Pricing TAIEX Options Based on Heston-Nandi Closed-Form GARCH Option Model

Yong-chern Su*, Chun-Ping Fong
*Department of Finance, National Taiwan University
China Trust

Abstract

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* Corresponding author, Email: ycsu@ccms.ntu.edu.tw, Address: 50 Lane 144 Sec. 4, Keelung Road, Taipei, Taiwan, Tel: 8862-27229803, Fax: 8862-23637714

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Many empirical studies have indicated that the assumption of Black-Scholes model exhibits systematic biases. In practice, Black-Scholes implied volatilities tend to differ across exercise prices and time to maturity. To overcome the shortcoming, many researchers have contributed to substantial new models. In this article, we employ Heston and Nandi (2000) closed-form GARCH option model in the TAIEX options pricing. As a benchmark model we choose the ad hoc BS model that has the flexibility of fitting to the strike and term structure of observed implied volatilities by using a separate implied volatility for each option. We found that the GARCH model has smaller out-of-sample valuation errors in pricing TAIEX options than the ad hoc BS model does.

1. Introduction

The Black-Scholes (1973) model assumes the asset price follows geometric Brownian motion and is log-normally distributed with constant volatility. Consequently, all options on the same asset should provide the same implied volatility. However, Fama (1965) and Mandelbrot (1966) found that stock returns exhibit both fat-tailed marginal distributions and volatility clustering. In practice, Black-Scholes implied volatilities tend to differ across exercise prices and time to maturity. Options that are deep in the money or out of the money have higher implied volatilities than at the money options. The failure of the Black-Scholes model to describe the structure of reported option prices is thought to arise from its constant volatility assumption. These features are interpreted as evidence of stochastic volatility of financial asset prices. To overcome the shortcoming, many researchers have contributed to substantial new models that incorporate stochastic volatility in the last two decades. It is thus interesting to examine whether the stochastic volatility option pricing models provide improvements to the Black-Scholes model.

Early amendments include the constant-elasticity-of-variance model by Cox (1975), the jump-diffusion model by Merton (1976), the compound-option model by Geske (1979), and the displaced-diffusion model by Rubinstein (1983). Empirically, these models face the difficulty that the variance rate is not observable. Latest amendments are two types: implied volatility and stochastic volatility. The latter include continuous-time stochastic models and discrete-time stochastic generalized autoregressive conditional heteroskedasticity (GARCH) models.

Continuous-time stochastic volatility models are effective for option pricing, but can be difficult to implement. Although these models assume that volatility is observable, it is very difficult to filter a continuous volatility variable from discrete observations. One alternative is to use implied volatilities computed from option prices. But empirically, this approach requires estimating many volatilities, one for every date and is computationally burdensome in a long time series of options records. In any case, continuous-time models must be augmented with nontrivial volatility estimation techniques.

Moreover, the continuous-time model can serve as the limit of a certain GARCH model. For example, Nelson (1990) showed that the GARCH (1, 1) model converged to a certain diffusion model. Duan (1996) argued that most of the existing bivariate diffusion models that had been used to model asset returns and volatility could be represented as limits of a family of GARCH models. As a special case, the particular GARCH option model proposed by Heston and Nandi (2000) was proved to contain Heston's (1993) stochastic volatility model as a continuous-time limit.

On the other hand, the GARCH model has an advantage over the continuous-time model in that the volatility is readily observable in the history of asset prices. As a result, it is possible to price an option only using the information from the observations of asset prices. In contrast, the continuous-time stochastic model has an inherent disadvantage that it assumes that volatility is observable, but it is impossible to exactly filter volatility from discrete observations of spot asset prices in a continuous-time stochastic volatility mode. Consequently, it is impossible to price an option solely on the basis of the history of asset prices. Since volatility is unobservable, one has to use the volatility implied from one option to value other options. Unfortunately, this method is not always feasible especially when the related options are thinly traded. Thus, the GARCH model is chosen over the continuous-time model when comparing the empirical performance of the stochastic option model and the discrete-time model.

Duan (1995) cited the econometric method – GARCH into discrete-time model and proposed the GARCH option pricing model to extend the Black-Scholes model. The ARCH model was first introduced by Engle (1982) and Bollerslev (1986) made an improvement as GARCH model. Under the GARCH process, the key hypothesis is conditional heterskedasticity and the variance is determined by a series of parameters and a sequence of random variables which are noise. To capture the negative correlation between returns and conditional volatility, Engle and Ng (1993) presented the NGARCH model. The general theory of GARCH option pricing, however, applies to the NGARCH model.

However, most GARCH models did not have closed-form solutions for option prices. These models were typically solved by Monte Carlo simulation (Engle and Mustafa (1992), Amin and Ng (1993), Duan (1995)) that can be slow and computationally intensive for empirical work. More recently, Hanke (1997) provided a network approach, Ritchken and Trevor (1999) provided a lattice approximation to value American options and Duan, Gauthier and Simonato (1999) provided Markov chain approach for GARCH process with single lags in the variance dynamics. Heston and Nandi (2000) developed a closed-form solution for European option values (and hedge ratios) in a GARCH model. The model allowing for multiple lags in the time series dynamics of the variance process and also for correlation between returns of the spot asset and variance did provide another choice to price options.

We employ Heston and Nandi (2000) closed-form GARCH model in the TAIEX options pricing. As a benchmark model we choose the ad hoc BS model of Dumas, Fleming and Whaley (1998, henceforth DFW) that has the flexibility of fitting to the strike and term structure of observed implied volatilities by using a separate implied volatility for each option. It is found that the GARCH model has smaller valuation errors (out-of-sample) than the ad hoc BS model.

The rest of this study organized as follows. Methodology including data description and the model we used are provided in Section 2. Section 3 reports the in-sample estimation through MLE and out-of-sample pricing results, while Section 4 concludes.

2. Data and Methodology

2.1 TAIFEX Options Market Overview

The Taiwan Futures Exchange launched an index option on the TAIEX starting at 12/24/2001, which can be used as a hedging tool for investors, especially institutional investors, to protect their investment position.

The TAIEX Options achieved a total of 1,566,446 contracts by the end of 2002, accounting for 19.72% of the market total for the year. According to Trade Data Global Services, the TAIFEX is in the 30th place in terms of trading volume of all products, edging up 6 places from 2001. A total of 139,575 contracts settled in 2002, including 70,604 index futures contracts and 68,971 TAIEX Options contracts which count for 50.58% and 49.42% respectively.

Following five index futures and options, TAIFEX launched equity options in early 2002. The product was debuted on 01/20/2003. In the first round of listing, five underlying stocks, namely Nan Ya, China Steel, UMC, TSMC, and Fubon were introduced. Periodic review will be carried out to add new contracts on different underlyings.

TAIFEX also have finished the drafts of contract specifications and trading rules for TSEC Taiwan 50 Index Futures, short-term interest rate futures and government bond futures. The TSEC Taiwan 50 Index Futures was launched at 06/30/2003 in coordination with the listing of TSEC Taiwan 50 ETF. The short-term interest rate futures and government bond futures is launched at 01/02/2004.

2.2 Data description

The sample of TAIEX options is from the Taiwan Futures Exchange (TAIFEX). The data set is formed by closing prices every Wednesday (or the next trading day if Wednesday is a holiday) from 07/01/2002 to 06/30/2003. Since many of the stocks in the TAIEX do not pay cash dividends, we assume the dividends to be zero and need not to subtract it from the current index level. For the risk free rate, 1-year time deposits interest rates reported on the board of Bank of Taiwan, interpolated to match the maturity of the option is used.

Three rules are applied to the dataset. First, we eliminate options with fewer than 6 or more than 60 days to expiration. The short-term options have substantial time decay that could interfere with one's being able to isolate the volatility parameters. The long-term options, on the other hand, are not included because they are not actively traded.

Second, only option records in which moneyness, K/S, lies between 0.9 and 1.1 are included in the sample. This excludes some very deep out-of-the-money and deep in-the-money options that are either infrequently traded.

Third, the options are ruled out if their prices do not satisfy the boundary condition:

$$S(t) \geq C(t,T) \geq \text{Max}(0, S(t) - PVD - Ke^{-r(T-t)}) \qquad (2.2.1)$$

The first inequality must hold because a call option enables one to buy one share of underlying asset and so the option can never be worth more than the asset itself. The second inequality must be satisfied since it ensures that there is no arbitrage opportunity.

The data set consists of 820 observations. The average number of options per day is 16 with a minimum of 7 and a maximum of 20. The whole sample is divided into 10 categories that are based on moneyness and the time to maturity. In terms of moneyness, the data set is divided into five categories: deep in-the-money options with $K/S < 0.95$, in-the-money options with $0.95 \leq K/S < 0.99$, at-the-money option with $0.99 \leq K/S < 1.01$, out-of-the-money options with $1.01 \leq K/S < 1.05$, and deep out-of-the-money options with $K/S \geq 1.05$. The terms to expiration are classified as (1) short-term (less than 30 days); (2) medium-term (between 30 to 60 days)

Table 1 provides the average call option prices based on moneyness and maturity. It shows that the average option price is a decreasing function of moneyness and an increasing function of days to maturity.

We find that the average option price of short-term deep out-of-the-money call options is the lowest while that of the medium-term deep in-the-money options is the highest. Out-of-the-money and deep out-of-he-money options count for 45% of total and short-term options takes up 50% alone. Before pricing options, we compute spot volatility. In computing the spot volatility, we use the daily history of spot index levels that trace back up to 1 year (252 days) before. We move the TAIEX index level windows from 07/01/2001 to 06/30/2003 to calculate the spot volatility on 07/01/2002 to 06/30/2003.

2.3 The Model

We price TAIEX options based on the closed-form GARCH option valuation model [Heston & Nandi (2000)]. In general, GARCH models are typically solved by simulation that can be slow and computationally intensive for empirical work. By contrast, the closed-form GARCH model provides an analytical solution and would be more powerful in option pricing.

A. The Closed-Form GARCH Option Pricing Model

The model has two basic assumptions. In the following section, we name this model as "H-N GARCH model." The first assumption is that the log-spot price follows a particular GARCH process.

Assumption 1: The spot asset price, S(t) (including accumulated interest or dividends) follows the following process over time steps of length Δ

$$\begin{aligned} \log(S(t)) &= \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t) \\ h(t) &= \omega + \sum_{i=1}^{p}\beta_i h(t-i\Delta) + \sum_{i=1}^{q}\alpha_i\left(z(t-i\Delta) - \gamma_i\sqrt{h(t-i\Delta)}\right)^2 \end{aligned} \quad (2.3.1)$$

r : the continuously compounded interest rate for the time interval Δ

z(t) : standard normal disturbance

h(t) : the conditional variance of the log return between t - Δ and is known from the information set at time t - Δ

As the αi and βi parameters approach zero, it is equivalent to the Black-Scholes model observed at discrete intervals.

Here we will focus on the first-order process (p=q=1). The first-order process is stationary with finite mean and variance if $\beta_1 + \alpha_1 \gamma_1^2 = 1$. In this model one can directly observe h(t + Δ) as a function of the spot price as follows:

$$h(t+\Delta) = \omega + \beta_1 h(t-\Delta) + \alpha_1 \frac{\left(\log(S(t)) - \log(S(t-\Delta)) - r - \lambda h(t) - \gamma_1 \sqrt{h(t)}\right)^2}{h(t)} \tag{2.3.2}$$

α_1 : the kurtosis of the distribution and a zero value implies a deterministic time varying variance

γ_1 : asymmetric influence of shocks; a large negative shock, z(t) raises the variance more than a large positive z(t)

In general the variance process h(t) and the spot return are correlated as follows:

$$\mathrm{Cov}_{t\text{-}\Delta}\left[h(t+\Delta), \log(S(t))\right] = -2\alpha_1 \gamma_1 h(t) \tag{2.3.3}$$

Given positive α_1 and γ_1 value for results in negative correlation between spot returns and variance.

The second assumption of the model concerns the pricing of options and other derivative securities. The spot price has a conditionally lognormal distribution over a single period. Since variance is not stochastic over this interval, Heston & Nandi assumed that the Black-Scholes-Rubinstein formula holds.

Assumption 2: The value of a call option one period prior to expiration obeys the Black-Scholes-Rubinstein formula.

Assumptions 1 and 2 allow us to derive the prices of all contingent claims that can be written as functions of the spot asset price. Since long-term options are functions of S(t) and h($t+\Delta$), and h($t+\Delta$) can be written as a function of S(t) in equation (2.3.4), this includes options of all maturities. Equation (2.3.1) is algebraically equivalent to

$$\begin{aligned} \log\left(S(t)\right) &= \log\left(S(t-\Delta)\right) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t) \\ h(t) &= \omega + \sum_{i=1}^{p}\beta_i h(t-i\Delta) + \sum_{i=1}^{q}\alpha_i\left(z^*(t-i\Delta) - \gamma_i^*\sqrt{h(t-i\Delta)}\right)^2 \end{aligned} \tag{2.3.4}$$

where, $\begin{cases} z^*(t) = z(t) + \left(\lambda + \frac{1}{2}\right)h(t) \\ \gamma_1^* = \gamma_1 + \lambda + \frac{1}{2} \end{cases}$

$z^*(t)$: standard normal distribution under the risk-neutral probabilities.

Formalize this property as the following proposition.

Proposition 1: The risk-neutral process takes the same GARCH form as equation (2.3.1) with λ replaced by $-1/2$ and γ_1 replaced by $\gamma_1^* = \gamma_1 + \lambda + 1/2$.

This proposition is trivial by noting that $z^*(t)$, γ_1^* and λ as defined above make the one period return from investing in the spot asset equal to the risk free rate. We proceed to solve for the generating function of the GARCH process (2.3.1) and use it to produce option prices. Let f(φ) denote the conditional generating function of the asset price

$$f(\phi) = E_t\left[S(T)^{\phi}\right]$$

This is also the moment generating function of the logarithm of S(T). The function f(φ) depends the parameters and state variables of the model, but these arguments are suppressed for notational convenience. We shall use the notation f*(φ) to denote the generating function for the risk-neutral process.

Proposition 2: The generating function takes the log-linear form

$$f(\phi) = S(T)^{\phi}\exp\begin{pmatrix} A(t;T,\phi) + \sum_{i=1}^{p} B_i(t;T,\phi)h(t+2\Delta - i\Delta) \\ + \sum_{i=1}^{q-1} C_i\left(z(t+\Delta-i\Delta) - \gamma_i\sqrt{h(t+\Delta-i\Delta)}\right)^2 \end{pmatrix} \quad (2.3.5)$$

where,

$$\begin{aligned} A(t;T,\phi) &= A(t+\Delta;T,\phi) + \phi\Delta + B_1(t+\Delta;T,\phi)\omega \\ &\quad - \frac{1}{2}\ln\left(1 - 2\alpha_1 B_1(t+\Delta;T,\phi)\right) \\ B_1(t;T,\phi) &= \phi(\lambda+\gamma_1) - \frac{1}{2}\gamma_1^2 + \beta_1 B_1(t+\Delta;T,\phi) \\ &\quad + \frac{\frac{1}{2}(\phi-\gamma_1)^2}{1 - 2\alpha_1 B_1(t+\Delta;T,\phi)} \end{aligned} \quad (2.3.6)$$

for the single lag (p=q=1) version and these coefficients can be computed recursively from the boundary conditions:

$$\begin{aligned} A(T;T,\phi) &= 0 \\ B_1(T;T,\phi) &= 0 \end{aligned} \quad (2.3.7)$$

Since the generating function of the spot price is the moment generating function of the logarithm of the spot price, f(iφ) is the characteristic function of the logarithm of the spot price.

Proposition 3: If the characteristic function of the log spot price is f(iφ) then

$$
\begin{aligned}
E_t\left[Max(S(t)-K,0)\right] = {} & f(1)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f(i\phi+1)}{i\phi\, f(1)}\right]d\phi\right) \\
& - K\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f(i\phi)}{i\phi}\right]d\phi\right)
\end{aligned}
\tag{2.3.8}
$$

where Re[ ] denotes the real part of a complex number. An option price is simply the discounted expected value of the payoff, Max(S(T)-K,0) calculated using the risk-neutral probabilities.

Corollary: At time t, a European call option with strike price K that expires at time T is worth

$$
\begin{aligned}
C = e^{-r(T-t)}E_t^*\left[Max(S(t)-K,0)\right] = {} & \frac{1}{2}S(t) \\
& + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f(i\phi+1)}{i\phi\, f(1)}\right]d\phi \\
& - Ke^{-r(T-t)}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f(i\phi)}{i\phi}\right]d\phi\right)
\end{aligned}
\tag{2.3.9}
$$

This completes the option pricing formula. As in the Black–Scholes formula, (2.3.9) can be written as (2.3.10). Delta of the call value is simply as equation (2.3.11). The other hedge ratios like the vega and the gamma can be calculated by straight differentiation in equation (2.3.9) and the expression for delta respectively. Put option values can be calculated using the put-call parity.

$$C = S(t)\left(\frac{1}{2} + \frac{e^{-r(T-t)}}{\pi \times S(t)}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi} f^*(i\phi + 1)}{i\phi\, f(1)}\right] d\phi\right) - Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}\right] d\phi\right) \tag{2.3.10}$$

$$\Delta = \frac{1}{2} + \frac{e^{-r(T-t)}}{\pi \times S(t)}\int_0^\infty \text{Re}\left[\frac{K^{-i\phi} f^*(i\phi + 1)}{i\phi\, f(1)}\right] d\phi \tag{2.3.11}$$

In contrast to the Black-Scholes formula, this formula is a function of the current asset price, S(t), and the conditional variance, h(t + Δ). Since h(t +Δ) is a function of the observed path of the asset price, the option formula is effectively a function of current and lagged asset prices. In contrast to continuous-time models, volatility is a readily observable function of historical asset prices and need not be estimated with other procedures.

B. Ad Hoc Black-Scholes Model

The measuring errors seem to be subjective for various purposes. One way to gauge the prediction errors is to measure them against a benchmark. For the BS model, volatility is constant across all exercise prices and maturities, although consistent theoretically is perhaps too restrictive in practice. Since he GARCH model has four more parameters than the BS model, it may have an unfair advantage over the BS model. To account for the sneer patterns in Black-Scholes implied volatilities, many market makers simply smooth the implied volatility relation across exercise prices and maturities, and then value options using the smoothed relation. To harmony with this practice, we follow DFW (1998) to fit the Black-Scholes model to the reported structure of option prices each week using the following model (ad hoc Black-Scholes model) to describe the Black-Scholes implied volatility.

$$\sigma = a_0 + a_1 K + a_2 K^2 + a_3 \tau + a_4 \tau^2 + a_5 K \tau \qquad (2.3.12)$$

σ is the implied volatility for an option of exercise price K and time to maturity τ . Obviously, applying the ad hoc Black-Scholes model in this context is internally inconsistent because the Black-Scholes model is based on the assumption of constant volatility. Nevertheless, the procedure is a variation of what is applied in practice as a mean of predicting option prices.

The empirical analysis follows the methodology laid out below. First, we use maximum likelihood method with rolling sample procedure to estimate the parameters of the GARCH process. As for the ac hoc Black-Scholes model, we imply the volatility from the model of DFW (1998) and find its correspondent in-sample estimated parameters. Second, we use these parameters to compute volatility. Finally, the volatility obtained by the in-sample GARCH option pricing model and the ac hoc Black-Scholes model are used to price the out-of-sample options and their pricing accuracy is examined.

3. Empirical Results

3.1 Maximum Likelihood Estimation

The empirical analysis focuses mainly on the single factor (one lag) version of the GARCH model. We set $\Delta = 1$ as daily index returns are used to model the evolution of volatility. Unlike continuous time stochastic volatility models wherein the volatility process is unobservable, all the parameters that enter the pricing formula can be easily estimated directly from the history of asset prices through maximum likelihood estimation (MLE).

The maximum likelihood method is applied to the TAIEX daily return series before GARCH option pricing because of the following reasons. First, a prerequisite to apply the GARCH option model is that the return series of the underlying asset behaves like a GARCH process. Next, the conditional volatility applied in the GARCH option pricing models is obtained directly from the TAIEX index return series instead of inferred from the market option prices. Therefore, we have to obtain the spot volatility before the in-sample fit and out-of-sample pricing errors can be examined. Third, the conditional volatility of the underlying asset can be easily filtered by the maximum likelihood method for the GARCH model.

Table 2 shows the maximum likelihood estimates of the GARCH model, both when γ_1 is non-zero and when it is restricted to zero, on the daily closing index levels and the futures prices of the shortest maturity contracts from 01/02/2001 to 12/31/2003. The parameters are quite similar across the analysis of spot and futures data. The parameter that measures the degree of mean reversion (as given by $\beta_1 + \alpha_1\gamma_1^2 = 1$) is 0.9475 from the cash/spot data and 0.9294 from the futures data. The volatility of volatility, as measured by α_1, is 1.46e-5 from the spot data and 2.51e-5 from the futures data. The annualized long-run mean of volatility /

standard deviation is 26.42% from the spot data and 29.93% for the futures data. The skewness parameter γ_1 are both positive indicating that shocks to returns and volatility is negatively correlated.

But while we use a likelihood ratio test, the symmetric versions for spot and futures data aren't rejected which imply that the negative correlations between returns and volatility of TAIEX and TAIFEX aren't significant.

The likely reasons for the lower asymmetric effect may be the introduction of futures and options. The Taiwan Futures Exchange (TAIFEX) launched its first product on July 21, 1998. In 2002, the average daily trading volume reached 32,033 contracts, a 66.39% growth from 2001. The TAIEX Options has shown an exponential growth from third quarter 2002 since first traded in December 2001, and the daily trading volume has arrived at 136,935 contracts on May 30, 2003. Index futures and index options trading improve the liquidity and depth of the spot market. They provide reliable information and lower the response of bad news. The other reason may be the inexpensive transaction costs attract the noise traders in the spot market and put them form spot market to futures or option market, either reduce the asymmetric effect.

Both symmetric and asymmetric versions are available to value the option prices. To simplify the procedure, we choose the asymmetric version for pricing.

3.2 In-Sample Estimation

Researchers who use time series models prefer longer sample period; that's to say, the more the sample periods, the better are the estimations. Since the distribution of stock returns varies with time, we may sacrifice the characteristics of volatility clustering or the accuracy of volatility estimations for a longer sample period. Thus, we use rolling sample procedure to update the parameters of the GARCH model. At each time we used the time-series of returns from the previous

252 days to filter the variance.

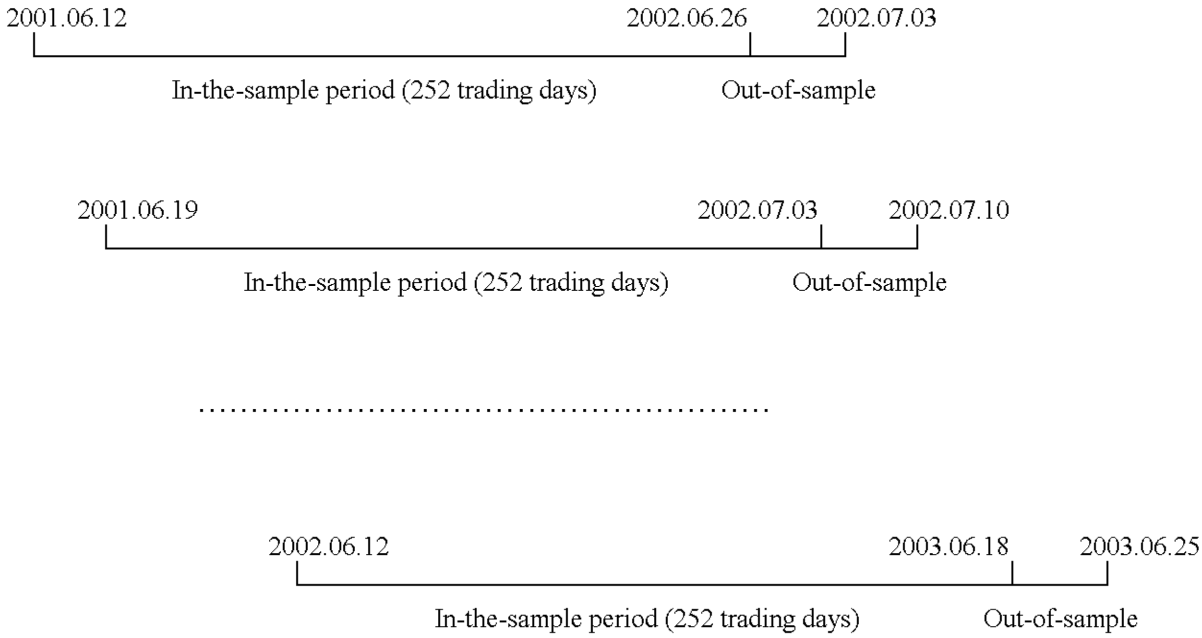

Table 3 represents the estimates form the updated GARCH model using maximum likelihood estimation. We then plug the parameter estimates obtained from the above MLEs into the options valuation formula to compute option values.

3.3 Out-of-Sample Pricing

We evaluate the updated out-of-sample prediction based on the following steps. For the GARCH option pricing model, we assume that the estimates are constant over one week. Then, we input the previous week's estimates to the GARCH option pricing model and compute current week's model-determined option prices. For the ad hoc Black-Scholes model, we use the implied volatility of the previous week to value the current option prices by assuming that the implied volatility does not change over one week. Both models predict one week ahead.

Let e (i, t) denote the model error in valuing option i at time t, e (i, t) is the difference between the model value of option and the market price of that option at time t. To assess the fitness, we present the mean absolute error (MAE), mean percentage error (MPE), and root mean squared error (RMSE) for each model respectively, where C_M is the market price of option and N is the number of options traded on Wednesday (or the next trading day).

$$\text{MAE} = \frac{1}{N}\sum \left| e(i,t) \right| \tag{3.3.1}$$

$$\text{MPE} = \frac{1}{N}\sum \frac{e(i,t)}{C_M} \tag{3.3.2}$$

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum e(i,t)^2} \tag{3.3.3}$$

Table 4 reports the out-of-sample valuation errors for the various models aggregated across all three out-of-sample periods. The aggregate root mean squared valuation errors are \$24.21 and \$14.42 for ad hoc BS and the updated GARCH model respectively. It also reports out-of-sample mean absolute error (MAE) that measures the absolute values of the valuation errors for all options. The aggregate out-of-sample MAE's are \$19.29 and \$12.00 for ad hoc BS and the updated GARCH model respectively. And MPE is 25.94% for ad hoc BS and 20% for the updated GARCH model.

The valuation errors by different option moneyness and maturity categories are shown in Table 5. Looking at the valuation errors by moneyness and maturity, we find that the GARCH model is able to value deep out-of-the-money options (K/S > 1.05 for calls) better for all maturities than the ad hoc BS.

For example, the RMSE for deep out-of-the-money calls (K/S > 1.05) that have less than thirty days to maturity is NT$10.26 for the updated GARCH versus NT$21.55 for the ad hoc BS. For near-the-money options, the results are mixed. For short-term (≦ 30 days to expiration) near-the-money (0.99 ≦ K/S < 1.01) call options, the ad hoc BS model has higher valuation errors than both versions of the GARCH. For medium-term (> 30 days) near-the-money calls, the updated GARCH has lower valuation errors than the ad hoc BS while for medium-term near-the-money call options.

In terms of maturity only, the percentage valuation errors under the GARCH tend to decrease with an increase in maturity, especially for out-of-the-money options. Short-term (≦ 30 days to expire) out-of-the-money options often tend to be the most difficult to value (in terms of percentage valuation error) under both GARCH and the ad hoc BS, especially for GARCH, the magnitude of valuation errors under the GARCH is substantially higher.

4. Conclusion

The empirical performance of the GARCH option pricing model and the ad hoc Black-Scholes model on TAIEX options has been evaluated in this article. The GARCH option pricing model outperforms the ad hoc Black-Scholes model in term of out-of-sample pricing according to the in-sample estimates. The ad hoc BS model uses a separate implied volatility for each option (specific to its strike and time to maturity) extracted from market prices and is designed to produce a very close fit to the shape of the implied volatilities across strike prices and maturities; also it is updated every period. In contrast, the GARCH model filters the volatility from the history of asset prices and uses rolling sample procedure to obtain its parameters.

Although the GARCH option pricing model outperforms the ad hoc Black-Scholes model, the magnitude of valuation errors of out-of-the-money options is substantially high and is most difficult to value. There are some likely explanations listed below.

First of all, model selection is subjective and could lead to determinated errors. Wang (1995) employed AGARCH model, EGARCH model, GJR-GARCH model, NGARCH and VGARCH asymmetric models to investigate the asymmetric effect in TAIEX return series. He found that GJR-GARCH (1,1) captured the asymmetry best. However, there is no closed-form solution for GJR-GARCH model provided.

Secondly, the data set is only sampled by closing prices every Wednesday (or the next trading day if Wednesday is a holiday) and used the time-series of returns from the previous 252 days to filter the variance. How to estimate in-sample is the key to price and it also could lead to estimated errors.

Third, in this article we filter the volatility from the history of asset prices through maximum likelihood estimation. However, the weakness of the method is that it contains only information in the historical TAIEX index prices. This information set is not necessarily the same as market option prices since the information in daily index time series is backward-looking while market option prices are forward- looking. In fact, when we try to value option prices in this way, the average pricing error is ridiculously high. Since the H-N GARCH model has a closed-form solution for option values, a natural candidate for parameter estimation is a non-linear least squares (NLS) procedure that tries to match model option values to observed option prices as closely as possible. These extensions are left for future research.

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Table 1 Sample Properties of TAIEX Options

	Moneyness	Time to maturity		Subtotal
	K/S	≤ 30	>30	
Deep in-the-money	< 0.95	362.7396	407.4598	0.2232
		71.3604	68.4616	
In-the-money	[0.95, 0.99)	197.9770	273.8046	0.2122
		53.2919	53.4470	
At-the-money	[0.99, 1.01)	116.6364	189.4889	0.1085
		37.7594	39.4914	
Out-of-the-money	[1.01, 1.05)	65.1425	131.9655	0.2122
		35.8102	40.7307	
Deep out-of-the-money	≥ 1.05	25.7922	76.8571	0.2439
		21.7110	31.7545	
Subtotal		0.4963	0.5037	1

The reported numbers are respectively the average option prices for each moneyness-maturity category while the proportion of the number of observations in this category to the number of all the call options in the whole sample are shown in parentheses. The sample period extends every Wednesday (or the next trading day if Wednesday is a holiday) from 07/01/2002 to 06/30/2003. S denotes the spot TAIEX index level and K is the exercise price.

Table 2-a Maximum Likelihood Estimation

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1+\alpha_1\gamma_1^2$	Log-Likelihood
GARCH(spot)	1.46E-05	0.9475	0.1605	0.00E+00	0.2162	26.42%	0.9475	2656.7008
	5.73E-06	0.0291	11.5490	2.91E-02	0.2867			
GARCH, γ_1 = 0(spot)	1.52E-05	0.9452		0.00E+00	0.0771	26.48%	0.9452	2656.3709
	5.84E-06	0.0296		6.25E-06	0.2750			
GARCH(futures)	2.51E-05	0.9294	0.1526	0.00E+00	0.1161	29.93%	0.9294	2576.6032
	7.28E-06	0.0254	8.5572	6.98E-06	0.2192			
GARCH, γ_1 = 0(futures)	2.64E-05	0.9260		0.00E+00	0.0193	29.99%	0.9260	2576.1945
	7.59E-06	0.0262		7.28E-06	0.2145			

Maximum Likelihood Estimates of the GARCH model with p = q = 1 and $\Delta = 1$ (day) using the spot/cash TAIEX levels and TAIFEX futures prices for the unrestricted ($\gamma_1 \neq 0$) and restricted ($\gamma_1 = 0$) model.

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t)$$

$$h(t) = \omega + \beta_1 h(t-\Delta) + \alpha_1\left(z(t-\Delta) - \gamma_1\sqrt{h(t-\Delta)}\right)^2$$

The log-likelihood function is $\sum_{t=1}^{T} -0.5\left(\log(h(t)) + z(t)^2\right)$, where T is the number of days in the sample. The daily closing index levels form 01/02/2001~ 12/31/2003 are used. The futures prices are those of the shortest maturity contracts. Number of observations = 741. Asymptotic standard errors appear in parentheses. θ is the annualized long run volatility (standard deviation) implied by the parameter estimates defined to be equal to $\sqrt{252(\omega+\alpha_1)/(1-\beta_1-\alpha_1\gamma_1^2)}$. $\beta_1+\alpha_1\gamma_1^2$ measures the degree of mean reversion in that $\beta_1+\alpha_1\gamma_1^2 = 1$ implies that the variance process is integrated

Table 2-b Maximum Likelihood Estimation

γ_1	α_1	β_1	γ_1	ω	λ	θ	$\beta_1+\alpha_1\gamma_1^2$	Log-Likelihood	# of observations
2001									
GARCH(spot)	2.00E-05	0.9442	0.3141	0.00E+00	0.8143	30.08%	0.9442	835.3569	244
	1.33E-05	0.0695	18.0704	2.27E-05	0.6541				
GARCH, γ_1 =0(spot)	1.79E-05	0.9497		0.00E+00	0.9588	29.95%	0.9497	835.6522	244
	1.20E-05	0.0585		7.15E-01	0.7149				
2002									
GARCH(spot)	2.70E-06	0.9618	10.2420	7.85E-06	5.0266	26.46%	0.9620	872.9950	248
	3.21E-06	0.0420	24.4445	1.42E-05	6.6105				
GARCH, γ_1 =0(spot)	1.84E-06	0.9791		3.93E-06	6.1007	26.38%	0.9791	872.5234	248
	2.14E-06	0.0324		1.08E-05	7.6993				
2003									
GARCH(spot)	4.98E-06	0.9688	0.1438	5.00E-10	-0.2665	20.06%	0.9688	947.4773	249
	5.83E-06	0.0470	43.4945	4.57E-06	0.5952				
GARCH, γ_1 =0(spot)	5.18E-06	0.9676		5.00E-10	-0.0564	20.08%	0.9676	947.5188	249
	4.81E-06	0.0416		4.65E-06	0.5568				

Maximum Likelihood Estimates of the GARCH model with p = q = 1 and $\Delta = 1$ (day) using the spot TAIEX levels for the unrestricted ($\gamma_1 \neq 0$) and restricted ($\gamma_1 = 0$) model in 2001, 2002, and 2003, respectively.

Table 3 Parameter Estimates From The Weekly Estimation

This table reports the parameter estimates from the weekly (every week) estimation of the GARCH model. Note however that the last Wednesday of the first half of each year appears in this sample. Variance, h(t +1) is drawn from the daily history (last 252 days) of index levels.

	α_1	β_1	γ_1	ω	λ		α_1	β_1	γ_1	ω	λ
2002/7/3	1.97E-05	0.9471	0.1041	1.32E-08	0.7700	2003/1/2	2.56E-06	0.9539	5.8589	1.59E-06	4.4741
2002/7/10	1.65E-05	0.9571	0.1304	6.40E-09	0.8369	2003/1/8	2.20E-06	0.9823	4.4209	2.40E-06	4.6492
2002/7/17	1.50E-05	0.9607	0.1858	0.00E+00	0.9229	2003/1/15	1.81E-05	0.9098	21.4740	6.51E-06	0.7467
2002/7/24	1.30E-05	0.9664	0.3459	0.00E+00	1.0178	2003/1/22	1.66E-05	0.9049	41.8601	3.74E-06	0.3094
2002/7/31	1.49E-05	0.9603	0.2085	3.30E-09	0.7320	2003/2/6	1.91E-06	0.8717	11.8400	3.36E-05	8.7195
2002/8/7	1.35E-05	0.9642	0.5731	0.00E+00	0.9164	2003/2/12	1.93E-05	0.9158	31.6137	9.00E-10	0.2592
2002/8/14	1.40E-05	0.9649	0.4237	0.00E+00	0.7370	2003/2/19	1.47E-05	0.8876	53.4416	7.47E-06	0.2475
2002/8/21	1.30E-05	0.9677	0.2766	0.00E+00	0.9233	2003/2/26	1.44E-05	0.8762	62.4654	7.51E-06	0.2060
2002/8/28	1.35E-05	0.9655	0.2685	4.90E-09	0.8417	2003/3/5	1.85E-05	0.9006	38.3234	4.64E-06	0.2327
2002/9/4	1.15E-05	0.9696	0.1936	0.00E+00	0.9233	2003/3/12	1.73E-06	0.9267	12.3677	1.87E-05	6.9007
2002/9/11	1.26E-05	0.9661	0.0877	1.80E-09	1.2842	2003/3/19	1.74E-05	0.8802	42.8563	9.90E-06	0.2473
2002/9/18	1.25E-05	0.9672	0.2678	1.50E-09	1.1539	2003/3/26	1.55E-05	0.8772	51.4939	1.08E-05	0.3184
2002/9/25	1.15E-05	0.9694	0.6534	1.10E-09	1.0680	2003/4/2	1.82E-05	0.8700	42.3614	1.31E-05	0.4087
2002/10/2	9.80E-06	0.9726	0.2516	8.00E-10	1.5096	2003/4/9	7.46E-06	0.8771	18.9099	2.99E-05	2.5697
2002/10/9	8.93E-06	0.9733	0.0458	8.00E-10	1.3761	2003/4/16	6.71E-06	0.8827	18.1032	2.90E-05	2.7294
2002/10/16	4.42E-06	0.9848	0.1923	1.90E-09	1.9691	2003/4/23	6.73E-06	0.8571	18.3082	3.66E-05	2.8350
2002/10/23	4.08E-06	0.9870	0.0929	8.00E-10	2.3312	2003/4/30	6.99E-06	0.8645	20.2257	3.42E-05	2.4467
2002/10/30	1.50E-06	0.9887	0.4763	1.81E-06	5.4352	2003/5/7	5.85E-06	0.8626	19.8262	3.62E-05	2.9462
2002/11/6	9.01E-06	0.8977	2.8554	2.73E-05	3.7375	2003/5/14	2.06E-06	0.8770	13.5197	3.52E-05	7.5079
2002/11/13	1.46E-06	0.9815	3.3396	4.15E-06	6.9573	2003/5/21	3.63E-06	0.8733	9.1573	3.62E-05	4.0754
2002/11/20	2.28E-06	0.9804	2.6369	3.79E-06	5.3143	2003/5/28	1.33E-05	0.8477	36.6811	2.82E-05	1.0250
2002/11/27	2.66E-06	0.9784	4.0774	4.21E-06	5.0757	2003/6/5	1.14E-05	0.8822	34.4599	2.14E-05	0.7801
2002/12/4	2.42E-06	0.9728	4.5939	6.04E-06	5.5949	2003/6/11	1.06E-05	0.8891	26.4099	2.17E-05	0.4099
2002/12/11	2.09E-06	0.9675	2.9622	7.61E-06	5.7287	2003/6/18	1.15E-05	0.9158	26.4825	1.18E-05	-0.0719
2002/12/18	3.17E-06	0.9803	5.1116	3.07E-06	4.0804	2003/6/25	1.37E-05	0.9384	28.2814	1.35E-06	-0.4861
2002/12/25	6.25E-06	0.9246	9.6908	2.19E-05	5.1631						
Averge	**9.80E-06**	**0.9302**	**14.3292**	**1.21E-05**	**2.3703**	Standard deviation	**5.77E-06**	**0.0450**	**17.0957**	**1.27E-05**	**2.3247**

Table 4 Aggregate Valuation Errors Across Various Models

Model	Measure Error		
	RMSE	MAE	MPE
GARCH	14.4197	12.0009	0.2000
ac hoc BS	24.2139	19.2910	0.2594

Table 5 Out-of-Sample Pricing Errors by Moneyness and Maturity

Moneyness K/S	Model	Time to maturity ≤ 30	Time to maturity > 30
Panel A: RMSE			
< 0.95	GARCH	10.2617	16.1409
	ac hoc BS	21.5490	27.7669
[0.95, 0.99)	GARCH	13.3932	18.7983
	ac hoc BS	23.0901	30.8379
[0.99, 1.01)	GARCH	12.9021	20.1889
	ac hoc BS	20.3140	31.1583
[1.01, 1.05)	GARCH	10.5416	18.5720
	ac hoc BS	17.0482	29.4288
≥ 1.05	GARCH	6.7572	14.7690
	ac hoc BS	11.6219	25.4467
Panel B: MAE			
< 0.95	GARCH	8.7161	14.0570
	ac hoc BS	16.6982	22.7068
[0.95, 0.99)	GARCH	12.1047	16.1891
	ac hoc BS	18.7341	26.5426
[0.99, 1.01)	GARCH	11.4444	18.1676
	ac hoc BS	16.5396	26.2916
[1.01, 1.05)	GARCH	9.0187	16.3402
	ac hoc BS	13.9909	24.8609
≥ 1.05	GARCH	5.2579	12.8144
	ac hoc BS	8.8900	21.1700
Panel C: RMPE			
< 0.95	GARCH	0.0264	0.0369
	ac hoc BS	0.0514	0.0581
[0.95, 0.99)	GARCH	0.0688	0.0650
	ac hoc BS	0.1065	0.1003
[0.99, 1.01)	GARCH	0.1140	0.1055
	ac hoc BS	0.1684	0.1430
[1.01, 1.05)	GARCH	0.2201	0.1487
	ac hoc BS	0.3269	0.1996
≥ 1.05	GARCH	0.8160	0.2202
	ac hoc BS	0.7534	0.2847

Figure 1 Daily TAIEX Index Level (from 07/01/2001 to 06/30/2003)

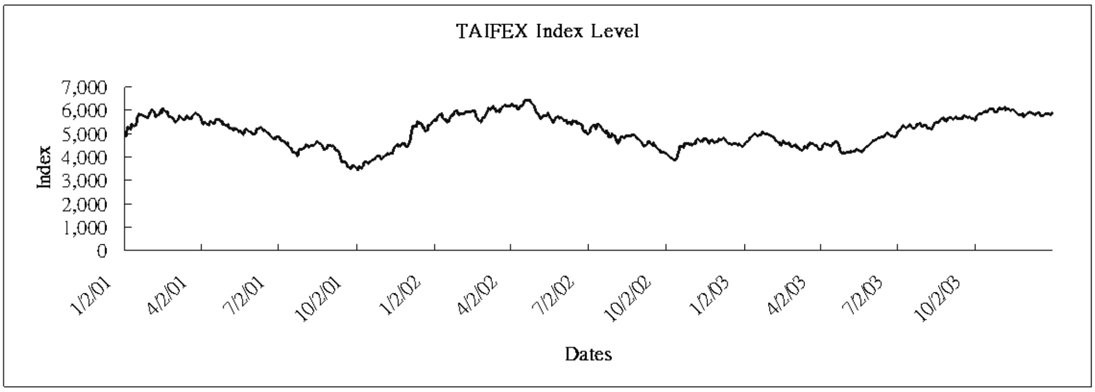

Figure 2 Daily TAIEX Index Return Rate (from 07/01/2001 to 06/30/2003)

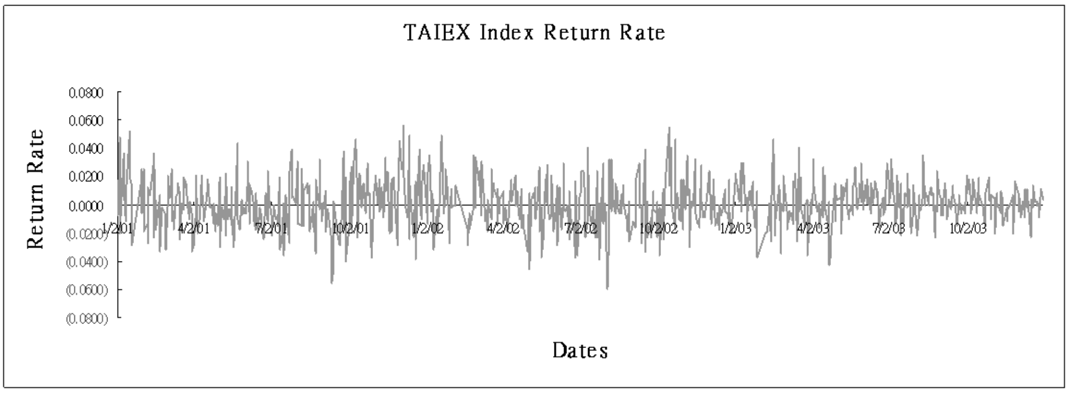

Figure 3 Probabilities of TAIEX Index Returns (from 01/02/2001~ 12/31/2003)

It shows than TAIFEX index returns distribute fat-tailed and leptokurtosis.

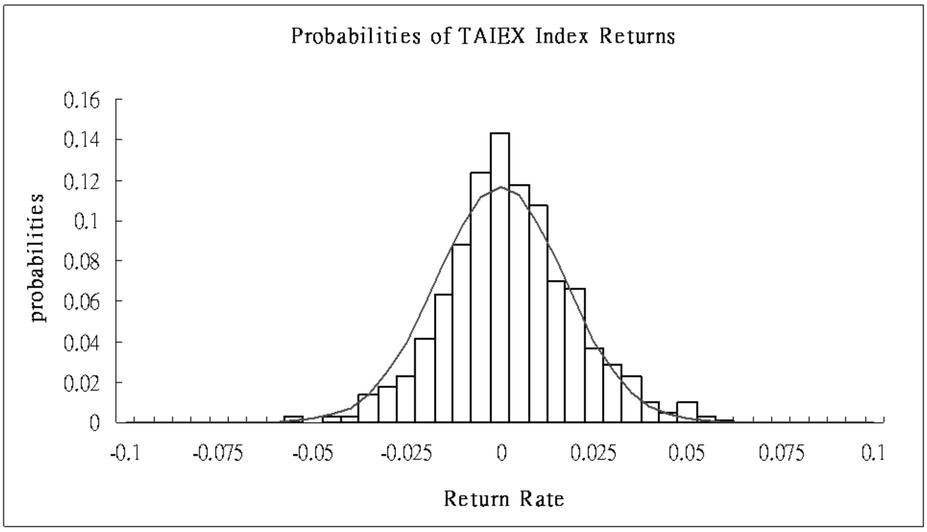

Figure 4 TAIEX Options Implied Volatilities

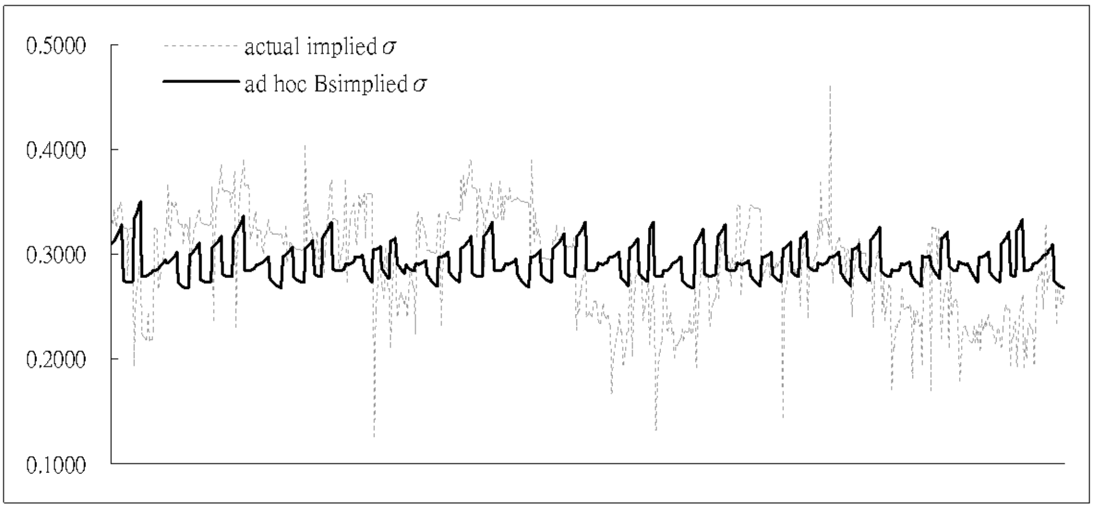

Figure 5-a Daily Annualized Spot Volatilities (asymmetric version)

This figure shows the daily annualized spot volatilities form the asymmetric GARCH model.

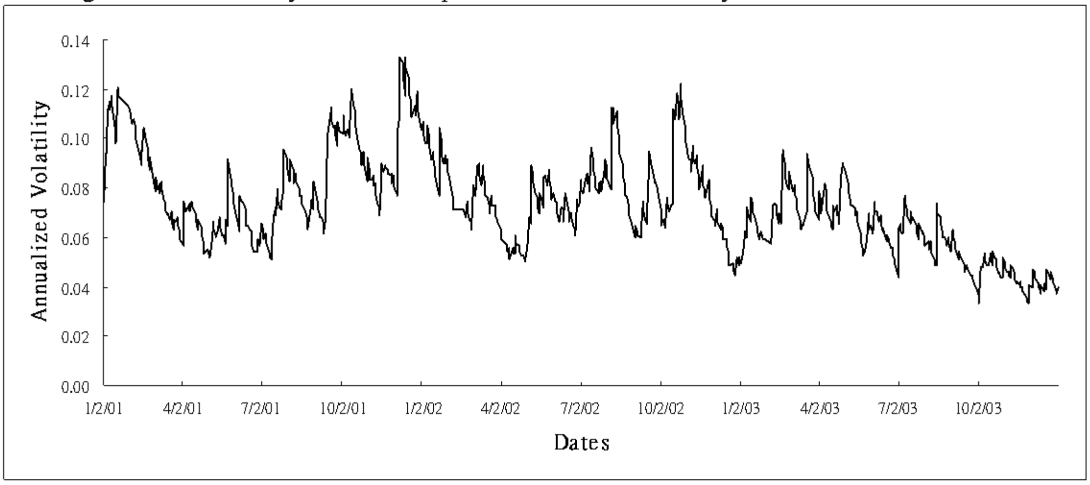

Figure 5-b Daily Annualized Spot Volatilities (symmetric version)

This figure shows the daily annualized spot volatilities form the symmetric GARCH model.

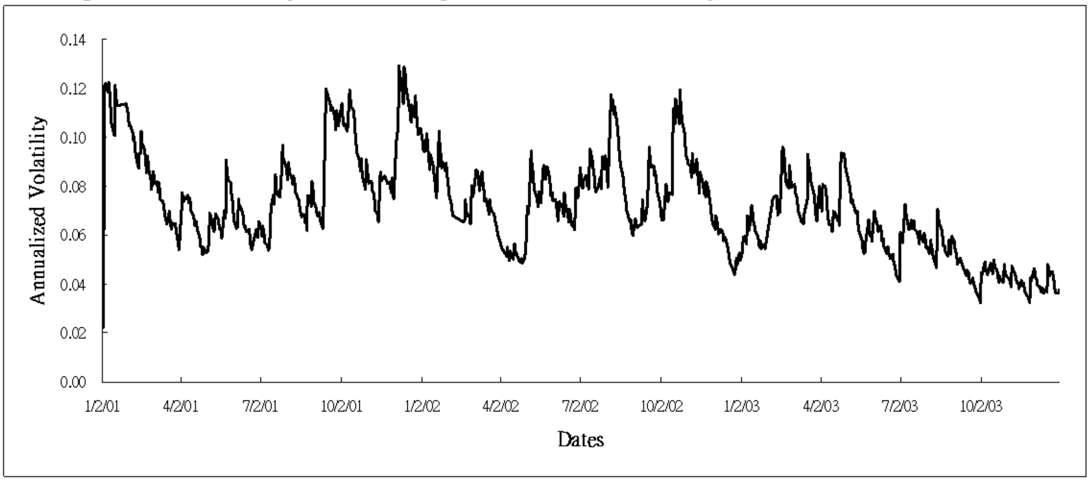

Figure 6 Relative Pricing Errors for Call Options (less than 30 days to maturity)

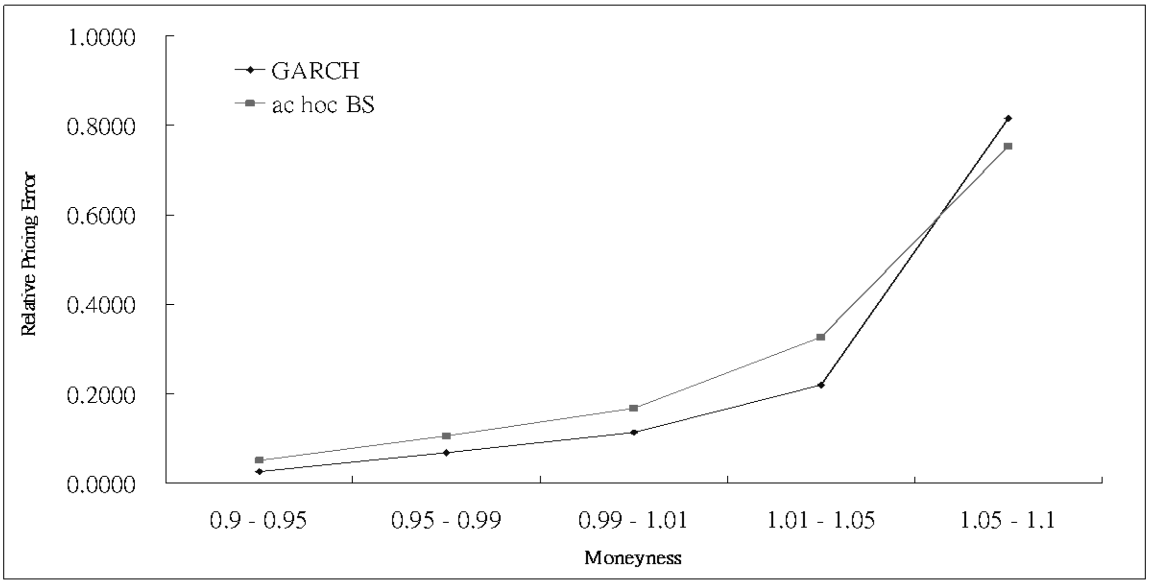

Figure 7 Relative Pricing Errors for Call Options (more than 30 days to maturity)