A Closed–Form Multi-Factor Binomial Interest Rate Model

By

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Abstract

This paper provides a multi-factor closed form binomial interest rate model. Ho-Lee, Black-Derman-Toy, Hull-White, are some of the binomial interest rate models that have found broad applications in valuing interest rate contingent claims. Recently, much research is seeking to extend the one factor model to multifactor models. However, to date, all multi-factor models are non-recombining interest rate models. These models are less accurate in valuing securities and calibrating to the market prices. This paper proposes a multi-factor closed form binomial interest rate model that is simple to implement and can capture a broad range of interest rate movements that are arbitrage-free. Empirical evidence supports the robustness of the model, which can be calibrated to 70 at-the-money swaptions with less than 1.3% average error.

I. Introduction

Tremendous progress has been made in the past 20 years in developing interest rate models that exhibit arbitrage-free movements. Many interest rate models have been proposed to value interest rate contingent claims. And these models are widely used in practice, for example, in portfolio management, trading, and risk management.

A closed-form binomial model is one popular class of interest rate models. The Ho–Lee and Hull-White models are examples of such a model class. These models are simple to implement and can be used for a broad range of applications. And these models provide a closed form solution to the entire discount function at each node point of the binomial lattice. Such close form solution enables us to value interest rate contingent claims at any future time and state of the binomial lattice and to value securities whose payoffs depends on the prevailing term structure of interest rates with significant computational efficiency.

However, to date, the closed form binomial models are confined to be one-factor models. They assume that all interest rates are instantaneously perfectly correlated, allowing the

yield curve only one degree of freedom of movement, moving up or down. Such an assumption is inconsistent with empirical observations, which have shown that most yield curve movements are explained by three principal movements: parallel, steepening and curvature. See Litterman and Scheinkman (1991) for the description of these movements.

The use of multi-factor interest rate model is important because one factor model may erroneously misstate value of many interest rate contingent claims. For example, consider a callable bond. Suppose that the time to expiration of the call provision is short relative to the underlying bond. A one-factor model would overstate the correlation between the stochastic bond price at the expiration of the option to the short term interest rates used to determine the present value of option payoff. As a result, the one factor model would tend to discount the payment when the bond is called at a rate lower than is appropriate and overstate the option value. Similarly, constant maturity swap pays a T period bond rate in exchange for a short-term rate. The erroneous assumption of perfect correlation between the T period bond rate and the short-term rate would affect the convexity value of the swap. A spread option is an option on the spread between the two interest rates. One factor model would significantly understate the

spread option value.   Multi-factor models are also important to valuing many non

marketable balance sheet items. For example, demand deposits, where the bank rate

depends on the stochastic yield curve shape and the single premium deferred annuities

whose interest payments depend on the insurers' investment returns. Detail descriptions

of these balance sheet items can be found in Ho and Lee (2004).

Further, a multi-factor closed form interest rate model can improve the calibration

procedure of the interest rate model. Specifically, the interest rate model can be

calibrated to a broader set of benchmark securities, beyond the caps and floors but

include also swaptions of different tenors and option expirations. Using a broader set of

benchmark securities for calibration, the interest rate model can better determine the

relative value of any contingent claims. More generally, when the interest rate model

can be effectively calibrated to options with early exercises, then any contingent claim

can be valued relative to a broad range of possible hedging instruments. This is

something that the propose model can effectively perform.

This paper provides a multi-factor closed form binomial interest rate model. The model

extends the Ho-Lee model (1986, 2004). The yield curve movements are based on the

yield curve movements implied by the benchmark securities market prices. The mean

reversion process of the interest rates is induced from the term structure of volatilities,

which is calibrated from the benchmark securities. This approach differs from the

Brennan and Schwartz model (1982) whose factors are based on the short rates and long

rates, where the short rate is assumed to mean revert to the long rate.

This paper also describes the relationships between the term structure of volatilities and

the yield curve movements. Finally, the paper describes a procedure in calibrating the

interest rate model to a set of 70 at the money swaptions. We show that the average

error is only 1.3%. And the results are stable over monthly observations of two years.

The paper is organized as follows. Section B provides the multi-factor binomial model.

Section C analyses the term structure of volatilities in the two-factor binomial model,

and it shows the relationship between yield curve movements and the term structure

volatility. Section D describes a method to calibrate the model to fit the current yield

curve and a set of benchmark securities, and provide the empirical results. Section E

summarizes the characteristics of the interest rate model. Finally, Section F contains the

conclusion.

II. Arbitrage-free Term Structure Movements

The model is a multi-factor recombining binomial model. The model specifies the discount function at each node of the binomial lattice such that these discount function movements are arbitrage-free. For clarity of exposition, we begin with a two factor model. The generalization from a two factor model to a multi-factor model is straightforward.

First we introduce some notations. The model is a 2 factor binomial model so there are $(n+1)^2$ nodes, denoted by $(i, j), i, j = 0, 1, \cdots n$, at time n. $P_{i,j}^n(T)$ is the price of a discount bond with time-to maturity T in state (i, j) at time n. $P(T)$ is the initial discount function with maturity T, which is observed in the market and is an input to the model. We assume that

(1). $P_{i,j}^n(0) = 1$ for all n and all i, j,

(2). $\lim_{T \to \infty} P_{i,j}^n(T) = 0$ for all n and all i, j.

The term structure movement is arbitrage-free if there is no arbitrage opportunity by holding a particular portfolio of the discount bonds at each node point of the binomial lattice. Such is the case, if the expected return of holding a discount bond of any

maturity T over any one period is the one period return. Specifically, according to

Harrison and Kreps (1979), we require that:

$$P_{i,j}^n(T) = \frac{1}{4} P_{i,j}^n(1) \left\{ P_{i+1,j+1}^{n+1}(T-1) + P_{i+1,j}^{n+1}(T-1) + P_{i,j+1}^{n+1}(T-1) + P_{i,j}^{n+1}(T-1) \right\} \tag{1}$$

Equation (1) should be satisfied for any $n = 0,1,2,\cdots$, $T = 1,2,\cdots$ and $i, j = 0,1,\cdots,n$.

The volatility structure of the model is generated by defining

$$P_{i,j}^n(T) = P_{0,0}^n(T) \cdot \alpha_1(n,T)^i \alpha_2(n,T)^j$$

for all $n > 0$, $T > 0$, and all $i, j \geq 0$. The original Ho-Lee model assumed 1-factor

model with factor α depending only on T.

By considering the nodes $(i, j),(i, j+1),(i+1, j),(i+1, j+1)$ at time n and the nodes

$(i, j),(i, j+1),(i, j+2),(i+1, j),(i+1, j+1),(i+1, j+2),(i+2, j),(i+2, j+1),(i+2, j+2)$

at time $(n+1)$ in the equation (1), we deduce the following consistency condition

which guarantees that the tree is well defined as a recombining tree :

$$\frac{\alpha_1(n,T)}{\alpha_1(n,1)} = \alpha_1(n+1,T-1),$$

$$\frac{\alpha_2(n,T)}{\alpha_2(n,1)} = \alpha_2(n+1,T-1).$$

for all $n, T \geq 1$.

Note that the above equations can be solved with the input data

$\delta_n^1 := \alpha_1(n,1), \delta_n^2 := \alpha_2(n,1)$ for $n = 1,2,\cdots$. The solution is given explicitly by

$$\alpha_1(n,T) = \frac{\alpha_1(1,T+n-1)}{\alpha_1(1,n-1)} = \delta_n^1 \delta_{n+1}^1 \ldots \delta_{T+n-1}^1 := d_{T+n-1,n}^1$$

$$\alpha_2(n,T) = \frac{\alpha_2(1,T+n-1)}{\alpha_2(1,n-1)} = \delta_n^2 \delta_{n+1}^2 \dots \delta_{T+n-1}^2 := d_{T+n-1,n}^2 \,,$$

where we define for $n > m$,

$$d_{n,m}^1 = \delta_n^1 \delta_{n-1}^1 \dots \delta_{m+1}^1 \delta_m^1 \quad \text{and} \quad d_{n,m}^2 = \delta_n^2 \delta_{n-1}^2 \dots \delta_{m+1}^2 \delta_m^2 \,.$$

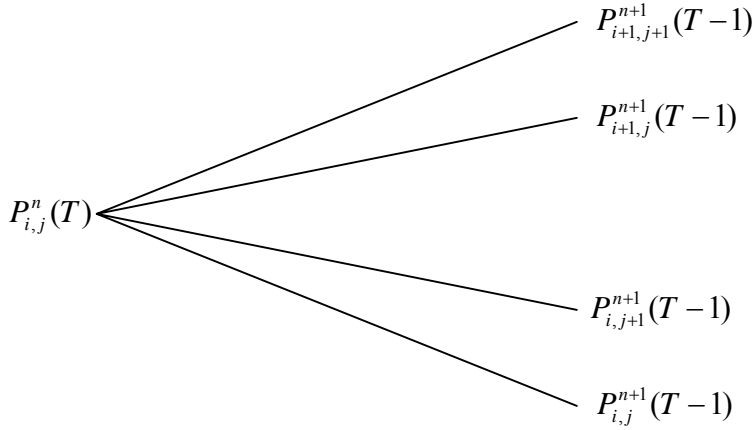**Figure1.** Basic building block of 2-factor binomial tree

Figure 1 shows the 2-factor binomial tree. Each move has the same risk-neutral probability which is 0.25. The proposition below provides the closed form solution of any zero coupon bond at each node point of the arbitrage-free binomial interest rate lattice.

Proposition 1. The Closed Form Solution for Bond Prices

Let $P(T)$ be the initial discount function. Then the arbitrage-free movement of the discount function is given by:

$$P_{i,j}^n(T) = 4\frac{P(T+n)}{P(n)} \left(\prod_{k=1}^n \frac{1+d_{n-1,k}^1}{1+d_{T+n-1,k}^1} \right) \left(\prod_{k=1}^n \frac{1+d_{n-1,k}^2}{1+d_{T+n-1,k}^2} \right) \left(d_{T+n-1,n}^1\right)^i \left(d_{T+n-1,n}^2\right)^j \tag{2}$$

where we set $d_{n-1,n}^1 = d_{n-1,n}^2 = 0$.

Alternatively, it can be expressed as

$$P_{i,j}^n(T) = \frac{P(T+n)}{P(n)} \times \left[\prod_{k=1}^{n} \left(G_{Tnk}^1\right)^{-1}\right](d_{T+n-1,n}^1)^{i-0.5n} \times \left[\prod_{l=1}^{n} \left(G_{Tnl}^2\right)^{-1}\right](d_{T+n-1,n}^2)^{j-0.5n} \qquad (2a)$$

where

$$G_{Tnk}^1 = \left[\left(d_{T+n-1,n}^1\right)^{-0.5} + \left(d_{T+n-1,n}^1\right)^{0.5} \times d_{n-1,k}^1\right] / (1+d_{n-1,k}^1)$$

$$G_{Tnl}^2 = \left[\left(d_{T+n-1,n}^2\right)^{-0.5} + \left(d_{T+n-1,n}^2\right)^{0.5} \times d_{n-1,l}^2\right] / (1+d_{n-1,l}^2)$$

$P_{i,j}^n(T)$ is the discount bond with \$1 principal and maturity T, at a future time n, in state

(i, j).   The term structure of volatilities are specified by δ_k^1 and δ_k^2 for k = 1,…

(n+T).

Proof:

From the construction we have

$$P_{i,j}^n(T) = P_{0,0}^n(T) \cdot \alpha_1(n,T)^i \alpha_2(n,T)^j = P_{0,0}^n(T) \cdot (d_{T+n-1,n}^1)^i (d_{T+n-1,n}^2)^j .$$

So it suffices to check that

$$P_{0,0}^n(T) = 4\frac{P(T+n)}{P(n)}\left(\prod_{k=1}^{n}\frac{1+d_{n-1,k}^1}{1+d_{T+n-1,k}^1}\right)\left(\prod_{k=1}^{n}\frac{1+d_{n-1,k}^2}{1+d_{T+n-1,k}^2}\right). \qquad (2b)$$

We use mathematical induction. For n=1 in Equation (2b), we have from equation (1)

with n=i=j=0 that

$$\frac{P(T+1)}{P(1)} = \frac{1}{4} P_{0,0}^1(T)\left[1 + \alpha_1(1,T) + \alpha_2(1,T) + \alpha_1(1,T)\alpha_2(1,T)\right]$$

$$= \frac{1}{4} P_{0,0}^1(T)\left[1 + d_{T,1}^1\right]\left[1 + d_{T,1}^2\right].$$

Hence we get

$$P_{0,0}^1(T) = 4\frac{P(T+1)}{P(1)}\frac{1}{\left[1 + d_{T,1}^1\right]\left[1 + d_{T,1}^2\right]}.$$

Therefore the formula for n=1 is valid. Now we suppose that the formula is satisfied for

n and we will check the formula for n+1. From the equation (1) we have

$$P_{i,j}^n(T+1)/P_{i,j}^n(1) = \frac{1}{4}\left\{P_{i+1,j+1}^{n+1}(T) + P_{i+1,j}^{n+1}(T) + P_{i,j+1}^{n+1}(T) + P_{i,j}^{n+1}(T)\right\}$$

$$= \frac{1}{4} P_{0,0}^{n+1}(T)\left(d_{T+n,n+1}^1\right)^i \left(d_{T+n,n+1}^2\right)^j \left(1 + d_{T+n,n+1}^1\right)\left(1 + d_{T+n,n+1}^2\right).$$

By the induction hypothesis, we get

$$P_{i,j}^n(T+1)/P_{i,j}^n(1) = \frac{P(T+n+1)}{P(n+1)}\left(\prod_{k=1}^n \frac{1+d_{n,k}^1}{1+d_{T+n,k}^1}\right)\left(\prod_{k=1}^n \frac{1+d_{n,k}^2}{1+d_{T+n,k}^2}\right)\left(d_{T+n,n+1}^1\right)^i \left(d_{T+n,n+1}^2\right)^j,$$

where we have used the fact $d_{T+n,n}/d_{n,n} = d_{T+n,n+1}$. Substituting this into above equation

we get

$$P_{0,0}^{n+1}(T) = 4\frac{P(T+n+1)}{P(n+1)}\left(\prod_{k=1}^n \frac{1+d_{n,k}^1}{1+d_{T+n,k}^1}\right)\left(\prod_{k=1}^n \frac{1+d_{n,k}^2}{1+d_{T+n,k}^2}\right)\frac{1}{\left(1+d_{T+n,n+1}^1\right)\left(1+d_{T+n,n+1}^2\right)}$$

$$= 4\frac{P(T+n+1)}{P(n+1)}\left(\prod_{k=1}^{n+1} \frac{1+d_{n,k}^1}{1+d_{T+n,k}^1}\right)\left(\prod_{k=1}^{n+1} \frac{1+d_{n,k}^2}{1+d_{T+n,k}^2}\right),$$

because $d_{n,n+1} = 0$ (by definition !). This is the required formula for n+1 and so the

proof is complete.

Q.E.D.

Equation (2) provides the specification of the discount function at each node point of a 2 factor binomial model. The extension of the model to any number of factors is straightforward. For an m factor model, we would have the term structure of volatilities be specified by m sets of movements $(\delta_k^1, \delta_k^2, ..., \delta_k^m)$.

$$P_{i_1,i_2,...i_m}^n(T) = 2^m \frac{P(T+n)}{P(n)} \left(\prod_{k=1}^{n} \frac{1+d_{n-1,k}^1}{1+d_{T+n-1,k}^1} \right) \left(\prod_{k=1}^{n} \frac{1+d_{n-1,k}^2}{1+d_{T+n-1,k}^2} \right) \cdots \left(\prod_{k=1}^{n} \frac{1+d_{n-1,k}^m}{1+d_{T+n-1,k}^m} \right) \times$$

$$\left(d_{T+n-1,n}^1 \right)^{i_1} \left(d_{T+n-1,n}^2 \right)^{i_2} \cdots \left(d_{T+n-1,n}^m \right)^{i_m}$$

It is straightforward to show that the above interest rate model is a direct extension of the Ho-Lee model where the instantaneous volatility of the interest rate movements are constant. Specifically, when δ_i^1 is constant for all i and δ_j^2 is equal to 1 for all j,

$$d_{n,m}^1 = \left(\delta^1 \right)^{n-m+1}, \qquad d_{n,m}^2 = 1 \tag{3}$$

If we substitute Equation(3) to Equation(2), we have

$$P_{i,j}^n(T) = 2 \frac{P(T+n)}{P(n)} \frac{\left[(1+(\delta^1)^{n-1})(1+(\delta^1)^{n-2}) \cdots (1+\delta^1) \right] (\delta^1)^{iT}}{(1+(\delta^1)^{T+n-1}) \cdots (1+\delta^1)^T} \tag{4}$$

Equation (4) is the Ho-Lee model.

III. Analysis of the Term Structure of Volatilities

In this section, we will use the model and analyze the interest rate movements in

relation to the term structure of volatilities. Specifically, we first express the interest rates into terms of the bond prices. We then study how the yield curve would move given the term structure of volatilities.

At the initial date, we have the discounted function $P(T)$. We define the yield of a discount bond with maturity T initially to be:

$$r(T) = -\frac{\ln P(T)}{T}$$

More generally, we can define the yield curve at each node point to be:

$$r_{i,j}^n(T) = -\ln P_{i,j}^n(T)/T$$

$$
\begin{aligned}
&= -\frac{1}{T}\ln\frac{P(T+n)}{P(n)} + \frac{\ln\prod_{k=1}^n G_{Tnk}^1}{T} + \frac{\ln\prod_{l=1}^n G_{Tnl}^2}{T} \\
&\quad - \frac{\ln d_{T+n-1,n}^1}{T}(i-0.5n) - \frac{\ln d_{T+n-1,n}^2}{T}(j-0.5n)
\end{aligned}
\tag{5}
$$

We define spot volatility of term T is the variance of the shifts of the yields for a two-factor binomial model.[1]

Equation (5) provides a clear depiction of the yield curve movements. The equation shows that the T period interest rate in time n and state (i, j) is specified by three terms.

The first term $-\frac{1}{T}\ln\frac{P(T+n)}{P(n)}$ is the T period forward rate at the horizon date n. The

second term $\frac{\ln\prod_{k=1}^n G_{Tnk}^1}{T} + \frac{\ln\prod_{l=1}^n G_{Tnl}^2}{T}$ is called the convexity drift, that we will

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[1] The variance of a two-factor binomial model has six terms($_4C_2$), each of which represents the spread between the two-factor binomial outcomes. If we replace r_{11}^1, r_{10}^1 by r_1^1 and r_{01}^1, r_{00}^1 by r_0^1 respectively, we will have $\left(r_1^1(T) - r_0^1(T)\right)^2/4$ which is spot volatility of a one-factor binomial model.

discuss later. The third term $-\dfrac{\ln d_{T+n-1,n}^1}{T}(i-0.5n)-\dfrac{\ln d_{T+n-1,n}^2}{T}(j-0.5n)$ describes the

movements of the yield curve. The third term shows that state (i, j) represents the yield

curve taking i and j steps with step sizes $-\dfrac{\ln d_{T+n-1,n}^1}{T}$ and $-\dfrac{\ln d_{T+n-1,n}^2}{T}$ respectively.

Therefore, the standard deviation of each of these two binomial movement is half that

step size. It follows that the variance of the yield curve movement over each step is the

sum of the variances of the two independent binomial movements. Specifically, we have

the forward volatility of term T given by

$$\sigma^{\mathrm{nf}}(\mathrm{T}) = \frac{1}{2\mathrm{T}}\left[\left(\ln d_{\mathrm{T}+n-1,n}^1\right)^2 + \left(\ln d_{\mathrm{T}+n-1,n}^2\right)^2\right]^{0.5}$$

Now we note that the spot volatility of term T is the forward volatility where n =1,
and therefore, the spot volatility is given by

$$\sigma(\mathrm{T}) = \frac{1}{2\mathrm{T}}\left[\left(\ln d_{\mathrm{T},1}^1\right)^2 + \left(\ln d_{\mathrm{T},1}^2\right)^2\right]^{0.5}$$

Note that since the movement is binomial for each of the factors, when we take the

expectation over the possible states of the world based on the risk neutral probabilities,

the third term is zero.

Now, we proceed to explain the term convexity drift. Consider taking the expectation

across all both sides of the equation (5), we can now conclude that the expected yield of

a T-period bond over an n-period horizon is the forward rate (for the T periods

starting from time n viewed from today) with a drift. The drift D is given by:

$$D = \left(\frac{1}{T}\right)\ln\left(\prod_{k=1}^{n} G_{Tnk}^{1}\right) + \left(\frac{1}{T}\right)\ln\left(\prod_{l=1}^{n} G_{Tnl}^{2}\right) \tag{6}$$

where

$$G_{Tnk}^{1} = \left[\left(d_{T+n-1,n}^{1}\right)^{-0.5} + \left(d_{T+n-1,n}^{1}\right)^{0.5} \times d_{n-1,k}^{1}\right]/(1+d_{n-1,k}^{1})$$

$$G_{Tnl}^{2} = \left[\left(d_{T+n-1,n}^{2}\right)^{-0.5} + \left(d_{T+n-1,n}^{2}\right)^{0.5} \times d_{n-1,l}^{2}\right]/(1+d_{n-1,l}^{2})$$

$$\tag{7}$$

But note that this drift is always positive and increases with volatilities. That is, the expected forward rate of any time period of delivery T and future date n is higher than the forward rate implied by the initial yield curve. And this spread increases with volatilities.

This result is shown by noting $0 < D$, by showing that G_{Tnk}^{1} and G_{Tnl}^{2} are greater than 1 for all k and l. Note that $\left(d_{T+n-1,k}^{1}\right)^{0.5} < 1$, $0 < d_{n-1,k}^{1}$, $\left(d_{T+n-1,l}^{2}\right)^{0.5} < 1$, $0 < d_{n-1,l}^{2}$.
Therefore, it follows that

$$\left[1 - d_{n-1,k}^{1}\left(d_{T+n-1,k}^{1}\right)^{0.5}\right]\left[1 - \left(d_{T+n-1,k}^{1}\right)^{0.5}\right] > 0 \tag{8}$$

$$\left[1 - d_{n-1,l}^{2}\left(d_{T+n-1,l}^{2}\right)^{0.5}\right]\left[1 - \left(d_{T+n-1,l}^{2}\right)^{0.5}\right] > 0$$

In expanding Equation (14) and simplify, we get

$$1 + d_{n-1,k}^{1}\left(d_{T+n-1,k}^{1}\right) > \left(d_{T+n-1,k}^{1}\right)^{0.5}\left(1 + d_{n-1,k}\right)$$

$$1 - d_{n-1,k}^{2}\left(d_{T+n-1,l}^{2}\right) > \left(d_{T+n-1,l}^{2}\right)^{0.5}\left(1 + d_{n-1,l}^{2}\right)$$

Then it follows that

$1 < G^1_{Tnk}$ and $1 < G^2_{Tnl}$

The results show that expected yield must rise above the forward rate because the expected returns of the bond increases with the volatility of interest rates as a result of the bond convexity. To ensure that the expected returns of the bond to be the same as the one period interest rate for any interest rate volatility, the convexity drift is needed to balance this convexity effect on the bond returns.

IV. Implementation of a multi-factor model: Calibrating the model to the prices of Benchmark Securities.

In this section we describe a method in implementing the n factor interest rate model. Specifically, we describe the procedure in calibrating the interest rate model to a set of benchmark securities. We also present empirical evidence to support that the two factor model can fit the market data better than the some of the standard one factor models.

Calibrating the model entails several steps.

  1. Benchmark securities. The main advantage of using an arbitrage-free interest rate model is to be able to fit the model to the observed term structure of interest rates and a set of interest rate options, called benchmark securities. This

calibration approach enables us to value an interest rate contingent claim relative to the spot curve and the benchmark securities. The spot curve represents the market determined time value of money. The benchmark securities represent the market pricing of interest rate volatilities. Therefore, if we can calibrate the model to the market time value of money and market price of interest rate risks, then we have a consistent valuation framework.

Since the multi-factor model allows for flexible movements of the yield curve, and the recombining binomial lattice allows for efficient computation, we can use a portfolio of caplets, floorlets and both American and European swaptions as the benchmark securities.

2. Specification of the term structure of interest rate. Any arbitrage-free interest rate model takes the observed spot yield curve as given. To calibrate the model, one methodology is to use the spot swap curve as the input to the model.

3. Pricing of caplets, floorlets and European swaptions. We observe that a caplet is equivalent to a put option written on a zero coupon bond and a swaption is

equivalent to an option (call or put) on a coupon bond. Hence we have a series

of zero bond put option prices or coupon bond option prices. We use the closed

form formula for bond option prices, which are completely determined by initial

discount function (which is a model input) and model volatility factors, which

are δ_n^1 *and* δ_n^2 , where n = 1, 2, 3...N, Determination of the volatility

factors δ_n^1 *and* δ_n^2 . We need to determine the volatility factors of the two factor

interest rate model using the market observed benchmark securities' prices. This

two-factor interest rate model assumes that at each instant the yield curve makes

two independent movements. Following the empirical results on the yield curve

movements, we can assume that the movements can be specified by

$\ln \delta^1(n) = -2f(n)\sigma^1(n)$ and $\ln \delta^2(n) = -2f(n)\sigma^2(n)$

4. For the calibration procedure we choose a functional form of the term structure

   of volatilities. The specific functional form is assumed to

   be $\sigma = (a+bt)\exp(-ct)+d$ . . We choose the function

   $\sigma(t) = (a+bt)\exp(-ct)+d$ where t is a time-to-maturity to represent the

   volatility curve because this function decays exponentially and, when t is

   relatively small, the curve may exhibit a hump, depending on the parameters a

   and b. This configuration of the volatility curve is observed in the

market. $\sigma^1(n) = (a_1 + b_1 n)\exp(-c_1 n) + d_1$ and $\sigma^2(n) = (a_2 + b_2 n)\exp(-c_2 n) + d_2$.

The constants a_j, b_j, c_j, d_j for j = 1, 2 are estimated from the benchmark securities market prices.

5. Calibration. The caplets', floorlets' and swaptions' observed market prices give us a set of equations to be satisfied by fitting the model volatility factors δ_n^1, δ_n^2 to the closed form pricing models of these benchmark securities. Such volatility factors are determined by non-linear search procedure in determining the parameters a_j, b_j, c_j, d_j j=1,2, such that the sum of squares of the errors between the observed and model prices of the benchmark securities is minimized. We use the following object function

$$\sum_{i=1}^{N} \left(\frac{MarketPrice_i - ModelPrice_i}{MarketPrice_i} \right)^2$$

where N is the number of benchmark securities.

Market Swaption Data:

The term structure of interest rates is the USD zero swap curve on July 31, 2002. The market swaption volatility surface is given by the Black volatilities quoted (in %).

Option Term (years)	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	44.40	36.50	32.90	29.70	27.60	25.70	25.40	24.00	23.40	23.30
2yr	31.20	28.80	27.10	25.30	24.20	22.90	23.00	21.80	21.30	21.10
3yr	27.00	25.30	24.30	23.10	22.20	21.30	21.10	20.40	20.00	19.90
4yr	24.00	22.80	21.90	21.20	20.70	19.90	19.70	19.10	18.70	18.60
5yr	22.30	21.30	20.80	19.90	19.40	18.70	18.50	18.00	17.70	17.40
7yr	19.80	19.10	18.50	18.10	17.60	16.90	16.70	16.30	16.00	15.80
10yr	17.40	16.40	15.90	15.60	15.10	14.50	14.50	14.10	13.90	13.70

The calibrated two yield curve movements are given by the estimated volatility parameters:

σ_1: $a_1 = 0.772079$, $b_1 = -0.168097$, $c_1 = 0.498474$, $d_1 = 0.113347$

σ_2: $a_2 = 0.023701$, $b_2 = -0.014509$, $c_2 = 0.003260$, $d_2 = 0.245125$

The first term $\sigma_1(n)$ is a curve that falls from 0.8 to and level on 0.1. The second term $\sigma_2(n)$ falls steadily and reaches zero in year 10. The results are consistent with the observed data that the term structure of volatilities declines with the term.

And the valuation error derived from the estimated swaption prices are given by the table below.

Swaption Matrix: 2-Factor Model percentage errors:

Option Term (years)	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	0.55	-1.67	1.35	0.35	-0.11	-1.60	1.97	0.46	1.62	4.47
2yr	-2.91	-1.05	0.29	-0.83	-0.55	-2.01	2.08	0.09	0.84	2.68
3yr	1.00	0.45	1.13	0.11	-0.30	-1.14	0.98	0.46	1.18	3.19
4yr	0.36	-0.77	-1.23	-1.21	-0.55	-1.59	0.18	-0.23	0.13	1.95
5yr	0.61	-0.72	-0.01	-1.54	-1.27	-2.27	-0.70	-0.95	-0.27	0.27
7yr	0.54	-0.35	-0.77	-0.29	-0.47	-2.03	-0.78	-0.92	-0.63	0.12

10yr	3.63	0.61	0.19	0.81	0.00	-1.77	0.41	-0.40	0.00	0.24

The result shows that the model fits the observed data quite well. Previous work using

the Brace, Gatarek and Musiela (1997) calibrates to only counter-diagonal of the swap

volatility surface, confining only to those swaptions where the option term plus the

swap tenor to be 31 years. When the 2 factor model is confined to one factor, the

average error exceeds 3%. The above model has been tested over 24 monthly data over

two year and the results are similar in all period as shown in the table below.

January 2002 are given in the table below. (Average percentage pricing error of 70 swaptions each month

from January 2000-The average percentage pricing error is defined as the square root of the mean squared

percentage errors.)

Average Error	Date	Average Error	Date
1.76	1/30/2004	1.58	12/31/2003
1.51	11/28/2003	1.75	10/30/2003
1.57	9/30/2003	1.75	8/29/2003
1.22	7/30/2003	1.46	6/30/2003
1.55	5/30/2003	1.73	4/30/2003
1.23	3/31/2003	1.61	2/28/2003
1.61	1/30/2003	1.70	12/30/2002
1.56	11/29/2002	1.69	10/30/2002
1.37	9/30/2002	1.63	8/30/2002
1.12	7/30/2002	1.55	6/28/2002
1.40	5/30/2002	1.49	4/30/2002
1.50	3/29/2002	1.51	2/28/2002

The results show that the average errors are less approximately 1.5%. This error is reasonably small,

especially when the market is quite volatile in this time period. In this period, the 5 year swap rate reached the high of 5.6% and the low of 2.8%. The one year Black volatility reached a high of 60.8% and a low of 26.1%. The average error remains quite stable despite the large changes in both the level and the volatility of the rates.

V. The characteristics and the applications of the model

The propose interest rate model has a number of useful characteristics for practical applications.

First, the model provides us with a closed form solution of term structures of interest rates at each node. Therefore, we can determine the interest rate yield curve at each node point, and thus, the model is computational efficient and minimizes errors in numerical approximation. By way of contract, the Black Derman and Toy model requires numerical construction of the spot rate for each node point on the lattice. And therefore, the yield curve at each node point has to be numerically calculated from the binomial lattice of the spot rates, a procedure that requires significant computation and can have significant numerical estimation errors.

Second, the two-factor binomial model takes the initial spot yield curve and initial term structure of volatilities as input data. The model does not specify the precise nature of

the mean reversion process of the interest rate. Instead, the mean reversion process is indirectly specified via the term structure of volatilities. As a result, the model is more flexible in calibrating to the observe benchmark securities prices. By way of contrast, Vasicek, Cox-Ingersoll-Ross and Hull-White assume that the short term interest rates follow a certain process. These models may not fit the observed benchmark securities prices satisfactorily.

Third, the two-factor binomial model is a discrete time and combining model and not a continuous time and non-combining model. Recently, Grant and Vora (1999) and Das and Sundaram (1999) have pointed out that discrete time models provide computational efficiency for practical implementation. Since we have to convert continuous time models into discrete time versions when we implement them for empirical testing or practical purposes, discrete time models can be used directly. Since the model is recombining, the model is more effective to price options (European and American) without using Monte-Carlo simulations. This model can be used to calculate the interest rate sensitivities of a large portfolio. By way of comparison, Brace, Gatarek, Musiela (BGM)(1997) and string model while provide flexibility in modeling the yield curve movements, the valuation of the securities require Monte-Carlo simulation method that

often lacks the pricing accuracy that recombining binomial model has.

Fourth, since the model can be calibrated to a large sample of benchmark securities of a broad range of interest rate options, the model can provide relative valuation of any contingent claims relative to its hedging instruments or an appropriate spectrum of liquid securities. Therefore, the model can provide more accurate pricing of the securities and the ratios to its hedging strategies.

Fifth, the model is closed form without any numerical procedure, like that of the Black Derman and Toy model, or any Monte-Carlo simulation to estimate the security prices. The model is similar to a Black model, where the closed form solution provides a direct mapping from the price to the volatilities. Therefore, the model can be used effectively as a standard model in an interest rate contingent claims management. The model's two factor movements can be used as inputs to value securities that are affected by the multi-factor movement as Black volatilities are used for the one factor models.

VI. Conclusions

This paper proposes a multi-factor binomial interest rate model. The model is a discrete time recombining lattice model that provides a closed form solution to the discount function at each node point. The interest rate movement has the Markovian property that exhibits a mean reversion derived by the term structure of volatilities. The multi-factor property enables the model to fit both the current yield curve and the volatility surface at the same time.

We have shown that such a model can have a broad range of applications in valuation interest rate contingent claims. The model can be calibrated to a large set of benchmark securities of different security types. Therefore the model offers a robust methodology to value interest rate contingent claims.

In this paper, we have discussed the implementation of the model for two factors. The research can be extended to the appropriateness of using higher order models. Such research will be left for future.

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