Pricing European Options on Stochastically Volatile Assets

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Abstract

Pricing of options on assets with stochastic volatililty is a very interesting problem in mathematical finance and it has widespread uses in the financial industry. Many researchers such as Hull and White (1987), Wiggins (1987), Stein and Stein (1991) and Willard (1996) have worked on the problem.

We consider the Hull and White model, suitably adjusted for the volatility process to follow an Ornstein Uhlenbeck process. The model is given by

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2}dB_t^{(2)}], \tag{1}$$

$$dV_t = -aV_t dt + dB_t^{(1)}, \tag{2}$$

where, X_t is the price of of an asset under an equivalent martingale measure at time t. V_t is the volatility process - an Ornstein Uhlenbeck process with a mean reversion force of a. r is the rate of interest which is a constant, $B_t^{(1)}$ and $B_t^{(2)}$ are the two independent standard Brownian motions driving the price process X_t and the volatility process V_t. We assume that the two processes are correlated with a correlation co-efficient ρ. Most of the work that has been done by others have concentrated on $\rho = 0$. We are interested in calculating the European call option, given by

$$X_0 e^{-r} E(e^{Y_t} - b)^+, \tag{3}$$

where, $Y_t = ln\frac{X_t}{X_0}$, X_0 is the current price and b is the strike price at which the option is calculated. However, in general, we can look at any function of X_t.

We first observe that, conditionally on the paths of $\{B_t^{(1)}; 0 \le t \le 1\}$ we have Y_1 following a normal distribution with mean (r - $\frac{1}{2}\sigma^2 P + \rho\sigma Q$) and variance $\sigma^2(1-\rho^2)P$ where,

$$P = \int_0^1 e^{kV_t} dt \quad \text{and} \quad Q = \int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)}.$$

To calculate the price of the option, we look at the two term expansion of $\Psi(P, Q)$, where
$\Psi(P, Q) = E[(e^{Y_1} - b)^+ | B_t^{(1)}; 0 \le t \le 1]$

$$= e^{(r - \frac{1}{2}\sigma^2\rho^2 E(P|Z) + \rho\sigma E(Q|Z))} \Phi(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}})$$

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$$-b\Phi(\frac{r-\frac{1}{2}\sigma^2 E(P|Z)+\rho\sigma E(Q|Z)-lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}). \quad (4)$$

We thus have, using the conditioning factor Z,

$$E(\Psi(P,Q)) = E\{E[\Psi(E(P|Z),E(Q|Z))]\} + \frac{\rho^2\sigma^2}{2}E\{\Psi_{22}(E(P|Z),E(Q|Z))\mathrm{Var}(Q|Z)\} + O(\sigma^3). \quad (5)$$

where, Ψ_{22} indicates the second derivative with respect to Q. Finally, we take the expectation over Z to obtain the price of the option. Now, conditionally on the random variable Z, B_s follows a Gaussian process. Thus it is easy to calculate the quantitites $E(P|Z)$, $E(Q|Z)$ and $\mathrm{Var}(Q|Z)$. Our choice of the conditioning factor Z is

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}. \quad (6)$$

The conditioning factor is similar to the one used by Rogers and Shi (1995) in valuing an Asian option and Basu (1999) to value bonds and reinsurance contracts with log-normal intensity.

The first term of the expansion alone is not accurate enough to approximate the price of the option and using this term alone would lead to an underestimate of the true value of the option. Thus, the second term in the expansion becomes important and is termed as the *correction factor*.

The price of the option is given as the sum of the two values.

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