Calibration of the Interest Rate Models: BDT, HL, String and BGM

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** This is a rough draft.

Abstract

Calibration is a powerful technique that fits the relative valuation models to the benchmark securities market observed prices in order to value other securities, in a relative sense, to these benchmark securities. In particular, the method ensures the arbitrage-free interest rate model to be consistent with the observed market yield curve, the market volatility surface, and other benchmark securities' prices to determine the basis spreads. The concept and the general methodology is applied to other markets beyond the interest rate markets and it is the key tool in developing hedging and arbitrage strategies, and pricing methodologies. This paper addresses how we determine interest rate option models such that the market will accept as the standard.

A. Introduction

The valuation of interest rate derivatives using an arbitrage-free interest rate model requires the inputs of the spot yield curve and the term structure of volatilities (or a volatility surface). Volatility surface has great importance to option pricing. Without volatilities, options simply reduce to some cash flows, which can be valued by the bond model. Volatilities affect the option value.

The valuation of options based on the term structure of volatilities or volatility surface that can be estimated from the observed time series of the bond yields. is problematic because the historical estimation of the yield volatilities is backward looking. The option value depends on future uncertainties and not on the volatility based on historical experience. The backward looking approach is appropriate only if we argue that the future uncertainty is the same as the historical uncertainty. But when the market anticipates higher uncertainty in the future, for example, when interest rates are subject to higher inflation rate uncertainty, then how is such market anticipation measured in evaluating interest rate options?

We have seen that the Black-Scholes model enables us to quote a stock option value not by the option price, but by the implied volatility which is the stock volatility used by the Black Scholes model to give the option price. We say XYZ option is traded at x% volatility, as a way to quote the price of the option via the Black-Scholes model. This price quote system has the advantage to express the option price in terms of the market's anticipation of the future risks. Can we quote bond options by the volatility value? This problem is much more complicated for interest rate options because we have seen that there are a number of interest rate option models. In order to agree on the quoted volatility to express the value of an option, we need to have models that the market will accept as the standard. How do we determine such models? These are two questions that this paper will address

B. Valuation of Interest Rate Derivatives Using Market Benchmark Prices

By introducing the pricing convention of benchmark securities using the Black model, we can now summarize the methodology step by step in valuing an interest rate derivative using market benchmark prices.

Step 1. Determining the set of benchmark securities

The selection of benchmark securities must depend on the purpose of the valuation. These securities should have tenors and other characteristics of the securities that we would like to value. The benchmark securities should have actively traded prices that are representative of the market assessment of their values. These prices may be determined by different traders, using different financial models and approaches in the market. Some firms may bet on interest rates falling while some firms bet on interest rates rising. For whatever reasons, the aggregation of their views is expressed by the prices of benchmark securities, and these prices are expressed in terms of the market volatility surface as market price quotes.

Table 1. An example of market volatility surface[1]

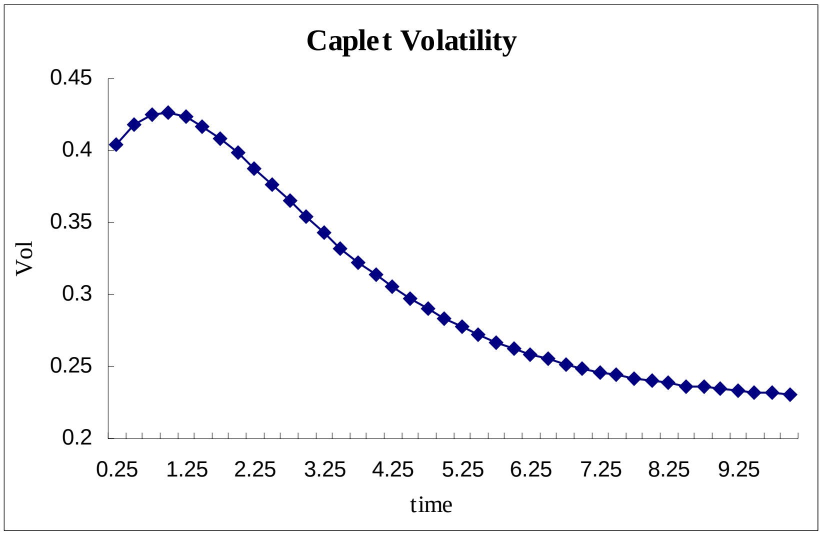
US Swaption vols: July 17, 2002

Option Term	Swap tenor					Cap volatility
	1 yr	3 yr	5 yr	7 yr	10 yr	
1 yr	37.2	29.3	25.4	23.7	22.2	42.5
2 yr	28.3	24.8	22.7	21.7	20.5	40.5
3 yr	25.0	22.9	21.3	20.5	19.4	34.6
4 yr	22.7	21.3	20.0	19.4	18.3	31.1
5 yr	21.5	20.2	18.9	18.3	17.2	28.7
7 yr	19.2	18.0	16.9	16.2	15.5	25.5
10 yr	16.8	15.5	14.6	14.1	13.6	22.6

For the following numerical example, we use a flat yield curve of 6% at continuously compounding rate. We assume a forward volatility curves in Figure 1. These volatility curves will be used as inputs to illustrate the implementation of the models that will be discussed below. The caplets are assumed to have a tenor of three months with strike at 6%.

[1] Source: Bloomberg, http://www.bloomberg.com/

Figure 1. Caplet volatilities calibration.

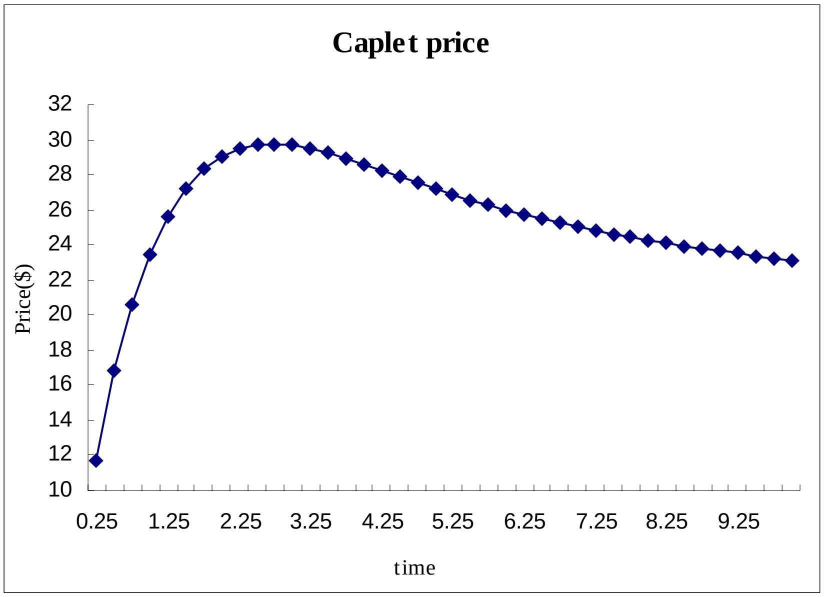


We have chosen the function $\sigma = (a + bt)\exp(-ct) + d$ to represent the volatility curve because this function decays exponentially and, when t is relatively small, the curve may exhibit a hump, depending on the parameters a and b. This configuration of the volatility curve is observed in the market.

Step 2. Calibration of the interest rate model

We first decide on the interest rate model that is most appropriate to value the derivative. A spot curve that can provide input to an interest rate model may be the swap curve or the Treasury rates. The choice is determined by how we would like to relatively value the derivatives. We use the Black models to translate the market volatility surface to the benchmark securities prices. (See Figure 2.)

Figure 2 Caplet prices obtained by the Black model.



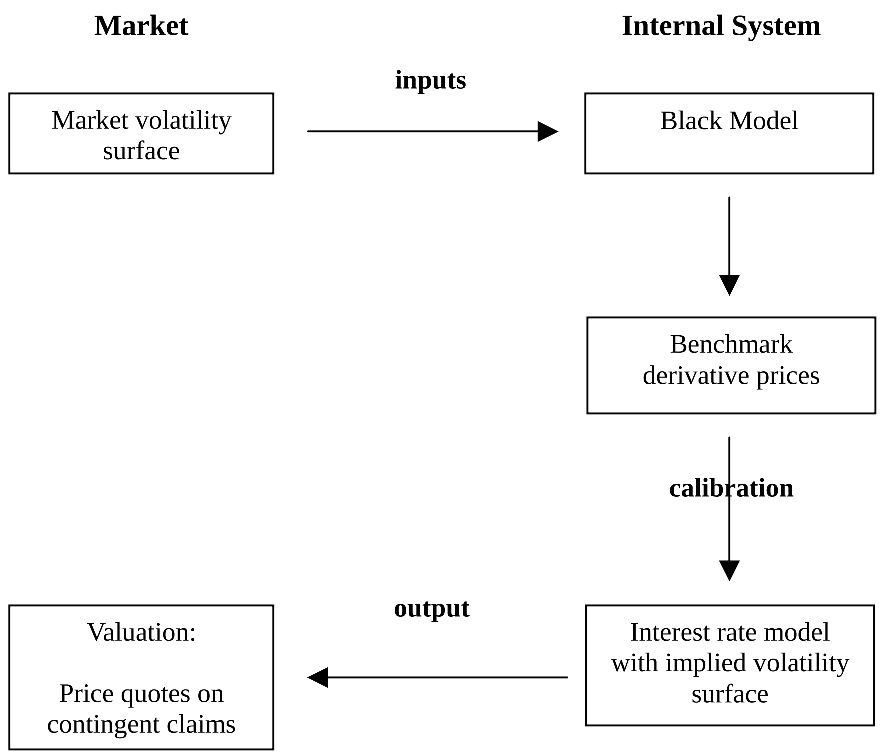
We now develop the valuation models of the benchmark securities, where these models are based on the arbitrage free interest rate movement model. For example, these valuation models may be using backward substitution methods in determining the cap and swaption prices. Using these valuation models, we can determine the set of implied volatilities as input for the interest rate model that best fits the model prices of the benchmark securities to their observed prices, calculated in Step 1. Once these implied volatilities are determined, we have calibrated the model.

Step 3. Valuing an interest rate derivative

Now we can apply the calibrated interest rate model to value an interest rate derivative, which may not be a benchmark security. In order to do so, we need to use the general tree construction procedure to determine the terminal and boundary conditions on the binomial lattice for the interest rate derivative. Then we use the valuation model to determine the derivative value. Note that, after we obtain the derivative value, we can continue to determine the sensitivities of the derivative value to small changes in market parameters, such as the interest rate level and the volatilities. We will discuss some of these sensitivities later in this paper.

Schematically, we can view the interest rate contingent claims are valued according to the following steps:

Figure 3 Valuation of interest rate derivatives using market benchmark prices



In completing the discussion of the valuation procedure of interest rate derivatives, we now discuss the advantages of using the arbitrage-free interest rate model in valuing interest rate derivatives. Given a bond option, can we use the discounted cash flow method to determine the bond option price? Once again, the discounted cash flow method fails to provide an accurate discount rate for the expected cash flow of the bond option. For this reason, the discounted cash flow method is not used for pricing bond options in this paper. We instead describe in this paper how we use an interest rate model and calibrate the model to some benchmark securities prices, and then use the model to value a bond option.

There are two other reasons why an arbitrage-free interest rate model is useful. The first reason is the use of benchmarks for accurate option pricing. Imagine you were a scientist in a laboratory, and you need to use a laser beam to target a tiny dot on a screen 10 yards away. One direct way is trial and error. You may continually adjust the laser beam until it hits the target. However, there is another way. You can first set up the instrument such that you can adjust the position of the laser accurately. Next, you determine several reference points (benchmarks) on the screen and adjust the positions of the laser so that it targets the reference points. Then measure the target point relative to the positions of the reference points and use this information to adjust (calibrate) your laser beam. Now you can accurately estimate the position of the dot since your laser beam has been calibrated using benchmark points relative to your target. Furthermore, if the target point moves slightly, you can continue to re-adjust the laser beam using the reference points. Arbitrage-free models value any derivative relative to a set of benchmark securities. To the extent that the benchmark securities are appropriately priced, the relative valuation procedure can provide an accurate pricing method.

Another useful aspect of an arbitrage-free interest rate model is that the model relates all the bonds and all the options in one framework. We no longer have to trade a bond option quoting the volatility of that particular bond volatility, as we would normally do if we treated a bond option like a stock option. Instead, we can quote the volatilities of the interest rates then all interest rate contingent claims can be valued in one consistent framework. For these reasons, the study of interest rate modeling is actively seeking to provide an accurate and efficient framework to value a broad class of derivatives. When an interest rate contingent claim is valued by this calibration approach, the implication is that this interest rate contingent claim can be replicated by the bonds that determine the spot yield curve and the benchmark securities at a cost of buying this portfolio (which may have to be dynamically adjusted over time) equaling the theoretical value of the security.

For these reasons, in the valuation of the interest rate contingent claims, the volatilities surface is as important as the determination of the spot yield curve. The volatilities surface specifies how the market prices the volatilities at each moment in time, much as the spot yield curve specifies the market time value of money.

In the following section we illustrate the calibration procedure for different interest rate models. We consider the valuation of caplets prices using forward volatilities in Figure 1 and market caplet prices (implied by the Black model) in Figure 2. We assume a flat rate of 6%.

C. Calibration of the Black, Derman and Toy Model

Up to now we have assumed that the term structure of volatility σ is known. We now discuss how it is determined so that the price of the benchmark securities implied by the BDT model are the same as the Black-Scholes prices. We illustrate the calibration procedure using caps with market parameters used in section B. Using three-month step sizes and at most ten-year maturity caps, we have 39 caplets as calibration instruments. We choose a term structure of forward volatilities of the form

$$\sigma(t) = (a + bt)\exp(-ct) + d$$

where the volatility parameters *a*, *b*, *c*, and *d* are chosen via the calibration procedure. The next stage is to choose a goodness of fit measure. In this example we choose

$$\sum_{i=2}^{40} \left(V_i - V_i^*\right)^2$$

where V_i is the market price (obtained using Black model) of the *i*th caplet and V_i^* is the corresponding price given by the BDT model. Note that the interest rates must also match the initial term structure.

Our objective is to use a nonlinear optimization routine to determine the volatility parameters (*a*, *b*, *c*, *d* of the term structure of volatilities above) in order to minimize the goodness of fit measure under the constraint of an initial term structure matching. Figure 4 below show the implied volatility obtained using this calibration procedure. Note that since the BDT model is lognormal, the implied volatility should be the same as given forward volatility. The error depicted by Figure 5 is primarily due to the step size used in this example and smaller step size will result in the same volatility functions.

Figure 4 Implied volatility of the BDT model.

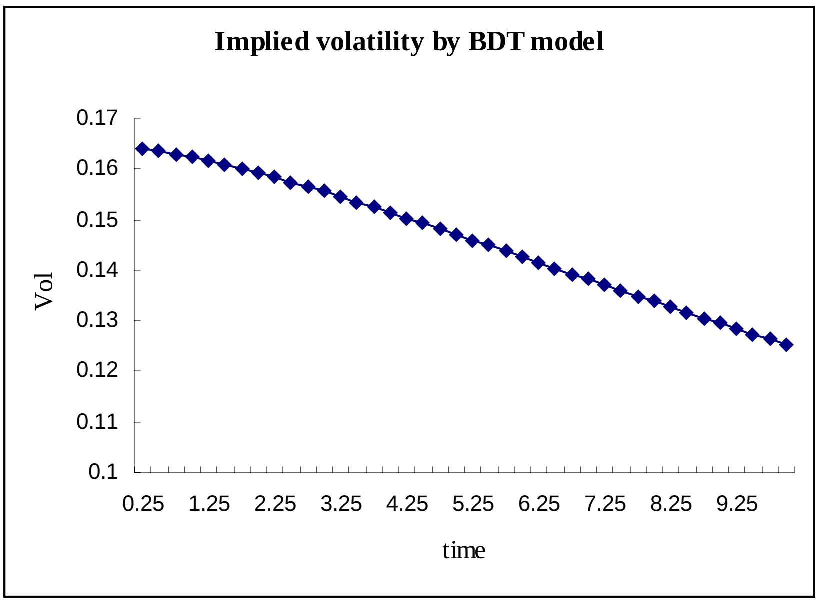
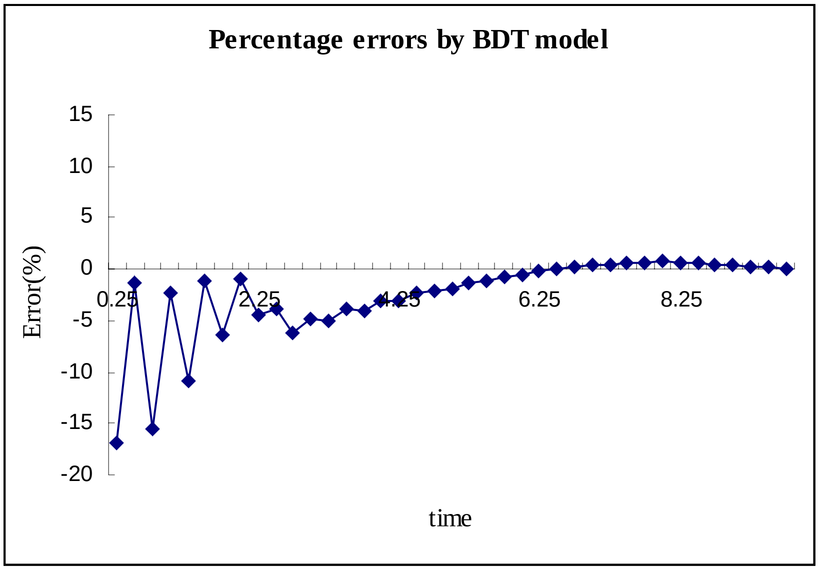


Figure 5 The percentage errors of the caplet prices obtained using the calibrated BDT model.



As expected, the BDT model can be fitted to value the caplets quite well using the term structure of volatilities. The inaccuracy of the model with nearby expiration date is simply because of the coarse step size of three months. For short dated derivatives, daily or weekly step size is more appropriate for the valuation models.

D. Calibration of the Ho-Lee Models

In this section we illustrate how the Extended Ho-Lee and the two factor Ho-Lee models can be calibrated to caplet prices obtained using Figure 1. An extension to *n*-factors directly follows the procedure outlined below. Using the general tree implementation procedure, the calibration procedure seeks to find an implied forward volatility function such that the extended Ho-Lee and two factor Ho-Lee model prices match the market prices quoted using the Black model.

In the extended Ho-Lee case, the volatility function enters the model through $\delta(n)$ given by

$$\ln \delta(n) = -2 f(n)\sigma(n)$$

where $f(n)$ are forward rates, in our example, equal to 6% for all *n*. We choose the functional form

$$\sigma(n) = (a + b \cdot n)\exp(-c \cdot n) + d$$

for the implied forward volatility. Using the goodness of fit function in section C, we obtain the optimal implied volatility parameters using a nonlinear optimization procedure. Figure 6 shows the implied volatilities resulting from the calibration procedure. Note that the resulting implied volatilities are estimated from a normal model and therefore they are different from the forward volatilities used as inputs in the Black model.

For the two-factor Ho-Lee model, we have

$$\ln \delta^1(n) = -2f(n)\sigma^1(n) \quad \text{and} \quad \ln \delta^2(n) = -2f(n)\sigma^2(n)$$

For our calibration procedure we choose

$$\sigma^1(n) = (a + b.n)\exp(-c.n) + d \quad \text{and} \quad \sigma^2(n) = e \quad \text{(a constant)}$$

Figure 6 and 7 shows the implied volatilities and percentage errors of caplet prices obtained using the calibration procedure, respectively.

Figure 6 Implied volatilities of the extended Ho-Lee and the two- factor Ho-Lee model.

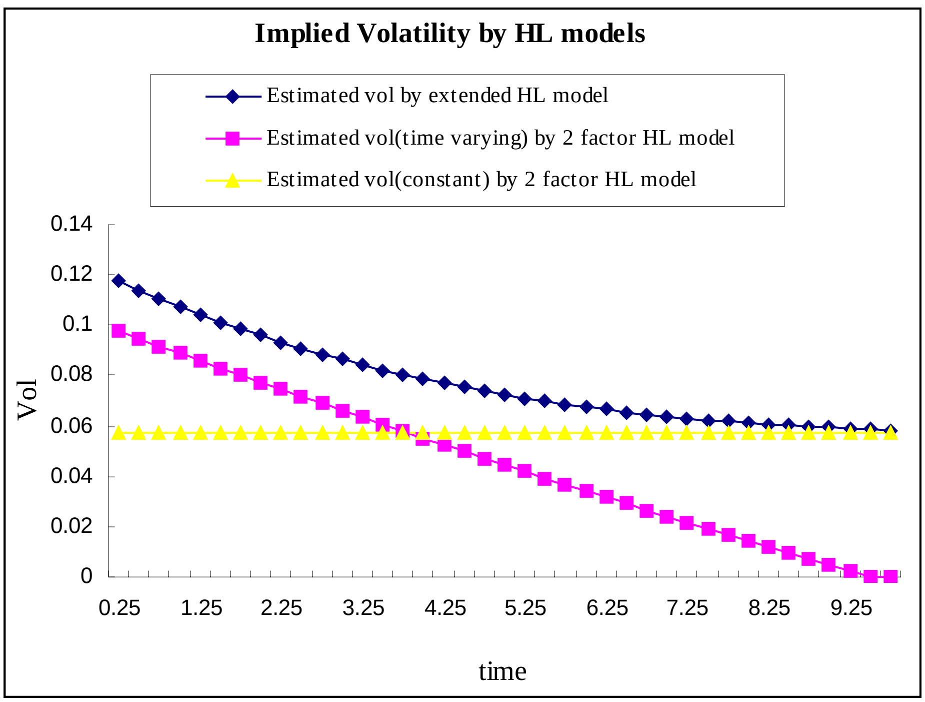
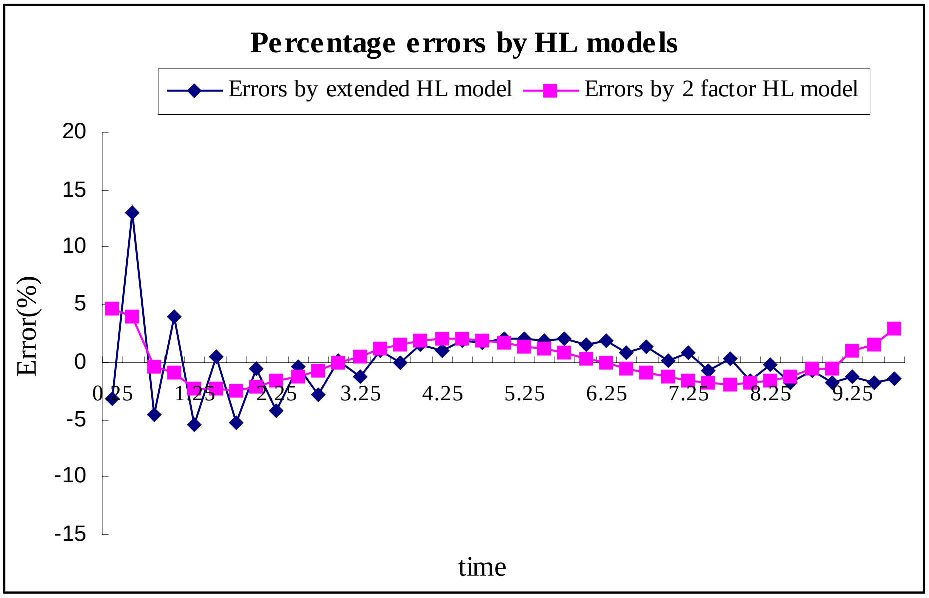


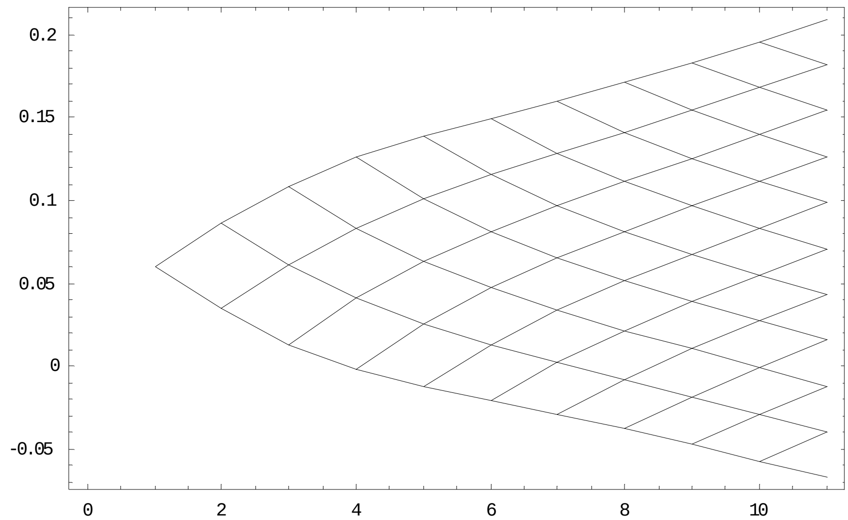
Figure 7 The percentage error of caplet prices obtained using the extended Ho-Lee model and the two-factor Ho-Lee Model.

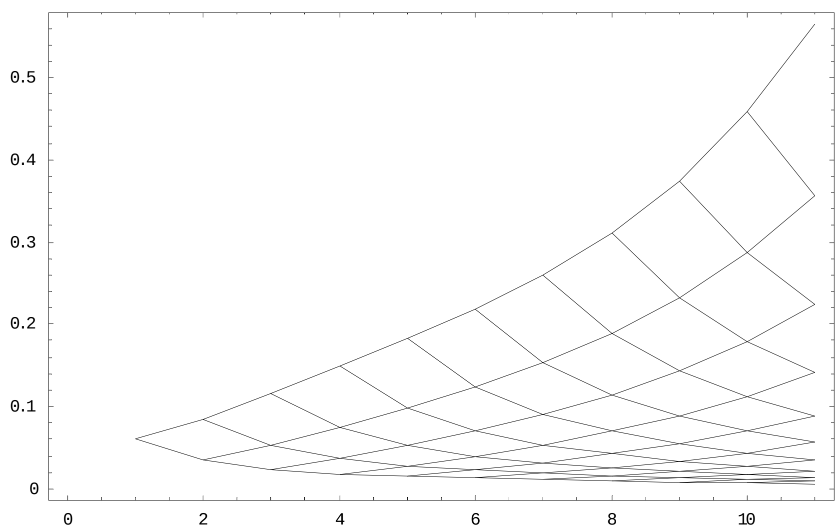


The results show that the Ho-Lee models can also fit the caplet prices quite well. Since we only use caplet prices and not the complete volatility surface, the extended Ho-Lee model does as well as the 2-factor Ho-Lee model. The result also shows that the implied volatilities as measured by normal models have similar magnitude as those measured by the lognormal model.

For comparison purposes, we draw a binomial lattice for the extended HL model (normal model) and the BDT model (lognormal model) in Figure 8.

Figure 8 The interest rate trees of the extended HL model versus the BDT model





E. Calibration of Longstaff, Santa-Clara and Schwartz "String" Model

The string model is given by:

$$dP(t,T^*) = r(t)P(t,T^*)dt + \sigma(t,T^*)^P P(t,T^*)dZ \qquad (1)$$

The string model is a multi-factor model where $\sigma(t,T^*)P(t,T^*)dZ$ is the vector formed by stacking correlated individual terms $\sigma_i(t,T_i^*)P(t,T_i^*)dZ_i$.

The volatilities and the correlations of the bond price return are calibrated to the observed term structure of volatilities and correlations. In our example we use the correlation matrix in Table 2.

Table 2. Correlation Matrix of the Interest Rates

	0.25	0.5	1	2	3	5	7	10	20	30
0.25	1.000	0.936	0.837	0.701	0.630	0.533	0.443	0.377	0.087	0.083
0.5	0.936	1.000	0.938	0.832	0.770	0.675	0.587	0.509	0.224	0.154
1	0.837	0.938	1.000	0.940	0.895	0.816	0.731	0.654	0.379	0.291
2	0.701	0.832	0.940	1.000	0.989	0.950	0.898	0.832	0.573	0.426
3	0.630	0.770	0.895	0.989	1.000	0.980	0.945	0.887	0.649	0.493
5	0.533	0.675	0.816	0.950	0.980	1.000	0.982	0.946	0.736	0.595
7	0.443	0.587	0.731	0.898	0.945	0.982	1.000	0.976	0.821	0.670
10	0.377	0.509	0.654	0.832	0.887	0.946	0.976	1.000	0.863	0.750
20	0.087	0.224	0.379	0.573	0.649	0.736	0.821	0.863	1.000	0.867
30	0.083	0.154	0.291	0.426	0.493	0.595	0.670	0.750	0.867	1.000

The results show that all the correlations are positive so that all the interest rates tend to move in the same direction. The long rates with terms over 10 years are highly correlated, which means that the rates from 10 year to 30 year range tend to move up and down together. The interest rates that are closer together along the yield curves have higher correlations. However, the correlations of the short term rates and the long term rates are relatively low.

The approach proposed by Longstaff, Santa-Clara and Schwartz (2000) is to solve for the implied correlation structure of the model that best fits the observed market prices of our caplet market prices. Instead, we specify the correlation structure exogenously and seek a term structure volatility that best fits caplet market prices. Let $\sigma(s)$ be the term structure of volatilities of proportional change in the bond price. This term structure of volatilities can be estimated from the term structure of spot volatilities and multiplied by the duration[2]. Specifically, we can write

$$\sigma\left(t,T^*\right) = y\left(t,T^*\right)\sigma^*\left(T^*-t\right)\cdot\left(T^*-t\right) \tag{2}$$

where $\sigma^*(\tau)$ is the estimated term structure of volatilities of the interest rates derived from the swaption, expiring over time Δ, with the tenor of the swap τ. $y(t,T^*)$ is the yield of a bond at time t maturing at calendar time T^*. $y(t,T^*)\sigma^*(T^*-t)$ is the standard deviation of the yield of a bond with maturity $(T-t)$ at time t. Since the duration of the $(T-t)$ year bond is $(T-t)$, assuming continuously compounding in the measure of yields, the right hand side of equation (2) is the standard deviation of the $(T-t)$ year bond, as required. If the yield is not measured in continuously compounding basis, then a modifier is used. The bond price dynamics follows

$$dP(t,T^*) = r(t)P(t,T^*)dt + \sigma(T^*-t)P(t,T^*)dZ(T^*-t). \tag{3}$$

Now we can re-write equation (3) in a discrete time formulation. Let us assume that the

[2] LSS proposed using the swaption prices.

step size be Δ. Let $k, j = 0,1,2,\ldots,m.$ For clarity of the exposition, we abuse the notations by writing $P(k, j)$, $\sigma(i)$, and $r(k)$ to mean $P(k\Delta, j\Delta)$, $\sigma(i\Delta)$ and $r(k\Delta)$ respectively. Then equation (3) can be written as:

$$P(k+1, j) = P(k, j)\exp\left[\left(r(k) - \frac{\sigma^2(j-k)}{2}\right)\Delta + \sigma(j-k)\sqrt{\Delta}Z(j-k)\right] \qquad (4)$$

where Z is a Brownian motion increment over a unit of time (i.e., a standard normal variate). Note that $P(0, j)$ is the initial discount function, price of a zero coupon bond for each maturity *j*. Now, using equation (4), we can simulate the arbitrage-free discount function movements, $P(k, j)$, where *k* denotes the time dimension of the evolution of the discount function movement and *j* is the calendar time of the maturity of the bond, or equivalently, (*j*-*k*) is the remaining maturity of the bond. In particular *P*(*k*, *k*+1) is the one period bond price at time *k*, and *r*(*k*) = -ln *P*(*k*, *k*+1) is the one period interest rate.

In this simulation, for illustrative purposes, we use the step size to be one year, Δ=1. To apply equation (4) iteratively, we first need to determine the initial discount function, which is derived from a flat initial yield curve assumption of 6%. Next we need to determine the Brownian process *Z*(*j*-*k*) for each time *j*. This is accomplished by taking the following steps.

We use the following definitions;
Z = vector of $\left(Z(1), Z(2), Z(3), Z(4)\right)$, $Z(i)$ is the price risk of a zero coupon bond with *i* year maturity at the end of the period and
$\mathbf{Z}_{uncorr}$ = vector of uncorrelated standard normal variates in each step in the Table 2.
M = Cholesky decomposition of the correlation matrix of the zero coupon bonds, which is the same as the correlation matrix of the spot yields, in Table 2. ***M*** is is an $n \times n$ matrix, defined to be $\mathbf{M}^{\mathbf{T}}\mathbf{M} = \Sigma$, where $\mathbf{M}^{\mathbf{T}}$ is the transpose of **M** and *Corr* is the correlation matrix of *n* risk sources. Then we have

$$\mathbf{Z} = \mathbf{M}^{\mathbf{T}} \times \mathbf{Z}_{uncorr} \qquad (5)$$

Using the correlation matrix

$$\Sigma = \begin{bmatrix} 1.000 & 0.940 & 0.895 & 0.856 \\ 0.940 & 1.000 & 0.989 & 0.970 \\ 0.895 & 0.989 & 1.000 & 0.990 \\ 0.856 & 0.970 & 0.990 & 1.000 \end{bmatrix}$$

extracted from Table 2, we can derive:

$$
\boldsymbol{M}^T = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.9400 & 0.3412 & 0.0000 & 0.0000 \\ 0.8950 & 0.4329 & 0.1075 & 0.0000 \\ 0.8560 & 0.4847 & 0.1307 & 0.1235 \end{bmatrix}
$$

Using a standardized normal distribution generator, we simulate the following 16 uncorrelated numbers:

Table 3 Table of Random Draws

$$
\mathbf{Z}_{uncorr} = \begin{bmatrix} 0.1589 & 0.4049 & 0.3110 & 0.5212 \\ -2.3733 & 0.8001 & 0.5650 & \\ -0.9183 & -0.1906 & & \\ 0.7583 & & & \end{bmatrix}
$$

Using equation (5), we can now generate the correlated random outcomes for the string model. The main characteristic of the string model is the use of correlated risk sources for each period along the entire yield curve. For this numerical example, for each time step, we have 5 sources of uncertainties at the beginning. After each step, the yield curve is shorten by one period and the number of uncertainties also falls accordingly. To generate one discount function movement scenario, we have the following random outcomes.

Table 4 Table of Uncertainties

T^*	t	1	2	3	4
1	$Z(1)$	0.1589	0.4049	0.3110	0.5212
2	$Z(2)$	–0.6603	0.6536	0.4851	
3	$Z(3)$	–0.9839	0.6883		
4	$Z(4)$	–1.0460			

Now, we proceed to determine the instantaneous volatilities of the bond price $\sigma(i)$ where *i* is the bond maturity. We begin with the observed spot volatilities quoted from the swaption market. $\sigma^*(T^* - t)$ are assumed as given by the first column in Table 5 below. Thus, at period *k*=1 the volatilities of bonds maturing at years *j*=2 and *j*=3 are given by $\sigma(1) =$ y(1) $\times \sigma^*(1) \times 1$ = 6.0% $\times$ 20% = 1.2% and $\sigma(2) =$ y(2) $\times \sigma^*(2) \times 2$ = 6.0%$\times$19%$\times$2 = 2.3%. Note that the yield is from the previous period *k* = 0. Table 5 is derived iteratively for *k*= 1, 2, 3 and 4. After deriving the column of volatilities for each *k*, we proceed to derive the zero coupon bond prices, that we will do next.

Table 5 Table of Term Structure of Volatilities

		t	1	2	3	4
$\sigma^*(1)$	20 %	$\sigma(1-k)$				
$\sigma^*(2)$	19 %	$\sigma(2-k)$	1.20 %			
$\sigma^*(3)$	18 %	$\sigma(3-k)$	2.28 %	1.16 %		
$\sigma^*(4)$	17 %	$\sigma(4-k)$	3.24 %	2.57 %	1.45 %	
$\sigma^*(5)$	16 %	$\sigma(5-k)$	4.08 %	3.82 %	2.62 %	1.22 %

The iterative process begins with the initial discount function presented in the first column where k = 0. From the one period bond price, we can calculate the one period interest rate. Now using the first column of the uncertainties and equation (4), we can derive the discount function for k= 1.

Table 6 Generating the Discount Function Movements, $P(k, j)$

T^* \ t	0	1	2	3	4	5
0	1.000					
1	0.942	1.000				
2	0.887	0.943	1.000			
3	0.835	0.873	0.930	1.000		
4	0.787	0.809	0.871	0.941	1.000	
5	0.741	0.753	0.819	0.892	0.954	1.000

Given the bond prices, we can determine the yields of the bonds by noting that the yield y is related to the bond price by $P(T) = \exp(-yT)$, where T is the bond maturity. At k=1, Table 7 shows that the prices of bonds maturing at j=2 and j=3 are given by

$$0.943 = 0.887\exp\left[0.06 - 0.012^2/2 + 0.012 \times 0.1589\right] \text{ and}$$

$$0.873 = 0.835\exp\left[0.06 - 0.023^2/2 + 0.023 \times 0.(-0.6603\right], \text{ respectively.}$$

Table 7 Deriving the Yields from the Bond Prices

T^* \ t	0	1	2	3	4
1 $y(1-k)$	6.0 %				
2 $y(2-k)$	6.0 %	5.82 %			
3 $y(3-k)$	6.0 %	6.77 %	7.25 %		
4 $y(4-k)$	6.0 %	7.08 %	6.89 %	6.09 %	
5 $y(5-k)$	6.0 %	7.08 %	6.65 %	5.73 %	4.76 %

Once again, we repeat the process. We begin with Table 4 period 2 (column 2), using the new set of bond price uncertainties, we now compute the instantaneous volatilities of the bond in period 2 using the updated bond yields. Then we derive the discount function for period 2 using equation (4) again. And we continue the process until we reach the period k = 5. The derived path of one period interest rates (6%, 5.82%, 7.25% 6.09%, 4.76%) is then used to determine the pathwise value of a security. Using the procedure by now familiar to the reader, we take the mean of all the pathwise values using a Monte Carlo simulation to determine the security value.

Thus far, we have assumed some observed term structure of volatilities as observed from the swaption market. We can also conduct a non-linear search for the best estimated term structure of volatilities by assuming the term structure has a certain functional form. Specifically, we assume that:

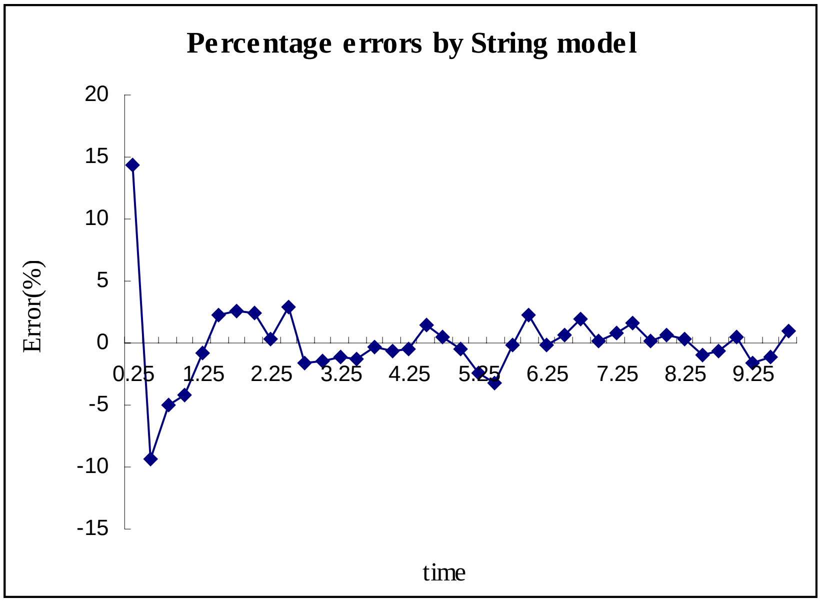
$$\sigma^*(T^* - t) = \left(a + b(T^* - t)\right)\exp\left(-c(T^* - t)\right) + d \tag{6}$$

and we can determine parameters a, b, c, and d, so that the model caplet prices fit the caplet market prices.

This procedure begins with specifying the discount function $P(T)$ using the prevailing zero coupon bond prices at the initial date. We assume some initial term structure of volatilities by assuming some values for a, b, and c for the specification of the term structure of volatilities. The correlations of the interest rates are based on the historical correlation matrix presented in Table 2.

Using equation (4) we simulate the term-structure of zero coupon bond prices which will be converted to forward rates in order to calculate the payoffs of the caplets. The value of $F(j+1, j) = \left(1/P(j+1, j) - 1\right)/\Delta t$ is the realized rate for time period between j and j+1 and this enables the caplet payoff at time j+1 to be calculated. This caplet payoff is discounted to time zero using the one period rates determined along that interest rate path. The estimated caplet value is the mean of the discounted payoffs. Now we use a non-linear optimization procedure to search for the optimized parameters *a*, *b*, and *c* such that the caplet prices are best fitted to the valuation model.

Figure 9 The percentage errors of the caplet prices by the string model.



The number of Monte Carlo simulation used is 1000 with quarterly step sizes. The result shows that the errors of the nearby caplets are higher because of the coarseness of the quarterly step size. Some errors can also be attributed to the Monte Carlo simulations that may require a higher number of simulations.

F. Calibration of the Brace-Gatarek-Musiela/Jamshidian Model (the LIBOR Market Model)

We implement a Monte Carlo process for BGM/J model with quarterly step sizes. Table 8 below shows the calibrated volatility obtained using Table 2. First we calculate instantaneous forward volatilities Λ_j using $\sigma_j^2 j = \sum_{i=1}^{j} \Lambda_{i-1}^2$ where σ_j is the Black volatility for a caplet that corresponds to the *j*th period. These Black volatilities are specified in Figure 1. The forward LIBOR rates $L(k, j)$ are given by:

$$L(k, j+1) = L(k, j)\exp\left[\left(\sum_{i=j+1}^{k} \frac{L(i, j)\Delta}{1+L(i, j)\Delta}\Lambda_{i-j-1}\Lambda_{k-j-1} - \frac{\Lambda_{k-j-1}^2}{2}\right)\Delta + \Lambda_{k-j-1}\sqrt{\Delta}Z\right] \qquad (7)$$

where Z is a random sample from a normal distribution with mean equal to 0 and variance equal to 1.

For illustrative purposes we consider the following example:

The path of the forward LIBOR $L(t, j)$ rates is shown in the Table below.

Table 8 Generating the Forward Libor Movements, $L(t, j)$

t			0	1	2	3	4
Z				−1.162	0.347	1.999	0.085
T*	σ_j	Λ_{j-1}					
1	20 %	20 %	6.00 %				
2	19 %	17.94 %	6.00 %	4.81 %			
3	18 %	15.81 %	6.00 %	4.96 %	5.21 %		
4	17 %	13.56 %	6.00 %	5.10 %	5.35 %	7.56 %	
5	16 %	11.14 %	6.00 %	5.27 %	5.50 %	7.47 %	7.50 %

The column σ_j presents the observed volatilities as observed from the caplets using the Black model. Λ_i are the instantaneous volatilities derived from σ_j. For this example, the initial yield curve is flat 6%. Therefore the forward one period rates for each year k is also 6%. At the end of the first period, a random draw is taken, giving $\widetilde{Z}$ = -1.162 and the Libor rate at the next period,

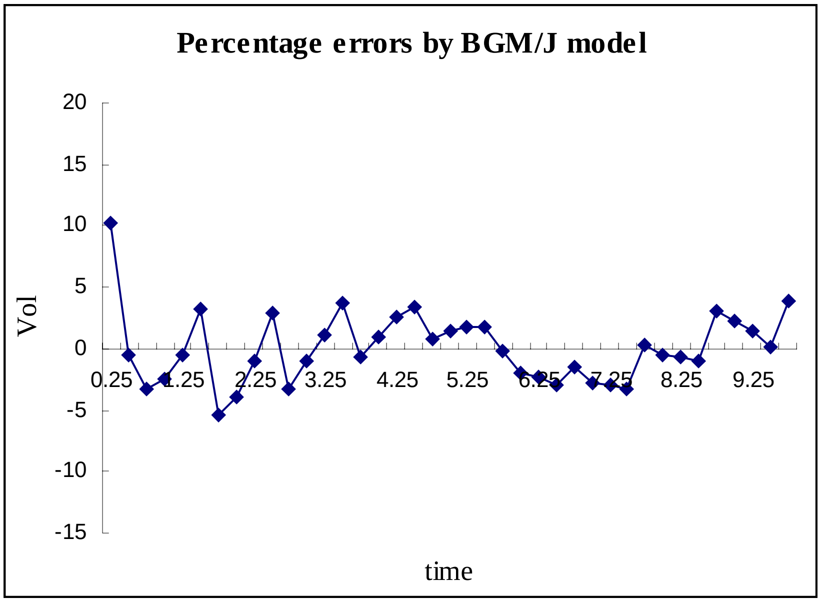
$$4.81\% = 6\% \times \exp\left(0.057\left(20\%\right)(17.94\%) + 0.057\left(17.94\%\right)^2 - (17.94\%)^2 / 2 + 17.94\%\left(-1.162\right)\right)$$

where $0.057 = 6\% / \left(1+6\%\right)$. Using equation (7) repeatedly for j = 1, 2, 3, 4, 5, we derive the second column of the forward LIBOR rates for the period k=1. Following this procedure, we can derive all the forward LIBOR rates as presented. Note that the forward LIBOR rates along the diagonal are in fact the simulated spot rates. These spot rates are the simulated LIBOR rates to be used for valuation. This interest rate path is then used for valuation in a way similar to other interest rate models.

In valuing the caplets, we use the interest rate path to calculate the payoff of the caplet. This caplet payoff is discounted to time zero along this interest rate path to determine the pathwise value. Finally, the estimated caplet value is the mean of the discounted payoffs for 1,000 interest rate path simulations.

The Figure 1 shows that instantaneous forward volatilities calculated as inputs to the model. The errors of the BGM model are presented in Figure 10, showing that the errors are comparable to the other models. The main advantage of this approach is not requiring the non-linear optimization procedure to calibrate the model to the caplet prices. Instead, the “calibration” of this market model is to calculate the instantaneous forward volatilities from the forward volatilities of the Black model before simulating the interest rates scenarios using the Monte-Carlo procedure.

Figure10 The percentage errors of the BGM/J caplet prices.



By contrast, Black model can price one benchmark security at a time and is not an interest rate model which is a general framework to value all interest rate contingent claims. All interest rate models that we have discussed require the calibration procedure to fit the benchmark securities' prices.

The BGM/J type of models assume that the LIBOR rates and swap rates follow a lognormal distribution as the Black model requires. Also, these models take the Black prices as given in much the same way as taking the spot yield curve as given. Therefore, these models are not constructed to evaluate the fair pricing of the caps/floors and swaptions. The models take these securities' prices as given and the models cannot be used to value these securities. In this sense, these models are calibrated to the market prices without using the non-linear search routines that are used in the Ho-Lee, BDT, Hull-White and other models.

Here we summarize the calibration results in terms of sum of squared residuals for comparison purposes. Even though we have many other measures to see which model has more fitting power, we choose the sum of squared residuals to this end, because this measure is intuitive and widely used in the finance area. We enumerate the sum of squared residuals of the five models in an increasing order, which are 7.36 (two factor Ho and lee), 17.73 (string), 18.51 (BGM), 20.11 (extended Ho and Lee) and 25.26 (BDT). Even though we do not conduct a formal statistical test to see whether those numbers are statistically different, we can easily see that two factor Ho and lee model has the least sum of squared residuals among the five interest rate models. Another thing which is worth mentioning is that the four models except BDT have been developed under the assumption that an initial term structure is given, so that we do not have to match the initial term structure when we simulate. However, since we should put the initial term structure matching as an equality constraint when we simulate the BDT model, the sum of squared residuals for BDT depends on how closely we empirically match the initial term structure to the given data. For example, if we apply a tight constraint when we simulate BDT, we might have a larger sum of squared residuals than what we report here. In this sense, we might not directly compare BDT with the other models in terms of the sum of squared residuals.

G. Applications of Interest Rate Models

A reasonable question to ask is: How do I choose the appropriate interest rate model for a task at hand? In this paper, we have introduced interest rate models in an order that follows the chronology of the discovery of interest rates models. Each interest rate model was proposed to solve the new challenges presented at the time. Now we should look back and discuss how the models fit together in order to solve our financial modeling problems.

There are various categories of financial modeling problems. We will present financial modeling problems and their solutions in different levels. On the highest level, understanding the term structure of interest rates (the yield curve) is important to economics. There is much research on the economic factors affecting the interest rate levels and movements. Often, these models are called the equilibrium models. These models study the supply and demand for funds in the economy and determine the equilibrium solutions to the preferences of the economic agents. They enable us to understand how macroeconomics affects interest rates and how interest rates relate to government fiscal policies and other macro-economic policies.

In financial markets, we are concerned with the valuation of interest rate contingent claims. The yield curve is used as benchmark to value other securities. In this case, we use the arbitrage-free interest rate movement models. These models provide a more accurate pricing because they specify the contingent claim price relative to the yield curve. Such models are partial equilibrium models because they do not require any notion of equilibrium solution to the yield curve, only the arbitrage free condition between the contingent claims and the yield curve is necessary.

Arbitrage-free interest rate models are also preference free, independent of the market agents' preferences, their investment and consumption behavior. However, the models need to be calibrated to fit the observed market volatilities, in addition to the yield curve. The market volatilities are represented by the prices of the benchmark securities, such as caps/floors, swaptions, and bond options.

Black models refer to closed form models that provide the price of these benchmark securities given the volatility assumptions. Therefore, these models are ideal to use for market convention in quoting these benchmark securities in terms of the market volatilities. Black models provide the exact mathematical relationships between the market quote on volatilities and the value of the benchmark derivatives.

Given the market quote of the volatilities, the market models translate those quotes to the prices of the benchmark derivatives. Now the arbitrage-free interest rate models are calibrated to fit these benchmark derivative prices by solving for the implied volatility surface that is applicable to the interest rate model. Different interest rate models would have (possibly only slightly) different implied volatility surfaces. For example, a lognormal interest rate model would assume the benchmark derivative prices are derived from a lognormal interest rate movement. A normal interest rate model would assume that the interest rates follow normal distributions. Irrespective of the assumptions of the models, they all seek to value the benchmark derivatives accurately and value other contingent claims relative to the benchmark derivatives values.

Within the arbitrage free interest rate models, we have the choice from a selection of models depending on the features of the models. There are two mathematical theorems that link the volatility surface to the interest rate models. The first theorem is the Heath, Jarrow, and Morton (HJM) model that provides an interest rate model as a solution to any volatility surface specified to the model. Therefore, HJM has reduced the problem of specifying an interest rate model to that of specifying the volatility surface. For a given volatility surface, HJM provides an interest rate model as a mathematical solution that would have the implied volatility surface to be identical to the given volatility surface. Using this specification, we can derive the instantaneous forward rates simulated in the future, and those rates would be consistent with the Ho-Lee model in the continuously time framework.

However the market provides the prices of the benchmark derivatives or the market volatility surface that is based on the Black models (for caps/floors, swaptions, etc.), and not a volatility surface based on a particular interest rate model. Therefore, we need to calibrate the implied volatility surface to the observed market prices. Calibrating the HJM interest rate model to the prices of the benchmark derivatives can be difficult because the HJM interest rate models are often non-recombining models. The model error of using Monte-Carlo simulations may lead to errors in the calibrating process.

The BGM/J model is the second mathematical theorem that provides a solution to the aforementioned problem. BGM/J shows that an interest rate model can be specified directly from the quoted volatility surface for a particular set of benchmark securities. The interest rate model is a multi-factor model derived from continuous time analysis.

Interest rate models derived from HJM or BGM/J are continuous time models. They are not binomial lattice models. Given a continuous time model, there is no specific procedure to provide a recombining lattice model. While binomial lattice models in the limit, as the step size of the lattice becomes arbitrarily small, becomes a continuous time model, the converse is not true in general. That is, there is no specific mathematical procedure that translates a continuous time model to a recombining binomial lattice. And there are advantages to use a recombining lattice model for particular purposes.

The most important attribute of recombining lattices is that they cover the states and time of the world by the nodes in a way not computationally prohibitive. Therefore, we have a way to describe the contingent claim, its price and behavior, at each node, and how the information of each node is related to other nodes. By way of contrast, when we simulate the interest rate movements as in the HJM, BGM/J, or string models, scenarios are

generated, and at each point on a scenario, we only know where that point came from and where it will go. There is no analytical relationship between any point of one scenario with another point of another scenario. For this reason, we will show that it is more difficult to solve multi-period optimal decisions in a tree that is not recombining. In the corporate finance section, we need to describe how corporate management makes optimal decisions, that are related not only to the prevailing market realities to the corporate managers but also related to the management's past and future decisions. In other words, corporate optimal decisions cannot be made myopically, looking only at the outcome of the next step. But decisions at all the nodes have to be made jointly.
Given the relatively inexpensive computational power, the step size used for practical use can be made very small. For illustrative purposes, academic research may use monthly step size to price a five-year bond, for example. But in practice, the step size can be daily. The computational requirement is still manageable in most cases. Therefore the error of assuming discrete outcome in the lattice model can be small.

For derivatives that require use of Monte Carlo simulations, the random scenarios can be generated from the lattice, as we have shown in the pathwise analysis. Therefore, a lattice can be viewed as a consistent valuation framework that can be used for selecting scenarios and for using backward substitution to value securities.

In summary, we provide below some of the issues in relation to using a normal recombining lattice interest rate model.

(1) Efficient in calibrating to benchmark securities

   a. Since the lattice approach can value a broad range of derivatives (including American options) accurately, we can use a broad range of derivatives to calibrate the model, instead of being confined to caps/floors and European swaptions.
   b. When a normal interest rate model is used, there is a concern of the probability of negative interest rates. This problem can be handled by using benchmark securities like floors with strike rate and the price being zero in calibrating the model. In fitting the interest rate model to ensure that the floors with zero strike price has no value would minimize the importance of negative interest rate scenarios in derivatives pricing.
   c. It is interesting to note that there were times, some floors on Yen rates with zero strike rate had positive value. Suggesting that the market perceives possible negative interest rate scenarios. A lognormal model that prohibits any possibility of negative interest rates would violate this observation in the market.
   d. Note also that some Black models assume the bond price to have a lognormal distribution. For example, if we extend the Black-Scholes model to the bond option model, then we would assume that the bond price follows a lognormal distribution. If we assume the price follows a lognormal distribution, and, since the log of the price is proportional to the yield by definition, we would implicitly assume that the interest rate is a normal process. Indeed, the string model suggested by Longstaff and Schwartz assumes implicitly that the

interest rates follow a normal model and negative interest rate scenarios are possible.

e. Finally, we must remember that this probability measure is a risk neutral measure and not the market probability measure- that is not the real measure. While normal model allows for negative interest rates, it does not assign any real probability to those scenarios to occur.

(2) Ensure consistency with simulations

a. Since scenarios can be selected from the lattice, random scenarios selected from the lattice can be ensured to be consistent with the pricing of the benchmark securities, as the interest rate model can be calibrated to all the benchmark securities' prices.

b. For the above reason, we can use the recombining lattice to calibrate the interest rate model, and then use the lattice to determine the Monte Carlo simulations.

(3) Simultaneous analysis of related decisions at all the nodes

a. Since we can relate all the information at each node to those at all other nodes, we can formulate optimal decisions made at all the nodes of a lattice simultaneously, resulting in a global optimal solution.

b. The main advantage of a normal model is that the interest rate model provides a closed form solution of the term structure of interest rate at each node point. Since many decisions require the knowledge of the entire term structure, the model provides an efficient method to determine the optimal decisions.

H. Conclusion

This paper considers an important issue in the implementation of an arbitrage-free interest rate model. The problem is to measure the future interest rate risks that determine the option price.

Often market implied volatilities are used to deal with this issue, where these implied volatilities are solved for using benchmark securities prices. We then discussed the market convention in the use of market models to provide the market quotes in terms of implied volatilities. And we discussed such conventions in more details with bond options, swaptions, and caps and floors.

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