Multiple Rational Bubbles*

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Abstract

We explore multiple rational bubbles with varied levels of confidence. Post-bursting, aggregate bubble size contracts, while surviving bubbles gain value. Greater confidence in a bubble increases its size, expanding aggregate bubble size but deflating others. Fragile bubble assets, collectively, might rival a stable one due to diversification benefits. Our stationary model provides practical insight into bubble behavior by offering a theoretical upper bound on aggregate bubble size, applicable to models handling price fluctuations, nonstationarity, and potential bubble formations in financial ecosystems. Our model has implications for crypto ETFs, CBDC issuance, and cryptocurrency regulations, illuminating diverse bubble dynamics.

JEL Classifications: D53, E44, E51, E58, G01

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1 Introduction

As famously stated in Minsky (1986), "everyone can create money; the problem is to get it accepted." With the recent rise of cryptocurrencies, this statement has become even more relevant. Perhaps a more pressing question today is whether we need so many money-like assets in the sense of classical bubbles (e.g., Tirole (1985)) and if so, what determines their market sizes. A (pure) bubble naturally arises in response to market imperfections, such as

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the absence of liquid vehicles for transferring wealth across different periods. However, it is unclear whether having more competing bubbles in the economy would provide greater benefits. Nevertheless, we observe a diverse range of bubbles, including classical asset bubbles such as gold and silver, as well as the recent explosion of cryptocurrencies. Is it desirable to have them in such abundance? How does competition among asset bubbles affect their pricing and survival? In this paper, we theoretically investigate these questions using a classical bubble framework with multiple heterogeneous bubbles.

To address these questions, we study an overlapping generations (OLG) exchange economy. The market in this economy is imperfect in the sense that it lacks investment opportunities for the agents to transfer wealth over time. Agents live for two periods and receive endowments when they are young and old. There exist multiple assets (henceforth referred to as "bubbles") that do not have any fundamental value. Agents prefer to transfer wealth to their older selves to increase consumption utility, leading to a demand for these bubbles.

As a bubble is intrinsically valueless, it may burst at any time if agents lose confidence in it (Weil (1987)). No one will want to hold a bubble once they expect it will not be accepted in the future. To this end, we model the bursting of a bubble as a random event with a certain probability. Because those bursting events are not synchronized, in our multi-bubble economy, a rational agent would naturally invest in a portfolio of bubbles, exploiting the benefits of diversification. This diversification motive, together with the agents' confidence, determines the current sizes of bubbles. Therefore, a basket of seemingly unstable bubble assets, such as an ETF of cryptocurrencies, may transfer liquidity more efficiently than a single, more stable bubble asset.

In equilibrium, confidence—the probability that a bubble will not burst—naturally emerges as the fundamental factor determining a bubble's size. In any multi-bubble economy, a bubble with a lower bursting probability tends to be larger. As agents gain confidence in one bubble, it becomes more attractive, leading to a decrease in the size of its competitors. On the other hand, with many bubbles in the market, we may reach a relatively accurate prediction of the size of the bubble market without knowing the specific confidence level of each bubble. Specifically, the size of the bubble market converges to that of savings in the presence of money, where agents can save without incurring significant risk. The law of large numbers and the fact that bubbles have no fundamental value dictate that a fully diversified portfolio of many small bubbles provides near-perfect insurance against the risk of bursting. Consequently, when bubbles burst, the growth of surviving bubbles offsets a significant portion of welfare loss. Therefore, using the lost market value of burst bubbles as a welfare indicator in a multi-bubble economy might significantly overstate the welfare loss.

In the analysis above, we focus on the stationary equilibrium where the size of each bubble depends solely on the surviving bubbles in the current period. This approach allows for tractable equilibrium characterizations and intuitive comparative statics. However, the nature of a stationary model means that bubble price predictions are stable, which can be at odds with the volatile prices observed in reality. On the other hand, in models without a stationarity requirement over the equilibrium, such as those with forward-looking prices, infinitely many self-fulfilling equilibria exist, and the model may fail to provide a strong prediction.

To address this dilemma, we extend our discussion to encompass history-dependent bubbles and stochastic shifts in bubble sizes in equilibrium. With infinitely many equilibria, we focus on characterizing the common features. In any equilibrium scenario, the collective size of any set of bubbles is (tightly) upper-bounded by the collective size of the bubble economy in the stationary equilibrium when only these bubbles survive. This result provides a tight upper bound (or a lower bound on confidence) for real-life bubbles with complex, non-stationary trajectories. Moreover, this principle does not negate the possibility of generating a positive net return in the bubble market. However, it suggests that as the bubble market expands, achieving high returns becomes progressively challenging, even when uncertainties exist over the true equilibrium bubble trajectories.

We also explore scenarios involving the creation of bubbles. The key proposition highlights that within any equilibrium, the combined size of existing bubbles cannot surpass the aggregate size of those bubbles in the stationary equilibrium. This holds universally, whether considering all bubbles together or individual bubbles within the equilibrium. Crucially, this proposition does not restrict the possibility of larger aggregate bubble sizes in subsequent periods due to new bubble additions but sets an upper limit based on the stationary equilibrium with specific surviving bubbles.

Drawing implications from our results, we can discuss the potential efficiency of decentralized cryptocurrencies. It is theoretically possible that maximum efficiency could be achieved in a scenario with a single centralized digital currency such as a CBDC when carefully managed by the central bank or government.[1] However, practical obstacles, including technical feasibility, security measures, and long-term stability, cast doubt on this possibility. Our results suggest that a more stable single centralized cryptocurrency may not necessarily dominate a basket of many cryptocurrencies unless the single centralized digital currency is entirely free from risk. A diversified portfolio made up of potentially risky cryptocurrencies may even be more stable than a relatively stable centralized digital

[1] Modeling CBDC as a bubble with usage value (rather than a pure bubble in our baseline model) does not change the main results.

currency.

Our paper is related to a vast literature on rational bubbles, following classical works such as Samuelson (1958) and Tirole (1985). Tirole (1985) shows that bubbles are sustainable in an OLG economy, based on Diamond (1965), in the presence of dynamic inefficiency, where agents cannot consume more in the future, even when they desire increased future consumption in the absence of a bubble.[2] While most papers in this strand study deterministic bubbles, Weil (1987) notably extends Tirole (1985)'s model by introducing a stochastic bubble that bursts with a certain probability. In that model, as the probability of bursting increases, sustaining the bubble becomes more challenging.[3] In real life, a multitude of bubbles can emerge, and there is also a growing literature studying multiple bubbles. Martin and Ventura (2012) study production economy models featuring multiple bubbles and sentiment fluctuations, demonstrating the emergence of macroeconomic boom-and-bust cycles, tied to the growth and busts of bubbles.[4] Extending this line of research, Martin and Ventura (2016) further explores the expansion of credit, examining the role of bubbles as collateral in the process.

Unlike existing literature, where the bursting of bubbles is often driven by a common factor or sentiment, our model incorporates idiosyncratic forces that influence bubble bursts. We extend Weil (1987)'s framework by introducing multiple bubbles with heterogeneous bursting events. Our model offers new insights into the literature, particularly regarding the diversification effect of investing in multiple bubbles and its implications for the size of individual and aggregate bubbles in the economy.

2 Setup

Consider an infinite-horizon, discrete-time exchange economy with overlapping generations of two-period-living agents. There is a single perishable consumption good. In each pe-

[2]Following this line of argument, whether agents are short-lived or long-lived, there is a vast literature contributing to the understanding the emergence and applications of rational bubbles under imperfect markets such as borrowing or credit constraints (e.g., Kocherlakota (1992), Grossman and Yanagawa (1993), Santos and Woodford (1997), Caballero and Krishnamurthy (2006), Kocherlakota (2008), Miao and Wang (2012), Galí (2014), Hirano and Yanagawa (2016)). For instance, Farhi and Tirole (2012) demonstrate the emergence of expansionary bubbles in a production economy in the presence of financial frictions, specifically limited pledgeability.

[3]Following the seminal papers, there is a growing literature that employs stochastic bubbles in various economic contexts. Miao and Wang (2018) study credit cycles with bubbles in a production economy featuring long-lived agents. Dong, Miao, and Wang (2020) demonstrate that, within a dynamic new Keynesian framework, bubbles can provide liquidity and consequently command a liquidity premium. Asriyan, Fornaro, Martin, and Ventura (2021) study optimal monetary policy in the presence of credit cycles driven by stochastic bubbles.

[4]For further details on additional applications of this type of model, refer to Martin and Ventura (2018).

riod, one unit mass of young agents arrives. They are homogeneous and endowed with a consumption good e_y when young and e_o when old. Each agent who arrives in period t maximizes the expected utility of a bundle of consumption $(c_{y,t}, c_{o,t})$ when young and old:

$$\mathrm{E}_t[u(c_{y,t}) + \beta u(c_{o,t})], \tag{1}$$

where $\beta < 1$ is the discount factor. The function $u(\cdot)$ is differentiable, increasing, strictly concave, and satisfies the Inada condition $u'(0) = \infty$. We further assume that the agent's elasticity of intertemporal substitution (EIS) is greater than or equal to one, that is, $xu'(x)$ is increasing. With this assumption, the agent is willing to decrease his consumption and increase his investment when expecting a higher return since the substitution effect over the consumption dominates the income effect.[5] Our analysis is general in the sense that we do not impose further assumptions on the utility function. Due to the concavity of the utility function, agents have incentives to smooth their consumption. Throughout the paper, we assume that e_y is sufficiently large to incentivize young agents to transfer the consumption good from the first period to the second period.

The economy begins with $n \in \mathbb{N}$ tradable assets that are intrinsically useless, meaning they do not offer any payoff other than capital gains.[6] Each asset has a unit supply. These assets are "bubbles," as they possess no intrinsic value but could have positive prices, with archetypal examples being cryptocurrencies.[7] For ease of notation, this collection of bubble assets is referred to as the "bubble sector."

Following Blanchard (1979), Blanchard and Watson (1982), and Weil (1987), we consider stochastic bubbles with random prices. Similar to Weil (1987), where a single bubble asset is featured, we assume that each bubble asset indexed by i (henceforth, the "i-th bubble") may lose its value next period with a constant probability $1 - q_i$ where q_i reflects agents' confidence in the asset. The usual interpretation is that agents' beliefs depend on sunspots or sentiment processes, which are extrinsic to the economy (e.g., Cass and Shell (1983), Weil (1987), Asriyan, Fuchs, and Green (2019)). For tractability, we further assume that the collapse of bubbles is independent across time and assets.[8]

[5]This assumption is adopted by papers in heterogeneous agent models, for instance, (Achdou, Han, Lasry, Lions, and Moll 2022) and (Light 2020), to establish the uniqueness of the equilibrium.

[6]For simplicity, we assume that there are no other means to transfer liquidity aside from bubbles, such as money. Bubbles still arise even in the presence of money as long as the interest rate is less than the growth rate of the economy (Wallace (1980)). Therefore, our results would remain qualitatively the same under such conditions.

[7]Cryptocurrencies may have inherent service values, but our focus is on the case of pure bubbles, where assets are solely used for storing liquidity.

[8]This assumption is not as restrictive as it appears to be. See Section 5 for a detailed discussion.

The model's timeline is as follows: The economy starts with n bubbles owned by the old generation. At the beginning of each period, the i-th bubble either continues to exist with a probability of q_i or collapses with a probability of $1 - q_i$. The old generation then sells all surviving bubbles to the young generation. At the end of the period, the old and the young generation consume the perishable consumption good. We refer to an economy as a n-bubble economy if there are n bubbles surviving at the trading stage.

Throughout this paper, we focus on the stationary equilibrium in which the size of a bubble remains constant across different periods if the set of surviving bubbles remains the same. Let $I \subset \{1, \ldots, n\}$ represent the index set of surviving bubbles in the current period. We denote by b_I^i the price of asset $i \in I$ when the index set of surviving bubbles is I. Since each asset has a unit supply, b_I^i also represents the size of bubble i. Therefore, we will use the terms "price" and "size" interchangeably.

The budget constraints of young agents are given by

$$c_y = e_y - \sum_{j \in I} b_I^i x_I^i; \tag{2}$$

$$c_o = e_o + \sum_{j \in I'} b_{I'}^i x_I^i; \tag{3}$$

$$c_y, c_o \geq 0, \tag{4}$$

where x_I^i is the young agent's holding of asset i. The index set I and I' represent the surviving bubbles in the current period and the subsequent period, respectively. Since we focus on the symmetric case, the market clearing conditions are

$$x_I^i = 1 \text{ for all } I \text{ and } i \in I. \tag{5}$$

For ease of notation, we denote by E the expectation under the information set of the young generation instead of including the information set in the form of conditional expectation. Then, the first-order condition for an interior maximum is

$$b_I^i u'(c_y) - \beta \mathrm{E}\left[b_{I'}^i u'(c_o)\right] = 0 \text{ for each } i \in I. \tag{6}$$

Without loss of generality, $b_{I'}^i = 0$ if $i \notin I'$. The surviving probability of a bubble is given by

$$P(i \in I' \mid i \in I) = q_i. \tag{7}$$

and

$$\text{for all } i \in I,\ i \in I' \text{ are mutually independent events.} \tag{8}$$

Stationary equilibrium is defined as follows:

Definition 1. *A stationary equilibrium is given by* $\{b_I^i\}_{I \subset \{1,\ldots,n\}, i \in I}$ *satisfying* $b_I^i > 0$ *and Eqs.* (2)-(8).[9]

Remark 1. *Two aspects of this definition are worth emphasizing. First, the equilibrium specifies bubble sizes in all possible collapsing scenarios. Second, in this definition, we require* $b_I^i > 0$ *to rule out the trivial situations where bubbles exist but have a value of zero.*

3 Two-Bubble Economy

To elucidate our primary findings and provide a clearer intuition, we begin by analyzing a simplified model featuring two bubbles. In contrast to standard bubble models, such as Tirole (1985) and Weil (1987), which feature a single bubble, our model can address the following questions: (i) Does an equilibrium exist where all bubbles are traded at positive prices? If such an equilibrium exists, is it unique? (ii) Is the bubble sector larger compared to a single-bubble economy? (iii) Are the bubble sizes reflecting their "fundamentals"? In other words, is the bubble with a lower collapse probability always larger in any economy? (iv) How does the size of an individual bubble and the bubble sector change when the agents' confidence over one bubble changes?

The stationary equilibrium in this case is a quadruple $\{b_1^1, b_2^2, b_{\{1,2\}}^1, b_{\{1,2\}}^2\}$, where b_1^1 represents the size (price) of the first bubble when the second bubble has collapsed, and $b_{\{1,2\}}^1$ represents the size of the first bubble when both bubbles have positive values, and so on. We denote by $B_{\{1,2\}}$, the size of the bubble sector in the two-bubble economy as the sum of the sizes of two surviving bubbles:

$$B_{\{1,2\}} = b_{\{1,2\}}^1 + b_{\{1,2\}}^2. \tag{9}$$

[9] Note that the demand for bubble assets $\{x_I^i\}$ and consumption choices $\{c_y, c_o\}$ are determined by the prices of bubble assets according to Eqs. (2)-(5). Therefore, for the sake of brevity, we omit them in the definition of equilibrium.

The equilibrium conditions are

$$b_1^1 u'(e_y - b_1^1) = \beta q_1 b_1^1 u'(e_o + b_1^1); \tag{10}$$
$$b_2^2 u'(e_y - b_2^2) = \beta q_2 b_2^2 u'(e_o + b_2^2); \tag{11}$$
$$b_{\{1,2\}}^1 u'(e_y - B_{\{1,2\}}) = \beta q_1 q_2 b_{\{1,2\}}^1 u'(e_o + B_{\{1,2\}}) + \beta q_1 (1 - q_2) b_1^1 u'(e_o + b_1^1); \tag{12}$$
$$b_{\{1,2\}}^2 u'(e_y - B_{\{1,2\}}) = \beta q_1 q_2 b_{\{1,2\}}^2 u'(e_o + B_{\{1,2\}}) + \beta q_2 (1 - q_1) b_2^2 u'(e_o + b_2^2). \tag{13}$$

Eqs. (10)-(11) are the first-order conditions for the young generation with only one bubble in the economy. Eqs. (12)-(13) are the first-order conditions with two bubbles in the economy. In the presence of two bubbles, agents consider the possibility that the price of a surviving bubble may change if the other one collapses in the subsequent period. This consideration influences agents' investment decisions, altering the size of the existing bubbles.

Our first result shows that a unique stationary equilibrium with positive bubble sizes exists as long as the bubbles' survival probabilities are above a threshold, denoted by $\underline{q}$. This can be interpreted as the minimum level of "confidence" in the bubble asset required for trading. Interestingly, this threshold coincides with the one found in an economy with a single stochastic bubble, as in Weil (1987). In other words, if a bubble can exist in a single-bubble economy, it can also exist regardless of the presence of other bubbles in an economy with multiple stochastic bubbles.

Proposition 1. *There exists a unique stationary equilibrium if and only if* $min(q_1, q_2) > \underline{q}$ *where*

$$\underline{q} = \frac{u'(e_y)}{\beta u'(e_o)}. \tag{14}$$

Proof. In the case of one surviving bubble (say, the first bubble), due to the continuity and the concavity of the utility function, there exists a unique $b_1^1 > 0$ that solves Eq. (10) if and only if $u'(e_y) < q_1 \beta u'(e_o)$, or equivalently, $q_1 > \underline{q}$ where the cutoff $\underline{q}$ is given by Eq. (14). Likewise for the second bubble, there exists a unique $b_2^2 > 0$ that solves Eq. (11) if and only if $q_2 > \underline{q}$.

In the case of two surviving bubbles (achievable when both q_1 and q_2 are greater than $\underline{q}$, as analyzed previously), we can demonstrate the existence of unique positive solutions for $b_{\{1,2\}}^1$ and $b_{\{1,2\}}^2$ given b_1^1 and b_2^2. To see this, Eqs. (12) and (13) together with Eq. (9) imply

$$\begin{aligned} B_{\{1,2\}} &\left[u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}})\right] \\ &= \beta q_1 (1 - q_2) b_1^1 u'(e_o + b_1^1) + \beta q_2 (1 - q_1) b_2^2 u'(e_o + b_2^2). \end{aligned} \tag{15}$$

The right-hand side of Eq. (15) is strictly positive since $b_1^1, b_2^2 > 0$. The left-hand side is zero when $B_{\{1,2\}}$ is zero, and diverges to infinity as $B_{\{1,2\}}$ approaches e_y. Furthermore, it is straightforward to show that the left-hand side increases in $B_{\{1,2\}}$ whenever it is nonnegative, because $u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}})$ increases in $B_{\{1,2\}}$. Therefore, the continuity of the left-hand side in $B_{\{1,2\}}$ implies that there exists a unique $B_{\{1,2\}} > 0$ satisfying Eq. (15). Given this $B_{\{1,2\}}$, there exists a unique $b^1_{\{1,2\}}, b^2_{\{1,2\}} > 0$ that satisfies Eq. (12) and Eq. (13), respectively. □

Following Proposition 1, we focus on the economically meaningful case in which both bubbles are stable enough to exist, i.e., $q_i > \underline{q}$ for both $i = 1, 2$. We then examine the size of the bubble sector of the two-bubble economy. Since each of the two bubbles collapses independently, the bubble sector becomes less risky due to the diversification effect, encouraging agents to invest more and consequently leading to a larger size compared to a single-bubble economy.

Proposition 2. *A two-bubble economy has a larger bubble sector and smaller individual bubbles than its one-bubble counterpart, i.e.,* $B_{\{1,2\}} > max(b_1^1, b_2^2)$*,* $b^1_{\{1,2\}} < b_1^1$ *and* $b^2_{\{1,2\}} < b_2^2$*.*

Proof. From Eq. (15), dropping the first term on the right-hand side and dividing both sides by $B_{\{1,2\}} > 0$, yields:

$$u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}}) - \beta q_2 (1 - q_1) \frac{b_2^2}{B_{\{1,2\}}} u'(e_o + b_2^2) > 0.$$

The left-hand side of the inequality is increasing in $B_{\{1,2\}}$. Furthermore, Eq. (11) implies that the left-hand side equals zero when $B_{\{1,2\}} = b_2^2$. Thus, it follows that $B_{\{1,2\}} > b_2^2$. By a similar argument, we can also show that $B_{\{1,2\}} > b_1^1$.

By Eq. (10) and Eq. (12), we have

$$b^1_{\{1,2\}}[u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}})] = b_1^1[u'(e_y - b_1^1) - \beta q_1 q_2 u'(e_o + b_1^1)].$$

Since $B_{\{1,2\}} > b_1^1$, the above implies that $b_1^1 > b^1_{\{1,2\}}$. Similarly, $b_2^2 > b^2_{\{1,2\}}$. □

Next, we analyze the relationship between bubble size and its probability of bursting. A bubble derives its value from its ability to transfer wealth across different time periods. The effectiveness of this intertemporal wealth transfer depends on its probability of bursting. A bubble with a lower probability of bursting has a stronger "fundamental", making it more attractive to agents and consequently resulting in a higher price. We demonstrate that this is generally true given our assumptions over the utility function.

Proposition 3. *A bubble with a lower probability of bursting is larger (equivalently, more expensive) than one with a higher probability of bursting, i.e., if* $q_2 \geq q_1$*, then* $b^2_{\{1,2\}} \geq b^1_{\{1,2\}}$*.*

Proof. From Eqs. (12) and (13), we have

$$b^1_{\{1,2\}} \left[u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}}) \right] = \beta q_1 (1 - q_2) b^1_1 u'(e_o + b^1_1);$$
$$b^2_{\{1,2\}} \left[u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}}) \right] = \beta q_2 (1 - q_1) b^2_2 u'(e_o + b^2_2),$$

which implies

$$\frac{b^1_{\{1,2\}}}{b^2_{\{1,2\}}} = \frac{q_1(1-q_2)}{q_2(1-q_1)} \cdot \frac{b^1_1 u'(e_o + b^1_1)}{b^2_2 u'(e_o + b^2_2)} = \frac{1-q_2}{1-q_1} \cdot \frac{b^1_1 u'(e_y - b^1_1)}{b^2_2 u'(e_y - b^2_2)}, \tag{16}$$

where the second equality is due to Eqs. (10)-(11). Then, since $1 - q_2 \leq 1 - q_1$ and $b^1_1 \leq b^2_2$, it must be $b^2_{\{1,2\}} \geq b^1_{\{1,2\}}$. □

We further analyze the comparative statics of the bubble sector and the size of each bubble when one bubble's bursting probability changes. Two bubbles substitute each other since they both help the agents to transfer their wealth to the future. If one bubble becomes less likely to collapse, it should become more attractive and its competitor should become less attractive. Moreover, the bubble sector as a whole should become more attractive to the agents since it becomes more stable. We show that these intuitive predictions are true in general.

Proposition 4. *If* q_i *increases,* $b^i_{\{1,2\}}$ *increases, while* $b^{-i}_{\{1,2\}}$ *(where* $-i$ *denotes the index of the other bubble than the i-th one) decreases. The aggregate size* $B_{\{1,2\}}$ *increases whenever* q_i *increases for any* $i = 1, 2$*.*

Proof. Without loss of generality, we focus on the case where q_1 increases. Consider an alternative economy where q_1 is increased to $\hat{q}$ while maintaining all other model primitives unchanged. We differentiate all the quantities related to the new equilibrium with the "hat" symbol. It is straightforward to show that $\hat{b}^1_1 > b^1_1$ using Eq. (10). By Eq. (15), we derive

$$\begin{aligned}
u'(e_y - \hat{B}_{\{1,2\}}) =& \beta\hat{q}q_2 u'(e_o + \hat{B}_{\{1,2\}}) + \beta\hat{q}(1-q_2)\frac{\hat{b}_1^1}{\hat{B}_{\{1,2\}}}u'(e_o + \hat{b}_1^1) \\
&+ \beta q_2(1-\hat{q})\frac{b_2^2}{\hat{B}_{\{1,2\}}}u'(e_o + b_2^2) \\
>& \beta q_1 q_2 u'(e_o + \hat{B}_{\{1,2\}}) + \beta q_1(1-q_2)\frac{\hat{b}_1^1}{\hat{B}_{\{1,2\}}}u'(e_o + \hat{b}_1^1) \\
&+ \beta q_2(1-q_1)\frac{b_2^2}{\hat{B}_{\{1,2\}}}u'(e_o + b_2^2) \\
>& \beta q_1 q_2 u'(e_o + \hat{B}_{\{1,2\}}) + \beta q_1(1-q_2)\frac{b_1^1}{\hat{B}_{\{1,2\}}}u'(e_o + b_1^1) \\
&+ \beta q_2(1-q_1)\frac{b_2^2}{\hat{B}_{\{1,2\}}}u'(e_o + b_2^2).
\end{aligned}$$

The first inequality is a result of $\hat{q} > q_1$ and $\hat{B}_{\{1,2\}}u'(e_o + \hat{B}_{\{1,2\}}) > b_2^2 u'(e_o + b_2^2)$ (due to Proposition 2 and the assumption that the EIS is greater than or equal to one). The second inequality is due to $\hat{b}_1^1 > b_1^1$. If we consider the left-hand side minus the right-hand side as a function of $\hat{B}_{\{1,2\}}$, it is increasing in $\hat{B}_{\{1,2\}}$ and is equal to 0 when $\hat{B}_{\{1,2\}} = B_{\{1,2\}}$. Thus it must be $\hat{B}_{\{1,2\}} > B_{\{1,2\}}$.

Moreover, we have

$$\begin{aligned}
\hat{b}^2_{\{1,2\}}&[u'(e_y - \hat{B}_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + \hat{B}_{\{1,2\}})] \\
&= \beta q_2(1-\hat{q})b_2^2 u'(e_o + b_2^2) + \beta q_2(\hat{q} - q_1)\hat{b}^2_{\{1,2\}}u'(e_o + \hat{B}_{\{1,2\}}) \\
&< \beta q_2(1-q_1)b_2^2 u'(e_o + b_2^2) \\
&= b^2_{\{1,2\}}[u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}})].
\end{aligned}$$

Since $u'(e_y - \hat{B}_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + \hat{B}_{\{1,2\}}) > u'(e_y - B_{\{1,2\}}) - \beta q_1 q_2 u'(e_o + B_{\{1,2\}})$, it must be $b^2_{\{1,2\}} > \hat{b}^2_{\{1,2\}}$, which in turn implies $\hat{b}^1_{\{1,2\}} > b^1_{\{1,2\}}$. □

4 General Bubble Economy

In this section, we generalize our model to a multi-bubble economy. We show that all results on the two-bubble economy derived in the previous section can be carried over to a multi-bubble economy. Moreover, we derive a limiting result showing that when the number of bubbles becomes large, each individual bubble becomes infinitesimal; regardless of the bursting probability of each bubble, the bubble sector converges to the same limiting size. We also extend the equilibrium definition to allow the equilibrium bubble size to be random and time-dependent. We demonstrate that the stationary equilibrium serves as a benchmark, as its bubble sector's size is the largest among all equilibria. These findings enable us to understand the welfare implications of bubble collapsing and the issuance of CBDC. All proofs in this section are relegated to the appendix.

4.1 Existence and Uniqueness of the Equilibrium

We first analyze the existence and the uniqueness of the stationary equilibrium. Similar to the two-bubble economy, the existence of a specific bubble depends only on its own bursting probability. Moreover, the existence does not change in the number of bubbles in the economy.

Proposition 1′. *There exists a unique stationary equilibrium if and only if* $\min_i\{q_i\} > \underline{q}$.

This result depends on the assumption that bubbles' collapses are independent events. Under this assumption, it is always worthwhile to invest a positive amount in a bubble since in some states of the world that bubble will be the last bubble left and the future generation will be willing to invest in that bubble. This leads to the non-zero size of the bubbles. From now on, we focus on the interesting situation by assuming $\min_i\{q_i\} > \underline{q}$.

4.2 Equilibrium Size of Bubbles

In this section, we extend the results from the two-bubble economy to the general model. Specifically, we analyze the following questions: (1) When there are more bubbles in the economy, how do the individual and aggregate bubble sizes change? (2) Is a bubble with a lower bursting probability always larger? (3) How will the bubble sizes change when the agents become more confident in a specific bubble?

Proposition 2′. *The aggregate bubble size becomes larger with more bubbles while each individual bubble becomes smaller, i.e., for any* $j \notin I$, $B_{I\bigcup\{j\}} > B_I$ *and* $b_I^i > b_{I\bigcup\{j\}}^i$ *for any* $i \in I$.

The equilibrium size of bubbles is determined by the riskiness of the investment, affected by two factors: the stability of the bubbles and the diversification effect. The presence of multiple bubbles offers opportunities for diversification, making them more valuable in aggregation because the risks associated with each bubble are independent.

Specifically, the portfolio of bubbles provides insurance against the collapse of individual bubbles. Enhanced diversification improves the investment opportunity set, increasing

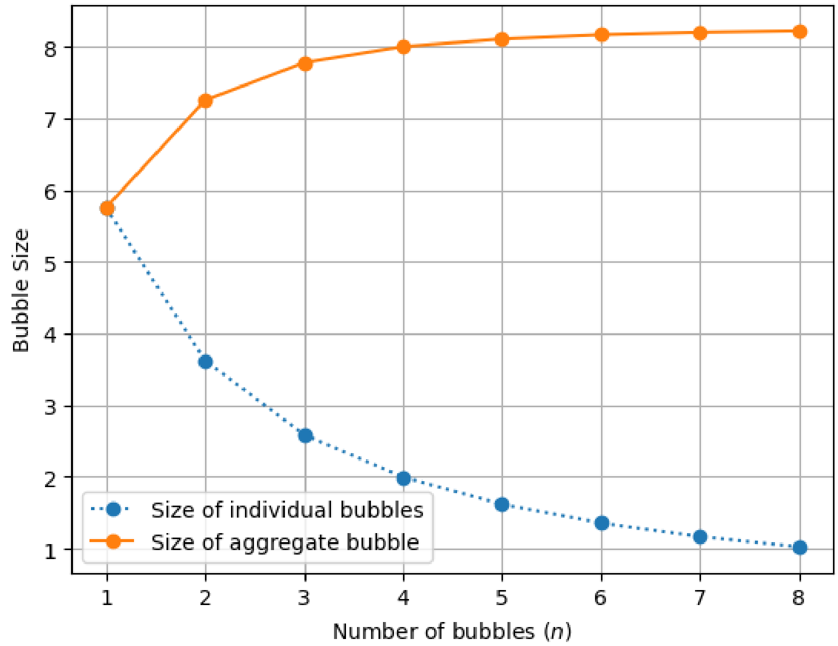

Figure 1: Individual and Aggregate Bubble Size with Homogeneous Bubbles (Parametric assumptions: $u(c) = c^{1-\gamma}$, $e_y = 20, e_o = 3, \gamma = 0.5, \beta = 0.98, q_i = 0.8$ for all $i \in \{1, 2, ...\}$)

agents' willingness to invest more money in bubbles. Consequently, the size of the bubble sector increases as the number of bubbles increases. On the other hand, the size of each individual bubble becomes smaller as bubbles substitute for each other. Moreover, since the young generation chooses between consumption and investment in this model, the young generation consumes less in an economy with more bubbles. Figure 1 illustrate this.

We also conduct comparative statics while fixing the number of bubbles. The emphasis is on the relative bubble size and how the sector changes when agents become more (or less) confident about a bubble. We establish that a bubble's bursting probability can indeed be considered as its "fundamental": In the equilibrium of any economy, a bubble with a lower bursting probability is always larger.

Proposition 3′. *A bubble with a lower bursting probability is larger than one with a higher bursting probability, i.e., if $i, j \in I$ and $q_i > q_j$, then $b_I^i > b_I^j$.*

Next, we examine the impact of changes in agents' confidence on individual bubble sizes and the overall bubble sector. Consistent with the two-bubble economy, when agents gain more confidence in a particular bubble, it expands, causing its competitors to shrink. The growth of the confident bubble dominates, resulting in an overall increase in the size of the bubble sector.

Proposition 4′. *Increased confidence in a specific bubble expands that bubble while shrinking its competitors, and expanding the bubble sector, i.e., if* $q_j > q_h$ *and* $j, h \notin I$*, then* $b^j_{I \bigcup \{j\}} > b^h_{I \bigcup \{h\}}$, $b^i_{I \bigcup \{h\}} > b^i_{I \bigcup \{j\}}$*, and* $B_{I \bigcup \{j\}} > B_{I \bigcup \{h\}}$ *for all* $i \in I$.

Figure 2 illustrates this.

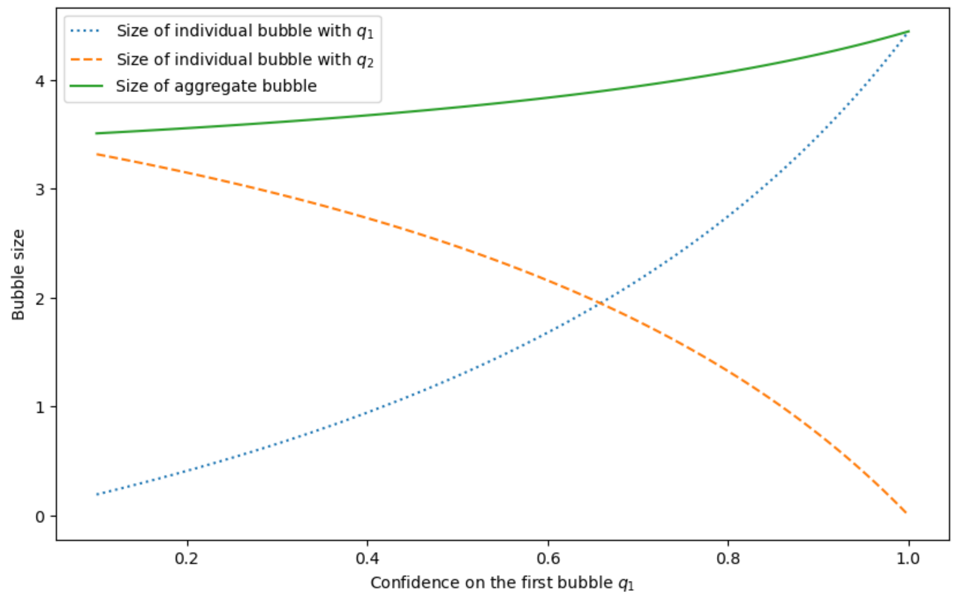

Figure 2: Bubble Sizes across Varying Levels of Confidence (Parametric assumptions: $u(c) = c^{1-\gamma}$, $e_y = 20, e_o = 3, \gamma = 0.5, \beta = 0.98, q_1 = 0.7, q_2 = 0.8$)

The next result examines the impact of an increased number of bubbles on the welfare of both older and younger generations.

Corollary 1. *Both the old and the young generation's expected utility is increasing in the number of bubbles in the economy.*

We conduct an explicit welfare comparison: with an increased number of bubbles in the economy, both the old and the young generations experience improved well-being. While the old generation obviously benefits from a larger bubble sector with more bubbles, it is intuitive, although not immediately evident, that the young generation is consistently better off due to changes in bubble prices with varying numbers of bubbles.

4.3 Limiting Results

In this section, we characterize the bubble sector and individual bubbles in an economy with a large number of bubbles. The analysis in previous sections establishes that the bubble sector expands with more bubbles (Proposition 2′). On the other hand, as the size of the bubble sector cannot exceed the endowment held by the young generation, the size of the bubble sector should approach an upper bound when the number of bubbles is large. Our analysis substantiates and expands this intuition by explicitly deriving the size of the bubble sector when the number of bubbles is large. Surprisingly, the limiting size of the bubble sector is "detail-free", i.e., the bubble sector's size converges to the *same* upper bound under any bursting probability specifications of bubbles. Specifically, in any economy, when the number of bubbles is large, the size of the bubble sector s satisfies

$$u'(e_y - s) = \beta u'(e_o + s). \tag{17}$$

The limiting size s has a natural interpretation. Consider a scenario where the agent has a perfect storage technology for the consumption good. Then eq. (17) is the first-order condition for the optimal amount of storage s. In other words, with many bubbles in the economy, the bubble sector essentially becomes "risk-free" regardless of the collapsing probabilities of individual bubbles. Figure 3 illustrates this. We summarize the discussion above into the following proposition.

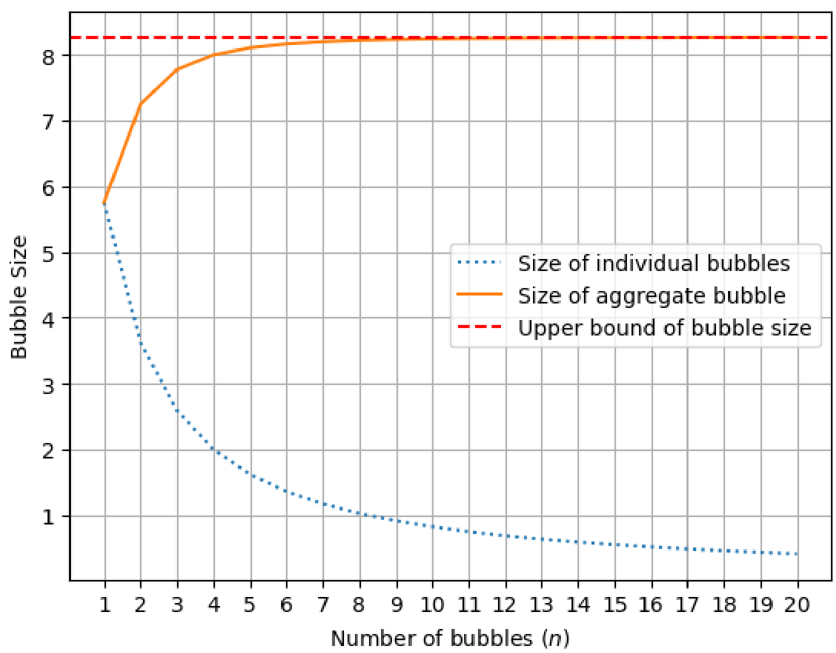

Figure 3: The Upper Bound of Bubble Size (Parametric assumptions: $u(c) = c^{1-\gamma}$, $e_y = 20, e_o = 3, \gamma = 0.5, \beta = 0.98, q_i = 0.8$ for all $i \in \{1, 2, ...\}$)

Proposition 5. *As $|I|$ diverges to ∞, the aggregate bubble size B_I converges to a positive* *constant s, which is the unique solution of Eq. (17). Each individual bubble size* b_I^i *converges to zero.*

This limiting result has welfare implications regarding bubbles' bursting in an economy with many bubbles. That is, in any economy with many bubbles, the welfare loss of bubbles' bursting might be small even when the lost market value of collapsing bubbles is large. In particular, when there are a substantial number of bubbles, there is a high likelihood that the bubble sector's size remains relatively stable compared to the previous period. This stability is not due to collapsed bubbles having infinitesimal market values. Instead, it occurs because surviving bubbles appreciate, counterbalancing a significant portion of the agents' losses. Utilizing the market value of collapsing bubbles as a welfare measure fails to capture this appreciation effect.

The welfare analysis above is "bubble specific" in the sense that if each asset can generate positive cash flow and the number of assets decreases in the economy, the agents will suffer substantial welfare loss. This difference can also be illustrated from the individual bubble's perspective. An asset that may generate positive cash flows always has a price bounded away from zero. Instead, in this economy, when the number of bubbles becomes large, each bubble becomes infinitesimal regardless of its bursting probability.

5 Stationary Equilibrium as a Benchmark

In previous sections, we analyze the stationary equilibrium of the baseline model, whose tractability leads to clear comparative statics and simple bubble dynamics. However, tractability comes with the cost of making simplifying assumptions over both the equilibrium concept and the model. Focusing on the stationary equilibrium is not without loss of generality since it fails to capture the transition phase of the economy. From the modeling perspective, the baseline model does not allow for (expected) bubble creation. If the agents expect new bubbles to be created, they may change their investment decisions over the existing bubbles. With these simplifications, our readers might wonder whether this model can provide insights into the more complicated bubble sector in reality. In this section, we provide a positive answer to this concern by demonstrating that the stationary equilibrium of the baseline model is an important benchmark. Specifically, in any non-stationary equilibrium or any stationary equilibrium with bubble creation, the size of the bubble sector is upper-bounded by the corresponding equilibrium bubble sector's size in the baseline model.

A standard approach addressing the concern about simplifying assumptions in a model involves extending the model or solving the model under a less restrictive equilibrium concept. However, this approach does not always lead to tractable models. In the rare cases with tractability, readers might still suspect the results are driven by specific clever setups rather than the robustness of the economic trade-offs. We approach this problem from a different angle by showing that our analysis of the baseline model is still informative even when the equilibrium concept or the model is mis-specified.

We provide a broad definition of the non-stationary equilibrium where the bubble sizes are random variables depending on the entire history of the bubble size dynamics. We show that in any equilibrium, the sum of the sizes of any subset of existing bubbles can only be smaller than the aggregate bubble size in the stationary equilibrium with that subset of bubbles. We also analyze the economy with a general bubble-creation function depending on the set of existing bubbles. We predict that the aggregate bubble size is always larger without creation.

The key insight behind our results is simple: rational expectation strongly disciplines the possible bubble dynamics. An equilibrium bubble-size process needs to satisfy the first-order conditions of the current and all future generations. Since agents' elasticity of inter-temporal substitution is greater than or equal to one, the willingness to invest is increasing in return. This implies the bubble cannot be too large. In a single-bubble economy, if the bubble is large in the current period, the first-order condition implies it needs to be larger in some future states. However, the larger future bubble needs to grow further, and this eventually leads to too large a bubble that cannot be sustained. The upper bound of the bubble size can be obtained when the surviving bubble's size does not change in all future states, which corresponds to the bubble size in the stationary equilibrium.[10] This result extends to the multi-bubble economy with some subtleties. With no restriction over the surviving bubbles' correlations, a bubble may become larger than its stationary size in the equilibrium. Yet the same argument applies to the aggregate bubble size. That is, the aggregate bubble size cannot be too large. The upper bound is provided by the corresponding aggregate bubble size in the stationary equilibrium.

The purposes of this section are twofold. It demonstrates that this stylized model serves as a nice benchmark for the more complicated reality. From a more technical perspective, we illustrate that under mild assumptions, any complicated, non-stationary bubble dynamics can be disciplined by a stationary, deterministic one. For the rest of the section, we provide details of our generalizations and discuss the results.

[10] In the appendix, we provide an example to illustrate that our result will not hold without this assumption.

5.1 Non-stationary Equilibrium

In this section, we generalize the equilibrium concept to allow for (1) random future bubble sizes and (2) non-stationary bubble size dynamics. To this end, we introduce the following new notations. Consider an n-bubble economy ($|I| = n$) in period $t = 0$. Let $b_t^i \geq 0$ represent the realized size of bubble i in period t, and let $b_t = (b_t^1, ..., b_t^n)$ represent an n-dimensional vector comprising all bubble size realizations in period t. If bubble i has collapsed by period t, then $b_t^i = 0$. The vector $H_t = \{b_0, \ldots, b_t\}$ represents the history of bubble size realizations up to period t. Without loss of generality, let $H_{-1} = \emptyset$. Lastly, we define $\tilde{b}_t(H_{t-1}) = (\tilde{b}_t^1, \ldots, \tilde{b}_t^n)$ as an n-dimensional vector comprising non-negative random variables that represent the sizes of bubbles in period t, dependent on the history up to period $t-1$. The random variable $\tilde{b}_t^i(H_{t-1})$ represents the i-th bubble's size in period t given the history up to period $t-1$. We use the "tilde" symbol here to explicitly differentiate random variables from their realizations.

Since the random variable $\tilde{b}_t(H_{t-1})$ describes the dynamics of bubbles, it needs to satisfy several properties. First, if a bubble collapses, its value remains zero forever. That is,

$$\tilde{b}_\tau^i = 0 \text{ if } b_{t-1}^i = 0 \text{ for all } \tau \geq t. \tag{18}$$

Furthermore, if bubble i did not collapse in period $t-1$, its probability of bursting in period t is $1 - q_i$. That is,

$$P(\tilde{b}_t^i > 0 \mid b_{t-1}^i > 0) = q_i \text{ for all } H_{t-1} \text{ with } b_{t-1}^i > 0. \tag{19}$$

We also assume that bubble collapses are independent events. That is,

$$\tilde{b}_t^i > 0 \text{ are mutually independent for all } i. \tag{20}$$

Since the agent's consumption is non-negative, it is impossible to have the bubble larger than the young generation's wealth. That is,

$$\text{The support of } \tilde{b}_t \text{ is a subset of } [0, e_y]^n. \tag{21}$$

This condition is implied by the assumption that $u'(0) = +\infty$. Yet we choose to state it explicitly since it plays an important role in establishing the upper-bound result. Finally, we have first-order conditions for any realizations of the bubble size. That is, for any

$(b_t^1, ..., b_t^n)$ in the support of $\tilde{b}_t^i(H_{t-1})$ at time $t-1$, for any $i \in \{1, 2..., n\}$ we have

$$b_t^i u'(e_y - \sum_{j=1}^{n} b_t^j) = \beta E \left(\tilde{b}_{t+1}^i(H_t) u'(e_o + \sum_{j=1}^{n} \tilde{b}_{t+1}^j(H_t)) \right) \text{ where } H_t = H_{t-1} \bigcup \{b_t\}. \quad (22)$$

Definition 2. *In an n-bubble economy, an equilibrium is a set of random variables* $\{\tilde{b}_t(H_{t-1})\}_{t=0}^{\infty}$ *that satisfies* (18)*,* (19)*,* (20)*,* (21) *and* (22)*.*

Several aspects of this definition deserve more discussion. The stationary equilibrium is a special case of this more general definition. Specifically, in the stationary equilibrium, $\tilde{b}_t(H_{t-1})$ only depends on positive realizations in b_{t-1} and b_t is uniquely determined by the positive coordinates in b_t. Moreover, in any n-bubble economy, there are infinitely many non-stationary bubble dynamics that satisfy this more general equilibrium definition. For example, in a single-bubble economy, if we focus on equilibrium with a deterministic surviving bubble size, there are infinitely many equilibria in which the surviving bubble's size approaches zero over time (asymptotically bubbleless as in (Tirole 1985)). With random surviving bubble size, we can fit any (realized) bubble size dynamics as long as the path stays below the stationary bubble size.[11]

The independence assumption is not a strong assumption under this equilibrium definition. Independence only requires that the collapse probability of one bubble does not change regarding the collapse of the other bubbles. However, it imposes no restrictions on the surviving bubbles' sizes. Since this generalized equilibrium definition allows for stochastic bubble size (fixing surviving bubbles) and explicit time dependency, we can construct an equilibrium where a bubble's size is negatively affected by bursting, i.e., an equilibrium where a surviving bubble shrinks after another bubble bursts.[12] Moreover, under this general definition of equilibria, the correlation between the sizes of two surviving bubbles can either be positive or negative.

Finally, it is helpful to connect the first-order condition in this definition to its more intuitive counterpart in the stationary equilibrium. Fixing time t, let I be the index set over the dimensions of the positive realizations in b_t. Moreover, we can denote $\tilde{b}_{t+1}^i(H_t)$ as $b_{I',t+1}^i$ when $\tilde{b}_{t+1}(H_t)$ has index set I'.[13] Let $B_{I,t} = \sum_{j=1}^n b_t^j$ and $B_{I',t+1} = \sum_{j=1}^n b_{I',t+1}^j$.[14]

[11]This can be done by constructing an equilibrium where the surviving bubble size has three possible realizations with corresponding probability specifications that satisfy the first-order condition: (1) jump up to the stationary bubble size, (2) transit to the bubble size specified by the realized path and (3) decrease to a size close to zero.

[12]Notice that in the stationary equilibrium, by Proposition 2′, a (surviving) bubble always grows when another bubble bursts.

[13]It happens with a probability of $\prod_{j \in I} q_j \prod_{r \in I/I'} (1 - q_r)$.

[14]We do not need to require $j \in I$ since $b_{I,t}^j \equiv 0$ when $j \notin I$.

The transformation leads to the familiar first-order condition

$$b^i_{I,t}u'(c_{y,t}) - \beta \mathrm{E}_t \left[b^i_{I',t+1} u'(c_{o,t+1}) \right] = 0 \tag{23}$$

where $c_{y,t} = e_y - B_{I,t}$ and $c_{o,t+1} = e_o + B_{I',t+1}$. We use the simplified first-order condition (23) in the following analysis.

Since we allow for random future bubble size and do not require the equilibrium to be stationary, a great many bubble dynamics can be rationalized as equilibrium outcomes. To give an extreme example, in a single-bubble economy, any trajectory of the bubble size can be supported by an equilibrium as long as the trajectory stays below the bubble size in the stationary equilibrium. Given this observation, our goal is to understand the bubble size dynamics that cannot be supported by the equilibrium given the model parameters. It turns out that the equilibrium bubble sizes of the stationary equilibrium provide important insight into this problem. Specifically, in any equilibrium, the sum of the bubble sizes of any subset of existing bubbles must be smaller than the aggregate bubble size in the stationary equilibrium with that subset of bubbles.

Proposition 6. *In any equilibrium, the sum of the sizes of any set of bubbles is upper-bounded by the aggregate bubble size in the stationary equilibrium with that set of bubbles, i.e., for any index set I and $\hat{I} \subset I$, $\sum_{i\in\hat{I}} b^i_{I,t} \leq B_{\hat{I}}$. Specifically, when $\hat{I} = I$, $B_{I,t} \leq B_I$; when $\hat{I} = \{i\}$, $b^i_{I,t} \leq b^i_i$.*

It is informative to consider two extreme cases of this result. Applying this result to all existing bubbles shows that the aggregate bubble size in any equilibrium must be smaller than the aggregate bubble size in the stationary equilibrium. Applying this result to a specific bubble indicates that in equilibrium, any bubble must be smaller than its stationary bubble size in a single bubble economy. Moreover, this proposition *does not* imply that the agent cannot achieve a positive net return over the bubble sector. This proposition instead suggests that the upper bound of the realized return is lower when the bubble sector is larger. In other words, the agent cannot achieve a high return over a bubble when it is already large.

5.2 Bubble Creation

In this section, we extend the model to address the situation where new bubbles can be created and added to the economy. Since the unexpected creation of bubbles will not change the economy described by the baseline model, here we focus on the expected creation of bubbles in a stationary manner. More specifically, we focus on a stationary equilibrium where both the bubble price and the creation decision are time-homogeneous and depend on the set of surviving bubbles. In each period, after the old generation sells bubbles to the young generation, new bubbles are created and distributed to the young generation for free before the economy enters the next period. When the index set of surviving bubbles is I, denote the index set of bubbles after creation by $C(I)$.

This general specification of bubble creation nests a wide range of bubble creation models. The baseline model with no bubble creation corresponds to the case where $C(I) = I$ for all I. It can also describe the situation where the government creates one additional bubble each period, independent of the existing bubbles, and distributes the newly-created bubble to the young generation. In this case, $C(I) = I \bigcup \{j\}$ for $j \notin I$ for all I. It also nests settings where firms create bubbles strategically for profit. If the prices of the bubbles only depend on the set of surviving bubbles I, it is without loss of generality to focus on the strategy depending on I. For example, consider a situation where, in each period, an entrepreneur is selected from the young generation, capable of creating bubbles at a fixed unit cost. After creation, they must distribute most of the bubbles to the young generation for free to build the market but can retain a fraction to sell for profit in the next period when they are old. In a stationary equilibrium, any optimal creation strategy can be summarized by $C(I)$. The key assumption is that the entrepreneur can only retain a small number of bubbles, limiting their impact on the economy to bubble creation.

To differentiate the equilibrium with bubble creation from the stationary equilibrium of the baseline model, we use the "check" symbol for the equilibrium with bubble creation. The first-order condition can be expressed as

$$\check{b}_I^i u'(e_y - \check{B}_I) - \beta \mathrm{E}_t \left[\check{b}_{I'}^i u'(e_o + \check{B}_{I'}) \right] = 0,$$

where the index set I and I' represent the surviving bubbles in the current period and the subsequent period, respectively. The key difference from the baseline model is that we need to differentiate the set of bubbles before and after the bubble creation. In this expression, the index set in the subscript represents the set of surviving bubbles *before* the bubble creation. The index set after bubble creation in the current period is $C(I)$ while the index set after bubble creation in the next period is $C(I')$.

In an economy with one-shot bubble creation (which is not stationary), the change in the existing bubbles' sizes is easy to predict. Since newly created bubbles substitute existing bubbles in transferring wealth, bubble creation will reduce the equilibrium size of existing bubbles. However, the intuition becomes less clear in an economy with stationary bubble creation. When the current bubble creation decision not only affects the bubble today but also the future bubble creation decisions, it is difficult to rule out the possibility of an equilibrium where bubble sizes stay large. We show that the sum of the sizes of any subset of existing bubbles is upper-bounded by the corresponding aggregate bubble size in the stationary equilibrium.

Proposition 7. *For any $\hat{I} \subseteq I$, $\sum_{i\in\hat{I}} \check{b}^i_I \leq B_{\hat{I}}$. Specifically, when $\hat{I} = I$, $\check{B}_I \leq B_I$; when $\hat{I} = \{i\}$, $\check{b}^i_i \leq b^i_i$.*

Applying this result to all existing bubbles shows that the aggregate bubble size in any equilibrium must be smaller than the aggregate bubble size in the stationary equilibrium. Applying this result to a specific bubble indicates that in equilibrium, any bubble must be smaller than its stationary bubble size in a single bubble economy. Another point worth noting is that this result is about the size of the existing bubbles. Since the newly created bubbles are distributed to the young generation for free, they do not contribute to the current aggregate bubble size $\hat{B}_I$. This result does not rule out the possibility that the aggregate bubble size in the next period, with the addition of new bubbles, is larger than B_I. Instead, our result implies that it will be bounded by $B_{I'}$, the stationary equilibrium of the baseline model with surviving bubbles I'.

5.3 Implications for Estimating Confidence Measures

In this section, we illustrate the importance of the upper bound results in applying the model to measure agents' confidence over bubble assets given the observed data. Our model provides a simple framework to quantify confidence focusing on the stationary equilibrium.[15] Yet stationarity is a strong requirement and the lack of it in the observations might invalidate the measurement. We use the results established in this section to justify using the stationary equilibrium in our framework to measure confidence when the true bubble dynamics are not necessarily stationary. Specifically, the confidence measurement in this framework remains valid when the true bubble dynamics are non-stationary as long as we interpret the measurement as the lower bounds of confidence rather than the accurate estimations.

Consider a two-bubble economy as an example. Suppose we can find a period of time in which both bubbles exist and the agents' confidence over the two bubbles does not change. The quantities of interest are q_1 and q_2, the agents' confidence over the first and the second bubble. From the observation, we can obtain the bubble sizes within that period $b^1_{\{1,2\}}$ and $b^2_{\{1,2\}}$, the discount factor β, and the agents' endowments e_y and e_o. From the analysis of

[15] In our model, stationarity means that each bubble's size only depends on the current set of bubbles

the two-bubble economy, we have four equations Eqs. (10)-(13) with four unknowns q_1, q_2, b_1^1 and b_2^2.[16] The following proposition shows that if there is a solution for the estimation problem, it must be unique.

Proposition 8. *If, given $b^1_{\{1,2\}}$ and $b^2_{\{1,2\}}$, there exists a solution (q_1, q_2, b_1^1, b_2^2) for the system of equations Eqs.* (10)-(13)*, then the solution is unique.*

Several concerns need to be addressed when interpreting the estimated values of confidence measure q_1 and q_2. The first concern relates to the scope of estimation. In this model, bubbles substitute each other. Thus, if a third bubble traded by the agents is omitted from the estimation, the measurement of q_1 and q_2 would be inaccurate. Second, the estimation of q_1 and q_2 crucially relies on assuming the bubble size dynamics we observe are from the stationary equilibrium, which is hard to test.

To address these concerns, we investigate the implications of the confidence measurement without the stationary assumption using the results from this section. Suppose the time interval of interest can be divided into $T+1$ periods from 0 to T. Denote the size of bubble i in period t by $b_{i,t}$. Note that if the bubble dynamics are strictly stationary, we should expect $b_{1,0} = \cdots = b_{1,T}$ and $b_{2,0} = \cdots = b_{2,T}$. Let $\bar{b}_i = max_t b_{i,t}$, the largest bubble size of bubble i in this time interval. By Proposition 6, this is upper-bounded by the stationary size of bubble i in a single-bubble economy. Moreover, by Proposition $2'$, the size of bubble i is increasing in q_i. Thus, the agent's confidence over the first (second) bubble must be higher than $\underline{q}_1$ ($\underline{q}_2$), which is the confidence over the first (second) bubble in the stationary equilibrium of a single bubble economy when the observed bubble size is $\bar{b}_1$ ($\bar{b}_2$). Applying Proposition 6 to the bubble sector leads to a joint constraint over the confidence. Let $\bar{B} = max_t(b_{1,t} + b_{2,t})$ be the largest aggregate bubble in the time interval and $\tau = max_t(b_{1,t} + b_{2,t})$ be the corresponding period. By Proposition 6, this is upper-bounded by the stationary size of the bubble sector in a two-bubble economy. Let $q_{1,\tau}$ and $q_{2,\tau}$ be the confidence levels in a two-bubble economy with bubble sizes $b_{1,\tau}$ and $b_{2,\tau}$ and the true confidence levels over two bubbles be q_1 and q_2. Proposition $2'$ provides a joint confidence bound over q_1 and q_2, the true confidence levels over two bubbles: either $q_1 \geq q_{1,\tau}$ or $q_2 \geq q_{2,\tau}$.

6 Discussion

The recent emergence of cryptocurrencies, notably Bitcoin, has sparked considerable interest and debate. Our paper aims to contribute to the discussion by studying the role of

[16]Notice that since both bubbles exist within the chosen period of time, b_1^1 and b_2^2 are not observable.

cryptocurrencies and CBDCs as pure bubbles that store value (or provide liquidity) and examining their welfare consequences.[17] In the following subsections, we discuss how our results can be applied to cryptocurrencies and CBDCs.

6.1 Cryptocurrencies

Cryptocurrency Exchange-Traded Funds (Crypto ETFs), which involve investing in cryptocurrencies such as Bitcoin, Ethereum, or a basket (or mix) of different cryptocurrencies, are gaining attention in the financial landscape. Our theoretical investigation into the dynamics of these bubbles can provide valuable insights into the desirability of a multitude of such bubble assets.

First, taking the stationary equilibrium as the benchmark, a crypto ETF can offer more stability than individual cryptocurrencies. The diversification effect, achieved through investing in a mix of cryptocurrencies such as Bitcoin, Ethereum, or other digital assets, plays a pivotal role. This inclusion minimizes the impact of bursting bubbles on the overall portfolio, providing resilience against market fluctuations. It is worth noting that the diversification effect of mixing cryptocurrencies is stronger than the diversification effect of mixing other assets due to the strong substitution effect among bubbles. That is, bubbles do not generate utility flows and their values are entirely derived from shifting agents' wealth into the future. Specifically, by Proposition 5, if a crypto ETF consists of many different digital assets, its price may still be stable even when one or several of its components burst. This proposition also predicts that the volatility difference between a single asset and an ETF (with multiple assets) is larger in the cryptocurrency asset class than in other asset classes. Therefore, Crypto ETFs can be an attractive option for investors seeking a more secure foothold in the volatile cryptocurrency landscape.

Second, the optimal ETF is determined by weighting it according to the market value of cryptocurrencies. In practice, this implies investors should choose a crypto ETF that includes more digital assets over a bitcoin-tracking ETF since investing in the former leads to a higher expected utility. As the cryptocurrency market expands, the difference in utility is expected to grow in tandem.

Third, the emergence of better bubbles (or more stable liquidity vehicles) may eventually

[17] Many papers in the literature focus on cryptocurrencies as tokens used by platforms (e.g., Cong, Li, and Wang (2021), Cong, Li, and Wang (2022)). There are also many papers focusing on characteristics deriving from blockchain systems, such as coin mining (e.g., Prat and Walter (2021), Biais, Bisière, Bouvard, Casamatta, and Menkveld (2023)). Another strand of literature focuses on the analysis and discussion of CBDC, including its design, impact on financial intermediation, stability, and its implications for the existing monetary system (e.g., Brunnermeier and Niepelt (2019), Fernández-Villaverde, Sanches, Schilling, and Uhlig (2021), Agur, Ari, and Dell'Ariccia (2022)).

substitute existing ones to a large extent. Propositions 3′-4′ demonstrate that more stable bubbles crowd out unstable bubbles by decreasing the weights of unstable ones in the bubble sector. For example, even in a scenario where cryptocurrencies may eventually become stable and popular, the ones used in the future do not necessarily have to be the existing ones in the market today. As better cryptocurrencies emerge over time, the existing ones may shrink, if not disappear.

Fourth, if we take into account the possibility that the prices of cryptocurrencies may not evolve as specified by the stationary equilibrium in our model, Proposition 6 implies that the stationary size of the bubble sector in our model is the tight upper bound on the size of crypto ETFs. Moreover, if we take the possibility of the inclusion of tokens created in the future, Proposition 7 indicates the same upper bound applies. This insight is crucial for investors, signaling that while there is potential for returns, it is inherently limited by the stationary equilibrium. In essence, there exists a realistic ceiling to the upside potential, guiding investors to form nuanced expectations and align their investment strategies accordingly. This insight can assist investors in navigating the crypto market, steering clear of overly optimistic projections.

6.2 CBDC

Drawing implications from our results, it becomes evident that the efficiency of decentralized cryptocurrencies is a complex issue. The concept of maximum efficiency theoretically favors a single centralized digital currency, such as a CBDC, when managed adeptly by a central authority. However, for this centralized digital currency model to work effectively, the central bank or government must optimize various aspects, such as monetary policy and regulatory measures, as CBDC may have various potential impacts on the banking system (e.g., Andolfatto (2018), Chiu, Davoodalhosseini, Jiang, and Zhu (2020), Fernández-Villaverde, Sanches, Schilling, and Uhlig (2021)). Nevertheless, the practical implementation of such a scenario raises significant concerns. Technical feasibility, for instance, presents challenges in creating a robust and scalable centralized digital currency infrastructure (e.g., Tian, Zhao, and Olivares (2023)). Moreover, ensuring the security of a single currency against potential threats and maintaining long-term stability in an ever-evolving financial landscape adds layers of complexity that cast doubt on the feasibility of achieving ultimate efficiency through centralization alone.

Our findings shed light on the dynamics between a single stable centralized digital currency and a basket of diverse cryptocurrencies such crypto ETFs. While one might assume that a stable centralized digital currency would dominate the market, our results suggest otherwise. Even a relatively stable centralized digital currency may not necessarily outperform a diversified portfolio of potentially risky cryptocurrencies. The key factor here is the presence of idiosyncratic risk. There is a growing literature that investigates portfolio strategies for cryptocurrencies in the presence of idiosyncratic risk (e.g., Platanakis, Sutcliffe, and Urquhart (2018), Petukhina, Trimborn, Härdle, and Elendner (2021)). For example, Tian, Zhao, and Olivares (2023) shows that portfolio diversification across 10 major cryptocurrencies can significantly improve investment results in terms of risk. If a centralized digital currency is entirely free from risk, it might hold an advantage, but such a scenario is challenging to achieve in practice. On the other hand, a diversified portfolio can leverage the inherent differences and risk profiles of various cryptocurrencies to create a more stable overall investment. This insight challenges conventional thinking and underscores the importance of considering risk as a fundamental component of cryptocurrency efficiency and stability.

In this respect, CBDCs may also be used as one of the possible ingredients in a well-diversified portfolio consisting of asset bubbles. For example, Usher, Reshidi, Rivadeneyra, and Hendry (2021) argue that CBDCs have the potential to function as a more effective alternative instrument or an outside option in the field of digital payments when contrasted with the current methods of regulation and legal enforcement. In conclusion, a diversified portfolio of various bubble assets including cryptocurrencies, despite its inherent risks, may provide greater stability. As the cryptocurrency landscape continues to evolve, it is crucial to consider these complexities when evaluating the efficiency and stability of different digital currency models.

7 Conclusion

In a world where the creation of money-like assets is no longer the exclusive domain of traditional financial institutions, the question of whether we need a plethora of such assets becomes increasingly pertinent. In this regard, we delve into the dynamics of multiple heterogeneous asset bubbles within an OLG economy with imperfect financial markets.

Our results provide valuable insights into the role and desirability of diverse bubble assets. While individual bubbles may inherently lack stability due to their absence of intrinsic value, the presence of multiple, diverse bubbles offers advantages through diversification, effectively mitigating the risks associated with their potential collapse. Our analysis reveals that as the number of bubbles increases, the speculative market as a whole expands, albeit with the diminishing significance of each individual bubble. Furthermore, bubbles with lower probabilities of bursting tend to be larger and more enduring, underscoring the importance of stability in attracting investor confidence. Ultimately, our findings suggest that within a sufficiently large market of diverse bubble assets, the bubble sector can effectively function as a substitute for a risk-free asset. This phenomenon has implications for the emergence and sustainability of cryptocurrencies.

While more centralized digital currencies, such as CBDCs, may theoretically offer optimal efficiency due to their stability, practical challenges and the potential for diversification benefits within a decentralized cryptocurrency landscape complicate the landscape. This illustrates that the appeal of multiple bubble assets may endure in our evolving financial ecosystem.

Appendix

Some Notations

To simplify the notation, define the transition probability

$$P(I' \mid I) = \prod_{l \in I'} q_l \prod_{r \in I \setminus I'} (1 - q_r).$$

This can be interpreted as the probability that the next period's bubbles have an index set I' when the current bubbles' index set is I. If $I' \not\subseteq I$, $P(I' \mid I) = 0$. $P(I \mid I) = \prod_{l \in I} q_l$ corresponds to the probability of no bursting. Denote $I \setminus \{i\}$ as I_{-i}.

Proof of Proposition 1′:

All propositions are established through induction over $|I|$. By Proposition 1, $min(q_i) > \underline{q}$ is necessary for the existence and uniqueness of the equilibrium. We hereby prove that it is also sufficient. The claim holds true when $|I| = 1$ because this is equivalent to a special case of Proposition 1 where there is only one active bubble. Now, suppose the claim in this proposition holds when $|I| = k$. Then, when $|I| = k + 1$, for each $i \in I$, the following first-order condition should be satisfied:

$$b_I^i \left[u'(e_y - B_I) - \beta u'(e_o + B_I) P(I \mid I) \right] = C_i, \tag{A.1}$$

where

$$C_i = \beta q_i \sum_{I' \subsetneq I_{-i}} P(I' \mid I_{-i}) \left[b^i_{I' \bigcup \{i\}} u'(e_o + B_{I' \bigcup \{i\}}) \right] > 0.$$

Note that C_i is well-defined in the sense that it is uniquely determined by the values of $b^i_{I'\bigcup\{i\}}$ and $B_{I'\bigcup\{i\}}$, which, in turn, are uniquely determined by the induction hypothesis ($|I'\bigcup\{i\}| \leq k$). Summing all the first-order conditions Eq. (A.1) for all $i \in \{1, 2, \ldots, k+1\}$ gives

$$B_I \left[u'(e_y - B_I) - \beta u'(e_o + B_I)P(I \mid I)\right] = \sum_{i=1}^{k+1} C_i. \tag{A.2}$$

Following a similar argument as the proof of Proposition 1, there is a unique $B_I > 0$ that solves Eq. (A.2). Given this B_I, for any $i \in I$, b^i_I is then uniquely determined by $b^i_I = B_I \frac{C_i}{\sum_{j=1}^{k+1} C_j}$. □

Proof of Proposition 2′:

We first prove the first claim of the proposition that, for any $j \notin I$, $B_{I\bigcup\{j\}} > B_I$. When $|I| = 1$, this claim is already proven by Proposition 2. Suppose the claim holds for all $|I| \leq k-1$. Summing all the first-order conditions for all $i \in I$, we have

$$u'(e_y - B_I) = \beta \sum_{I' \subseteq I} P(I' \mid I) \left[\frac{B_{I'}}{B_I} u'(e_o + B_{I'})\right] \tag{A.3}$$

where $B_\emptyset$ is defined to be 0. Applying the same argument to the case with $I\bigcup\{j\}$, we have

$$\begin{aligned} u'(e_y - B_{I\bigcup\{j\}}) =& \beta \sum_{I' \subseteq I\bigcup\{j\}} P(I' \mid I\bigcup\{j\}) \left[\frac{B_{I'}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'})\right] \\ =& \beta \Bigg\{ (1-q_j) \sum_{I' \subseteq I} P(I' \mid I) \left[\frac{B_{I'}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'})\right] \\ &+ q_j \sum_{I' \subseteq I} P(I' \mid I) \left[\frac{B_{I'\bigcup\{j\}}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'\bigcup\{j\}})\right] \Bigg\} \\ >& \beta \Bigg\{ (1-q_j) \sum_{I' \subseteq I} P(I' \mid I) \left[\frac{B_{I'}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'})\right] \\ &+ q_j \sum_{I' \subsetneq I} P(I' \mid I) \left[\frac{B_{I'}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'})\right] \\ &+ q_j P(I \mid I) u'(e_o + B_{I\bigcup\{j\}}) \Bigg\}. \end{aligned} \tag{A.4}$$

The inequality in Eq. (A.4) is due to the induction hypothesis, i.e., $B_{I'\bigcup\{j\}} > B_{I'}$ because $|I'| \leq k-1$ for all $I' \subsetneq I$ (note that $|I| = k$). Considering both sides of the inequality as functions of $B_{I\bigcup\{j\}}$, the left-hand side of the inequality increases in $B_{I\bigcup\{j\}}$, while the right-hand side decreases. Furthermore, if $B_{I\bigcup\{j\}}$ had the same value as B_I, the right-hand side of the inequality would match the right-hand side of Eq. (A.3), requiring the left-hand side of the inequality to equal the right-hand side of the inequality. This contradiction implies that, for the inequality to hold, it must be $B_{I\bigcup\{j\}} > B_I$.

We now prove the second claim that $b_I^i > b^i_{I\bigcup\{j\}}$ for any $i \in I$. When $|I| = 1$, this claim is already proven by Proposition 2. Suppose the claim holds for all $|I| \leq k-1$. For $|I| = k$, the first-order condition for b_I^i can be written as

$$b_I^i u'(e_y - B_I) = \beta \sum_{I' \subseteq I\ ,i\in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}).$$

For $j \notin I$, from the first-order condition of $b^i_{I\bigcup\{j\}}$, we have

$$\begin{aligned} b^i_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{j\}}) &= \beta \bigg[(1-q_j) \sum_{I' \subseteq I\ ,i\in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\ &\quad + q_j \sum_{I' \subseteq I\ ,i\in I'} P(I' \mid I) b^i_{I'\bigcup\{j\}} u'(e_o + B_{I'\bigcup\{j\}}) \bigg] \\ &< \beta \bigg[(1-q_j) \sum_{I' \subseteq I\ ,i\in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\ &\quad + q_j \sum_{I' \subsetneq I\ ,i\in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\ &\quad + q_j P(I \mid I) b^i_{I\bigcup\{j\}} u'(e_o + B_I) \bigg] \\ &= \beta \bigg[\sum_{I' \subsetneq I\ ,i\in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\ &\quad + (1-q_j) P(I \mid I) b^i_I u'(e_o + B_I) \\ &\quad + q_j P(I \mid I) b^i_{I\bigcup\{j\}} u'(e_o + B_I) \bigg]. \end{aligned}$$

The inequality follows from the induction hypothesis and $B_{I\bigcup\{j\}} > B_I$ for any I and $j \notin I$. Moreover, $u'(e_y - B_{I\bigcup\{j\}}) > u'(e_y - B_I)$. The inequality and the first-order condition for b^i_I together imply

$$b^i_{I\bigcup\{j\}}[u'(e_y - B_I) - \beta q_j P(I \mid I) u'(e_o + B_I)] < b^i_I[u'(e_y - B_I) - \beta q_j P(I \mid I) u'(e_o + B_I)].$$

Since $u'(e_y - B_I) - \beta q_j P(I \mid I) u'(e_o + B_I) > 0$, it must be $b^i_I > b^i_{I\bigcup\{j\}}$. □

Proof of Proposition 3′:

Proof. For each $i \in I$, the following first-order condition should be satisfied:

$$b_I^i \left[u'(e_y - B_I) - \beta u'(e_o + B_I)P(I \mid I)\right] = C_i, \tag{A.5}$$

where

$$C_i = \beta q_i \sum_{I' \subsetneq I_{-i}} P(I' \mid I_{-i}) \left[b^i_{I' \bigcup\{i\}} u'(e_o + B_{I' \bigcup\{i\}})\right].$$

Because $u'(e_y - B_I) - \beta u'(e_o + B_I)P(I \mid I)$ is fixed, $C_i > C_j$ implies $b_I^i > b_I^j$. Therefore, we only need to show that $q_i > q_j$ implies $C_i > C_j$. The claim is already proven when $|I| = 2$ in the two-bubble economy. Suppose the result holds for $|I| = k$. We now show that the conclusion holds for $|I| = k+1$. We perform a term-by-term comparison by identifying each $I'_{-i} \subsetneq I_{-i}$ with a $I'_{-j} \subsetneq I_{-j}$ using the following constructions: If $j \notin I'_{-i}$, let $I'_{-j} = I'_{-i}$; if $j \in I'_{-i}$, let $I'_{-j} = (I'_{-i} \setminus \{j\}) \bigcup \{i\} \subsetneq I_{-j}$.

Now we compare the corresponding terms in C_i (related to I'_{-i}) and C_j (related to I'_{-j}). If $j \notin I'_{-i}$, $I'_{-i} = I'_{-j}$. Since $q_i > q_j$, by Proposition 4′, $b^i_{I'_{-i}\bigcup\{i\}} > b^j_{I'_{-j}\bigcup\{j\}}$ and $B_{I'_{-i}\bigcup\{i\}} > B_{I'_{-j}\bigcup\{j\}}$.[18] Moreover, $q_i(1-q_j) > q_j(1-q_i)$. Thus,

$$\beta q_i P(I'_{-i} \mid I_{-i}) b^i_{I'_{-i}\bigcup\{i\}} u'\left(e_o + B_{I'_{-i}\bigcup\{i\}}\right) > \beta q_j P(I'_{-j} \mid I_{-j}) b^i_{I'_{-j}\bigcup\{j\}} u'\left(e_o + B_{I'_{-j}\bigcup\{j\}}\right).$$

If $j \in I'_{-i}$, $I'_{-i}\bigcup\{i\} = I'_{-j}\bigcup\{j\}$. From the induction hypothesis $b^i_{I'_{-i}\bigcup\{i\}} > b^j_{I'_{-j}\bigcup\{j\}}$. Thus the same inequality holds. This implies that for $|I| = k+1$, $q_i > q_j$ implies $C_i > C_j$ and the proof is complete. □

Proof of Proposition 4′:

Proof. First, consider the change in the aggregate bubble size. When $|I| = 0$, this observation trivially holds. Suppose the claim holds for all $|I| \leq k$. When $|I| = k+1$, from the

[18] The argument is not tautological since the proof of Proposition 4′ does not depend on Proposition 3′

first order condition we have

$$\begin{aligned}
u'(e_y - B_{I\bigcup\{j\}}) =& \beta \sum_{I'\subseteq I\bigcup\{j\}} P(I' \mid I\bigcup\{j\}) \frac{B_{I'}}{B_{I\bigcup\{j\}}} u'(e_o + B_{I'}) \\
=& \beta\Bigg[(1-q_j)\sum_{I'\subseteq I} P(I' \mid I)\frac{B_{I'}}{B_{I\bigcup\{j\}}}u'(e_o + B_{I'}) \\
&+ q_j \sum_{I'\subseteq I} P(I' \mid I)\frac{B_{I'\bigcup\{j\}}}{B_{I\bigcup\{j\}}}u'(e_o + B_{I'\bigcup\{j\}})\Bigg] \\
>& \beta\Bigg[(1-q_h)\sum_{I'\subseteq I} P(I' \mid I)\frac{B_{I'}}{B_{I\bigcup\{j\}}}u'(e_o + B_{I'}) \\
&+ q_h \sum_{I'\subseteq I} P(I' \mid I)\frac{B_{I'\bigcup\{j\}}}{B_{I\bigcup\{j\}}}u'(e_o + B_{I'\bigcup\{j\}})\Bigg] \\
>& \beta\Bigg[(1-q_h)\sum_{I'\subseteq I} P(I' \mid I)\frac{B_{I'}}{B_{I\bigcup\{j\}}}u'(e_o + B_{I'}) \\
&+ q_h \sum_{I'\subsetneq I} P(I' \mid I)\frac{B_{I'\bigcup\{h\}}}{B_{I\bigcup\{h\}}}u'(e_o + B_{I'\bigcup\{h\}}) \\
&+ q_h P(I \mid I) u'(e_o + B_{I\bigcup\{j\}})\Bigg].
\end{aligned} \tag{A.6}$$

By Proposition 2′, $B_{I'\bigcup\{j\}}u'(e_o+B_{I'\bigcup\{j\}}) > B_{I'}u'(e_o+B_{I'})$. Moreover, $q_j > q_h$. These facts imply the first inequality. The second inequality follows from the induction hypothesis (i.e., $B_{I'\bigcup\{j\}} > B_{I'\bigcup\{h\}}$ for $I' \subsetneq I$). Regarding the leftmost term and the rightmost term of (A.6) as functions of $B_{I\bigcup\{j\}}$. The left-hand side is increasing in $B_{I\bigcup\{j\}}$ and the right-hand side is decreasing in $B_{I\bigcup\{j\}}$. When $B_{I\bigcup\{j\}} = B_{I\bigcup\{h\}}$, both sides are equal to $u'(e_y - B_{I\bigcup\{h\}})$. Thus, for the inequality to hold, it must be that $B_{I\bigcup\{j\}} > B_{I\bigcup\{h\}}$ and the claim holds for $|I| = k+1$.

Next, consider the comparison between $b^i_{I\bigcup\{j\}}$ and $b^i_{I\bigcup\{h\}}$. The situation when $|I| = 1$ has been analyzed in the two-bubble economy. Suppose the claim holds for all $|I| \leq k$. For

$|I| = k+1$,

$$
\begin{aligned}
& b^i_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{j\}}) \\
=& \beta[(1-q_j) \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\
& + q_j \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'\bigcup\{j\}} u'(e_o + B_{I'\bigcup\{j\}})] \\
<& \beta[(1-q_h) \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\
& + q_h \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'\bigcup\{j\}} u'(e_o + B_{I'\bigcup\{j\}})] \\
<& \beta[(1-q_h) \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\
& + q_h \sum_{I' \subsetneq I \ , i \in I'} P(I' \mid I) b^i_{I'\bigcup\{h\}} u'(e_o + B_{I'\bigcup\{h\}}) \\
& + q_h P(I \mid I) b^i_{I\bigcup\{j\}} u'(e_o + B_{I\bigcup\{h\}})].
\end{aligned} \tag{A.7}
$$

By Proposition 2′, $b^i_{I'} > b^i_{I'\bigcup\{j\}}$ and $B_{I'} < B_{I'\bigcup\{j\}}$. Moreover, $q_j > q_h$. These facts lead to the first inequality. Notice that when $I' \subseteq I$, $|I'| \leq k$. From the induction hypothesis, $b^i_{I'\bigcup\{h\}} > b^i_{I'\bigcup\{j\}}$. Moreover, from the first part of the proof, $B_{I'\bigcup\{j\}} > B_{I'\bigcup\{h\}}$. These two facts together imply the second inequality and $b^i_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{j\}}) > b^i_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{h\}})$. Thus,

$$
\begin{aligned}
& b^i_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{h\}}) \\
<& \beta[(1-q_h) \sum_{I' \subseteq I \ , i \in I'} P(I' \mid I) b^i_{I'} u'(e_o + B_{I'}) \\
& + q_h \sum_{I' \subsetneq I \ , i \in I'} P(I' \mid I) b^i_{I'\bigcup\{h\}} u'(e_o + B_{I'\bigcup\{h\}}) \\
& + q_h P(I \mid I) b^i_{I\bigcup\{j\}} u'(e_o + B_{I\bigcup\{h\}})].
\end{aligned} \tag{A.8}
$$

Subtract $q_h P(I \mid I) b^i_{I\bigcup\{j\}} u'(e_o + B_{I\bigcup\{h\}})]$ from both sides of (A.8) and plug in the first order condition regarding $b^i_{I\bigcup\{h\}}$ to get

$$
\begin{aligned}
& b^i_{I\bigcup\{j\}} [u'(e_y - B_{I\bigcup\{h\}}) - \beta q_h P(I \mid I) u'(e_o + B_{I\bigcup\{h\}})] \\
<& b^i_{I\bigcup\{h\}} [u'(e_y - B_{I\bigcup\{h\}}) - \beta q_h P(I \mid I) u'(e_o + B_{I\bigcup\{h\}})]
\end{aligned} \tag{A.9}
$$

Since $u'(e_y - B_{I\bigcup\{h\}}) - \beta q_h P(I \mid I) u'(e_o + B_{I\bigcup\{h\}}) > 0$, it must be $b^i_{I\bigcup\{h\}} > b^i_{I\bigcup\{j\}}$. Moreover, since $B_{I\bigcup\{j\}} \geq B_{I\bigcup\{h\}}$ and for all $i \in I$, $b^i_{I\bigcup\{h\}} > b^i_{I\bigcup\{j\}}$, it must be $b^i_{I\bigcup\{j\}} > b^h_{I\bigcup\{h\}}$. □

Proof of Corollary 1:

Proof. The old generation's utility is increasing since B_I is increasing in $|I|$. For the young generation, it is sufficient to show that for any $j \notin I$,

$$\begin{aligned} & u(e_y - B_{I \bigcup \{j\}}) + \beta \sum_{I' \subseteq I \bigcup \{j\}} P(I' \mid I \bigcup \{j\}) u(e_o + B_{I'}) \\ > & u(e_y - B_I) + \beta \sum_{I' \subseteq I} P(I' \mid I) u(e_o + B_{I'}). \end{aligned} \tag{A.10}$$

Since u is concave and $B_{I \bigcup \{j\}} > B_I$

$$u(e_y - B_I) - u(e_y - B_{I \bigcup \{j\}}) = \int_{e_y - B_{I \bigcup \{j\}}}^{e_y - B_I} u'(x) dx < \left(B_{I \bigcup \{j\}} - B_I\right) u'(e_y - B_{I \bigcup \{j\}}).$$

Moreover,

$$\begin{aligned} & \beta \sum_{I' \subseteq I \bigcup \{j\}} P(I' \mid I \bigcup \{j\}) u(e_o + B_{I'}) - \beta \sum_{I' \subseteq I} P(I' \mid I) u(e_o + B_{I'}) \\ = & \beta q_j \sum_{I' \subseteq I} P(I' \mid I) \int_{e_o + B_{I'}}^{e_o + B_{I' \bigcup \{j\}}} u'(x) dx \\ > & \beta q_j \sum_{I' \subseteq I} P(I' \mid I) \left(B_{I' \bigcup \{j\}} - B_{I'}\right) u'(e_o + B_{I' \bigcup \{j\}}). \end{aligned}$$

Thus, it is sufficient to show

$$\beta q_j \sum_{I' \subseteq I} P(I' \mid I) \left(B_{I' \bigcup \{j\}} - B_{I'}\right) u'(e_o + B_{I' \bigcup \{j\}}) - \left(B_{I \bigcup \{j\}} - B_I\right) u'(e_y - B_{I \bigcup \{j\}}) > 0. \tag{A.11}$$

(A.11) is equivalent to

$$D_1 - D_2 > 0.$$

where

$$
\begin{aligned}
D_1 =& B_I u'(e_y - B_{I\bigcup\{j\}}) - \beta q_j \sum_{I' \subseteq I} P(I' \mid I) B_{I'} u'(e_o + B_{I'\bigcup\{j\}}) \\
>& B_I u'(e_y - B_I) - \beta q_j \sum_{I' \subseteq I} P(I' \mid I) B_{I'} u'(e_o + B_{I'}) \\
=& \beta(1 - q_j) \sum_{I' \subseteq I} P(I' \mid I) B_{I'} u'(e_o + B_{I'}) \\
=& B_{I\bigcup\{j\}} u'(e_y - B_{I\bigcup\{j\}}) - \beta q_j \sum_{I' \subseteq I} P(I' \mid I) B_{I'\bigcup\{j\}} u'(e_o + B_{I'\bigcup\{j\}}) = D_2.
\end{aligned}
$$

The second line is from $B_{I\bigcup\{j\}} > B_I$ and the concavity of $u(\cdot)$. The third and fourth lines are from the first-order condition. This establishes the proposition. □

Proof of Proposition 5:

Proof. It is sufficient to establish the convergence result in the symmetric case when each bubble collapses with probability q. This is because from the established results, B_I is larger (smaller) than the bubble sector of a symmetric case with $q = \min_i\{q_i\}$ ($q = \max_i\{q_i\}$).

Fixing q, we denote by b_k the size of each individual bubble in the economy with k symmetric bubbles (due to the bursting of other existing bubbles). Notice that kb_k is increasing in k (Proposition 2′) and bounded above. Moreover, $xu'(x)$ is increasing in x. This implies that $kb_k u'(e_o + kb_k)$ is increasing in k with a finite upperbound. Thus, for any $\epsilon > 0$, there exists N such that for $m, n \geq N$, $\frac{nb_n u'(e_o+nb_n)}{mb_m u'(e_o+mb_m)} \geq 1 - \epsilon$. Let $m = N^2$, since $\binom{n}{i+1} = \binom{n-1}{i} \cdot \frac{n}{i+1}$, from the first-order condition we have

$$
\begin{aligned}
u'(e_y - mb_m) =& q\beta \sum_{i=0}^{m-1} \binom{m-1}{i} q^i (1-q)^{m-1-i} \frac{b_{i+1}}{b_m} u'(e_o + (i+1)b_{i+1}) \\
=& \beta \sum_{i=1}^{m} \binom{m}{i} q^i (1-q)^{n-i} \frac{ib_i}{mb_m} u'(e_o + ib_i) \\
>& \beta \sum_{i=N}^{m} \binom{m}{i} q^i (1-q)^{n-i} \frac{ib_i}{mb_m} \frac{u'(e_o + ib_i)}{u'(e_o + mb_m)} u'(e_o + mb_m) \\
\geq& \beta(1 - \eta(N))(1-\epsilon) u'(e_o + mb_m)
\end{aligned}
\tag{A.12}
$$

where $\eta(N)$ is the probability of having less than N successes in a binomial distribution with success probability q and $m = N^2$ number of trials. We have

$$\eta(N) < \binom{N^2}{N}(1-q)^{N^2-N} \cdot \sum_{i=0}^{N} q^i(1-q)^{N-i} < \binom{N^2}{N}(1-q)^{N^2-N}.$$

Clearly, this quantity goes to 0 as $N \to \infty$. Moreover, Eq. (A.12) holds for arbitrary ϵ when N is large. Let $s = lim_{n\to\infty} nb_n$ be the limiting size of the bubble sector. It must be $u'(e_y - s) \geq \beta u'(e_o + s)$.

For the upperbound, notice that $\binom{n-1}{i} b_{i+1} u'(e_o + (i+1)b_{i+1}) < \binom{n}{i+1} b_n u'(e_o + nb_n)$ for any integer i smaller than n. Thus,

$$\begin{aligned} b_n u'(e_y - nb_n) =& q\beta \sum_{i=0}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i} b_{i+1} u'(e_o + (i+1)b_{i+1}) \\ =& \beta \sum_{i=0}^{n-1} \binom{n}{i+1} q^{i+1} (1-q)^{n-1-i} \frac{i+1}{n} b_{i+1} u'(e_o + (i+1)b_{i+1}) \\ <& \beta \sum_{i=0}^{n-1} \binom{n}{i+1} q^{i+1} (1-q)^{n-1-i} b_n u'(e_o + nb_n) \\ =& \beta[1 - (1-q)^n] b_n u'(e_o + nb_n) \end{aligned} \tag{A.13}$$

Thus, for any n, $u'(e_y - nb_n) < \beta u'(e_o + nb_n)$. Thus, it must be $u'(e_y - s) \leq \beta u'(e_o + s)$. This means the aggregate bubble size nb_n converges to s.

To see that the individual bubble size converges to zero, it is sufficient to analyze the situation where one bubble's surviving probability is q and all other bubbles' surviving probability is $\hat{q}$, which is slightly above $\underline{q}$. We aim to show that when the number of bubbles becomes large, the size of the bubble with surviving probability q converges to zero. To simplify the notation (since we only focus on one specific bubble), Let b_{n+1} be the size of that bubble (survive with probability q) and B_{n+1} be the size of the bubble sector when that bubble and n other bubbles (with surviving probability $\hat{q}$) in the economy. For $m = N^2$, we have

$$\begin{aligned} & b_m[u'(e_y - B_m) - \beta q \hat{q}^m u'(e_o + B_m)] \\ <& \beta q(1 - \hat{q}^m - \hat{\eta}(N)) b_{N+1} u'(e_o + B_{N+1}) + \beta q \hat{\eta}(N) b_1 u'(e_o + B_1) \end{aligned} \tag{A.14}$$

where $\hat{\eta}(N) < \binom{N^2}{N}(1-\hat{q})^{N^2-N}$ is the probability of having less than N successes in a binomial distribution with success probability $\hat{q}$ and $m = N^2$ number of trials. Suppose b_n does not converge to zero. Since it is decreasing with a lower bound, it converges to $d > 0$.

Take $N \to \infty$,

$$du'(e_y - s) \leq \beta q du'(e_o + s). \tag{A.15}$$

This cannot be true since $q < 1$. Contradiction. By Proposition 3, we can conclude that fixing $i \in I$, $b_I^i \to 0$ as $|I| \to \infty$. □

Proof of Proposition 6:

Proof. To illustrate the key insight, we first prove $B_{I,t+1} \leq B_I$ via induction and then discuss how the argument can be modified slightly to establish $\sum_{i \in \hat{I}} b_{I,t}^i \leq B_{\hat{I}}$.

Consider a simplified situation of a single-bubble economy where the future bubble size is deterministic. With only one bubble, we can simplify our notation from $B_{\{1\},t} = b_{1,t}$ to b_t. Let $\theta_{t+1} = \frac{b_{t+1}}{b_t}$ be the return of the bubble between period t and $t+1$. For any $b_t > 0$, the first-order condition can be written as

$$u'(e_y - b_t) - \beta q \theta_{t+1} u'(e_o + \theta_{t+1} b_t) = 0.$$

Since the first-order condition must hold for all t, we may drop the time subscript when there is no confusion and write the first-order condition as

$$u'(e_y - b) - \beta q \theta u'(e_o + \theta b) = 0.$$

Fixing θ, the left-hand side increases with b. Thus, for any given θ, there exists a unique bubble size b satisfying the first-order condition. Let's denote this unique solution as $b(\theta)$. The bubble size in the stationary equilibrium corresponds to $b(1)$.

We now show that $b(\theta)$ is increasing in θ. For any constant b, note that

$$\beta q \theta u'(e_o + \theta b) = \frac{\beta q}{b} \frac{\theta b}{e_o + \theta b} (e_o + \theta b) u'(e_o + \theta b),$$

is increasing in θ.[19] If $\hat{\theta} > \theta$, we have

$$u'(e_y - b(\theta)) - \beta q \hat{\theta} u'(e_o + \hat{\theta} b(\theta)) < 0.$$

Since the left-hand side approaches ∞ when $b \to e_y$ and there exists a unique b satisfying the first-order condition, it must be $b(\hat{\theta}) > b(\theta)$.

We now show that in any equilibrium, $b_t \leq b(1)$ for all t. If not, then it must be $b_t = b(\theta_{t+1})$ with $\theta_{t+1} > 1$ since $b(\theta)$ is increasing. Thus, $b_{t+1} > b_t$. Define function $\theta(b)$

[19] Here we use the assumption that $xu'(x)$ is increasing for all x.

to be the inverse function of $b(\theta)$. $\theta(b)$ is well-defined since $b(\theta)$ is increasing. Consider $\theta_{t+2} = \theta(b_{t+1})$. If $\theta(b_{t+1})$ does not exist (due to b_{t+1} not in the range of $b(\cdot)$), it cannot be an equilibrium since there is no return to justify the bubble size b_{t+1}. However, if $\theta(b_{t+1})$ exists, since $\theta(b)$ is increasing, $\theta_{t+2} > \theta_{t+1} > 1$. Iterating forward, we have $b_{t+n} > \theta_{t+1}^n b_t$, which diverges to infinity when $n \to \infty$ and it cannot be an equilibrium since the bubble size should be smaller than the endowment e_y.

This argument extends to the situation of random bubble size on the path where the bubble does not collapse. In this case, the first-order condition is

$$u'(e_y - b_t) - \beta E_t[\theta_{t+1} u'(e_o + \theta_{t+1} b_t)] = 0.$$

Using the independence assumption, we can represent the above equation as follows:

$$u'(e_y - b_t) - \beta q E_t[\theta_{t+1} u'(e_o + \theta_{t+1} b_t) \mid \theta_{t+1} > 0] = 0.$$

Suppose there exists an equilibrium in which $b_t > b(1)$ for some t on a sample path. Since $\theta u'(e_o + \theta b)$ is increasing in θ, there must exist a realization of θ_{t+1} such that $\theta_{t+1} \geq 1 + \epsilon_{t+1}$ where $\epsilon_{t+1} > 0$ is the unique solution of the following equation:

$$u'(e_y - b_t) - \beta q(1 + \epsilon_{t+1}) u'[e_o + (1 + \epsilon_{t+1}) b_t] = 0. \tag{A.16}$$

On that equilibrium path, $b_{t+1} > b_t > b(1)$. From the first-order condition of b_{t+1}, since $u'(e_y - b_{t+1}) > u'(e_y - b_t)$ and $u'(e_o + \theta b)$ is decreasing in b, there must exist a realization of θ_{t+1} such that $\theta_{t+2} \geq 1 + \epsilon_{t+2} > 1 + \epsilon_{t+1}$.[20] Iterating forward to see that there exists a sample path where $\lim_{n\to\infty} b_{t+n} \to \infty$. Contradiction.

Note that a similar bound applies to the size of one bubble in an n-bubble economy. Specifically, for any $i \in I$, the first-order condition is

$$b_{I,t}^i u'(e_y - B_{I,t}) - \beta \mathrm{E}_t \left[b_{I',t+1}^i u'(e_o + B_{I',t+1}) \right] = 0.$$

Using the independence assumption and noting that $b_{I,t}^i \leq B_{I,t}$ and $b_{I',t+1}^i \leq B_{I',t+1}$, the first order condition implies that

$$u'(e_y - b_{I,t}^i) - \beta q E_t[\theta_{t+1} u'(e_o + \theta_{t+1} b_{I,t}^i) \mid \theta_{t+1} > 0] \leq 0.$$

By a similar argument, $b_{I,t}^i < b_i^i$ for all I, t and $i \in I$.

[20]The inequality $1 + \epsilon_{t+2} > 1 + \epsilon_{t+1}$ is due to Eq. (A.16) and $b_{t+1} > b_t$.

We proceed by induction over $|I|$. Suppose $B_{I,t} \leq B_I$ for all $|I| \leq m-1$. For $|I| = m$, the first-order conditions implies that

$$\begin{aligned} & B_{I,t}\left[u'(e_y - B_{I,t}) - \beta P(I \mid I) E_t[\theta_{t+1} u'(e_o + \theta_{t+1} B_{I,t}) \mid I' = I]\right] \\ =& \beta \sum_{I' \subsetneq I} P(I' \mid I) E_t[B_{I',t+1} u'(e_o + B_{I',t+1}) \mid I'], \end{aligned}$$

where $\theta_{t+1} = \frac{B_{I,t+1}}{B_{I,t}}$. Notice that the expectation on the left-hand side corresponds to the situation where no bubble bursts and I', the index set of the next period, is the same as I. The terms on the right-hand side correspond to the situation where at least one bubble bursts in the $(t+1)th$ period. By the induction hypothesis and the first-order conditions of the stationary equilibrium,

$$\begin{aligned} & \beta \sum_{I' \subsetneq I} P(I' \mid I) E_t[B_{I',t+1} u'(e_o + B_{I',t+1}) \mid I'] \\ \leq & \beta \sum_{I' \subsetneq I} P(I' \mid I) B_{I'} u'(e_o + B_{I'}) \\ =& B_I\left[u'(e_y - B_I) - \beta P(I \mid I) u'(e_o + B_I)\right]. \end{aligned}$$

This implies

$$\begin{aligned} & B_{I,t}\left[u'(e_y - B_{I,t}) - \beta P(I \mid I) E_t[\theta_{t+1} u'(e_o + \theta_{t+1} B_{I,t}) \mid I]\right] \\ \leq & B_I\left[u'(e_y - B_I) - \beta P(I \mid I) u'(e_o + B_I)\right] \end{aligned} \tag{A.17}$$

Suppose there exists an equilibrium such that in some scenarios $B_{I,t} > B_I > 0$. Fix a $B_{I,t} > B_I$, we have,

$$\begin{aligned} & \beta P(I \mid I)\left[E_t[\theta_{t+1} u'(e_o + \theta_{t+1} B_{I,t}) \mid I] - u'(e_o + B_I)\right] \\ \geq & u'(e_y - B_{I,t}) - u'(e_y - B_I) \\ >& 0. \end{aligned}$$

We have shown that $\theta_{t+1} u'(e_o + \theta_{t+1} B_{I,t})$ is increasing in θ_{t+1} and is decreasing in $B_{I,t}$. Moreover, since $B_{I,t} > B_I$, $u'(e_o + B_{I,t}) < u'(e_o + B_I)$. Thus, there must exist a sample path such that $\theta_{t+1} \geq 1 + \epsilon_{t+1}$ where $\epsilon_{t+1} > 0$ is the unique solution of

$$(1 + \epsilon_{t+1}) u'(e_o + (1 + \epsilon_{t+1}) B_{I,t}) = u'(e_o + B_I).$$

Since $\theta_{t+1} > 1$, $B_{I,t+1} = B_{I,t}\theta_{t+1} > B_{I,t}$. With a similar argument, there exists a sample path such that $\theta_{t+2} \geq 1 + \epsilon_{t+2}$ where $\epsilon_{t+2} > 0$ is the unique solution of

$$(1 + \epsilon_{t+2})u'(e_o + (1 + \epsilon_{t+2})B_{I,t+1}) = u'(e_o + B_I).$$

Moreover, since $B_{I,t+1} > B_{I,t}$, it must be $\epsilon_{t+2} > \epsilon_{t+1}$. Iterate forward, there exists a sample path such that $B_{I,t+n} > \theta_{t+1}^n B_{I,t}$. This implies that along the chosen sample path, $\lim_{n\to\infty} B_{I,t+n}$ diverges to infinity. Contradiction.

Following a similar argument, we can show that for any I and $\hat{I} \subset I$, $\sum_{i\in\hat{I}} b_{I,t}^i \leq B_{\hat{I}}$.

□

To complete the discussion, we hereby show that the agents' elasticity of intertemporal substitution greater than one ($xu'(x)$ increasing) is necessary for Proposition 6. Specifically, there exist situations with the agents' EIS smaller than 1 where the aggregate bubble size in the stationary equilibrium is smaller than its counterpart in other equilibria.

We consider an oscillating equilibrium in a single-bubble economy as in Weil (1987) where the agents' utility function is CRRA with relative risk aversion γ and EIS $\frac{1}{\gamma}$. In the equilibrium, if the surviving bubble is large in the current period, it will be small in the next period (if it survives), or vice versa. By Proposition 6, when $\gamma \leq 1$, no such equilibrium exists. We hereby show that when γ is large, such equilibrium exists and when the bubble is large, its size is larger than the bubble size in the unique stationary equilibrium.

Proposition A.1. *In a single-bubble economy with CRRA utility, when γ is large enough, there exists an oscillating equilibrium where the surviving bubble's size oscillates between b_l and b_h and $b_h > b(1)$ where $b(1)$ corresponds to the bubble size in the stationary equilibrium.*

Proof. By Grandmont (1985) Lemma 1.2., $\theta b(\theta)$ is increasing in θ, the return over the bubble (given the bubble does not collapse). If there exists an oscillating equilibrium, since $b_h > b(1)$, when the current bubble size is b_l, $\theta > 1$ and this implies $b_h > b(1)$.

By Grandmont (1985) Equation (4.10), with CRRA utility, a sufficient condition for an oscillating equilibrium is

$$\gamma > 2 \cdot \frac{e_o + b(1)}{b(1)} + \frac{e_o + b(1)}{e_y - b(1)} \cdot \gamma$$

where

$$b(1) = \frac{(\beta q)^{\frac{1}{\gamma}} e_y - e_o}{1 + (\beta q)^{\frac{1}{\gamma}}}.$$

This is equivalent to

$$[1 - (\beta q)^{\frac{1}{\gamma}}]\gamma > \frac{2(\beta q)^{\frac{1}{\gamma}}(e_y + e_o)}{1 + (\beta q)^{\frac{1}{\gamma}}}.$$

The condition can be satisfied by taking γ large. $\square$

Proof of Proposition 7:

Proof. We prove by contradiction with induction over $|\hat{I}|$ from 1 to $|I|$ for arbitrary I. First, consider the situation where $|\hat{I}| = 1$. Let $\hat{I} = \{i\}$. The first-order conditions for the bubble $i \in I$ is

$$\check{b}_I^i u'(e_y - \check{B}_I) = \beta \sum_{I' \subseteq C(I)} P(I' \mid C(I)) \left[\check{b}_{I'}^i u'(e_o + \check{B}_{I'}) \right]. \tag{A.18}$$

If $i \notin I'$, $\check{b}_{I'}^i \equiv 0$. $C(I)$ is the index set of bubbles after the creation in the state I. $\check{B}_I = \sum_{j \in I} \check{b}_I^j$ is the value of the bubble sector when the surviving bubbles' index set is $C(I)$ after creation; $\check{B}_{I'} = \sum_{j \in I'} \check{b}_{I'}^j$ is the value of the bubble sector when the surviving bubbles' index set is $C(I')$ after creation.

Let $\theta(I') = \frac{\check{b}_{I'}^i}{\check{b}_I^i}$. From Eq. (A.18), we have

$$u'(e_y - \check{B}_I) = \beta \sum_{I' \subseteq C(I)} P(I' \mid C(I)) \left[\theta(I') u'(e_o + \check{B}_{I'}) \right]. \tag{A.19}$$

This implies

$$\begin{aligned} u'(e_y - \check{b}_I^i) \leq & \check{b}_I^i u'(e_y - \check{B}_I) \\ \leq & \beta \sum_{I' \subseteq C(I)} P(I' \mid C(I)) \left[\theta(I') u'(e_o + \theta(I') \check{b}_I^i) \right], \end{aligned}$$

where the first inequality is because $\check{b}_I^i < \check{B}_I$, the second inequality is due to Eq. (A.19), $u'(e_o + \check{B}_{I'}) \leq u'(e_o + \check{b}_{I'}^i)$, and the definition of $\theta(I')$.

Following a similar argument as in the proof of Proposition 6 , if $\check{b}_I^i > b_i^i$, there must exist a $I' \subseteq C(I)$ where $\theta(I') \geq 1 + \epsilon$ with $\epsilon > 0$. This leads to contradiction and it must be $\check{b}_I^i \leq b_i^i$. That is, the bubble size of a bubble with surviving probability q_i in any equilibrium with bubble creation must be (weakly) smaller than that bubble in a single-bubble economy without creation.

Suppose $\sum_{i \in \hat{I}} \check{b}_I^i \leq B_{\hat{I}}$ holds for all $|\hat{I}| \leq k$. Fix $|\hat{I}| = k + 1$, let $\theta(I') = \frac{\sum_{i \in \hat{I}} \check{b}_{I'}^i}{\sum_{i \in \hat{I}} \check{b}_I^i}$. We have

$$
\begin{aligned}
&\sum_{i\in\hat{I}} \check{b}_I^i \left[u'(e_y - \sum_{i\in\hat{I}} \check{b}_I^i) - \beta \sum_{\hat{I}\subseteq I'\subseteq C(I)} P(I' \mid C(I))\theta(I')u'(e_o + \theta(I') \sum_{i\in\hat{I}} \check{b}_I^i) \right] \\
\leq &\sum_{i\in\hat{I}} \check{b}_I^i \left[u'(e_y - \check{B}_I) - \beta \sum_{\hat{I}\subseteq I'\subseteq C(I)} P(I' \mid C(I))\theta(I')u'(e_o + \check{B}_{I'}) \right] \\
= &\beta \sum_{I'\subseteq C(I), \hat{I}\nsubseteq I'} P(I' \mid C(I)) \left[\sum_{i\in\hat{I}} \check{b}_{I'}^i u'(e_o + \check{B}_{I'}) \right] \\
\leq &\beta \sum_{I'\subseteq C(I), \hat{I}\nsubseteq I'} P(I' \mid C(I)) \left[\sum_{i\in\hat{I}} \check{b}_{I'}^i u'(e_o + \sum_{i\in\hat{I}} \check{b}_{I'}^i) \right] \\
\leq &\beta \sum_{I'\subsetneq I} P(I' \mid C(I)) \left[B_{I'} u'(e_o + B_{I'}) \right] \\
= &B_{\hat{I}} \left[u'(e_y - B_{\hat{I}}) - \beta P(\hat{I} \mid \hat{I}) u'(e_o + B_{\hat{I}}) \right].
\end{aligned}
$$

The first inequality is due to $\sum_{i\in\hat{I}} \check{b}_I^i \leq \check{B}_I \equiv \sum_{i\in I} \hat{b}_I^i$ and $\theta(I') \sum_{i\in\hat{I}} \check{b}_I^i = \sum_{i\in\hat{I}} \check{b}_{I'}^i \leq \check{B}_{I'} \equiv \sum_{i\in I'} \check{b}_{I'}^i$. The first equality is due to the first-order conditions regarding $\hat{b}_I^i$ for all $i \in \hat{I}$ and the independence assumption. The second inequality is due to $\sum_{i\in\hat{I}} \check{b}_{I'}^i \leq \check{B}_{I'}$.[21] The third inequality is from the induction hypothesis. The second equality is implied by the first-order condition of the baseline model. If $\sum_{i\in\hat{I}} \check{b}_I^i > B_{\hat{I}}$, the argument from the proof of Proposition 6 applies and on some sample paths, the bubble size will diverge, which cannot be an equilibrium. □

Proof of Proposition 8:

Proof. We prove the claim by contradiction. Note that by the first order conditions Eqs. (10)-(11), b_1^1 is uniquely determined by q_1 and b_2^2 is uniquely determined by q_2. Moreover, when $q_1 = q_2$, $b_1^1 = b_2^2$. Thus, given e_y, e_o, β and $u(\cdot)$, there exists an strictly increasing function $f(\cdot)$ such that $b_1^1 = f(q_1)$ and $b_2^2 = f(q_2)$. Thus, we can focus on analyzing the uniqueness of q_1 and q_2.

Suppose there are two tuples (q_1, q_2) and $(\check{q}_1, \check{q}_2)$ that satisfies Eqs. (12)-(13). It cannot be $q_1 = \check{q}_1$ since the first-order conditions would imply $q_2 = \check{q}_2$. Without loss of generality, assume $q_1 q_2 > \check{q}_1 \check{q}_2$. Then, the assumption together with Eq. (12) implies $\check{q}_1(1-\check{q}_2)f(\check{q}_1) > q_1(1-q_2)f(q_1)$. Also, the assumption together with Eq. (13) implies $\check{q}_2(1-\check{q}_1)f(\check{q}_2) >$

[21]Note that if $i \notin I'$, $\check{b}_{I'}^i \equiv 0$.

$q_2(1-q_1)f(q_2)$. Thus, when $q_1 > \breve{q}_1$, it must be that $q_2 > \breve{q}_2$, or vice versa. By Proposition 4, however, it must be either $q_1 > \breve{q}_1$, $q_2 < \breve{q}_2$ or $q_1 < \breve{q}_1$, $q_2 > \breve{q}_2$. This contradicts. □

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