What Does the Market Price Risk Tell us in the Single Factor Interest Rate Model ?

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Abstract. We show that the market price of risk has an important role of keeping the consistency in several well-known interest rate models.

1. Introduction

The empirical behaviour of interest rates has been widely studied (see for instance Fama and French (89), (91)) and various empirical regularities have been observed. Empirical distributions of bond yields are highly volatile but not highly skewed and expected excess returns are close to zero, however the slope of the term structure predicts a relatively large amount of the variation in excess returns. In particular, there is a strong negative correlation between the slope of the yield curve and excess returns to long bonds: long rates tend to fall when the slope becomes steeper.

Interest rate models driven by standard Brownian motions seem unable to account for these stylised facts. This has been blamed on the form of the market price of risk used to transform between the physical pricing measure and the risk-neutral pricing measure. In the case of affine term structure models, Dai and Singleton (2002) found problems in using the standard form of the market price of risk traditionally used in these models, described by Dai and Singleton (2000) and Duffie and Kan (1996)), to explain empirical behaviour.

We study the problem from a different perspective. Lévy processes have been widely used to fit both to returns distributions under the physical measure and in derivative pricing models under the risk-neutral measure. Time series properties of empirical returns processes have investigated by modelling them as time-changed Brownian motions (Sato (2001) and Carr and Wu (2000)), which also reduce to modelling by Lévy processes. Lévy processes are suitable for these applications since they can capture the heavy tailed behaviour displayed by asset returns processes, fitting both to returns distributions and to implied volatility smiles implied from options prices.

Popular models use the generalised hyperbolic process (Eberlein (2000)) based on the hyperbolic distribution of Barndorff-Nielsen (1978), the normal inverse Gaussian

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process (Barndorff-Nielsen (1995)) and the variance-gamma process (Madan, Carr and Chang (1998)).

Models which attempt to relate the physical measure to the risk neutral measure must specify the market prices of risk that determine the relationship between the two measures. Although the general form of the change of measure is well known, few papers have investigated the change of measure for interest rate models and their implications for the way a model can fit to empirical facts about the behaviour of rates. Unlike models based on Wiener processes, where there may be a unique martingale measure, the jump component of a Lévy process allows different in-equivalent specifications of Lévy-Sheffer changes of measure.

In the context of a Vasicek type Lévy interest rate model, we investigate changes of measure of various types of Lévy-Sheffer type, which generalises the Esscher transform. Depending on the choice of Lévy-Sheffer type, we find that some Lévy models are quite tractable for financial modelling and some other types of models produce the same effects of the multi-factor model. Furthermore, we find that if the true data generating process of interest rates have jumps with stochastic market price risk, the short rate process is equivalent to the polynomials of some state variable.

2. Change of Measure

2.1. Changes of Measure of Exponential Type. Consider a class

$$\mathcal{P} = \{\mathbb{P}_z \mid z \in \Theta\}, \ \Theta \subseteq \mathbb{R}^k$$

of probability measures on $(\Omega, \mathcal{F})$. The class $\mathcal{P}$ is called an exponential family on $(\Omega, \mathcal{F})$ if there exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that for all $z \in \Theta$ there is a measure $\mathbb{P}_z \in \mathbb{R}$ such that $\mathbb{P}_z^t \ll \mathbb{P}^t$ for all $t \geq 0$ and

$$\frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} = f_t(z)\, q_t \exp\left(u_t'(z)\, L_t\right), \tag{1}$$

where $f_t(z)$ and $u_t(z)$ are deterministic functions and q_t and L_t are stochastic processes (Küchler and Sørensen(1994)). Lévy-Sheffer changes of measures are of this general exponential form.

2.2. Time Inhomogeneous Lévy-Sheffer Changes of Measure. For a Lévy process L we can define a change of measure by

$$\frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} = \exp(zL_t - t\varphi(z)), \tag{2}$$

the natural exponential family generated by L. This defines the Esscher transform, discussed below.

The definition can be generalised (Schoutens and Teugels (1998)). We define a Lévy-Sheffer change of measure to be of the form

$$\frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} = f(z)^t \exp\left(u(z)\, L_t\right) \tag{3}$$

where L is 1-dimensional and $f(z) = \frac{1}{\Psi(-iu(z))}$ is obtained from the characteristic function Ψ of L. Note that

$$f(z)^t \exp(u(z) L_t) = \exp(u(z) L_t - t\varphi(u(z))) \tag{4}$$

so that when $u(z) = z$ we recover the natural exponential family.

A Lévy-Sheffer change of measure is strongly related to Lévy-Sheffer polynomial systems (Schoutens and Teugels (1998)).

Define $K_t = \int_0^t u_s(z)\, \mathrm{d}L_s$ then,

$$\mathbb{E}[\exp(hK_t)] = \mathbb{E}\left[\exp\left(\int_0^t hu_s(z)\, \mathrm{d}L_s\right)\right] = \exp\left(\int_0^t \varphi^L(hu_s(z))\, \mathrm{d}s\right) \tag{5}$$

where φ^L is the log-moment generating function of L. So the log-moment generating function $\varphi^K(h)$ of KK_t is $\varphi^K(h) = \int_0^t \varphi^L(hu_s(z))\, \mathrm{d}s$. Hence we can define a change of measure by

$$\frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} = \exp\left(K_t - \varphi^K(1)\right) \tag{6}$$

$$= \exp\left(\int_0^t u_s(z)\, \mathrm{d}L_s - \int_0^t \varphi^L(u_s(z))\, \mathrm{d}s\right) \tag{7}$$

We call this a time inhomogeneous Lévy-Sheffer change of measure. We have also assumed that the Lévy process L always satisfies the integrability condition.

3. Model

We start Vasicek type Levy model. Under P, we assume that short rate follows

$$dr(t) = a(\overline{r} - r(t))dt + \sigma \mathrm{d}L_t \tag{8}$$

where we denote by $L = (L_s)_{s\geq 0}$ a Levy process, a stochastic process with stationary and independent increments which is continuous in probability and satisfies $L_0 = 0$ a.s. The strong solution of (8) is

$$r(t) = (1 - e^{-at})\overline{r} + e^{-at} r(0) + \sigma e^{-at} \int_0^t e^{av} \mathrm{d}L_v \tag{9}$$

The price of the pure discount is

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)\mathrm{d}u\right) |\mathcal{F}_t\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T r(u)\mathrm{d}u\right) \frac{\mathbf{dQ}}{\mathbf{dP}} |\mathcal{F}_t\right]$$

where the Radon-Nikodym derivative is not unique. In next section, we provide a bond pricing formula using the time inhomogeneous Lévy-Sheffer changes of measure

4. BOND PRICING

Theorem 1. *When the interest rate follows (8) and the Radon-Nikodym derivative has a form of time inhomogeneous Lévy-Sheffer changes of measure, then time t price of a discount bond that promises to pay one unit currency at maturity T is*

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right)$$

where

$$\begin{aligned} A(t,T) &= \exp\left(-\overline{r}(T-t)\right) \\ &\times \exp\left(\frac{\left(e^{-aT} - e^{-at}\right)\left(\overline{r} - r(0)\right) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\overline{r} + e^{-at}r(0)\right)}{a}\right) \\ &\times \exp\left(\int_t^T \varphi^L\left(f(v) + u_v(z)\right)\mathrm{dv} - \int_t^T \varphi^L\left(u_v(z)\right)\mathrm{dv}\right) \end{aligned}$$

and

$$B(t,T) = \frac{\left(e^{-a(T-t)} - 1\right)}{a}$$

5. STOCHASTIC LÉVY-SHEFFER CHANGES OF MEASURE: $\alpha-$STABLE LEVY PROCESS

Time inhomogeneous Lévy-Sheffer changes of measures are of the general exponential form. This section focuses on the stochastic version of .Lévy-Sheffer changes of measures. This gives the stochastic feature of the market price of risk. To do this we need some preliminaries.. We use notations from Jacod and Shiryaev (1986). We modify the theorem of Kallsen and Shiyaev (2001).

Theorem 2. *Let L be an $\alpha-$stable Levy motion. Moreover let $M \equiv H \cdot L$ for some nonnegative process H such that $\int_0^t H_s^\alpha ds \rightarrow\ \infty$ as $t \rightarrow\ \infty$. Then there exists a filtration $(\widehat{\mathcal{F}}_\theta)_{\theta\in\mathbb{R}_+}$ on $(\Omega, \mathcal{F})$, a process $(\widehat{L}_\theta)_{\theta\in\mathbb{R}_+}$, and a finite time change $(\widehat{T}_t)_{t\in\mathbb{R}_+}$ such that (1) $\widehat{L}$ is a $(\widehat{T}_t)_{t\in\mathbb{R}_+}$ adapted Lévy-process on $\left(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_\theta)_{\theta\in\mathbb{R}_+}, \mathbb{P}\right)$ with Law($\widehat{L}$)=Law(L) and (2) $M = (\widehat{L}_{\widehat{T}_t})_{t\in\mathbb{R}_+}$.If we define the $\left(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_\theta)_{\theta\in\mathbb{R}_+}, \mathbb{P}\right)$-time change $(\widehat{T}_\theta)_{\theta\in\mathbb{R}_+}$ by $T_\theta \equiv \inf\{t \in \mathbb{R}_+ : \int_0^t H_s^\alpha ds > \theta\}$, we may choose $\widehat{\mathcal{F}}_\theta = \mathcal{F}_{T_\theta}$, and $\widehat{T}$ as the inverse time change of T. In particular, $\widehat{T}_t = \int_0^t H_s^\alpha ds$.Then the following equalities hold:*

$$\begin{aligned} \frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} &= \exp\left(z\int_t^T H_v \mathrm{d}L_s - \int_t^T \varphi^L\left(zH\right)_s \mathrm{d}s\right) \\ &= \exp\left(z\int_t^T \mathrm{d}\widehat{L}_{\widehat{L}_{\widehat{T}_s}} - \int_t^T \varphi^{\widehat{L}_{\widehat{T}_s}}\left(z\right)\mathrm{d}s\right) \\ &= \exp\left(z\int_t^T H_v \mathrm{d}L_s - \int_t^T \mathcal{L}_{\widehat{T}_s}^{Q(z)}\left(\varphi^L\left(z\right)\right)\mathrm{d}s\right) \end{aligned} \qquad (10)$$

The new class of measure $\mathbf{Q}(\mathbf{z})$ *are absolutely continuous with respective to* $\mathbf{P}$ *and is defined by*

$$\frac{\mathrm{d}\mathbf{Q}(\mathbf{z})_t}{\mathrm{d}\mathbf{P}_t} = \exp\left(z\widehat{L}_{\widehat{T}_t} + \widehat{T}_t\varphi^L(z)\right)$$

We apply this result to bond pricing.

Theorem 3. *When the interest rate follows (8) with* L $\alpha-$*stable Levy Process and the Radon-Nikodym derivative has a form of stochastic Lévy-Sheffer changes of measure, then time* t *price of a discount bond that promises to pay one unit currency at maturity* T *is*

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right)$$

where

$$\begin{aligned} A(t,T) &= \exp\left(-\overline{r}(T-t)\right) \\ &\times \exp\left(\frac{\left(e^{-aT} - e^{-at}\right)\left(\overline{r} - r(0)\right) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\overline{r} + e^{-at}r(0)\right)}{a}\right) \\ &\times \exp\left(\int_t^T \mathcal{L}_{\widetilde{T}_v}^{\widetilde{Q}(z)}\left(\varphi^L(1)\right)\mathrm{d}v - \int_t^T \mathcal{L}_{\widehat{T}_v}^{Q(z)}\left(\varphi^L(z)\right)\mathrm{d}v\right) \end{aligned}$$

,

$$B(t,T) = \frac{(e^{-a(T-t)} - 1)}{a}$$

and $\mathcal{L}_{\widetilde{T}_v}^{\widetilde{Q}(z)}$ *is the Laplace cumulant of* $\widetilde{T}_v$ *under the measure* $\widetilde{Q}(z)$, *where the new class of measure* $\widetilde{\mathbf{Q}}$ *are absolutely continuous with respective to* $\mathbf{P}$ *and is defined by*

$$\begin{aligned} \frac{\mathrm{d}\widetilde{\mathbf{Q}}_t}{\mathrm{d}\mathbf{P}_t} &= \exp\left(\widehat{L}_{\widetilde{t}_t} - \widetilde{T}_t\varphi^L(1)\right) \\ &= \exp\left(\int_t^T G_v L_v - \int_t^T \varphi^L(G)_s\,\mathrm{d}s\right) \end{aligned}$$

and

$$\widehat{T}_t = \int_0^t G_s^\alpha \mathrm{d}s,\ G_v = f(v) + zH_v$$

Although $\alpha-$stable Levy Process has some advantage of the representation of the bond pricing formula, there is still a problem of the computation $\widetilde{T}_v$. This means that it is quite restrictive to choose the process H for obtaining tractable pricing formula.

6. STOCHASTIC AND TIME VARYING VOLATILITY MODEL

In this section, we incorporate stochastic or time varying volatility into the single factor model. Under P, short rate is

$$dr(t) = a(\overline{r} - r(t))dt + \sigma dY_t \tag{11}$$

where

$$Y_t \equiv L_{T_t}$$

and T_t is an increasing right-continuous process with left limit such that for each fixed t, the random variable T_t is a stopping time with respect to $(\widehat{\mathcal{F}}_\theta)_{\theta\in\mathbb{R}_+}$. Suppose that T_t is finite $\mathbb{P}$ a.s. for all t$\geqq 0$ and that $T_t \to \infty$ as t$\to \infty$. Then the family of stopping time $\{T_t\}$ defines a stochastic time change.

6.1. Time Varying Market Price of Risk.

Theorem 4. *When the interest rate follows (11) and the Radon-Nikodym derivative has a form of time inhomogeneous Lévy-Sheffer changes of measure, then time t price of a discount bond that promises to pay one unit currency at maturity T is*

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right)$$

where

$$\begin{aligned} A(t,T) &= \exp\left(-\overline{r}(T-t)\right) \\ &\times \exp\left(\frac{\left(e^{-aT} - e^{-at}\right)(\overline{r} - r(0)) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\overline{r} + e^{-at}r(0)\right)}{a}\right) \\ &\times \exp\left(\int_t^T \mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L\left(f(v) + u_v(z)\right)\right)\mathrm{d}v - \int_t^T \mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L\left(u(z)\right)\right)\mathrm{d}v\right) \end{aligned}$$

and

$$B(t,T) = \frac{(e^{-a(T-t)} - 1)}{a}$$

6.2. Stochastic Market Price of Risk.

Theorem 5. *When the interest rate follows (11) with L $\alpha-$stable Levy Process and the Radon-Nikodym derivative has a form of stochastic Lévy-Sheffer changes of measure, then time t price of a discount bond that promises to pay one unit currency at maturity T is*

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right)$$

where

$$A(t,T) = \exp\left(-\overline{r}(T-t)\right) \times \exp\left(\frac{\left(e^{-aT}-e^{-at}\right)\left(\overline{r}-r(0)\right)-\left(e^{-a(T-t)}-1\right)\left((1-e^{-at})\overline{r}+e^{-at}r(0)\right)}{a}\right) \times \exp\left(\int_t^T \mathcal{L}_{\widetilde{T}_v}^{\widetilde{Q}(z)}\left(\mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L(1)\right)\right)\mathrm{d}v - \int_t^T \mathcal{L}_{\widetilde{T}_v}^{\widetilde{Q}(z)}\left(\mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L(z)\right)\right)\mathrm{d}v\right)$$

,

$$B(t,T) = \frac{\left(e^{-a(T-t)}-1\right)}{a}$$

and $\mathcal{L}_{\widetilde{T}_v}^{\widetilde{Q}(z)}$ *is the Laplace cumulant of* $\widetilde{T}_v$ *under the measure* $\widetilde{Q}(z)$, *where the new class of measure* $\widetilde{\mathbf{Q}}$ *are absolutely continuous with respective to* $\mathbf{P}$ *and is defined by*

$$\frac{\mathrm{d}\widetilde{\mathbf{Q}}_t}{\mathrm{d}\mathbf{P}_t} = \exp\left(\widehat{L}_{\widetilde{t}_t} - \widetilde{T}_t\varphi^L(1)\right) = \exp\left(\int_t^T G_v L_v - \int_t^T \varphi^L(G)_s \,\mathrm{d}s\right)$$

and

$$\widehat{T}_t = \int_0^t G_s^\alpha \mathrm{d}s,\ G_v = f(v) + zH_v$$

7. Stochastic Lévy-Sheffer Changes of Measure: Quasi-Left-Continuous Semimartingale

In this section, we provide a more general approach of producing multi-factor bond pricing formula in the single factor setting. To do this, we choose a general process of Radon-Nikodym in semimartingale process. Among the semimartingale process, Levy process is quasi-left-continuous. It means that the process has no fixed time discontinuity. Quasi-left-continuous semimartingales simplify the type of the Radon-Nikodym. The following theorem can be seen in Kallsen and Shiyaev (2001).

Theorem 6. *Let* H *be the predictable process such that* $H \cdot L$ *is exponentially special, where* L *is the quasi-left-continuous Levy process. Then* $K^L(H)$ *is the exponential compensator of* $H \cdot L$. *Then, we can define a change of measure by*

$$\frac{\mathrm{d}\mathbb{P}_z^t}{\mathrm{d}\mathbb{P}^t} = \exp\left(H \cdot L - \Phi^L(H)\right) \tag{12}$$

where

$$\Phi^L(H) = \varphi^L(H) \cdot t$$

,

$$\varphi^L(H)_t = H_t b + \frac{1}{2}H_t^2 c + \int_{\mathbf{R}} (e^{H_t x} - 1 - h(x)x)v(dx)$$

and $h : \mathbf{R} \to \mathbf{R}$ is some truncated function.

In this case, the bond price is

$$\begin{aligned} P(t,T) &= \exp\left(-\overline{r}(T-t) + \frac{(e^{-aT} - e^{-at})(\overline{r} - r(0))}{a}\right) \\ &\times \exp\left(\frac{(e^{-a(T-t)} - 1)\left(r(t) - (1 - e^{-at})\overline{r} - e^{-at}r(0)\right)}{a}\right) \\ &\times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^T f(s)\mathrm{d}L_{s_v}\right)\exp\left(\int_t^T H\mathrm{d}L_{s_v} - \int_t^T \varphi^L(H_s)\mathrm{d}s\right)|\mathcal{F}_t\right] \end{aligned}$$

Note that the following relationship holds:

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^T f(s)\mathrm{d}L_{s_v}\right)\exp\left(\int_t^T H\mathrm{d}L_{s_v} - \int_t^T \varphi^L(H_s)\mathrm{d}s\right)|\mathcal{F}_t\right] \\ = & \mathbb{E}^{\overline{\mathbf{Q}}}\left[\exp\left(\int_t^T \left(H_s f(s)c + \left(e^{f(s)} - 1\right)\int_{\mathbf{R}}\left(e^{H_s x}\right)v(\mathrm{d}x)\right)\mathrm{d}s\right)|\mathcal{F}_t\right] \qquad (13) \\ & \times \exp\left(\int_t^T \left(f(s)b + \frac{f(s)^2 c}{2} - \int_{\mathbf{R}}\left((g(x) - h(x))\,x\right)v(\mathrm{d}x)\right)\mathrm{d}s\right) \end{aligned}$$

where $\mathbb{E}[\cdot]$ and $\mathbb{E}^{\overline{\mathbf{Q}}}[\cdot]$ denote expectations under measure $\mathbf{P}$ and $\overline{\mathbf{Q}}$, respectively. The new class of measure $\overline{\mathbf{Q}}$ are absolutely continuous with respective to $\mathbf{P}$ and is defined by

$$\frac{\mathrm{d}\overline{\mathbf{Q}}_t}{\mathrm{d}\mathbf{P}_t} = \exp\left(\int_t^T (f(s) + H_s)\,\mathrm{d}L_{s_v} - \int_t^T \varphi^L(f(s) + H_s)\mathrm{d}s\right)$$

Note that unless the Levy measure is simple, the analytic solution of the expectation is not easy. We provide an example of a tractable case in the following proposition.

Proposition 7. *When L is Brownian motion and Z follows as*

$$dZ_t = \mu(Z_t)dt + \sigma(Z_t)dL_t$$

where Z is k-dimensional Markov process, $\mu(Z_t)$ is $k \times 1$ vector and $\sigma(Z_t)$ is $k \times k$ matrix. Then define

$$H_t = \mathbf{b}'Z_t + \mathbf{c}$$

,

$$\mu(Z_t) = \mathbf{a} - \kappa Z_t$$

and

$$\left(\sigma(Z_t)\sigma(Z_t)'\right)_{ii} = \alpha_i + \beta_i' Z_t$$
$$\left(\sigma(Z_t)\sigma(Z_t)'\right)_{ij} = 0, \ i \neq j$$

, then time t price of a discount bond that promises to pay one unit currency at maturity T is

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t) - b(t,T)'Z_t + c(t,T)\right)$$

where

$$\begin{aligned} A(t,T) &= \exp\left(-\overline{r}(T-t)\right) \\ &\times \exp\left(\frac{\left(e^{-aT} - e^{-at}\right)(\overline{r} - r(0)) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\overline{r} + e^{-at}r(0)\right)}{a}\right) \\ &\times \exp\left(\int_t^T \left(f(s)b + \frac{f(s)^2 c}{2}\right) \mathrm{d}s\right) \end{aligned}$$

,

$$B(t,T) = \frac{(e^{-a(T-t)} - 1)}{a}$$

and $b(t,T)$ and $c(t,T)$ are determined by the following ordinary differential equations:

$$\frac{\partial b(t,T)}{\partial t} = f(t)c\mathbf{b} - \boldsymbol{\kappa}' b(t,T) - \frac{\beta b(t,T)' b(t,T)}{2}$$
$$\frac{\partial c(t,T)}{\partial t} = f(t)c\mathbf{c} - b(t,T)'\mathbf{a} - \frac{\mathbf{a} b(t,T)' b(t,T)}{2}$$

with boundary conditions $b(T,T) = 0$, and $C(T,T) = 0$.

Note that the analytic forms of $b(t,T)$ and $c(t,T)$ can be obtained only under the special assumption. As seen in above proposition, when interest rates follow the affine type of continuous semimartingales with stochastic market price risk, the affine framework hold only under the affine type of market price of risk. As we expect naturally, when the market price has a form of quadratic processes as in Leippold and Wu (1998) and Ahn, Dittmar, and Gallant (2001), analytic forms of the bond pricing formula can be obtained. The affine framework, however, collapses in this case. Therefore, the affine model of Duffie and Kan (1996) is quite restrict in choosing the market price risk.

8. POSITIVE INTEREST RATE MODEL

In this section, we provide a new class of interest rate model in which positive interest rates are generated. It is well-known that CIR process or Bessel process produce positive interest rate. This, however, is only for the model tractability.. We perform the same purpose in the some Levy settings. Let L be a subordinator and $\tau = \tau(q)$ independent exponential time with parameters $q > 0$ Then the process $L^{(q)}$ takes values in $[0, \infty]$ and is given by

$$L_t^{(q)} = L_t \text{ if } t \in [0, \tau),\ L_t^{(q)} = \infty \text{ if } t \in [\tau, \infty)$$

which is called a subordinator killed at rate q. When the subordinator has zero killing rate,it is called a strict subordinator. Under P, we model short rate as

$$dr(t) = a(b - r(t))dt + \sigma \mathrm{d}L_t \tag{14}$$

where we denote by $L = (L_s)_{s \geq 0}$ a strict subordinator and $a > 0$.In addition, we assume that $b = \overline{r} - c > 0, c > 0$ for forcing the interest rates to fluctuate around mean reversion parameter $\overline{r}$. Then the short rate has always positive value and does not explode. Note that the killing rate zero subordinator satisfies $\mathbb{P}\,(L_t < \infty)$=1 for all t and

$$E\left[\exp\left(zL_t\right)\right] = \exp\left(t\varphi(z)\right)$$

where

$$\varphi(z) = \int_{]0,\infty[} \left(1 - e^{-zx}\right) v(dx)$$

As before, the strong solution of (8) is

$$r(t) = (1 - e^{-at})\overline{r} + e^{-at} r(0) + \sigma e^{-at} \int_0^t e^{av} \mathrm{d}L_v \tag{15}$$

The price of the pure discount is

$$\begin{aligned} P(t,T) &= \mathbb{E}^{\mathbb{Q}}[\exp\left(-\int_t^T r(u)\mathrm{d}u\right) |\xi_t] \\ &= \mathbb{E}^{\mathbb{P}}[\exp\left(-\int_t^T r(u)\mathrm{d}u\right) \frac{\mathbf{dQ}}{\mathbf{dP}} |\xi_t] \end{aligned}$$

where the Radon-Nikodym derivative is not unique, as before. We use time change process for modelling stochastic volatility.

Proposition 8. *Assume the measure change is of the form of time Inhomogeneous Lévy-Sheffer changes of measure. When interest rate follows:*

$$dr(t) = a(b - r(t))dt + \sigma \mathrm{d}L_{T_t} \tag{16}$$

where L_t is the strict subordinator and the economic time elapsed in t is given by the integrated CIR process, $T = \{T_t, t \geq 0\}$, where

$$T_t = \int_0^t y_s \mathrm{d}s$$

where

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dw_t$$

then the time t price of a discount bond that promises to pay one unit currency at maturity T is

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t)\right)$$

where

$$\begin{aligned} A(t,T) &= \exp\left(-\overline{r}(T-t) + \frac{\left(e^{-aT} - e^{-at}\right)\left(\overline{r} - r(0)\right) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\overline{r} + e^{-at}r(0)\right)}{a}\right) \\ &\times \exp\left(\int_t^T \mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L\left(f(v) + u_v(z)\right)\right)\mathrm{d}v - \int_t^T \mathcal{L}_{T_s}^{Q(z)}\left(\varphi^L\left(u(z)\right)\right)\mathrm{d}v\right) \end{aligned}$$

and

$$B(t,T) = \frac{(e^{-a(T-t)} - 1)}{a}$$

and where

$$\mathcal{L}_{T_s}^{Q(z)}(z) = \frac{\exp(\frac{\kappa^2\eta t}{\lambda})\exp\left(-\frac{2y_0 z}{\kappa + \gamma\cosh\left(\frac{\gamma t}{2}\right)}\right)}{\left(\cosh\left(\frac{\gamma t}{2}\right) + \frac{\kappa\sinh\left(\frac{\gamma t}{2}\right)}{\gamma}\right)^{\frac{2\kappa\eta}{\lambda^2}}}$$

and where

$$\gamma = \sqrt{\kappa^2 + 2\lambda^2 z}$$

In next proposition, we provide an important example for modelling interest rates.

Proposition 9. *Assume the measure change is of the form of stochastic Lévy-Sheffer changes of measure as in Theorem 6 and jumps consist of only small size. When interest rate follows:*

$$dr(t) = a(b - r(t))dt + \sigma\mathrm{d}L_t \tag{17}$$

where L_t is the strict subordinator and

$$dH_t = -\kappa H_t dt + dw_t$$

then the time t price of a discount bond that promises to pay one unit currency at maturity T is

$$P(t,T) = \exp\left(A(t,T) + B(t,T)r(t) + D(t,T)H_t^2 + G(t,T)H_t + F(t,T)\right)$$

where

$$A(t,T) = \exp\left(-\overline{r}(T-t) + \frac{\left(e^{-aT} - e^{-at}\right)\left(\overline{r} - r(0)\right) - \left(e^{-a(T-t)} - 1\right)\left((1 - e^{-at})\overline{r} + e^{-at} r(0)\right)}{a}\right)$$

$$B(t,T) = \frac{(e^{-a(T-t)} - 1)}{a}$$

and where $D(t,T)$, $G(t,T)$ *and* $F(t,T)$ *are obtained by solving following differential equation:*

$$\frac{\partial D(t,T)}{\partial t} = \alpha_2 - D(t,T)\kappa - 2D^2(t,T) = 0$$

$$\frac{\partial G(t,T)}{\partial t} = \alpha_1 - G(t,T)\kappa - 2D(t,T)G(t,T) = 0$$

$$\frac{\partial F(t,T)}{\partial t} = \alpha_0 + D(t,T) - \frac{G^2(t,T)}{2} = 0$$

with the boundary conditions $D(T,T) = 0, G(T,T) = 0$, *and* $F(T,T) = 0$, *and where*

$$\alpha_0 = \left(e^{f(s)} - 1\right)\int_{\mathbf{R}} v(\mathrm{d}x)$$

$$\alpha_1 = \left(e^{f(s)} - 1\right)\int_{\mathbf{R}} xv(\mathrm{d}x)$$

$$\alpha_2 = \left(e^{f(s)} - 1\right)\int_{\mathbf{R}} x^2 v(\mathrm{d}x)$$

9. CONCLUSION

As seen in proposition 9, if true data generating processes of interest rates have jumps with stochastic market price of risk, the short rate process is equivalent to the polynomial type of some state variable. In this setting, the quadratic model of Leippold and Wu (1998) and Ahn, Dittmar, and Gallant (2001) is a special case of the general Levy model where $\int_{\mathbf{R}} x^k v(dx) = 0, k \geqq 3$. Furthermore, affine models with jumps does not preserve affinity of the model under $\mathbb{P}$ in the case of the stochastic market price of risk. When true data generating process of interest rate has jumps with stochastic market price of risk, the quadratic models fit market data well compared with the affine model.. However, this is not because the quadratic models are better than the affine model, but because the approximation of jumps is more accurate in the quadratic models.

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