with the exception of Smith (2019) and Chabakauri, Yuan, and Zachariadis (2022), who study the informational role of volatility derivatives in static asymmetric information frameworks. Smith (2019) develops an incomplete market model in which investors receive private signals on both the mean and the variance of the stock payoff and focuses on the variance risk premium in his analysis. However, differently from our key finding, the variance risk premium does not switch sign in his static model. Chabakauri, Yuan, and Zachariadis (2022) finds that volatility derivatives make incomplete markets effectively complete, and their prices reflect the shadow value of information. In our model, the presence of volatility derivatives completes the markets too. However, differently from them, and all the other works above, we consider a dynamic symmetric information framework focusing on the asset pricing implications of volatility disagreement.

The remainder of the paper is organized as follows. Section 2 introduces our model with volatility disagreement. Section 3 presents our results on the stock market, while Section 4 focuses on the volatility derivatives market. In Section 5 we study the return comovement across markets. In Section 6, we extend our baseline model to incorporate an aggregate volatility bias, before concluding in Section 7. Appendix A contains all the proofs, Appendix B discusses the effects of pure variance shocks, and Appendix C discusses the parameter values employed in our figures and tables.

2 Model

In this section, we present a simple and tractable pure-exchange economy in which two types of investors disagree about future volatility. A salient feature of our model is the presence of volatility derivatives market, which allows investors with different future volatility expectations to trade on their beliefs and speculate against each other.

2.1 Securities Market

We consider a continuous-time economy with horizon T. In this economy, three securities are available for trading: a riskless bond, a risky stock (representing the aggregate stock market), and a volatility derivative. The riskless (zero-coupon) bond, with its time-t price denoted by Z_t, is in zero net supply with a constant rate of return r. The stock, with its time-t price denoted by S_t, is in positive net supply of one unit and is a claim to the risky payoff D_T at horizon T, so $S_T = D_T$. The stock payoff is the time T realization of the fundamental (cashflow news) process D_t with dynamics

$$\frac{dD_t}{D_t} = \mu dt + \sqrt{V_t} d\omega_{1t}, \tag{1}$$

$$dV_t = \kappa \left(\overline{V} - V_t\right) dt + \sigma \sqrt{V_t} d\omega_{2t}, \tag{2}$$

where μ is the constant mean growth rate and V_t is the stochastic variance of the fundamental process. The positive constants κ, $\overline{V}$, σ, control the mean reversion speed, long-run mean, and the volatility of the fundamental variance process, respectively. Two sources of risk, represented by independent Brownian motions ω_{1t} and ω_{2t} defined on the objective probability measure $\mathbb{P}$, capture the *cashflow risk* and *variance risk*, respectively. We assume $D_0 > 0$ and $V_0 > 0$, and the parameter restriction $2\kappa\overline{V} > \sigma^2$ so that the fundamental variance process V_t is positive in finite time.[3] The stock price S_t and the stock return variance $\upsilon_t \equiv \text{Var}_t \left[dS_t/S_t\right]/dt$ are determined in equilibrium.

To complete the securities market, as a third security, we consider a series of zero net supply volatility derivatives whose payoff depend on the risky stock's future return variance. Toward that, we introduce instantaneous variance swap contracts that are initiated at each time t with maturity over the next instant $t + dt$. Like any swaps, these variance swaps require zero upfront payment at their initiation time t. At their maturity date $t + dt$, an investor who has a long position in this contract receives $\upsilon_t dt + d\upsilon_t$, and in return, pays the variance swap rate $y_t dt$.[4] The variance swap rate y_t is endogenously determined at the contract initiation time t.

Remark 1 (Further discussion on volatility derivatives). The most common financial instruments for getting direct volatility exposure in real world are volatility derivatives such as variance swaps and VIX options/futures. To achieve volatility exposures, investors can also form portfolio of equity options, such as straddles. The exact choice of financial instrument may depend on several factors, which include investors' preferences toward the direct vs. indirect volatility exposure, their complex portfolio management capabilities, and counterparty risk tolerance. In our model, we consider variance swaps as they allow for direct volatility exposures and have simple linear payoffs, which simplifies our analysis. That said, we highlight that the specific choice of volatility derivative is irrelevant for asset prices in our model. Any volatility derivative is sufficient to complete the securities market, leading to a unique state price density in equilibrium, which in turn can be used to recover the prices of other derivative contracts, such as long maturity variance swaps.

[3]Our framework allows for a general correlation between the fundamental process D_t and its variance V_t. To keep the model parsimonious we set such correlation to zero. A fundamental process with stochastic variance is consistent with empirical findings in Schorfheide, Song, and Yaron (2018) and Pettenuzzo, Sabbatucci, and Timmermann (2020), documenting heteroskedasticity in cashflow growth rates and is also commonly employed in asset pricing models, particularly in the long-run risk models (e.g., Bansal and Yaron (2004)).

[4]The variance swaps typically have notional amounts to scale the derivative payoffs. Since, in our model, the swap notional amount of the swap does not play any role, we normalize it to 1.

2.2 Investors' Beliefs

In this economy, we have two types of investors who agree on the fundamental D_t and its fundamental variance V_t at each time t. Investors are assumed to know the fundamental mean growth rate μ and the volatility coefficient of the fundamental variance process σ, but they have different beliefs about the expected future variance. The h-type investors misperceive the expected change in the fundamental variance as

$$\mathrm{E}_t^h\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \tfrac{1}{2}\delta V_t dt,$$

whereas the ℓ-type investors misperceive it as

$$\mathrm{E}_t^\ell\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] - \tfrac{1}{2}\delta V_t dt.$$

The positive constant $\delta \geq 0$ controls the *volatility disagreement* in our model since the difference in investors' subjective variance expectations is given by $\mathrm{E}_t^h\left[dV_t\right] - \mathrm{E}_t^\ell\left[dV_t\right] = \delta V_t dt$.[5] This specification implies that each i-type investor, $i = h, \ell$, perceives the fundamental variance as

$$dV_t = \kappa_i\left(\overline{V_i} - V_t\right)dt + \sigma\sqrt{V_t}d\omega_{2t}^i,$$

where $\kappa_h = \kappa - \delta/2$, $\kappa_\ell = \kappa + \delta/2$, and $\overline{V_i} = \overline{V}\kappa/\kappa_i$ are positive constants and ω_2^i is a standard Brownian motion under the i-type investor's subjective probability measure $\mathbb{P}^i$, with the relations $d\omega_{2t}^h = d\omega_{2t} - (1/2\sigma)\delta\sqrt{V_t}dt$, and $d\omega_{2t}^\ell = d\omega_{2t} + (1/2\sigma)\delta\sqrt{V_t}dt$. To ensure the equilibrium stock price admits a real solution in our model, we impose the parameter restriction of $\kappa_h > \sqrt{2}\sigma$, which also guarantees the fundamental variance being mean-reverting under investors' subjective expectations. Given the above beliefs, we interpret h-type investors as *high-fear* investors since they have higher and more persistent variance expectations than ℓ-type investors, who we refer to as *low-fear* investors.[6] Moreover, we interpret the objective expectation is coming from the estimation process of an econometrician in this economy.[7]

[5]Analogously, the volatility disagreement δ controls the equally-weighted standard deviation of variance expectations, which is equal to $(\delta/2)V_t dt$. Moreover, using our equilibrium quantities of Section 3, it is easy to show the parameter δ also controls the wealth-share weighted disagreement in variance expectations, which is sometimes referred to as the "market view" (Heyerdahl-Larsen and Illeditsch (2021)). See Andrei, Hasler, and Jeanneret (2019b) for a recent work that models the forecast dispersion as a square root process.

[6]As highlighted in Introduction, our terminology for high- and low-fear is motivated by the CBOE's volatility index, VIX, being commonly referred to as "fear index" in financial press and industry. Our modeling of volatility disagreement with high- and low-fear investors is also akin to the settings with persistently optimistic and pessimistic investors that is commonly employed in growth rate disagreement models (e.g., Detemple and Murthy (1994), Basak (2000), Buraschi and Jiltsov (2006), Gârleanu, Panageas, and Zheng (2023)). See, also Andrei, Carlin, and Hasler (2019a) for a model in which two investors disagree about a persistence parameter.

[7]Since the equally-weighted average of the subjective variance expectations coincides with the econometrician's expectation, there is no volatility bias at the aggregate level in our baseline specification. However, in Section 6, we extend our model to allow for asymmetric beliefs around the objective one, generating an aggregate volatility bias.

2.2.1 Discussion on Modeling Volatility Expectations

It is important to note that our specification of investors' beliefs about future volatility is consistent with the classic argument of Merton (1980). This argument claims that, when a given variable can be observed with sufficiently high frequency, the second-moment of that variable can be estimated more accurately than its first-moment. In line with this argument, investors in our model can perfectly estimate, and hence do not disagree on, the current level of the fundamental variance V_t. However, they disagree on the expectation (i.e., the conditional first-moment) of future variance, since, as claimed above, it is more difficult to precisely estimate such a quantity.

Our choice of modeling disagreement about future volatility is also consistent with survey evidence. For instance, in a survey of chief financial officers (CFOs) in U.S. corporations, Graham and Harvey (2001) document a cross-sectional average dispersion for the volatility expectations on the next year S&P 500 returns to be 4.6%. Similar findings of volatility disagreement are also present in Amromin and Sharpe (2014) and Kaplanski et al. (2016), who employ different survey data.

The disagreement in our model is proportional to the fundamental variance in order to capture that more uncertainty amplifies disagreement. This is economically meaningful and also in line with the findings in Ben-David, Graham, and Harvey (2013), who document that investors' expectations widen in periods of increased contemporaneous volatility. One could plausibly entertain alternative formulations of volatility expectations and still be in line with the evidence. For instance, a more general non-symmetric beliefs could be modeled as $\mathrm{E}_t^i\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \left(\alpha^i + \beta^i V_t\right) dt$ for $i = h, \ell$ such that $\alpha^h \neq \alpha^\ell$, or $\beta^h \neq \beta^\ell$. Alternatively, one could also consider extrapolative beliefs driven by past variance shocks to introduce slow-moving average expectations as in Lochstoer and Muir (2022). These alternative considerations typically lead to additional state variables in equilibrium and complicates the analysis. In this paper, we abstract away from these more complicated settings to focus on the equilibrium implications of volatility disagreement and trading in a simpler framework.

Regarding the source of disagreement, one plausible economic channel is that investors employ different models to estimate future volatility. Moreover, investors may utilize data from different sample periods or at different frequencies. An alternative source of different volatility expectations could be related to behavioral biases, such as overconfidence and miscalibration. For instance, Ben-David, Graham, and Harvey (2013) show that CFOs in their survey overestimate the precision of their own forecasts and underestimate the variance of risky processes. In particular, they find the average volatility expectation to be around 7%, a much lower value than the historical realized volatility, which suggests a downward volatility bias on average, a finding also supported by Barrero (2022) and Boutros et al. (2024). To incorporate this behavior in our model, in Section 6 we generalize our setting allowing for an aggregate volatility bias.

2.3 Investors' Preferences and Optimization

Each investor is initially endowed with the same number of stock shares and no bonds, and no variance swap contracts, so that their initial wealth is the same across types, $W_{h0} = W_{\ell 0} = W_0$. At each point in time t, i-type investor, $i = h, \ell$, chooses an admissible dynamic portfolio strategy, defined by the number of bonds α_{it}, the number of shares in the stock ψ_{it}, and the number of variance swap contracts θ_{it} to hold, so as to maximize her logarithmic preferences defined over the value of her wealth at the horizon date T,

$$\max_{\{\alpha_{it},\psi_{it},\theta_{it}\}_{t=0}^{T}} \mathrm{E}^i \left[\ln W_{iT}\right],$$

subject to her dynamic budget constraint

$$dW_{it} = \alpha_{it} dZ_t + \psi_{it} dS_t + \theta_{it} \left(\upsilon_t dt + d\upsilon_t - y_t dt\right), \tag{3}$$

where E^i denotes the expectation under the i-type investor's subjective probability measure $\mathbb{P}^i$.[8] According to (3), an investor's wealth evolves over time depending on the returns on her portfolio holdings, including the variance swaps. In particular, a positive (negative) θ_{it} indicates that the investor is long (short) in the variance swap contract at time t, thus, she is a volatility buyer (seller).

3 Stock Market

In this section, we study the equilibrium properties of the stock market. As novel predictions, we find that on average higher volatility disagreement leads to lower stock market valuation but higher and more volatile stock return variance.

Equilibrium in our economy with volatility disagreement is defined in a standard way. The economy is said to be in equilibrium if the stock price S_t, the variance swap rate y_t, and each i-type investor's, $i = h, \ell$, consumption W_{iT} and portfolio strategies $(\alpha_{it}, \psi_{it}, \theta_{it})$ are such that (i) all investors choose their optimal consumption and portfolio strategies given prices and beliefs, (ii) the goods market clear at time T, $W_{hT} + W_{\ell T} = D_T$, (iii) the bond, the stock, and the variance swap market clear at all times $t \in [0, T]$, $\alpha_{ht} + \alpha_{\ell t} = 0$, $\psi_{ht} + \psi_{\ell t} = 1$, and $\theta_{ht} + \theta_{\ell t} = 0$, respectively.

[8]In our setting, the fact that consumption occurs only at time T allows variance shocks to be priced even with time-separable preferences. This is in contrast to settings with intertemporal consumption in which variance shocks are not priced unless one considers more complex time-inseparable preferences (e.g., Bansal and Yaron (2004)). Solving a model with a stochastic volatility and volatility disagreement in a heterogeneous agent setting is a more challenging task under time-inseparable preferences, and is beyond the scope of this paper. As we will show, our setting leads to tractable closed-from solutions for all our economic quantities in equilibrium. Other dynamic asset pricing models with no intertemporal consumption include Kogan et al. (2006), Pástor and Veronesi (2012), Basak and Pavlova (2013), and Buffa and Hodor (2023).

The availability of a volatility derivative at each point in time makes financial markets dynamically complete. This implies the existence of a unique state price density, which allows us to employ standard martingale methods (Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989)) to solve each investor's optimization problem and apply market clearing conditions to obtain equilibrium quantities. The equilibrium is characterized by two state variables: the exogenous fundamental variance V_t, and the endogenous wealth-share of the high-fear investors $\mathrm{w}_t \equiv W_{ht}/(W_{ht} + W_{\ell t})$. In our analysis, we are interested in how economic quantities behave on average so that they can be more easily mapped into empirical predictions, and compared with existing evidence. To this end, we primarily focus on the state in which these state variables are at their respective long-run means, $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$, capturing the average state in our economy, and we refer to it as the *"steady state."* Moreover, to appreciate the equilibrium implications of volatility disagreement, we will often make comparisons with the equilibrium in an otherwise identical economy where all investors are either high-fear, low-fear, or have the same unbiased variance expectations. These benchmark economies arise as special cases in our model by setting $\mathrm{w}_t = 1$, $\mathrm{w}_t = 0$, or $\delta = 0$, respectively.

3.1 Stock Price

Proposition 1 (**Equilibrium stock price**). *The equilibrium stock price in the economy with volatility disagreement is given by*

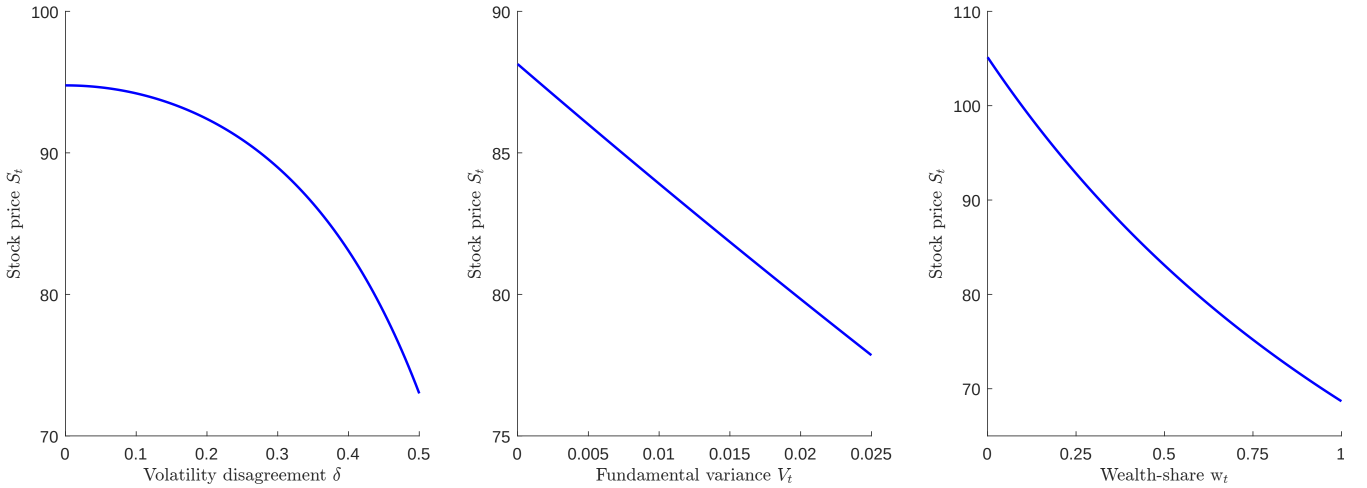
$$S_t = D_t e^{\mu(T-t)} \frac{1}{e^{r(T-t)}} \frac{1}{\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}}, \tag{4}$$

where the wealth-share of high-fear investors w_t follows

$$d\mathrm{w}_t = \delta^2 \mathrm{w}_t (1-\mathrm{w}_t)(\bar{\mathrm{w}} - \mathrm{w}_t) \frac{1}{\sigma^2} V_t dt + \delta \mathrm{w}_t (1-\mathrm{w}_t) \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}, \tag{5}$$

with $\bar{\mathrm{w}} = 1/2$ denoting its long-run mean, and the positive deterministic functions $A_i(t)$ and $B_i(t)$ for $i = h, \ell$, are provided in Appendix A. Consequently, a higher volatility disagreement δ leads to a lower stock price S_t at the steady state.

The equilibrium stock price in (4) can be described in three terms. The first term $D_t e^{\mu(T-t)}$ is the expected stock payoff. In the second term, $e^{r(T-t)}$ captures the stock payoff's time discount, and the last term captures its *risk discount*. The risk discount is determined by the (wealth-share) weighted average of each investor's *subjective risk discount* term $e^{A_i(t)+B_i(t)V_t}$. Since high-fear investors have higher and more persistent variance expectations than low-fear investors, they are more sensitive to variance shocks. Thus, they have a higher subjective risk discount, i.e., $A_h(t) > A_\ell(t)$ and $B_h(t) > B_\ell(t)$.



Panel A. Effect of δ Panel B. Effect of V_t Panel C. Effect of w_t

Figure 1. Stock price. These panels plot the equilibrium stock price S_t against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The parameter values follow from Table C1 of Appendix C.

The wealth distribution affects the risk discount, and thus the stock price, because the differences in investors' perceptions about future uncertainty leads to different investments in the stock and volatility derivatives. This portfolio heterogeneity creates a room for wealth transfers such that investors whose beliefs are more in line with realized shocks get relatively wealthier in equilibrium. Since investors' stock demands are functions of their wealth, as they get relatively wealthier, their variance expectations and resulting subjective risk discounts affect the stock price more. The dynamics of the wealth-share in (5) reveals that, in equilibrium, the wealth-share distribution follows a mean-reverting process. Due to the beliefs being symmetric around the objective variance expectation, the wealth-share fluctuates around its long-run mean $\bar{\mathrm{w}} = 1/2$. Changes in w_t crucially depend on the volatility disagreement parameter δ, as this is the source of heterogeneity that leads investors to hold different positions in securities. Since investors agree on cashflow shocks, the equilibrium wealth-share distribution is only driven by the variance shocks ω_{2t} that investors disagree on, so that following positive (negative) variance shocks, high-fear (low-fear) investors get relatively wealthier.

Proposition 1 also shows that, on average, the equilibrium stock price decreases in volatility disagreement, as also illustrated in Figure 1. A higher volatility disagreement increases the mean-preserving spread about future volatility expectations, thus leading to high-fear (low-fear) investors to have higher (lower) subjective risk discount. Since high-fear investors are more sensitive to risk, the increase in their subjective risk discount is greater than the decrease in that of the low-fear investors, leading to a reduction in the overall stock demand, and through market clearing, of its equilibrium price. Figure 1 illustrates that the stock price decreases also in the fundamental variance and the wealth-share of high-fear investors.[9] A higher V_t leads to a more uncertain stock payoff, and thus to greater subjective risk discount terms for *both* types of investors, who are now willing to hold the stock only if its price is lower. Moreover, as w_t increases, high-fear investors become more dominant in the economy, and their relatively higher risk discount is reflected more in the stock price.

3.2 Stock Return Variance

Proposition 2 (**Equilibrium stock return variance**). *The equilibrium stock return variance in the economy with volatility disagreement is given by*

$$\upsilon_t = V_t + \left[\sigma B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \left(\Lambda_{ht} - \Lambda_{\ell t}\right) \frac{1}{\sigma}\right]^2 V_t, \tag{6}$$

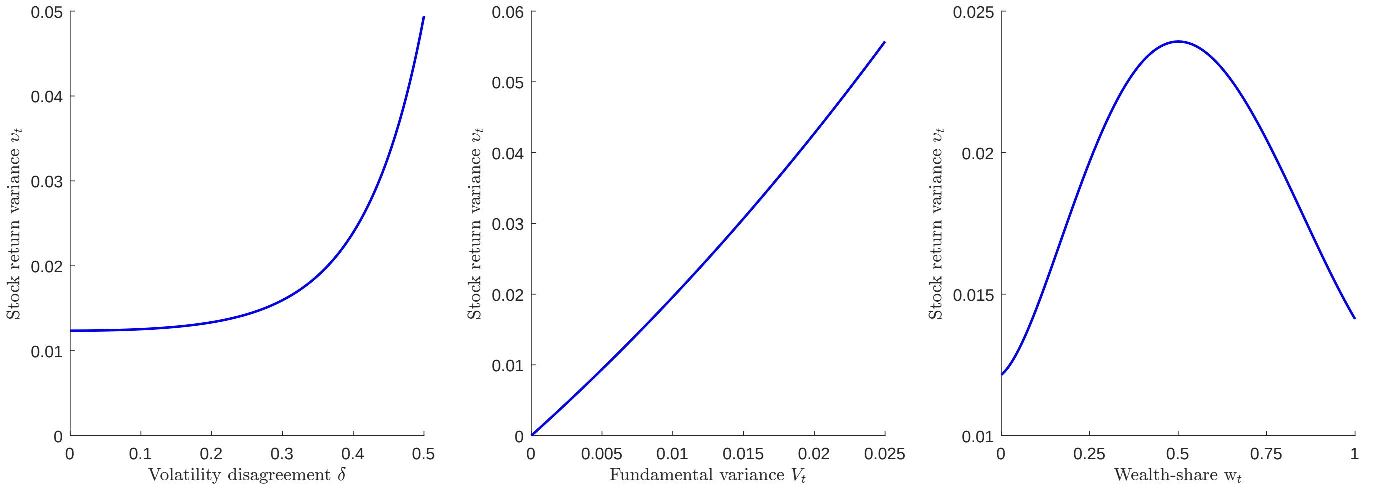
where the wealth-share of high-fear investors w_t *is as in Proposition 1 and the positive processes* Λ_{it} *for* $i = h, \ell$, *and* B_t *are given by*

$$\Lambda_{it} = \frac{e^{A_i(t)+B_i(t)V_t}}{\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + \left(1 - \mathrm{w}_t\right) e^{A_\ell(t)+B_\ell(t)V_t}}, \qquad B_t = \mathrm{w}_t \Lambda_{ht} B_h\left(t\right) + \left(1 - \mathrm{w}_t \Lambda_{ht}\right) B_\ell\left(t\right). \tag{7}$$

Consequently, a higher volatility disagreement δ *leads to a higher stock return variance* υ_t *at the steady state.*

The equilibrium stock return variance in (6) consists of two terms. The first term captures the uncertainty in stock returns due to fluctuations in the cashflow news D_t. Since investors agree on cashflow shocks, this term is simply equal to the variance of the cashflow news. The second term is due to fluctuations in the risk discount and takes a much richer form since the overall risk discount is driven by both the fundamental variance V_t and the relative wealth distribution w_t (Proposition 1). The extent to which state variables V_t and w_t induce fluctuations in the risk discount and make stock returns more volatile depends on the quantities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$, respectively. We refer to B_t as *variance elasticity*, since it captures the rate of decrease in the stock price following a unit increase in V_t, i.e., $B_t = -\partial \ln S_t / \partial V_t$. As (7) shows, B_t fluctuates between the (deterministic) investor-specific variance elasticities $B_h(t)$ and $B_\ell(t)$, driving their subjective risk discounts. Whether B_t is closer to $B_h(t)$ or to $B_\ell(t)$ depends on the (stochastic) weight $\mathrm{w}_t \Lambda_{ht}$ where the function Λ_{it} tells us how much i-type investors discount future cashflow more than the average in the economy. Henceforth,

[9] The effects of V_t and w_t on the stock price can also be obtained analytically in a straightforward way using the stock price expression (4). Even though Figure 1 illustrates these effects at the steady state, they are actually more general and hold for any state. Moreover, when we plot the effects of V_t, we keep the other state variable w_t fixed, and vice versa. However, as their dynamics show, these state variables are positively correlated: a positive variance shock ($d\omega_{2t} > 0$) increases both V_t and w_t. In Appendix B, we study the effects of such "pure variance shocks."



Panel A. Effect of δ | Panel B. Effect of V_t | Panel C. Effect of w_t

Figure 2. Stock return variance. These panels plot the equilibrium stock return variance υ_t against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The parameter values follow from Table C1 of Appendix C.

we refer to Λ_{it} as the *relative risk discount* of i-type investors. Since high-fear investors discount future cashflows more heavily, it follows that $\Lambda_{ht} > \Lambda_{\ell t}$. Since the difference in the relative risk discounts $\Lambda_{ht} - \Lambda_{\ell t}$ captures the rate of decrease in the stock price following a unit increase in w_t, i.e., $\Lambda_{ht} - \Lambda_{\ell t} = -\partial \ln S_t / \partial \mathrm{w}_t$, we refer to it as *wealth-share elasticity*.

Looking at the effects of volatility disagreement δ, we find that it tends to increase the equilibrium stock return variance as highlighted in Proposition 2 and illustrated in Figure 2. This result is due to a direct and an indirect effect reinforcing each other in equilibrium. The direct effect refers to the fact that a higher volatility disagreement amplifies wealth transfers following variance shocks. Thus, it increases the uncertainty about which investor type will be more dominant and have greater impact on the stock price next period, leading to higher return volatility. The indirect effect, instead, works through the elasticities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$. A higher volatility disagreement means that high-fear investors have higher and more persistent variance expectations, whereas low-fear investor have lower and less persistent variance expectations. This implies that $A_h(t)$ and $B_h(t)$ increase, while $A_\ell(t)$ and $B_\ell(t)$ decrease, leading to a higher variance and wealth-share elasticities. Therefore, by making the stock price more sensitive to fundamental variance and wealth-share fluctuations, the indirect effect goes in the same direction as the direct effect, overall leading to more volatile stock returns.

Figure 2 also illustrates that the stock return variance increases in the fundamental variance V_t (Panel B) and is non-monotonically related to the wealth-share of high-fear investors w_t (Panel C). The former result might seem unsurprising as it also arises in a benchmark economy without disagreement since more volatile cashflow news translates into a more volatile stock returns. However, under volatility disagreement, there is a new channel through which V_t amplifies the return variance υ_t. A higher V_t also increases the variance and wealth-share elasticities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$ since, as discussed above, high-fear investors are more sensitive to risk. Panel C, instead, shows that the stock return variance is hump-shaped in the investors' wealth-shares. The left (right) end-point of the curve represents the equilibrium return variance in a benchmark economy with only low-fear (high-fear) investors. Thus, we see that under volatility disagreement, return volatility can be much higher than its benchmark counterparts. In particular, it reaches its highest values close to the long-run mean of the wealth-share distribution $\bar{\mathrm{w}} = 1/2$, since that is when variance shocks lead to most wealth transfers. Indeed, when the wealth distribution is extremely skewed towards one investor type, there is less room for wealth transfers and hence less uncertainty about whose subjective variance expectations will matter more for asset prices going forward.

3.3 Volatility of Variance

In addition to its level, the volatility of the stock return variance also plays a major role in the volatility derivatives market. The following Proposition presents the equilibrium volatility of stock return variance, which corresponds to the diffusion term of the process $d\upsilon_t = \mu_{\upsilon t} dt + \sigma_{\upsilon t} d\omega_{2t}$.[10]

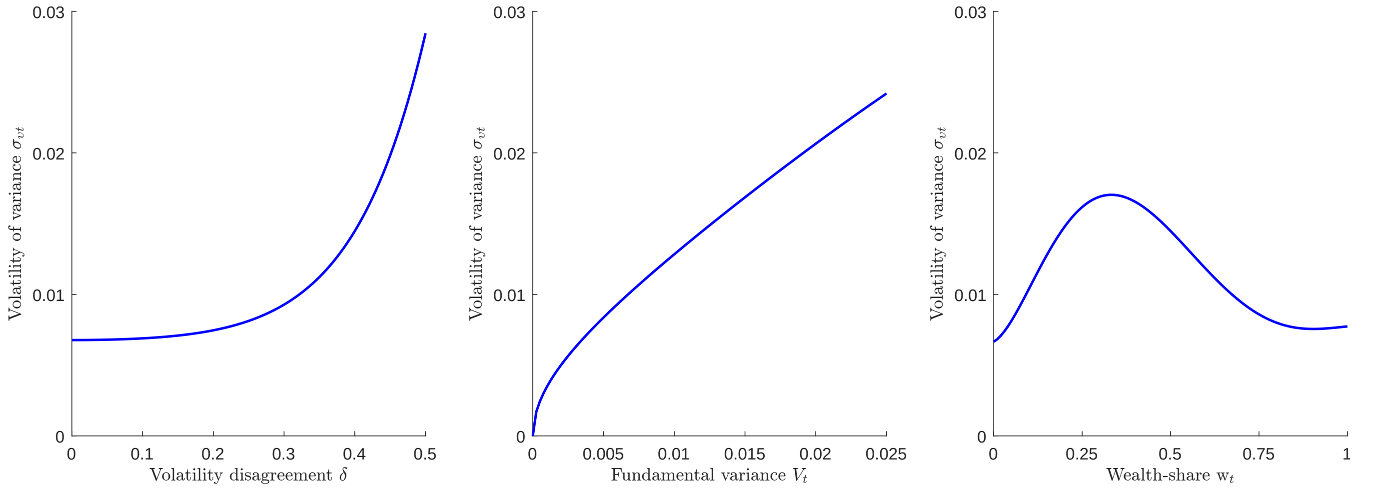
Proposition 3 (**Equilibrium volatility of stock return variance**). *The equilibrium volatility of stock return variance in the economy with volatility disagreement is given by*

$$\begin{aligned}\sigma_{\upsilon t} = {} & \sigma\sqrt{V_t} + \left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\frac{1}{\sigma}\right]^2 \sigma\sqrt{V_t} + 2\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\frac{1}{\sigma}\right] \\ & \times \left[\sigma\sigma_{Bt} + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\sigma_{\Lambda ht} - \sigma_{\Lambda \ell t} + \delta\left(1-2\mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\frac{1}{\sigma}\sqrt{V_t}\right)\frac{1}{\sigma}\right] V_t, \end{aligned} \tag{8}$$

where the wealth-share of high-fear investors w_t *is as in Proposition 1, the elasticities* B_t *and* $\Lambda_{ht} - \Lambda_{\ell t}$ *are as in Proposition 2, and* $\sigma_{\Lambda it}$ *and* σ_{Bt}*, denoting the diffusion coefficients of the processes* Λ_{it} *and* B_t*, respectively, are provided in Appendix A.*

Since the underlying "asset" of a variance swap is the stock return variance, its volatility $\sigma_{\upsilon t}$, which for brevity we refer to as *volatility of variance*, captures the "risk" in investing in a volatility derivative. Proposition 3 shows that the equilibrium volatility of variance takes a complex formulation under volatility disagreement. To better understand its behavior, Figure 3 plots it against volatility

[10] Since the cashflow news process D_t and its variance V_t are uncorrelated in our model, the stock return variance dynamics is driven only by the variance shocks ω_{2t}. Our analysis shows that our main results hold in a more general setting when the fundamental process and its variance are negatively correlated and hold when these processes are positively correlated, provided that such correlation is not excessively large.



Panel A. Effect of δ Panel B. Effect of V_t Panel C. Effect of w_t

Figure 3. Volatility of stock return variance. These panels plot the equilibrium volatility of variance σ_{vt} against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The parameter values follow from Table C1 of Appendix C.

disagreement and the two state variables in our economy. At the steady state, the volatility of variance is increasing in the disagreement δ (Panel A) and in the fundamental variance V_t (Panel B). These effects arise due to the same economic channels affecting the equilibrium return variance v_t, and suggest that investing in volatility derivatives can become particularly risky during periods of high disagreement and high volatility. Panel C, instead, shows that the volatility of variance is hump-shaped in the investors' wealth-shares. Although this is in line with the corresponding effect on the return variance, the volatility of variance reaches its highest value when the wealth distribution is more skewed towards the low-fear investors.

Our analysis in this section not only lays the foundation for studying the volatility derivatives market in Section 4, but it also provides significant contributions to the literature. For instance, wealth transfers in our model arise due to volatility disagreement, and are thus driven by shocks to the second-moment of asset returns. Therefore, the effect of wealth transfers on asset prices in our model are new and different than those in the existing literature. Moreover, theoretical works studying second-moment disagreement employ static settings, which limits their ability to analyze economic quantities that are intrinsically dynamic, such as the stock return variance and volatility of variance. Furthermore, they focus on disagreement about current realized volatility, as opposed to future expected volatility, and find that higher disagreement leads to higher stock price, the opposite of what we find. So, to the best of our knowledge, our findings on the effects of the volatility disagreement on the stock price, stock return variance, and the volatility of variance are all novel.

4 Volatility Derivatives Market

In this section, we study the equilibrium implications of our model for the volatility derivatives market. We first demonstrate that the variance risk premium is negative on average, but it turns positive when low-fear investors are more dominant in the economy. We show that, on average, higher volatility disagreement increases the magnitude of the variance risk premium and the variance swap rate. We uncover a hump-shape relation between the trading activity in the variance swaps and volatility disagreement. Equally surprising, but consistent with empirical evidence, we also find that investors on average trade fewer volatility derivatives in more volatile periods. Our theory is able to reconcile the empirical finding that during market turmoils, while investors trade fewer volatility derivatives, the variance risk premium is positive with its magnitude rising in volatility.

4.1 Variance Risk Premium

We begin our analysis of the volatility derivatives market by studying the variance risk premium (VRP), i.e., how much investors expect to be compensated (if VRP > 0) or to give up (if VRP < 0) to hold an asset whose payoff is positively exposed to variance shocks. Because of their different expectations on V_t, high-fear and low-fear investors have different views about future stock return variance, and thus perceive the VRP differently. Since the VRP is not a quantity that is directly observed in the market, it is instructive to distinguish between the objective VRP that an econometrician in this economy would estimate, $\pi_{\upsilon t} \equiv \mathrm{E}_t\left[d\upsilon_t\right]/dt - \mathrm{E}_t^*\left[d\upsilon_t\right]/dt$, where E_t^* denotes the conditional expectation under the risk-neutral measure, and the subjective VRPs perceived by the two investor types, $\pi_{\upsilon t}^i \equiv \mathrm{E}_t^i\left[d\upsilon_t\right]/dt - \mathrm{E}_t^*\left[d\upsilon_t\right]/dt$, for $i = h, \ell$.

Proposition 4 (**Equilibrium variance risk premium**)**.** *The equilibrium objective variance risk premium estimated by an econometrician in the economy with volatility disagreement is given by*

$$\pi_{\upsilon t} = -\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]\sqrt{V_t}\sigma_{\upsilon t} + \delta\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}\sigma_{\upsilon t}, \tag{9}$$

whereas the subjective variance risk premia perceived by high-fear and low-fear investors are given by

$$\pi_{\upsilon t}^h = -\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]\sqrt{V_t}\sigma_{\upsilon t} + \delta\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}\sigma_{\upsilon t}, \tag{10}$$

$$\pi_{\upsilon t}^\ell = -\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]\sqrt{V_t}\sigma_{\upsilon t} - \delta \mathrm{w}_t\frac{1}{\sigma}\sqrt{V_t}\sigma_{\upsilon t}, \tag{11}$$

respectively, where the wealth-share of high-fear investors w_t *is as in Proposition 1, the elasticities* B_t *and* $\Lambda_{ht}-\Lambda_{\ell t}$ *are as in Proposition 2, and the volatility of variance* $\sigma_{\upsilon t}$ *is as in Proposition 3.*

To highlight the distinct behavior of the VRP under volatility disagreement, we first discuss its behavior in a benchmark economy with only high-fear or low-fear investors. In such an economy, homogeneous investors bear all the market risk because in equilibrium they must hold the stock (since it is in positive net supply), but do not trade any volatility derivatives (since they are in zero net supply). Therefore, positive variance shocks represent "bad times" as they decrease the investors' marginal utility by reducing their wealth through stock investments. Since investors are risk-averse, they are willing to *pay* a premium to hold assets that are positively exposed to variance shocks, leading to a negative subjective VRP in equilibrium. Moreover, given that high-fear investors have higher and more persistent variance expectations, they would be willing to pay a higher premium to insure against variance shocks. Thus, the subjective variance risk premium would be lower in an economy with only high-fear investors than in one with only low-fear investors, i.e., $\bar{\pi}^h_{vt} < \bar{\pi}^\ell_{vt} < 0$.[11]

Under volatility disagreement, the equilibrium subjective variance risk premia differ from their benchmark economy counterparts in significant ways. Heterogeneous expectations lead to trading in volatility derivatives, where high-fear investors become volatility buyers and low-fear investors volatility sellers. This implies that low-fear investors face variance risk not only through their long stock positions but also through their short variance swap positions, making them require a higher expected compensation to provide insurance against variance risk. Thus, in equilibrium their subjective VRP become more negative than that in a benchmark economy, $\pi^\ell_{vt} < \bar{\pi}^\ell_{vt} < 0$.[12] In contrast, the variance risk high-fear investors face through their long stock positions is now partially offset by their long positions in the volatility derivative. Since they receive an insurance against variance risk, their subjective VRP becomes less negative than that in a benchmark economy, and possibly positive, $\bar{\pi}^h_{vt} < \pi^h_{vt}$. Therefore, the ability to trade volatility derivatives induces a "risk transfer" from high-fear to low-fear investors, which causes the subjective VRP of low-fear investors to become lower than that of high-fear investors, $\pi^\ell_{vt} < \pi^h_{vt}$.

More formally, the subjective VRP expressions in (10) and (11) exhibit the standard "price of risk times quantity of risk" form that comes from the covariance between the equilibrium subjective state price densities and the variance swap payoff. The quantity of variance risk is captured by the volatility of variance σ_{vt}, discussed in Section 3, whereas terms that multiply it constitute the subjective market prices of variance risk.

The subjective market price of variance risk, and hence the associated VRP, are driven by two economically distinct components: a hedging component and a speculative component. The hedging

[11] In the benchmark economy with only i-type investors, the subjective VRP is given by $\bar{\pi}^i_{vt} = -\sigma^2 B_i(t)\left[1 + \sigma^2 B_i^2(t)\right] V_t$.

[12] We note that due to their short volatility positions, low-fear investors' subjective VRP (with opposite sign) corresponds to their "expected profit" from selling an additional volatility derivative rather than the premium they would be willing to pay to insure against variance shocks.

component of the VRP, corresponding to the common first term in (10) and (11), reflects the investors' desire to hedge stock price fluctuations that are driven by variance shocks. The ensuing demand for variance swaps, enabling the hedging, creates price pressure that, in equilibrium, leads to a negative hedging component. In particular, the square bracket in (10) and (11) shows more explicitly that there are two hedging motives: one is related to fluctuations in the fundamental variance V_t, the other to fluctuations in the wealth-share w_t, both of which contribute to the uncertainty in the stock price (Proposition 1).

In contrast, the speculative component of the VRP, corresponding to the second term in (10) and (11), differs across high-fear and low-fear investors, as it reflects the opposite positions that they take in the variance swaps. In particular, the desire of the investors to speculate on their beliefs—that is, for high-fear investors to hold long positions in the variance swaps, and for low-fear investors to hold short positions—makes the volatility derivative "over-valued" from the perspective of low-fear investors and "under-valued" from the perspective of high-fear investors. So the positive speculative component in (10) captures the perceived undervaluation induced by low-fear investors, whereas the negative speculative component in (11) captures the perceived overvaluation induced by high-fear investors. Proposition 4 also reveals that the sizes of the perceived misvaluations are proportional to the wealth-share of the investors causing them, i.e., $(1 - \mathrm{w}_t)$ for the perceived undervaluation in (10) and w_t for the perceived overvaluation in (11).

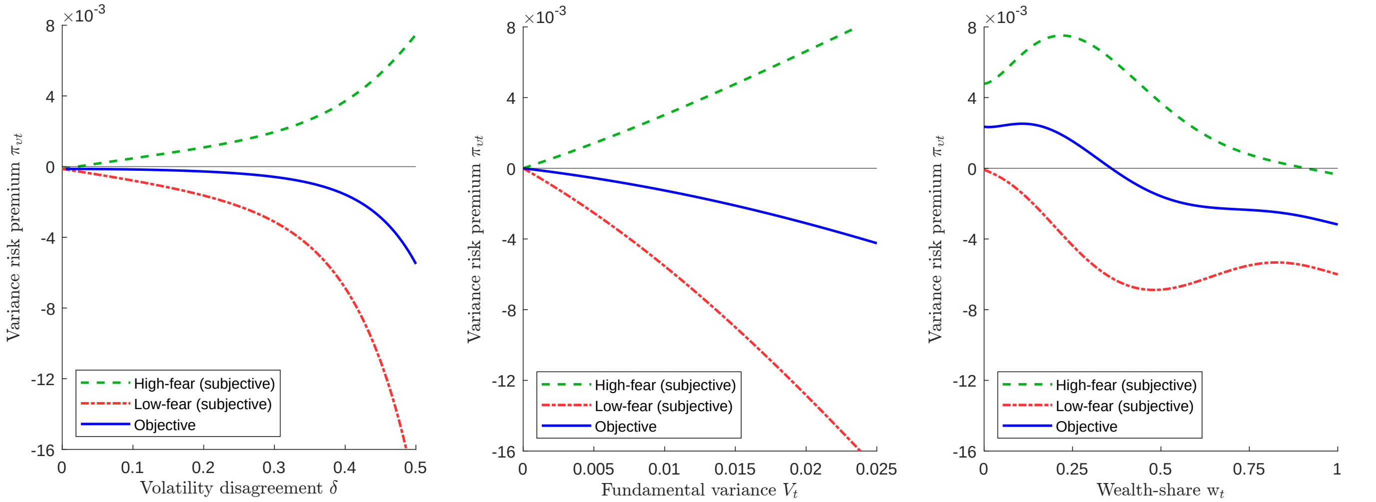
The objective VRP in (9) is the variance risk premium measured by an econometrician in this economy and turns out to be the simple average of the two subjective ones.[13] Therefore, while the hedging component remains the same, the speculative component can become positive or negative depending on which investor type has a stronger price impact. Indeed, the second term in (9) shows that when the wealth-share of high-fear investors is higher than that of low-fear investors, $\mathrm{w}_t > \bar{\mathrm{w}}$, the VRP becomes more negative; when, instead, $\mathrm{w}_t < \bar{\mathrm{w}}$, it becomes less negative, and possibly positive. The following corollary formalizes this key finding.

Corollary 1 (**Positive variance risk premium**). *If volatility disagreement is sufficiently high, $\delta > \delta^*$, the objective variance risk premium becomes positive, $\pi_{vt} > 0$, when the wealth-share of high-fear investors is sufficiently low, $\mathrm{w}_t < \mathrm{w}_t^* < \bar{\mathrm{w}}$, where*

$$\mathrm{w}_t^* \equiv \frac{e^{A_\ell(t)+B_\ell(t)V_t}\left(\delta\bar{\mathrm{w}} - \sigma^2 B_\ell\left(t\right)\right)}{e^{A_h(t)+B_h(t)V_t}\left(\delta\left(1-\bar{\mathrm{w}}\right) + \sigma^2 B_h\left(t\right)\right) + e^{A_\ell(t)+B_\ell(t)V_t}\left(\delta\bar{\mathrm{w}} - \sigma^2 B_\ell\left(t\right)\right)}, \tag{12}$$

and δ^* is provided in Appendix A.

[13] To be more precise, the objective VRP is the weighted average of the subjective VRP of high- and low-fear investors, where the weights are given by $\bar{\mathrm{w}}$ and $1-\bar{\mathrm{w}}$, respectively. Since in our baseline setting $\bar{\mathrm{w}} = 1/2$, the weighted average becomes a simple average.



Panel A. Effect of δ Panel B. Effect of V_t Panel C. Effect of w_t

Figure 4. Variance risk premium. These panels plot the equilibrium variance risk premium against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The solid blue lines represent the variance risk premium π_{vt} estimated by an econometrician, the dashed green (dash-dotted red) lines represent the variance risk premium perceived by the high-fear (low-fear) investor. The parameter values follow from Table C1 of Appendix C.

The fact that the objective VRP can become positive, which is consistent with the empirical evidence, is intriguing and somewhat puzzling as it tell us that in some states an econometrician may estimate a positive expected compensation for holding an asset that is positively exposed to variance risk. The intuition for this counter-intuitive result is as follows. Since low-fear investors underestimate future fundamental variance, it means that they overestimate the expected returns that they would make by selling variance swaps. As they become more dominant in the economy, the supply of variance swaps increases while its demand decreases, hence the expected return of providing volatility insurance must go down for the derivatives market to clear. Therefore, when w_t is sufficiently low, the underestimation of future risk by low-fear investors can make their expected returns negative from the perspective of an econometrician. So, although low-fear investors always expect to "make money" from their derivative positions, when they hold most of the wealth in the economy, the variance swap rate they agree to receive becomes lower than the objective expected future variance, resulting in a positive VRP.

Figure 4 shows that, at the steady state, the objective VRP is negative, and its magnitude increases with the volatility disagreement δ (Panel A) and the fundamental variance V_t (Panel B). These effects come from the hedging component of the VRP since at the steady state ($\mathrm{w}_t = \bar{\mathrm{w}}$) the perceived misvaluations of the two investor types fully offset each other. As δ or V_t increases, the stock price becomes more volatile, thus intensifying the investors' desire to hedge against stock market fluctuations. In order for the derivatives market to clear the VRP must become more negative. Notably, with sufficient disagreement δ, both investor types expect to profit from their volatility trading, as illustrated by the subjective VRP of high-fear investors (green dashed line) being positive and the subjective VRP of low-fear investors (red dot-dashed line) being negative. Finally, Panel C graphically shows that the objective VRP, despite being negative at the steady state, becomes positive when low-fear investors become sufficiently wealthy, as discussed earlier.

4.2 Variance Swap Rate

We next investigate the price that volatility buyers agree to pay to volatility sellers for swap contracts.

Proposition 5 (**Equilibrium variance swap rate**). *The equilibrium variance swap rate in the economy with volatility disagreement is given by*

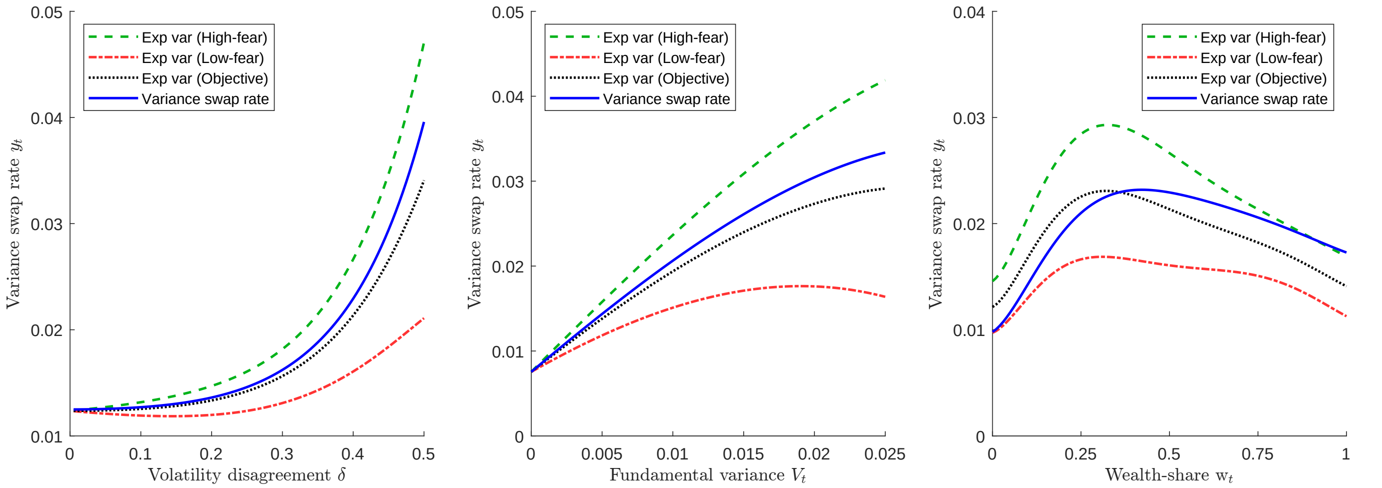
$$y_t = \upsilon_t + \mu_{\upsilon t} - \pi_{\upsilon t}, \tag{13}$$

where the stock return variance υ_t is as in Proposition 2, the variance risk premium $\pi_{\upsilon t}$ is as in Proposition 4, and the mean change of the stock return variance $\mu_{\upsilon t}$ is provided in Appendix A.

In our model, the equilibrium variance swap rate is determined by market clearing, i.e., it is the rate that makes volatility buyers' demand coincides with volatility sellers' supply in the volatility derivatives market. Naturally, it also satisfies a standard no-arbitrage condition, which makes it equal to the risk-neutral expectation of the future stock return variance.[14] To better understand the equilibrium behavior of the variance swap rate, we plot it along with the future return variance expectations $\upsilon_t + \mathrm{E}_t^i\left[d\upsilon_t\right]/dt$ in Figure 5. At the steady state, the variance swap rate is positively related to the volatility disagreement δ (Panel A) and the fundamental variance V_t (Panel B). These effects arise because a higher δ or V_t leads to an increase not only in the current stock return variance (Figure 2) but also in the expected future variance. Therefore, for the derivatives market to clear, and for the trading in the variance swap contract to occur, the variance swap rate must go up. In addition, as discussed above, an increase in either quantity also leads to a more negative VRP, causing the variance swap rate (blue solid line) to further increase above the objective expected future variance $\upsilon_t + \mu_{\upsilon t}$ (black dotted line).

Figure 5 also shows that the variance swap rate is hump-shaped in the investors' wealth shares (Panel C). This non-monotonicity comes from the corresponding hump-shaped behavior of the ex-

[14] To see this, note that the sum of first two terms in (13) is the (objective) expectation of the future return variance $\upsilon_t + \mathrm{E}_t\left[d\upsilon_t\right]/dt$. Subtracting the variance risk premium yields the risk-neutral expectation $y_t = \upsilon_t + \mathrm{E}_t^*\left[d\upsilon_t\right]/dt$. We note that since volatility is not a traded asset $\mu_{\upsilon t} - \pi_{\upsilon t}$ does not need to coincide with $r\upsilon_t$.



Panel A. Effect of δ Panel B. Effect of V_t Panel C. Effect of w_t

Figure 5. Variance swap rate. These panels plot the equilibrium variance swap rate y_t and the expected future variance $v_t + \mathrm{E}_t^i [dv_t]/dt$ against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The dotted black lines represent the future variance estimated by an econometrician, the dashed green (dash-dotted red) lines represent the future variance perceived by the high-fear (low-fear) investor. The parameter values follow from Table C1 of Appendix C.

pected future variance. Moreover, Panel C also reveals that as high-fear, or low-fear, investors become more dominant in the economy, the variance swap rate converges to their subjective future variance expectations. In particular, when most of the wealth is held by the group of investors that who underestimate the future variance (i.e., low-fear investors), they trade at a swap rate which is lower than the objective expected future variance (leading to a positive VRP, as discussed above).

4.3 Variance Swap Holdings

Having established the key properties of the "price" that facilitate investors' trading in variance swaps, we next study their equilibrium holdings in these derivatives contracts.

Proposition 6 (Equilibrium variance swap holdings). *The equilibrium variance swap holdings in the economy with volatility disagreement are given by*

$$\theta_{ht} = \delta \mathrm{w}_t (1 - \mathrm{w}_t) \frac{1}{\sigma} \frac{1}{\sigma_{vt}} S_t \sqrt{V_t}, \qquad \theta_{\ell t} = -\theta_{ht}, \tag{14}$$

where the stock price S_t and the wealth-share of high-fear investors w_t are as in Proposition 1 and the volatility of variance σ_{vt} is as in Proposition 3.

In a benchmark economy with homogeneous investors, despite their desire to hedge against variance risk, variance swaps are not traded in equilibrium. This is not the case with volatility disagreement. As Proposition 6 demonstrates, investors trade volatility by holding opposite positions in variance swap contracts, with high-fear investors become volatility buyers, $\theta_{ht} > 0$, and low-fear investors becoming volatility sellers, $\theta_{\ell t} < 0$.[15] High-fear investors' holdings θ_{ht} also coincide with the total number of long positions in the variance swap contract (i.e., the open interest) at time t, and since these derivatives are renewed at each time t, it also captures the volatility trading activity.

In our model, due to the investors' logarithmic preferences, the equilibrium portfolio holdings admit the standard (multivariate) mean-variance efficient portfolio representation, $\boldsymbol{\theta}_{it}/W_{it} = \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\pi}_t^i$.[16] Therefore, the variance swap holdings are pinned down not only by the risk and return attributes of the variance swap contracts, i.e., the volatility of variance σ_{vt} and the subjective VRP π_{vt}^i, but also by their covariance with the stock returns. As discussed above, volatility derivatives have a dual role, since they allow investors to hedge their stock positions, and to speculate on their different volatility expectations. So, to shed light on these two roles, we decompose the investors' variance swap holdings (14) into a hedging component θ_{it}^H and a speculative component θ_{it}^S, such that $\theta_{it} = \theta_{it}^H + \theta_{it}^S$.

To identify the speculative component, we consider a fictitious (measure-zero) investor who has the same beliefs and preferences as the i-type investors but can only invest in the riskless bond and the volatility derivatives. Since such an investor does not hold any stock positions, by construction, her volatility derivative holdings do not have a hedging component and thus reflect only the speculative one. It is easy to show that the variance swap holdings of this investor are given by the (univariate) mean-variance efficient portfolio $\theta_{it}^S/W_{it} = \sigma_{vt}^{-2}\pi_{vt}^i$, thus driven solely by the volatility derivative's subjective risk premium and variance. Subtracting the speculative components of high- and low-fear investors from their respective variance swap holdings in (14) yields the hedging components

$$\theta_{ht}^H = \frac{\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}}{\sigma_{vt}}\sqrt{V_t}\mathrm{w}_t S_t,$$
$$\theta_{\ell t}^H = \frac{\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}}{\sigma_{vt}}\sqrt{V_t}\left(1-\mathrm{w}_t\right)S_t.$$

[15] In our model, investors are either volatility buyers or sellers at all times, which is consistent with the evidence in Cheng (2019), who finds systematic patterns for what type of investors are long or short in the volatility derivative (VIX futures) market. In particular, Cheng (2019) documents that in recent times, systematically, dealers and asset managers have long positions, while hedge funds have short positions in the volatility derivative market. Thus, given this evidence, one could interpret high-fear investors in our model as mutual fund managers who are worried about the volatility risk inherent in their long stock positions and low-fear investors as hedge fund managers, who short volatility derivatives to profit from variance risk premium being negative on average.

[16] In this representation, $\boldsymbol{\theta}_{it}$ is the i-type investors' portfolio vector (in terms of dollars invested in the stock and numbers of volatility derivative contracts), $\boldsymbol{\pi}_t^i$ is the vector of subjective risk premia on the stock and the variance swap, and $\boldsymbol{\Sigma}_t$ is the variance covariance matrix of all security returns.

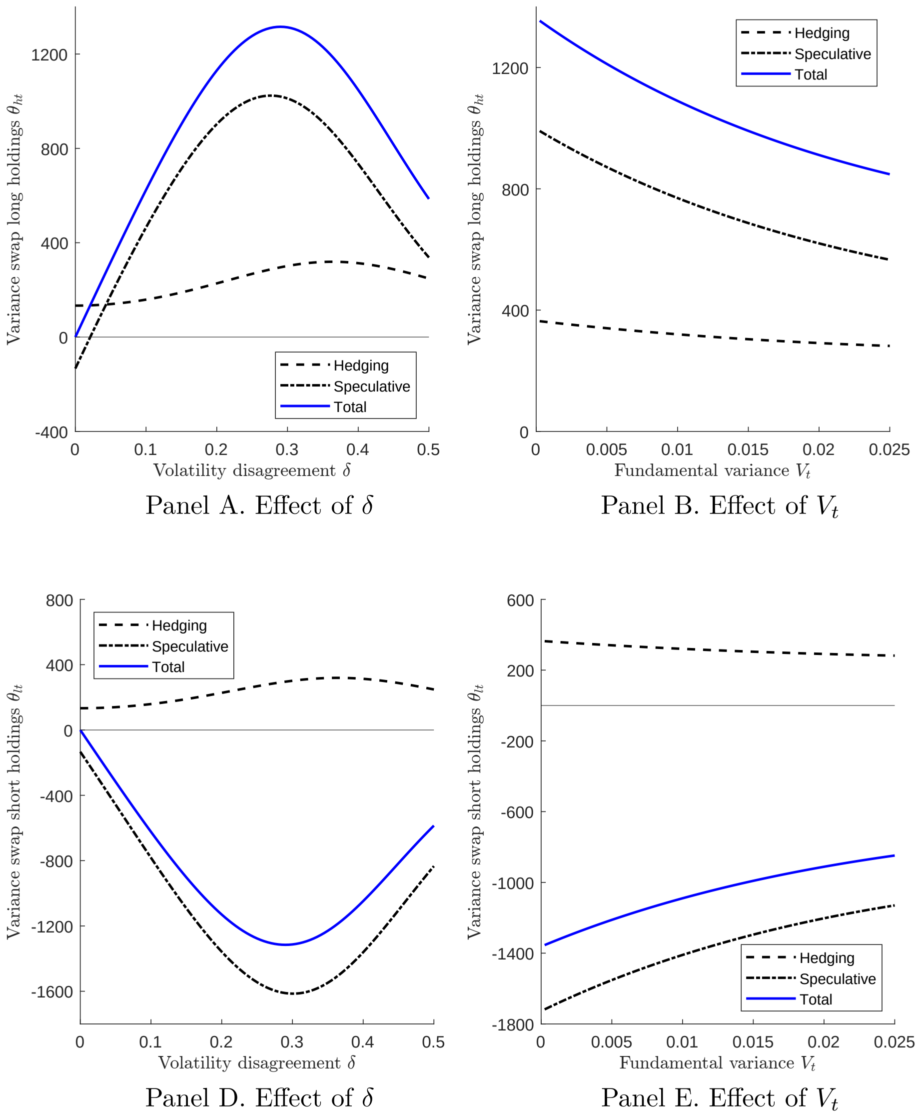


Figure 6. Variance swap holdings. The top panels plot the equilibrium variance swap holdings of high-fear investor against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The bottom panels plot the equilibrium variance swap holdings of low-fear investor against the same quantities. The solid blue lines represent the total holdings, the dashed (dash-dotted) black lines represent the hedging (speculative) part of the total holdings. The parameter values follow from Table C1 of Appendix C.

Figure 6 illustrates the equilibrium behavior of investors' total variance swap holdings, including their hedging and speculative components. Panel A reveals a notable, and perhaps unexpected, hump-shaped relation between the volatility disagreement and the variance swap open interest, which seem to be driven mostly by the speculative component. This non-monotonic relation occurs because of two opposing effects that disagreement induces, and can be described in terms of the mean-variance portfolio structure. First, a higher disagreement increases investors' expected profits from trading variance swap contracts by increasing the magnitude of their subjective VRP (see Figure 4). Therefore, everything else equal, this effect leads to larger variance swap holdings in equilibrium. Second, a higher disagreement increases the riskiness of the variance swap contracts (σ_{vt} goes up), and it does so in a convex way (see Figure 3), leading to smaller variance swap holdings. For low levels of disagreement, the former effect dominates, inducing a positive relation between volatility disagreement and the variance swap open interest. However, for high levels of disagreement, the relation inverts because the variance swap becomes too risky (the effect of volatility of variance dominates).[17]

Figure 6, Panel B, depicts another notable result: at the steady state, the variance swap open interest decreases with the fundamental variance V_t. This negative relation might seem puzzling at first as it implies that, on average, investors hold and trade less volatility derivatives in high volatility periods, during which arguably there is more volatility risk to be hedged. This effect can be explained by the sharp increase in the riskiness of variance swaps that happens during these periods, inducing the investors to trade less derivatives. Figure 3 indeed shows that the volatility of the swap contract σ_{vt} heightens in volatile times. In line with this intuition, our analysis confirms that if we were to shut down the volatility of variance channel, we would obtain a positive relation between θ_{ht} and V_t, suggesting that investors would trade more volatility derivatives in more volatile times if their riskiness were to remain the same.

All panels in Figure 6 show that the hedging components of the variance swap holdings are positive for both investor types. This is not surprising since both high- and low-fear investors have long stock positions, which are negatively impacted by variance shocks. Panel C and F show the hedging component tend to increase with the investor's wealth-shares, to then decreases when they hold almost all the wealth in the economy. Intuitively, as investors get wealthier, they increase their stock positions, and consequently their hedging demand. However, once they hold most of the wealth, the effective heterogeneity and the resulting wealth transfers diminish, reducing the hedging needs. Moreover, since both high- and low-fear investors have positive hedging demands, the magnitude of the speculative component of high-fear investors is lower than that of low-fear investors.

[17]We note that the hump-shape relation is not due to the investors wealth levels W_{it}. As investors get wealthier, they invest more in the stock, which increases their exposure to variance risk. Therefore, they increase their variance swap holdings without affecting the hump-shaped relation. Indeed, we obtain a similar hump-shaped relation when considering the fraction of wealth invested in the variance swap contracts, θ_{it}/W_{it}.

4.4 Discussion of Key Predictions

It is worth at this point to highlight several noteworthy contributions of our analysis in this section. To the best of our knowledge, our implications for the effects of volatility disagreement on the variance risk premium, variance swap rate, and volatility derivative holdings are all novel. Importantly, the economics around the variance risk premium in our model significantly differs from those in the existing literature, which are predominantly based on representative agent settings, and thus unable to generate our "risk transfer" and "expected profit" effects (e.g., Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), Eraker and Wu (2017), Atmaz (2022), Lochstoer and Muir (2022)). Furthermore, several key implications of our model are supported by existing empirical evidence. For example, the variance risk premium being negative on average is well-documented (e.g., Bakshi and Kapadia (2003), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009)).[18] Our finding that variance risk premium becomes more negative with higher uncertainty is also consistent with observed patterns (e.g., Barras and Malkhozov (2016)).

Our model also reconciles an otherwise puzzling empirical evidence in Cheng (2019), which finds that spikes in market volatility is accompanied with reductions in volatility derivative holdings. Moreover, Cheng (2019, 2020) find that during periods of heightened volatility and market turmoils, the variance risk premium is largely positive, its magnitude rises in volatility, and investors trade fewer volatility derivatives. Our model is able to reconcile these patterns jointly by demonstrating that these are likely to occur when the market is dominated by investors who underestimate future volatility. To further zoom in on this phenomena, Table 1 reports key economic quantities, focusing on the case in which low-fear investors are more dominant in the economy ($\mathrm{w}_t = 0.25$). It shows that when the uncertainty increases, the stock price goes down, and stock return volatility increases, as typically occurs during market turmoils. Since the market is dominated by investors who underestimate future volatility, the variance swap rate (VSR) becomes much lower than the expected future variance (EV), generating a larger positive VRP. Notably, the last column shows that periods with larger positive VRP also coincide with periods of low volatility trading, consistent with the documented behavior.

According to our model prediction, for the VRP to turn positive, low-fear investors must become sufficiently dominant in the economy. This means that before observing the VRP turning sign, the economy must have experienced a sequence of negative variance shocks, giving rise to a sufficiently long period of low volatility in the market. This additional prediction of our theory is in line with extended "calm" periods preceding two of the most notable times in which the VRP has turned positive, the 2008 financial crises and the 2020 Covid-19 pandemic. The rapid surge in volatility associated with the outset of these events has led the positive VRP to spike (Cheng (2019, 2020)), as our theory predicts.

[18] Choi, Mueller, and Vedolin (2017) reports a similar negative VRP for the bond market.

Table 1. Effects of heightened volatility when low-fear investors are dominant. This table reports the equilibrium quantities when the economy is dominated by low-fear investors, $\mathrm{w}_t = 0.25$, by varying the fundamental variance V_t from its long-run mean $\overline{V}$ to its value that is 6 standard deviation (st.d.) higher than its mean. The stock return volatility $\sqrt{\upsilon_t}$, the expected variance (EV) $\upsilon_t + \mu_{\upsilon t}$, and the variance swap rate (VSR) y_t are in annualized percentage points; the variance risk premium (VRP) $\pi_{\upsilon t}$ is in annualized percentage squared (basis) points. All other parameter values are as in Table C1.

Variance V_t	Stock Price S_t	Volatility $\sqrt{\upsilon_t}$	EV $\upsilon_t + \mu_{\upsilon t}$	VSR y_t	VRP $\pi_{\upsilon t}$	OI θ_{ht}
$\overline{V}$	92.82	14.03	2.25	2.10	**15**	**786**
$\overline{V} + 1 \times$ st.d.	90.23	18.63	3.49	3.17	**31**	**571**
$\overline{V} + 2 \times$ st.d.	87.68	22.72	4.82	4.29	**53**	**425**
$\overline{V} + 4 \times$ st.d.	82.70	30.38	7.58	6.50	**108**	**251**
$\overline{V} + 6 \times$ st.d.	77.88	37.89	9.91	8.20	**170**	**159**

5 Cross-Market Comovement

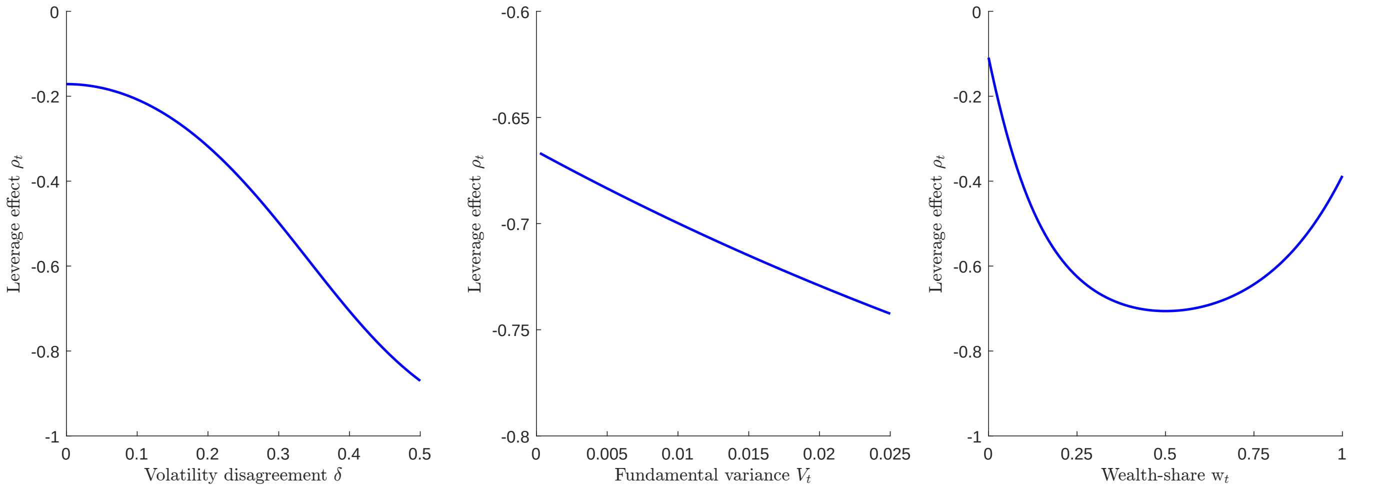
In our model, investors' portfolios consist of two risky securities: the stock and the variance swap. It is interesting to see how the returns of these securities comove. To this end, we study the equilibrium correlation between stock and variance swap returns, which we denote by ρ_t. Since variance swap returns are given by $\upsilon_t dt + d\upsilon_t - y_t dt$, this correlation coincides with $\mathrm{Corr}_t \left[dS_t/S_t, d\upsilon_t\right]/dt$, i.e., the correlation between stock returns and changes in the stock return variance. Following an established empirical literature, when this correlation is negative—a robust feature of the data—we refer to it as the "leverage effect."[19] Proposition 7 presents the equilibrium leverage effect in our economy.

Proposition 7 (**Equilibrium leverage effect**)**.** *The equilibrium leverage effect in the economy with volatility disagreement is given by*

$$\rho_t = -\frac{\sigma^2 B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)}{\sqrt{\sigma^2 + \left[\sigma^2 B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\right]^2}}, \tag{15}$$

where the wealth-share of high-fear investors w_t is as in Proposition 1, the elasticities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$ are as in Proposition 2. Consequently, a higher volatility disagreement δ leads to a stronger leverage effect ρ_t at the steady state.

[19]This terminology stems from the fact that due to financial leverage, a decline in the stock price leads to a higher debt-to-equity ratio and consequently to an increase in the stock volatility (e.g., Black (1976), Christie (1982)). More recent works on the leverage effect include Bandi and Renò (2012), Aït-Sahalia, Fan, and Li (2013), Andersen, Bondarenko, and Gonzalez-Perez (2015), Hu, Jacobs, and Seo (2022). The negative correlation between the stock returns and changes in stock return variance is sometimes also referred to as the "volatility feedback effect" (e.g., French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992)).



Panel A. Effect of δ Panel B. Effect of V_t Panel C. Effect of w_t

Figure 7. Leverage effect. These panels plot the equilibrium leverage effect ρ_t against the volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel A), against the fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$ (Panel B), and against the high-fear investors' wealth-share w_t when $V_t = \overline{V}$ (Panel C). The parameter values follow from Table C1 of Appendix C.

In our model, the leverage effect arises in equilibrium because a positive variance shock decreases the stock price, while increasing the stock return variance. Therefore, such shock leads to a negative stock return and a positive variance swap return. The magnitude of the leverage effect is determined by the sensitivity of stock returns to variance shocks, which is driven by V_t and w_t along with their elasticities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$. Due to the stochastic nature of these quantities, the leverage effect becomes time-varying under volatility disagreement. Therefore, one of our key contributions here is to complement the existing vast literature on the leverage effect by identifying investors' volatility disagreement as one of its economic determinant and a source of time-variation.

As highlighted in Proposition 7 and illustrated in Figure 7, Panel A, we find that a higher volatility disagreement δ is associated with a stronger leverage effect. This result is due to amplified wealth transfers and elasticities, as discussed in detail in the context of Proposition 2. Figure 7, Panel B, shows that the leverage effect becomes stronger during high volatility periods, consistent with the empirical evidence in Bandi and Renò (2012) and Andersen, Bondarenko, and Gonzalez-Perez (2015). In contrast, in a benchmark economy with only i-type investors, the leverage effect is deterministic and does not vary with the fundamental variance V_t.[20] This difference arises because, under volatility disagreement, V_t amplifies the return variance υ_t by increasing the elasticities B_t and $\Lambda_{ht} - \Lambda_{\ell t}$, since high-fear investors are more risk-sensitive than low-fear investors.

[20]When only i-type investors are present in the economy, the leverage effect is given by $\bar{\rho}_t = -\sigma^2 B_i(t) / \sqrt{\sigma^2 + [\sigma^2 B_i(t)]^2}$, which is obtained from (15) by setting w_t to 1 or 0.

Figure 7 also illustrates that the leverage effect is non-monotonically related to the wealth share of high-fear investors w_t. The left (right) end-point of the curve in Panel C is the leverage effect in the benchmark economy with only low-fear (high-fear) investors. Therefore, the leverage effect becomes stronger under volatility disagreement when there is a large dispersion in investors' wealth. Indeed, when w_t is around $\bar{\mathrm{w}} = 0.5$, positive variance shocks lead to larger wealth transfers to high-fear investors, and hence larger decline in the stock price and larger increase in the stock return variance. We also note that the magnitude of the leverage effect for our baseline calibration is consistent with its empirical estimates, which are typically found to be within the range of -0.50 to -0.90 (e.g., Broadie, Chernov, and Johannes (2009), Aït-Sahalia, Fan, and Li (2013), Andersen, Bondarenko, and Gonzalez-Perez (2015)).

6 Aggregate Volatility Bias

Thus far we have studied an economy in which investors' subjective variance expectations were symmetric around the objective one. This setting was sufficient to demonstrate our key insights on the effects of volatility disagreement. In this section, we extend our model by considering *asymmetric* subjective variance expectations to study the implications of an aggregate volatility bias.

We introduce the aggregate volatility bias into our framework in a straightforward manner. Specifically, we modify investors' subjective variance expectations in Section 2.2 by adding a common component to both investor types:

$$\mathrm{E}_t^h\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \left(\beta + \tfrac{1}{2}\delta\right)V_t dt, \qquad \mathrm{E}_t^\ell\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \left(\beta - \tfrac{1}{2}\delta\right)V_t dt.$$

Under this more general specification, the constant β controls the *aggregate volatility bias* in the economy. When the bias is absent ($\beta = 0$) we revert to our baseline economy characterized by only volatility disagreement. To ensure that, as in our main economy, high-fear (low-fear) investors' volatility expectations are higher (lower) than the objective one, we impose the parameter restriction of $-\delta/2 < \beta < \delta/2$.[21] With this generalization, in addition to the volatility disagreement, our model can also address the documented average bias in variance expectations in surveys (e.g., Graham and Harvey (2001), Ben-David, Graham, and Harvey (2013), Amromin and Sharpe (2014), Barrero (2022), Boutros et al. (2024)), as also discussed in Section 2.2.1. The next proposition characterizes the equilibrium in our extended economy.

[21] The parameter restriction ensuring that the equilibrium stock price admits a real solution now becomes $\kappa - (\beta + \delta/2) > \sqrt{2}\sigma$.

Proposition 8 (**Equilibrium with aggregate volatility bias**). *The equilibrium in the economy with volatility disagreement and aggregate volatility bias is characterized as in our baseline economy where the perceived persistencies of variance shocks κ_i are reduced by β, and the wealth-share's long-run mean $\bar{\mathrm{w}}$ is reduced by β/δ.*

Proposition 8 confirms that in the economy with volatility disagreement and bias, the equilibrium quantities have similar structures to those in our baseline economy with only disagreement. The key difference between the two settings is captured by the equilibrium long-run mean of the high-fear investors' wealth share, which becomes lower with a positive aggregate bias, and higher with a negative bias. For instance, when $\beta < 0$, high-fear investors' wealth is on average higher than that of low-fear investors'. This finding is intuitive since high-fear investors' beliefs are relatively more accurate under a downward bias, as a consequence, they accumulate more wealth on average.[22] We next investigate how the aggregate volatility bias affects the stock market.

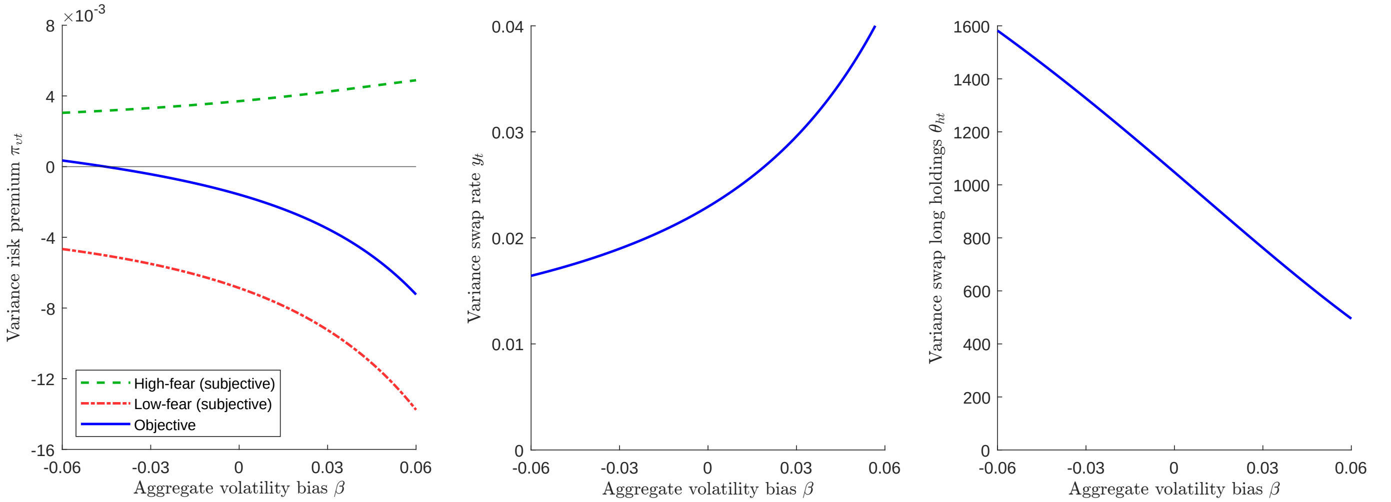
Proposition 9 (**Aggregate volatility bias and the stock market**). *A larger aggregate volatility bias β leads to: (i) a lower stock price S_t, (ii) a higher stock return variance υ_t, and (iii) a stronger leverage effect ρ_t.*

Proposition 9 reveals that similar to the effects of volatility disagreement, a larger aggregate volatility bias leads to a lower stock price, a higher stock return variance, and a stronger leverage effect. However, the underlying mechanisms of volatility disagreement and volatility bias are notably different. As volatility disagreement increases, implying a higher mean-preserving spread between the investors' subjective expectations, high-fear investors' risk discount gets larger, whereas that of low-fear investors gets smaller (Section 3). However, as the aggregate volatility bias grows, *all* investors perceive the future variance to be higher and more persistent, increasing the risk discounts of both investor types.[23] Perceiving the future cashflow as riskier, both investor types are less willing to hold the stock, thus driving its equilibrium price down. Higher volatility expectations also make the stock price more sensitive to variance shocks, thus causing stock returns to become more volatile and the leverage effect stronger.

We conclude this section by discussing the effects of the aggregate volatility bias on the volatility derivatives market. Our findings are illustrated in Figure 8. As the aggregate volatility bias increases, the objective VRP decreases, the subjective VRP perceived by high-fear (low-fear) investors increases

[22] This behavior is consistent with the survival effects demonstrated in the literature (e.g., Kogan et al. (2006), Yan (2008)). For instance, in the polar case of $\beta = \delta/2$ ($\beta = -\delta/2$), low-fear (high-fear) investors have correct beliefs and thus would dominate the economy eventually for a sufficiently large horizon T (i.e., their wealth-shares converges to unity in the long run). In all other relevant cases of our model, $-\delta/2 < \beta < \delta/2$, both types of investors have incorrect volatility expectations, so neither of them dominates the economy in the long run.

[23] Since the effect of bias β goes in the same direction for both investor types, the results in Proposition 9 hold for all states, and not just the steady state.



Panel A. Effect on π_{vt} Panel B. Effect on y_t Panel C. Effect on θ_{ht}

Figure 8. Effects of aggregate volatility bias in the volatility derivatives market. These panels plot the effects of aggregate volatility bias β on the equilibrium variance risk premium π_{vt} (Panel A), variance swap rate y_t (Panel B), and variance swap long holdings θ_{ht} (Panel C) when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$. In Panel A, the solid blue line represents the objective risk premium π_{vt} observed by an econometrician, the dashed green (dash-dotted red) lines represent the subjective risk premium perceived by the high-fear (low-fear) investor. The parameter values follow from Table C1 of Appendix C.

(decreases), and the variance swap rate increases. The (objective) expected compensation for providing insurance against variance risk increases because all investors perceive future volatility as higher. A higher expectation of future volatility also causes the "price" of the variance swap contracts to go up. The divergence of the subjective VRP, instead, is due to a more volatile stock return variance, which amplifies the differences in the speculative components of these premia.

Figure 8, Panel C, shows a novel and somewhat surprising result: a higher aggregate volatility bias leads to lower volatility derivative holdings and trade in equilibrium. This monotonic behavior arises because the stock price becomes more sensitive to variance shocks under a larger volatility bias, leading to a more volatile return variance σ_{vt}. Because of this increased riskiness of the swap contracts, risk-averse investors find it optimal to hold fewer volatility derivatives in equilibrium. While, as discussed in Section 4, a higher disagreement increases both the expected return and riskiness of the volatility derivatives—resulting in a hump-shaped relation with their holdings—a larger volatility bias only affects the riskiness. This finding further highlights how volatility disagreement and bias have economically different effects on equilibrium outcomes.

7 Conclusion

In this paper, we study the equilibrium properties of a tractable dynamic asset pricing model in which investors with different future volatility expectations trade a riskless asset, a risky stock, and a volatility derivative. The presence of volatility derivatives allows investors to hedge their volatility exposures and speculate on their differing beliefs. The model delivers closed-from expressions and generates numerous predictions for the volatility derivatives market, as well as the stock market.

We uncover that higher volatility disagreement leads, on average, to a lower stock market valuation, a higher and more volatile market volatility, and a stronger leverage effect. It also leads to a more negative variance risk premium and a higher variance swap rate. Moreover, trading activity in volatility derivatives increases at first and then decreases with the level of disagreement. Importantly, our model proposes a possible mechanism why the variance risk premium changes sign, highlighting that this is more likely to happen when the market underestimates future volatility. Consistent with empirical evidence, we also find that investors hold and trade fewer volatility derivatives, and the leverage effect becomes stronger during more volatile periods. Exploring the implications of an aggregate volatility bias in an extension of our model, we show that a larger bias tends to amplify variance shocks, and reduces the equilibrium trading activity in volatility derivatives. To the best of our knowledge, we are the first to incorporate volatility disagreement in a dynamic equilibrium setting, allowing us to simultaneously reconcile the puzzling empirical evidence that during market turmoils, the variance risk premium is positive, its magnitude rises in volatility, and investors trade fewer volatility derivatives.

Appendix A: Proofs

This appendix provides the proofs of all Propositions 1–9, Corollary 1, and Lemma A1 introduced in this Appendix. To prevent repetition, we solve the more general version of our model presented in Section 6, which additionally features the aggregate volatility bias β. The relevant economic quantities in our baseline economy arise as the special case when $\beta = 0$ in the following proofs.

Proof of Proposition 1. We first solve investors' optimization problem using standard martingale methods to determine the equilibrium in our complete market economy. Each i-type investor's static optimization problem, $i = h, \ell$, becomes

$$\max_{W_{iT}} \mathrm{E}^i \left[\ln W_{iT}\right], \qquad \text{subject to} \qquad \mathrm{E}^i \left[\xi_{iT} W_{iT}\right] \leq \xi_{i0} W_{i0}, \tag{A.1}$$

where ξ_{it} is the i-type investor's subjective state price density. The consistency relation across investors' subjective beliefs implies $\xi_{ht} = L_t \xi_{\ell t}$, where L_t is the likelihood ratio process given by

$$L_t = \left.\frac{d\mathbb{P}^\ell}{d\mathbb{P}^h}\right|_t = e^{-\int_0^t \delta \frac{1}{\sigma}\sqrt{V_u} d\omega^h_{2u} - \frac{1}{2}\int_0^t \left(\delta \frac{1}{\sigma}\sqrt{V_u}\right)^2 du},$$

with dynamics

$$\frac{dL_t}{L_t} = -\delta \frac{1}{\sigma}\sqrt{V_t} d\omega^h_{2t}. \tag{A.2}$$

The first order conditions of the static optimization problem (A.1) gives the optimal terminal wealth of each i-type as $W_{iT} = \lambda_i^{-1}\xi_{iT}^{-1}$, where the Lagrange multiplier λ_i solves the static budget constraint with equality and is given by $\lambda_i^{-1} = \xi_{i0} W_{i0}$.

Imposing the goods market clear condition $W_{hT} + W_{\ell T} = D_T$ along with the consistency relation $\xi_{hT} = L_T \xi_{\ell T}$, leads to the h-type investor's subjective state price density at time-T as

$$\xi_{hT} = D_T^{-1}\left(\lambda_h^{-1} + \lambda_\ell^{-1} L_T\right). \tag{A.3}$$

The subjective state price density at an earlier time $t < T$ is determined through the relation $\xi_{ht} = e^{r(T-t)}\mathrm{E}^h_t\left[\xi_{hT}\right]$, which yields

$$\xi_{ht} = e^{r(T-t)}\left(\lambda_h^{-1}\mathrm{E}^h_t\left[D_T^{-1}\right] + \lambda_\ell^{-1}\mathrm{E}^h_t\left[D_T^{-1} L_T\right]\right).$$

We use Lemma A1 at the end of this Appendix to compute the above expectations. By taking $a = -1$ and $b = 0$ in Lemma A1, we obtain the first expectation as

$$M_{ht} \equiv \mathrm{E}^h_t\left[D_T^{-1}\right] = D_t^{-1} e^{-\mu(T-t)} e^{A_h(t) + B_h(t) V_t},$$

along with its dynamics

$$\frac{dM_{ht}}{M_{ht}} = -\sqrt{V_t}d\omega_{1t} + \sigma B_h(t)\sqrt{V_t}d\omega_{2t}^h. \tag{A.4}$$

and by taking $a = -1$ and $b = 1$ in Lemma A1, we obtain the second expectation as

$$M_{\ell t} \equiv \mathrm{E}_t^h\left[D_T^{-1}L_T\right] = D_t^{-1}L_t e^{-\mu(T-t)}e^{A_\ell(t)+B_\ell(t)V_t},$$

along with its dynamics

$$\frac{dM_{\ell t}}{M_{\ell t}} = -\sqrt{V_t}d\omega_{1t} + \left(\sigma^2 B_\ell(t) - \delta\right)\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h, \tag{A.5}$$

where the positive deterministic functions $A_i(t)$ and $B_i(t)$ for $i = h, \ell$ are given by

$$A_i(t) = \frac{2\kappa\overline{V}}{\sigma^2}\ln\frac{2\eta_i e^{\frac{1}{2}(\kappa_i+\eta_i)(T-t)}}{(\kappa_i+\eta_i)\left(e^{\eta_i(T-t)}-1\right)+2\eta_i}, \qquad B_i(t) = 2\frac{e^{\eta_i(T-t)}-1}{(\kappa_i+\eta_i)\left(e^{\eta_i(T-t)}-1\right)+2\eta_i},$$

with the positive constants $\kappa_h = \kappa - \left(\beta + \frac{1}{2}\delta\right)$, $\eta_h = \sqrt{\kappa_h^2 - 2\sigma^2}$, $\kappa_\ell = \kappa - \left(\beta - \frac{1}{2}\delta\right)$, and $\eta_\ell = \sqrt{\kappa_\ell^2 - 2\sigma^2}$.

Next, we apply Itô's Lemma to $\xi_{ht} = e^{r(T-t)}\left(\lambda_h^{-1}M_{ht} + \lambda_\ell^{-1}M_{\ell t}\right)$ using (A.4) and (A.5), and obtain

$$\frac{d\xi_{ht}}{\xi_{ht}} = -rdt - \sqrt{V_t}d\omega_{1t} + \left[\sigma^2\frac{\lambda_h^{-1}M_{ht}B_h(t) + \lambda_\ell^{-1}M_{\ell t}B_\ell(t)}{\lambda_h^{-1}M_{ht} + \lambda_\ell^{-1}M_{\ell t}} - \frac{\delta\lambda_\ell^{-1}M_{\ell t}}{\lambda_h^{-1}M_{ht} + \lambda_\ell^{-1}M_{\ell t}}\right]\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h. \tag{A.6}$$

To express the above dynamics in terms of investors' wealth share, we determine their wealth as

$$W_{ht} = \frac{1}{\xi_{ht}}\mathrm{E}_t^h\left[\xi_{hT}W_{hT}\right] = \frac{1}{\xi_{ht}}\lambda_h^{-1}, \qquad W_{\ell t} = \frac{1}{\xi_{\ell t}}\mathrm{E}_t^\ell\left[\xi_{\ell T}W_{\ell T}\right] = \frac{1}{\xi_{\ell t}}\lambda_\ell^{-1},$$

which along with the consistency relation $\xi_{ht} = L_t\xi_{\ell t}$, give the wealth share of the h-type investor as

$$\mathrm{w}_t = \frac{W_{ht}}{W_{ht} + W_{\ell t}} = \frac{\lambda_h^{-1}}{\lambda_h^{-1} + \lambda_\ell^{-1}L_t}, \tag{A.7}$$

and

$$\frac{\lambda_h^{-1}M_{ht}}{\lambda_h^{-1}M_{ht} + \lambda_\ell^{-1}M_{\ell t}} = \frac{\lambda_h^{-1}e^{A_h(t)+B_h(t)V_t}}{\lambda_h^{-1}e^{A_h(t)+B_h(t)V_t} + \lambda_\ell^{-1}L_t e^{A_\ell(t)+B_\ell(t)V_t}} = \mathrm{w}_t\Lambda_{ht},$$

where investors' relative risk discount term Λ_{it} is as in (7). Therefore, by also using the representation of the stochastic variance elasticity B_t in (7), we simply rewrite the dynamics in (A.6) as

$$\frac{d\xi_{ht}}{\xi_{ht}} = -rdt - \sqrt{V_t}d\omega_{1t} + \left[\sigma^2 B_t - \delta\left(1 - \mathrm{w}_t\right)\left(1 - \mathrm{w}_t\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\right)\right]\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h.$$

Matching the above dynamics to the h-type investor's subjective dynamics $d\xi_{ht}/\xi_{ht} = -rdt - m_{1t}^h d\omega_{1t} - m_{2t}^h d\omega_{2t}^h$, gives her perceived market prices of risks as

$$m_{1t}^h = \sqrt{V_t}, \qquad m_{2t}^h = -\left[\sigma B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \left(\Lambda_{ht} - \Lambda_{\ell t}\right) \frac{1}{\sigma}\right] \sqrt{V_t} + \delta \left(1 - \mathrm{w}_t\right) \frac{1}{\sigma} \sqrt{V_t}. \tag{A.8}$$

Using the consistency relations between the objective measure and the h-type investor's subjective measure, which imply $m_{1t} = m_{1t}^h$ and $m_{2t} = m_{2t}^h - (\beta + \delta/2)\sqrt{V_t}/\sigma$, we obtain the market prices of risks for the objective shocks ω_1 and ω_2 as

$$m_{1t} = \sqrt{V_t}, \qquad m_{2t} = -\left[\sigma B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \left(\Lambda_{ht} - \Lambda_{\ell t}\right) \frac{1}{\sigma}\right] \sqrt{V_t} + \delta \left(\bar{\mathrm{w}} - \mathrm{w}_t\right) \frac{1}{\sigma} \sqrt{V_t}. \tag{A.9}$$

Moreover, applying Itô's Lemma to the wealth share (A.7) using the dynamics of L_t in (A.2) gives the subjective dynamics

$$d\mathrm{w}_t = \delta^2 \mathrm{w}_t \left(1 - \mathrm{w}_t\right)^2 \frac{1}{\sigma^2} V_t dt + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}^h,$$

which after $d\omega_{2t}^h = d\omega_{2t} - \left(\beta + \frac{1}{2}\delta\right) \frac{1}{\sigma} \sqrt{V_t} dt$ substituted in becomes the objective dynamics as in (5) with the constant long-run mean $\bar{\mathrm{w}} = 1/2 - \beta/\delta$.

By no arbitrage, the stock price satisfies $\xi_{ht} S_t = \mathrm{E}_t^h \left[\xi_{hT} D_T\right]$. Using (A.3) and the martingale property of L_t under $\mathbb{P}^h$ gives the expectation as $\mathrm{E}_t^h \left[\xi_{hT} D_T\right] = \lambda_h^{-1} + \lambda_\ell^{-1} L_t$. Substituting the h-type investor's subjective state price density in terms of the wealth share

$$\xi_{ht} = D_t^{-1} e^{-(\mu - r)(T-t)} \left(\lambda_h^{-1} + \lambda_\ell^{-1} L_t\right) \left[\mathrm{w}_t e^{A_h(t) + B_h(t) V_t} + \left(1 - \mathrm{w}_t\right) e^{A_\ell(t) + B_\ell(t) V_t}\right],$$

immediately gives (4). We also note that applying Itô's Lemma to the stock price (4) yields the dynamics $dS_t/S_t = \mu_{St} dt + \sigma_{S1t} d\omega_{1t} + \sigma_{S2t} d\omega_{2t}$, where the diffusion coefficients are given by

$$\sigma_{S1t} = \sqrt{V_t}, \qquad \sigma_{S2t} = -\left[\sigma B_t + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \left(\Lambda_{ht} - \Lambda_{\ell t}\right) \frac{1}{\sigma}\right] \sqrt{V_t}. \tag{A.10}$$

Property that a higher volatility disagreement δ leads to a lower stock price at the steady state, follows from the partial derivative of (4) with respect to δ. This property holds if and only if $\frac{\partial}{\partial \delta} \left[e^{A_h(t) + B_h(t)\overline{V}} + e^{A_\ell(t) + B_\ell(t)\overline{V}}\right] > 0$. Differentiating yields

$$\frac{\partial}{\partial \delta} \left[e^{A_h(t) + B_h(t)\overline{V}} + e^{A_\ell(t) + B_\ell(t)\overline{V}}\right] = e^{A_h(t) + B_h(t)\overline{V}} \frac{\partial}{\partial \delta} \left[A_h(t) + B_h(t) \overline{V}\right] + e^{A_\ell(t) + B_\ell(t)\overline{V}} \frac{\partial}{\partial \delta} \left[A_\ell(t) + B_\ell(t) \overline{V}\right].$$

To show this is always positive, we use the fact that the deterministic functions $A_i(t)$ and $B_i(t)$ convexly decrease in κ. Since $\kappa_h < \kappa_\ell$, we immediately obtain $e^{A_h(t) + B_h(t)\overline{V}} > e^{A_\ell(t) + B_\ell(t)\overline{V}}$. Moreover, due to convexity, we obtain $\frac{\partial}{\partial\delta}\left[A_h(t)+B_h(t)\overline{V}\right] > |\frac{\partial}{\partial\delta}\left[A_\ell(t)+B_\ell(t)\overline{V}\right]|$ since an increase in δ means a mean-preserving spread around $\kappa-\beta$ for κ_h and κ_ℓ. That is, a higher disagreement leads to more increases in $A_h(t)$ and $B_h(t)$ than the decreases in $A_\ell(t)$ and $B_\ell(t)$, respectively. □

Proof of Proposition 2. Using the diffusion coefficients in (A.10), we immediately obtain the equilibrium stock return variance using $\upsilon_t = \mathrm{Var}_t\left[dS_t/S_t\right]/dt = \sigma_{S1t}^2+\sigma_{S2t}^2$ as in (6).

Property that a higher volatility disagreement δ leads to a higher stock return variance at the steady state, follows from the partial derivative of (6) with respect to δ. This property holds if and only if $\frac{\partial}{\partial\delta}\left[\sigma B_t+\delta\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{4\sigma}\right]>0$. Differentiating yields

$$\frac{\partial}{\partial\delta}\left[\sigma B_t+\delta\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{4\sigma}\right]=\sigma\frac{\partial}{\partial\delta}B_t+\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{4\sigma}+\delta\left(\frac{\partial}{\partial\delta}\Lambda_{ht}-\frac{\partial}{\partial\delta}\Lambda_{\ell t}\right)\frac{1}{4\sigma}, \tag{A.11}$$

where

$$\frac{\partial}{\partial\delta}\Lambda_{ht}=2\frac{e^{A_\ell(t)+B_\ell(t)\overline{V}}\frac{\partial}{\partial\delta}e^{A_h(t)+B_h(t)\overline{V}}-e^{A_h(t)+B_h(t)\overline{V}}\frac{\partial}{\partial\delta}e^{A_\ell(t)+B_\ell(t)\overline{V}}}{\left[e^{A_h(t)+B_h(t)\overline{V}}+e^{A_\ell(t)+B_\ell(t)\overline{V}}\right]^2},$$
$$\frac{\partial}{\partial\delta}\Lambda_{\ell t}=2\frac{e^{A_h(t)+B_h(t)\overline{V}}\frac{\partial}{\partial\delta}e^{A_\ell(t)+B_\ell(t)\overline{V}}-e^{A_\ell(t)+B_\ell(t)\overline{V}}\frac{\partial}{\partial\delta}e^{A_h(t)+B_h(t)\overline{V}}}{\left[e^{A_h(t)+B_h(t)\overline{V}}+e^{A_\ell(t)+B_\ell(t)\overline{V}}\right]^2}.$$

As we show in the proof of Proposition 1, $\frac{\partial}{\partial\delta}\left[A_h(t)+B_h(t)\overline{V}\right]>\frac{\partial}{\partial\delta}\left[A_\ell(t)+B_\ell(t)\overline{V}\right]$, which implies $\frac{\partial}{\partial\delta}\Lambda_{ht}>0$, $\frac{\partial}{\partial\delta}\Lambda_{\ell t}<0$, and thus $\frac{\partial}{\partial\delta}\Lambda_{ht}-\frac{\partial}{\partial\delta}\Lambda_{\ell t}>0$ in (A.11). Next, we show the term $\frac{\partial}{\partial\delta}B_t$ in (A.11) is also positive. To see this, note that $B_t=\mathrm{w}_t\Lambda_{ht}B_h(t)+\left(1-\mathrm{w}_t\Lambda_{ht}\right)B_\ell(t)=\mathrm{w}_t\Lambda_{ht}B_h(t)+\left(1-\mathrm{w}_t\right)\Lambda_{\ell t}B_\ell(t)$. Thus, at the steady state

$$\frac{\partial}{\partial\delta}B_t=\frac{1}{2}\left[\Lambda_{ht}\frac{\partial}{\partial\delta}B_h(t)+\Lambda_{\ell t}\frac{\partial}{\partial\delta}B_\ell(t)+B_h(t)\frac{\partial}{\partial\delta}\Lambda_{ht}+B_\ell(t)\frac{\partial}{\partial\delta}\Lambda_{\ell t}\right].$$

Since, $\Lambda_{ht}>\Lambda_{\ell t}$ and $\frac{\partial}{\partial\delta}B_h(t)>\left|\frac{\partial}{\partial\delta}B_\ell(t)\right|$, we have $\left(\Lambda_{ht}\frac{\partial}{\partial\delta}B_h(t)+\Lambda_{\ell t}\frac{\partial}{\partial\delta}B_\ell(t)\right)>0$. Since $B_h(t)>B_\ell(t)$ and $\frac{\partial}{\partial\delta}\Lambda_{ht}>\left|\frac{\partial}{\partial\delta}\Lambda_{\ell t}\right|$, we also have $\left(B_h(t)\frac{\partial}{\partial\delta}\Lambda_{ht}+B_\ell(t)\frac{\partial}{\partial\delta}\Lambda_{\ell t}\right)>0$, proving $\frac{\partial}{\partial\delta}B_t>0$, which along with the earlier relation $\frac{\partial}{\partial\delta}\Lambda_{ht}-\frac{\partial}{\partial\delta}\Lambda_{\ell t}>0$ proves the right hand side of (A.11) is positive. □

Proof of Proposition 3. To determine the dynamics of the equilibrium stock return variance, we first apply Itô's Lemma to Λ_{it} and B_t using their representations in (7). After straightforward but lengthy algebra, we obtain the dynamics for Λ_{it} as $d\Lambda_{it}=\mu_{\Lambda_i t}dt+\sigma_{\Lambda_i t}d\omega_{2t}$, with the diffusion term

$$\sigma_{\Lambda_i t}=\Lambda_{it}\left[\sigma B_i(t)-\left(\sigma B_t+\delta\mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right)\right]\sqrt{V_t},$$

and the drift term

$$\mu_{\Lambda_i t} = \Lambda_{it}\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]^2 V_t$$
$$- \Lambda_{it}\delta\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\frac{1}{\sigma}\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right] V_t$$
$$+ \Lambda_{it}\left(\kappa_i-\kappa\right) B_i\left(t\right) V_t - \Lambda_{it}\sigma B_i\left(t\right)\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right] V_t.$$

Similarly, we obtain the dynamics $dB_t = \mu_{Bt}dt + \sigma_{Bt}d\omega_{2t}$, with the diffusion term

$$\sigma_{Bt} = \left(B_h\left(t\right) - B_\ell\left(t\right)\right)\left(\mathrm{w}_t\sigma_{\Lambda_h t} + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\Lambda_{ht}\frac{1}{\sigma}\sqrt{V_t}\right),$$

and the drift term

$$\mu_{Bt} = \mathrm{w}_t\left(B_h\left(t\right) - B_\ell\left(t\right)\right)\left[\mu_{\Lambda_h t} + \delta\sigma_{\Lambda_h t}\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t} + \delta^2\Lambda_{ht}\left(1-\mathrm{w}_t\right)\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\frac{1}{\sigma^2}V_t\right]$$
$$+ \mathrm{w}_t\Lambda_{ht}\dot{B}_h\left(t\right) + \left(1-\mathrm{w}_t\Lambda_{ht}\right)\dot{B}_\ell\left(t\right),$$

where $\dot{B}_i\left(t\right) = dB_i\left(t\right)/dt = -1 + \kappa_i B_i\left(t\right) - \frac{1}{2}\sigma^2 B_i^2\left(t\right).$

Finally, we apply Itô's Lemma to stock return variance (6) to obtain $dv_t = \mu_{vt}dt + \sigma_{vt}d\omega_{2t}$, where the diffusion term as in (8) and the drift term

$$\mu_{vt} = \kappa\left(\overline{V} - V_t\right) + \left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]^2\kappa\left(\overline{V} - V_t\right)$$
$$+ 2\mu_{Qt}\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right] V_t$$
$$+ 2\sigma\sigma_{Qt}\left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\right]\sqrt{V_t} + \sigma_{Qt}^2 V_t, \tag{A.12}$$

where μ_{Qt} and σ_{Qt} are defined as

$$\sigma_{Qt} = \sigma\sigma_{Bt} + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\sigma_{\Lambda_h t} - \sigma_{\Lambda_\ell t} + \delta\left(1-2\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma}\sqrt{V_t}\right)\frac{1}{\sigma},$$
$$\mu_{Qt} = \sigma\mu_{Bt} + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\mu_{\Lambda_h t} - \mu_{\Lambda_\ell t}\right)\frac{1}{\sigma} + \delta^2 \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(1-2\mathrm{w}_t\right)\left(\sigma_{\Lambda_h t} - \sigma_{\Lambda_\ell t}\right)\frac{1}{\sigma^2}\sqrt{V_t}$$
$$+ \delta^3 \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(1-2\mathrm{w}_t\right)\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma^3}V_t - \delta^3 \mathrm{w}_t^2\left(1-\mathrm{w}_t\right)^2\left(\Lambda_{ht}-\Lambda_{\ell t}\right)\frac{1}{\sigma^3}V_t.$$

□

Proof of Proposition 4. The equilibrium variance risk premium is given by the conditional covariance of the state price density growth with the variance swap payoff

$$\pi_{vt}dt = -\frac{d\xi_t}{\xi_t}\left(\upsilon_t dt + d\upsilon_t - y_t dt\right) = -\frac{d\xi_t}{\xi_t}d\upsilon_t = m_{2t}\sigma_{vt}dt,$$

which becomes (9) after substituting m_{2t} in (A.9). Similarly, the subjective variance risk premium of i-type investors, $i = h, \ell$, is given by $\pi^i_{vt}dt = m^i_{2t}\sigma_{vt}dt$, which becomes (10) for high-fear investors after substituting m^h_{2t} in (A.8). Moreover, using the consistency relation $m^\ell_{2t} = m^h_{2t} - (\delta/\sigma)\sqrt{V_t}$, we obtain (11) for low-fear investors. □

Proof of Corollary 1. The objective variance risk premium (9) becomes positive if and only if

$$\delta\left(\bar{\mathrm{w}} - \mathrm{w}_t\right)\frac{1}{\sigma} > \left[\sigma B_t + \delta \mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\frac{1}{\sigma}\right],$$

under the positivity of σ_{vt}. Note that when $\delta = 0$, the above inequality never holds, implying that VRP never becomes positive in the benchmark economy. For $\delta > 0$, after substituting investors' relative risk discount terms Λ_{it} and the stochastic variance elasticity B_t in (7), we see that the above inequality holds if and only if

$$0 > \left[\sigma^2 B_h(t) + \delta\left(1-\bar{\mathrm{w}}\right)\right]\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + \left[\sigma^2 B_\ell(t) - \delta\bar{\mathrm{w}}\right]\left(1-\mathrm{w}_t\right)e^{A_\ell(t)+B_\ell(t)V_t},$$

or $\mathrm{w}_t < \mathrm{w}^*_t$, where w^*_t is as in (12). Note that it is also easy to show that $\mathrm{w}^*_t < \bar{\mathrm{w}}$. To ensure $\mathrm{w}^*_t > 0$, we also need to have $\delta\bar{\mathrm{w}} - \sigma^2 B_\ell(t) > 0$. Given that $\delta\bar{\mathrm{w}}$ increases while $\sigma^2 B_\ell(t)$ decreases in δ, there exist a disagreement level δ^* such that for $\delta > \delta^*$, the last inequality holds. Moreover, δ^* solves $\delta^*\bar{\mathrm{w}} = \sigma^2 B_\ell(t;\delta^*)$, and a sufficient condition for this to hold for all t is $\delta^* = 2\left[\beta - \kappa + \sqrt{\kappa^2 + 2\sigma^2}\right]$ since the maximum value of $\sigma^2 B_\ell(t)$ is $\kappa - \left(\beta - \frac{1}{2}\delta\right) - \sqrt{\left[\kappa - \left(\beta - \frac{1}{2}\delta\right)\right]^2 - 2\sigma^2}$. □

Proof of Proposition 5. By no-arbitrage, the equilibrium variance swap rate is given by equating the risk-neutral expectation of the variance swap payoff to zero, $\mathrm{E}^*_t\left[\upsilon_t dt + d\upsilon_t - y_t dt\right] = 0$, which yields (13) after substituting $\mathrm{E}^*_t\left[d\upsilon_t\right] = \mathrm{E}_t\left[d\upsilon_t\right] - \pi_{vt}dt = \left(\mu_{vt} - \pi_{vt}\right)dt$, where μ_{vt} is as in (A.12). □

Proof of Proposition 6. To determine the equilibrium holdings in variance swaps, we begin with the observation that for $i = h, \ell$, i-type investor's discounted wealth process satisfies $\xi_{it}W_{it} = \mathrm{E}^i_t\left[\xi_{iT}W_{iT}\right] = \lambda_i^{-1}$, which implies the dynamics $d\left(\xi_{it}W_{it}\right) = 0$ under their subjective measure $\mathbb{P}^i$. Matching this dynamics to their discounted budget constraint

$$d\left(\xi_{it}W_{it}\right) = \xi_{it}\left[\psi_{it}S_t\sigma_{S1t} - W_{it}m^i_{1t}\right]d\omega_{1t} + \xi_{it}\left[\psi_{it}S_t\sigma_{S2t} + \theta_{it}\sigma_{vt} - W_{it}m^i_{2t}\right]d\omega^i_{2t},$$

yields the system of two equations in two unknowns, ψ_{it} and θ_{it},

$$\left[\psi_{it} S_t \sigma_{S1t} - W_{it} m^i_{1t}\right] = 0,$$
$$\left[\psi_{it} S_t \sigma_{S2t} + \theta_{it}\sigma_{\upsilon t} - W_{it} m^i_{2t}\right] = 0,$$

where m^i_{jt} is i-type investor's perceived market price of risk for the Brownian motion ω^i_j, which satisfy $d\xi_{it}/\xi_{it} = -rdt - m^i_{1t} d\omega_{1t} - m^i_{2t} d\omega^i_{2t}$. Solving the above system of equations yields the variance swap contract holdings $\theta_{ht} = W_{ht}\left[\sigma_{S1t} m^h_{2t} - \sigma_{S2t} m^h_{1t}\right]/\sigma_{S1t}\sigma_{\upsilon t}$. Substituting (A.8), (A.10), and $W_{ht} = \mathrm{w}_t S_t$ yields θ_{it} as in (14). $\square$

Proof of Proposition 7. The equilibrium leverage effect is given by

$$\rho_t dt = \frac{\mathrm{Cov}_t\left[dS_t/S_t, d\upsilon_t\right]}{\sqrt{\mathrm{Var}_t\left[dS_t/S_t\right]}\sqrt{\mathrm{Var}_t\left[d\upsilon_t\right]}} = \frac{\sigma_{S2t}\sigma_{\upsilon t}}{\sqrt{\sigma^2_{S1t} + \sigma^2_{S2t}}\sqrt{\sigma^2_{\upsilon t}}} dt,$$

which immediately yields (15) after substituting (A.10).

Property that a higher volatility disagreement δ leads to a stronger leverage effect at the steady state, follows from the partial derivative of (15) with respect to δ. This property holds if and only if $\frac{\partial}{\partial\delta}\left[\sigma B_t + \delta\left(\Lambda_{ht} - \Lambda_{\ell t}\right)\frac{1}{4\sigma}\right] > 0$, which is shown to hold in the proof of Proposition 2. $\square$

Proof of Proposition 8. The quantities in the economy with aggregate bias are already obtained in the proofs of Propositions 1–7. $\square$

Proof of Proposition 9. Property (i), which states that the stock price is decreasing in aggregate volatility bias, follows from the partial derivative of (4) with respect to β. This property holds if and only if $\frac{\partial}{\partial\beta}\left[\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}\right] > 0$. Since $\frac{\partial}{\partial\beta}\left[A_h(t) + B_h(t)\, V_t\right] > 0$ and $\frac{\partial}{\partial\beta}\left[A_\ell(t) + B_\ell(t)\, V_t\right] > 0$, the above inequality holds.

Property (ii), which states that the stock return variance is increasing in aggregate volatility volatility bias, follows from the partial derivative of (6) with respect to β. This property holds if and only if $\sigma_{S1t}\frac{\partial}{\partial\beta}\sigma_{S1t} + \sigma_{S2t}\frac{\partial}{\partial\beta}\sigma_{S2t} > 0$. Knowing that $\sigma_{S1t} = \sqrt{V_t}$ implies the first term is zero. Knowing that $\frac{\partial}{\partial\beta}\sigma_{S2t} < 0$ and $\sigma_{S2t} < 0$ implies the positivity of the second term.

Property (iii), which states that the leverage effect gets stronger in volatility bias, follows from the partial derivative of (15) with respect to β. This property holds if and only if $\frac{\partial}{\partial\beta}[\sigma^2 B_t + \delta\mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\Lambda_{ht} - \Lambda_{\ell t}\right)] > 0$. This inequality holds, since similar to the case of δ, one can show that $\frac{\partial}{\partial\beta}B_t > 0$ and $\frac{\partial}{\partial\beta}\left(\Lambda_{ht} - \Lambda_{\ell t}\right) > 0$. $\square$

Lemma A1. *Let the processes D_t and L_t be as in* (1) *and* (A.2)*. Then, for numbers a and b, the conditional joint moment generating function of* $\ln D_T$ *and* $\ln L_T$ *under $\mathbb{P}^h$, denoted by $M_t(a,b)$, is given by*

$$M_t(a,b) = \mathrm{E}_t^h\left[D_T^a L_T^b\right] = D_t^a L_t^b e^{a\mu(T-t)} e^{A(t;a,b)+B(t;a,b)V_t}, \tag{A.13}$$

with its dynamics by

$$\frac{dM_t(a,b)}{M_t(a,b)} = a\sqrt{V_t}d\omega_{1t} + \left(\sigma^2 B(t;a,b) - b\delta\right)\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h, \tag{A.14}$$

where the deterministic functions are

$$A(t;a,b) = \frac{2\kappa\overline{V}}{\sigma^2}\ln\frac{2\eta e^{\frac{1}{2}(\tilde{\kappa}+\eta)(T-t)}}{(\tilde{\kappa}+\eta)\left(e^{\eta(T-t)}-1\right)+2\eta}, \tag{A.15}$$

$$B(t;a,b) = \left(a(a-1) + b(b-1)\delta^2\frac{1}{\sigma^2}\right)\frac{\left(e^{\eta(T-t)}-1\right)}{(\tilde{\kappa}+\eta)\left(e^{\eta(T-t)}-1\right)+2\eta}, \tag{A.16}$$

with the constants $\tilde{\kappa} = \kappa + \left(b-\frac{1}{2}\right)\delta - \beta$ and $\eta = \sqrt{\tilde{\kappa}^2 - a(a-1)\sigma^2 - b(b-1)\delta^2}$.

Proof of Lemma A1. We use the standard transform analysis to compute the conditional joint moment generating function (A.13). We first apply Itô's Lemma using (1) and (A.2) to obtain

$$\frac{dD_t^a L_t^b}{D_t^a L_t^b} = \left[a\mu + \frac{1}{2}\left(a(a-1) + b(b-1)\delta^2\frac{1}{\sigma^2}\right)V_t\right]dt + a\sqrt{V_t}d\omega_{1t} - b\delta\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h,$$

for numbers a and b. Next, we rewrite the process $D_t^a L_t^b$ in terms of the martingale G and the finite variation process N by defining $D_t^a L_t^b = G_t N_t$ with

$$\frac{dG_t}{G_t} = a\sqrt{V_t}d\omega_{1t} - b\delta\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h, \qquad \frac{dN_t}{N_t} = \left[a\mu + \frac{1}{2}\left(a(a-1) + b(b-1)\delta^2\frac{1}{\sigma^2}\right)V_t\right]dt,$$

which implies $\mathrm{E}_t^h\left[D_T^a L_T^b\right] = \mathrm{E}_t^h\left[G_T N_T\right]$. Next, we define $d\omega_{1t}^G = d\omega_{1t} - a\sqrt{V_t}dt$ and $d\omega_{2t}^G = d\omega_{2t}^h + b\delta\frac{1}{\sigma}\sqrt{V_t}dt$ with the likelihood ratio process

$$\left.\frac{d\mathbb{P}^G}{d\mathbb{P}^h}\right|_t = G_t = e^{\int_0^t a\sqrt{V_u}d\omega_{1u} - \int_0^t b\delta\sigma^{-1}\sqrt{V_u}d\omega_{2u}^h - \frac{1}{2}\int_0^t\left(a^2+b^2\delta^2\sigma^{-2}\right)V_u du},$$

and by changing the measure to $\mathbb{P}^G$, we obtain the required expectation as $\mathrm{E}_t^h\left[D_T^a L_T^b\right] = D_t^a L_t^b \mathrm{E}_t^G\left[\frac{N_T}{N_t}\right]$, where $\frac{N_T}{N_t} = e^{\int_t^T(a\mu + cV_u)du}$ with the constant $c \equiv \frac{1}{2}\left[a(a-1) + b(b-1)\delta^2\sigma^{-2}\right]$. Since the dynamics of V under measure $\mathbb{P}^G$ becomes a standard square-root process $dV_t = \left(\kappa\overline{V} - \tilde{\kappa}V_t\right)dt + \sigma\sqrt{V_t}d\omega_{2t}^G$, where $\tilde{\kappa} \equiv \kappa + \left(b-\frac{1}{2}\right)\delta - \beta$, using the standard moment generating function of the square-root process, we obtain (A.13) where the deterministic functions solve the ODEs

$$\begin{aligned}
\frac{d}{dt}A(t;a,b) &= -\kappa\overline{V}B(t;a,b), \\
\frac{d}{dt}B(t;a,b) &= -c+\tilde{\kappa}B(t;a,b)-\frac{1}{2}\sigma^2B^2(t;a,b),
\end{aligned}$$

with $A(T;a,b)=B(T;a,b)=0$, whose solutions are as in (A.15) and (A.16) under $\tilde{\kappa}^2>2c\sigma^2$.

The subjective dynamics of the $M_t(a,b)$ in (A.14) is obtained by applying Itô's Lemma to (A.13) while employing the dynamics in (1), (2), (A.2), and the ODEs above. □

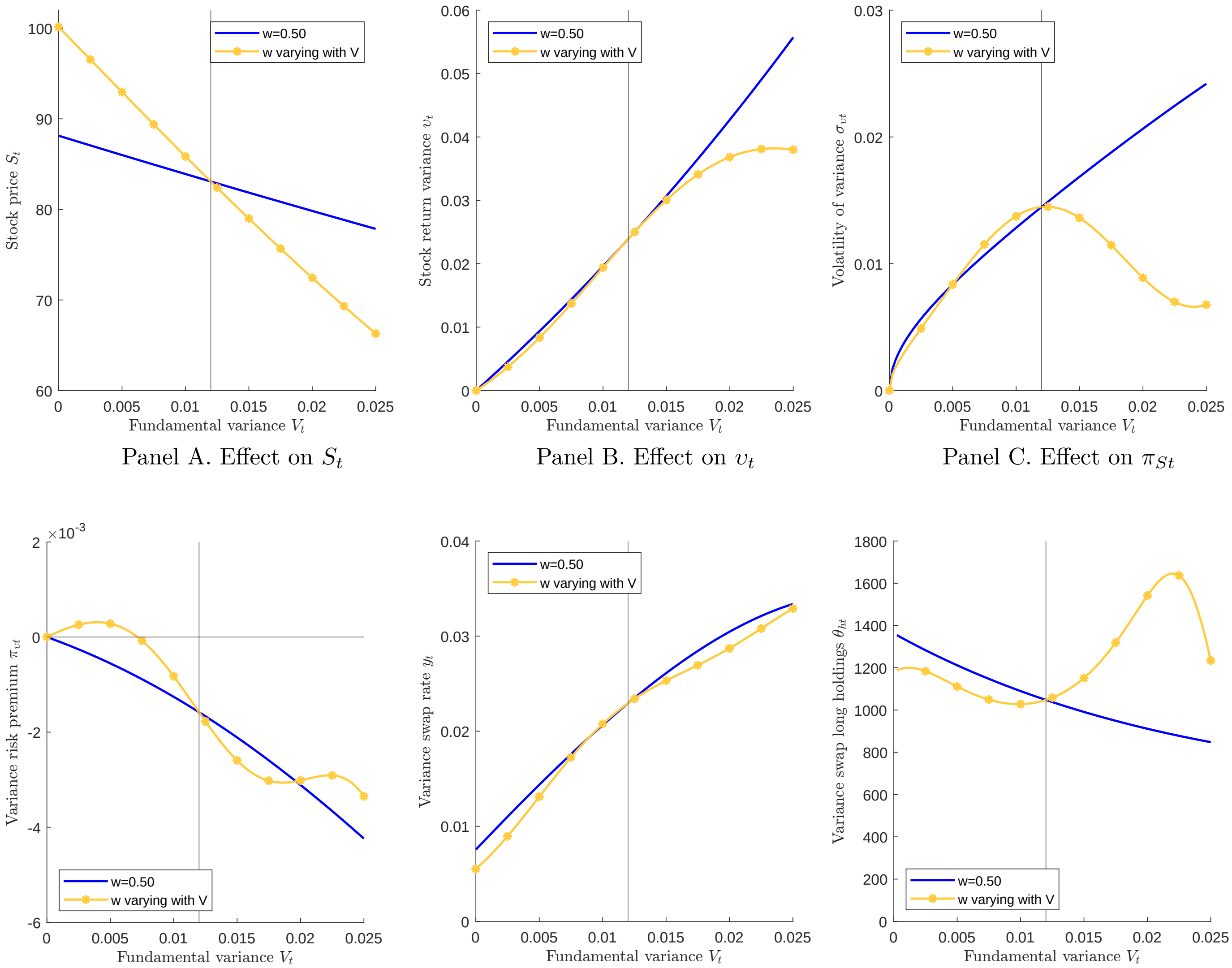
Appendix B: Effects of Pure Variance Shocks

In our main analysis, when we plot the effects of fundamental variance or the wealth-share, we kept the other state variable fixed. However, as the dynamics (2) and (5) show, these state variables are positively correlated, $dV_t d\mathrm{w}_t>0$. Thus, following a positive shock, $d\omega_{2t}$, not only the fundamental variance increases but also the high-fear investors' wealth-share. To better understand the effects of such *pure variance shocks*, in Figure B1 we plot the equilibrium quantities by varying the wealth-share along with the fundamental variance.[24]

Figure B1, Panel A, illustrates that the stock price is more sensitive to the pure variance shocks than to the changes in the fundamental variance. By increasing the wealth share of the high-fear investors w_t and the fundamental variance V_t simultaneously, a positive pure variance shock leads to an economy in which high-fear investors are relatively more dominant in more volatile times. Since these investors are more sensitive to risk, they are willing to hold stock only if its price is lower, leading to these amplified effects. Panel B also shows that pure variance shocks lead to mostly similar behavior for the stock return variance. However, as Panel C illustrates, a sufficiently large pure variance shocks can lead to lower volatility of variance. This behavior arises because now a positive shock is associated with an increase in the relative dominance of the high-fear investors, resulting in a reduction in the "effective disagreement," since there is less room for wealth transfers in the future. This reduced wealth transfer risk leads to less volatile stock return variance.

We also see that, pure variance shocks lead to non-linear but mostly similar economic behaviors for the variance risk premium and variance swap rate as depicted in our earlier Figures. That said, as

[24]We measure the effects of pure variance shocks as follows. Starting from steady state, $V_t=\overline{V}$ and $\mathrm{w}_t=1/2$, and using the dynamics (2) and (5), we obtain the changes in the state variables as $dV_t=\sigma\sqrt{\overline{V}}d\omega_{2t}$ and $d\mathrm{w}_t=(1/4)\,\delta\,(1/\sigma)\sqrt{\overline{V}}d\omega_{2t}$. This implies a relation between these two quantities as $\mathrm{w}_t=(1/2)+(1/4)\,\delta\left(1/\sigma^2\right)\left(V_t-\overline{V}\right)$, after integrating and lagging by dt. By substituting this relation into the economic quantities, and plotting them by varying V_t, we obtain the effects of pure variance shocks that we illustrate in Figure B1.



Panel A. Effect on S_t Panel B. Effect on υ_t Panel C. Effect on π_{St}

Panel D. Effect on $\pi_{\upsilon t}$ Panel E. Effect on y_t Panel F. Effect on θ_{ht}

Figure B1. Effects of pure variance shocks. These panels plot the effects of pure variance shocks $d\omega_{2t}$ on the equilibrium stock price S_t (Panel A), stock return variance υ_t (Panel B), volatility of variance $\sigma_{\upsilon t}$ (Panel C), variance risk premium $\pi_{\upsilon t}$ (Panel D), variance swap rate y_t (Panel E), and variance swap long holdings θ_{ht} (Panel F) with dotted yellow lines. The solid blue lines represent the effects of fundamental variance when $\mathrm{w}_t = 0.5$. The vertical lines represents the long-run mean of the fundamental variance $\overline{V}$. The parameter values follow from Table C1 of Appendix C.

Panel F shows, positive pure variance shocks can lead to an increase in the variance swap holdings. This result occurs because such a positive shock is associated with an increase in the wealth share of the high-fear investors and decrease in the riskiness of the swap contract, $\sigma_{\upsilon t}$, as depicted in Panel C, leading to more swap holdings for risk-averse investors in equilibrium.

Table C1. Parameter values. This table reports the parameter values used in our numerical illustrations.

Parameter	Symbol	Value
Interest rate	r	0.01
Fundamental mean growth rate	μ	0.02
Fundamental variance long-run mean	$\overline{V}$	0.012
Fundamental variance mean reversion speed	κ	0.35
Fundamental variance volatility	σ	0.06
Volatility disagreement	δ	0.40
Payoff time	T	50
Current time	t	25

Appendix C: Parameter Values

In this Appendix, we discuss the parameter values employed in our Figures. The interest rate and the fundamental mean growth rate do not affect any of our key quantities, apart from the stock price level, so we simply them to $r = 1\%$ and to $\mu = 2\%$, consistently with data and other works in the literature. We set the long-run mean of the fundamental variance to $\overline{V} = 1.2\%$, implying the average volatility of $\sqrt{\overline{V}} = 11\%$, which is consistent with the time-series average of the aggregate dividend volatility as reported in Beeler and Campbell (2012). We set the fundamental variance mean reversion speed, which is also the mean reversion speed of the realized variance of our benchmark model in the limit, $T \to \infty$, to $\kappa = 0.35$, since it roughly corresponds to the reported first-order auto-correlation of 0.70 for realized variance in Bollerslev, Tauchen, and Zhou (2009). We set the fundamental variance volatility parameter to the corresponding volatility value in the stochastic volatility model estimation of Chernov and Ghysels (2000, Table 2), $\sigma = 6\%$. We set the disagreement parameter $\delta = 0.40$ so that the stock return volatility at the steady state of our main model is comparable to the average volatility of the S&P 500, 15.5%. Finally, we set the time to maturity $T - t$ to 25 years so that model horizon is comparable to the duration of the aggregate stock market.[25] To that end, we set $T = 50$ years and take the model evaluation time to be $t = 25$ years. This procedure yields the parameter values in Table C1.

[25]Most researchers find the stock market duration to be around 20-30 years using the classic dividend growth model, which implies the stock duration as the average price-dividend ratio. See, for example, the recent work of Van Binsbergen (2020) who finds that the aggregate stock market duration to lie somewhere between 20 and 50 years.

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