CreditRisk+: Extension and Application

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Summary. The CreditRisk+ [CSFP, 1997] methodology, attractive in its analytic tractability and small data requirement, has limited application because of its sector independency assumption. Attempts have been dedicated to relax this impractical assumption but none of them seems to have reached the goal. This paper proposes a simple way of extending the original model so as to accomodate sector correlation. Our model is flexible enough to cover various covariance structures. One advantage of the approach is that existing numerical algorithms designed for the classical CreditRisk+ model can be reused with little modification.

A simulation technique for the CreditRisk+ model, introduced in [Glasserman, 2003], is also applicable to our model. Then, using the simulated losses as input, we can find an optimal portfolio allocation by minimizing conditional Value-at-Risk (CVaR) as proposed in [Andersson *et al.*, 2001].

Case study shows that our model outperforms other CreditRisk+ variants which allows sector dependency. Simulation error is very small compared to analytic results and the optimization significantly reduces portfolio credit risk.

Keywords: *CreditRisk+, Sector correlation, Simulation, Importance Sampling, Conditional Value-at-Risk, Optimization.*

1 CreditRisk+ for Correlated Sectors

CreditRisk+ model, since its debut in 1997 [CSFP, 1997], has received great attention from the financial industry and improvements have been proposed in the literature. [Gundlach and Lehrbass, 2003] is a good reference to review recent issues.

CreditRisk+ models probability of default as a linear combination of gamma-distributed risk factors, referred to as sectors. So the probability of default, p_i^s, conditional on sectors has the form

$$p_i^s = p_i \left(w_{0i} + \sum_{k=1}^{K} w_{ki} S_k \right)$$

where p_i is the expected probability of default, w_{ki} are weights on sectors, and $S_k,\ k = 1, \ldots, K$ are sector values, which are gamma distributed with

$$E[S_k] = 1 \quad \text{and} \quad V[S_k] = \sigma_k^2. \tag{1}$$

One major pitfall of the original model is that it assumes S_k are independently distributed, which is unreal and impractical: Sectors, being normally defined as industry sectors, are in general highly correlated, so defining sctors as industries and assuming they are independent underestimates credit risk. Though, in theory, correlation between obligors can be incorporated through sector weights, there is no intuitive way to determine them.

Various approaches to address this issue have been introduced in the literature; [Bürgisser *et al.*, 1999], [Giese, 2003], [Lesko *et al.*, 2003] to list a few. However, their ability to incorporate sector correlation is limited to a narrow range of covariance structure and sector correlation still remains as a critical implementing issue for the CreditRisk+ model.

Among the models taking sector dependency into account, the hidden gamma model [Giese, 2003] is of particular interest. While other models attempt to incorporate sector dependency with a reduced number of sectors, one sector in [Akkaya *et al.*, 2003], for example, the hidden gamma model adds a common risk factor which affects all sectors and so determines covariance structure of sectors. This is consistent with the economic intuition for sector correlation: a macro economic factor affects all industry sectors and results in correlation. The hidden gamma model has the form

$$S_k = \sigma_k^2 (Y_k + \hat{Y}) \tag{2}$$

where $\hat{Y} \sim Gamma(\hat{\theta}, 1)$ is the common risk factor, and $Y_k \sim Gamma(\theta_k, 1)$ are sector specific risk factors, which is independent of $\hat{Y}$. Equation (1) and $\sigma_k^2 \geq 0$ define the bounds on $\hat{\theta}$:

$$0 \leq \hat{\theta} \leq \min_k \left\{ \frac{1}{\sigma_k^2} \right\} \tag{3}$$

This condition, which might be overlooked at the first glance, in fact, considerably restricts the covariance structure the model is able to depict. The correlation coefficient between sector k and l is given by

$$\rho_{kl} = \hat{\theta} \sigma_k \sigma_l \tag{4}$$

Suppose that sector k has the largest variance and sector l has the smallest variance and $\sigma_k / \sigma_l = 5$, then

$$\rho_{kl} \leq \frac{\sigma_l}{\sigma_k} = 0.2.$$

Thus, if sector variances vary in a wide range, the hidden gamma model becomes valid only when correlations are low.

Note that the hidden gamma model can be restated in the form of the classical CreditRisk+ model, now with $K+1$ sectors:

$$p_i^s = p_i \left(w_{0i} + \sum_{k=1}^{K+1} w_{ki} \hat{S}_k \right) \tag{5}$$

where [1]

$$\begin{aligned}
\hat{S}_k &\sim Gamma\left(\frac{1}{\sigma_k^2} - \hat{\theta}, \sigma_k^2\right), \quad k = 1, \ldots, K, \\
\hat{S}_{K+1} &\sim Gamma\left(\hat{\theta}, 1\right), \quad \text{and} \\
w_{(K+1)i} &= \sum_{k=1}^{K} w_{ki} \sigma_k^2
\end{aligned}$$

1.1 Generalization of the Hidden Gamma Model

In the hidden gamma model, It is not a necessary requirement that the coefficients of Y_k and $\hat{Y}$ be the same as σ_k^2. We can generalize the hidden gamma model as follows.

$$S_k = \delta_k Y_k + \gamma_k \hat{Y}, \qquad k = 1, \ldots, K \tag{6}$$

where

$$Y_k \sim Gamma\,(\theta_k, 1) \tag{7}$$
$$\hat{Y} \sim Gamma\,(\hat{\theta}, 1) \tag{8}$$

In this generalized model, S_k is no longer gamma distributed as long as $\delta_k \neq \gamma_k$, which is a deviation from a key assumption of the CreditRisk+. However, newly defining $K+1$ gamma distributed sectors, p_i^s can be written as a linear combination of those sectors and the model still lies within the framework of the CreditRisk+:

$$p_i^s = p_i \left(w_{0i} + \sum_{k=1}^{K+1} w_{ki} \hat{S}_k \right) \tag{9}$$

where

[1] One might argue that $\sum_{k=0}^{K+1} w_{ki} \neq 1$ is a violation of the model. A gamma distributed random variable has the following property:

$$g_k \sim Gamma\,(\theta_k, 1) \rightarrow \beta_k g_k \sim Gamma\,(\theta_k, \beta_k),$$

which is why β_k is called a scale parameter. Thus, sum of weights being 1 is not a hard constraint.

$$\hat{S}_k \sim Gamma\,(\theta_k, \delta_k)\,, \quad k = 1, \ldots, K, \tag{10}$$

$$\hat{S}_{K+1} \sim Gamma\,\left(\hat{\theta}, 1\right), \quad \text{and} \tag{11}$$

$$w_{(K+1)i} = \sum_{k=1}^{K} w_{ki}\gamma_k \tag{12}$$

$\hat{S}_k,\ k = 1, \ldots, K$ can be interpreted as sector specific risk factors and $\hat{S}_{K+1}$ as a macro economic risk factor, and the degree by which a sector is affected by the macro economic risk factor is determined by δ_k and γ_k. From this point of view, we refer to the model as *CreditRisk+ with a common risk factor* or simply as a *common factor model*, for convenience.

The main advantage of the common factor model is that it enables us to incorporate sector correlation in a intuitive way and still retains the form of the original CreditRisk+. This allows us to faciliate numerical algorithms developed for the CreditRisk+ model with minor change. Among them, a generalized version of the numerical procedure in [Haaf *et al.*, 2003], suited for the common factor model, is presented in Appendix A.

The common factor model has the forms of expectation and covariance matrix

$$E[S_k] = \delta_k\theta_k + \gamma_k\hat{\theta}, \tag{13}$$

$$V[S_k] = \delta_k^2\theta_k + \gamma_k^2\hat{\theta}, \tag{14}$$

$$Cov[S_k, S_l] = \gamma_k\gamma_k\hat{\theta}. \tag{15}$$

Therefore, appropriately choosing the parameters, various covaraince structures can be described by the model.

1.2 Calibration of the Common Factor Model

Now it remains to identify the parameters of the common factor model. The sector covariance matrix consists of $K(K+1)/2$ elements and the common factor model has $3K+1$ parameters. Combined with the constraint $E[S_k] = 1$, we have $2K + 1$ degrees of freedom to specify the covariance matrix.

Define the distance between an observed covariance matrix and the estimated covariance matrix as the residual sum of squares:

$$f(\theta_k, \delta_k, \gamma_k, \hat{\theta}) = \sum_{k=1}^{K}\left(\sigma_k^2 - \delta_k^2\theta_k - \gamma_k^2\hat{\theta}\right)^2 + \sum_{k=1}^{K}\sum_{l=1}^{k-1}\left(\sigma_{kl} - \gamma_k\gamma_l\hat{\theta}\right)^2. \tag{16}$$

The parameters can be estimated by minimizing this distance under constraints:

$$\min_{\theta_k, \delta_k, \gamma_k, \hat{\theta}} f(\theta_k, \delta_k, \gamma_k, \hat{\theta})$$

such that

$$\delta_k \theta_k + \gamma_k \hat{\theta} = 1, \quad k = 1, \ldots, K,$$
$$\theta_k \geq 0, \quad \hat{\theta} \geq 0, \quad k = 1, \ldots, K,$$
$$\delta_k \geq 0, \quad k = 1, \ldots, K,$$
$$\sum_{k=1}^{K} w_{ki} \gamma_k \geq 0, \quad \text{for each obligor } i.$$

The first constraint is from $E[S_k] = 1$ and the last constraint is from $w_{(K+1)i} \geq 0$.

If variances can be more accurately measured than covariances, an alternative would be to exactly match the variances and then estimate parameters by minimizing the second term of $f(\cdot)$.

2 Minimizing CVaR within the CreditRisk+ Framework

CVaR as an optimization criterion for credit portfolios, first proposed in [Andersson *et al.*, 2001], has also been studied in [Jobst, 2001]. The results indicate that this criterion is well-justified and leads to meaningful results. The main advantage of the methodology is, for a simulation-based risk model, the optimization problem is formulated as a linear programming and easily solved. The formulation of CVaR minimization with various practical constraints can be found in [Han and Park, 2006].

To take advantage of the methodology in [Andersson *et al.*, 2001], we need to draw random samples from the loss distribution defined by the common factor model. In the CreditRisk+, loss from an obligor is given by

$$X_i = \nu_i \cdot D_i,$$

where ν_i is the net exposure of obligor i and D_i is a conditional Poisson random variable with $E[D_i] = p_i^s$. So the (conditional) loss function of the portfolio under scenario j is given in terms of Poisson random sample, $D_j = \{D_{j1}, \ldots, D_{jN}\}$ by

$$f(x, D_j) = \sum_{i=1}^{N} (\nu_i \cdot D_{ji})\, x_i, \tag{17}$$

where x_i is the portfolio weight of obligor i in terms of a multiple of the current net exposure. D_i can be efficiently simulated by deploying the importance sampling technique proposed in [Glasserman, 2003]. The basic idea is to adjust the gamma and Poisson distributions so that the expectation of the loss becomes a desired extreme loss such as VaR. Importance sampling is especially well suited for the CreditRisk+ framework in which the desired loss can be found analytically. Otherwise, an approximation method, such as quadratic approximation, whould be employed. Importance sampling procedure for the common factor model is summarized in Appendix B.

3 Case Studies

In this section, the common factor model is compared to the classical CreditRisk+ model, the hidden gamma model, and the compound gamma model. For details of the last two models, refer to [Giese, 2003]. Obligor data of Korea Credit Gurantee Fund (KCGF) was collected for the test. KCGF is a public financial institution, whose objective is to lead the balanced development of the national economy by extending credit guarantee services for the liabilities of promising enterprises which lack tangible collateral. The CreditRisk+ methodology, which models only default events, is particularly well suited for the fund since most obligors are small to mid-size businesses, for which market-to-market approach is inappropriate. As the instruments are purely based on obligor's credit, they are characterized by high default probability and low recovery rate; about 7% of default rate and 25% of recovery rate in average.

The fund, as of the end of 2006, has 213,000 obligors. The size of loans varies from tens of millions to billions summing up to 24,999,480 million in the local currency (won). The fund classifies their obligors into six industry sectors; light industry (LIGHT), heavy industry (HEAVY), construction, wholesale and retail (RETAIL), service (SERVICE), and others (OTHERS), and we defined the sectors as such.

The probability of default and the net exposure were provided by the fund, and the covariance matrix of the sectors was estimated from 60 monthly observations drawn from January 2001 to December 2005 (Figure 1). Default rates are calculated based on the default events occured for one year to each observation date, so the montly drawn samples were overlapped. This gives rise to autocorrelation and underestimation of variance. It, however, turned out that, in our sample, increasing the sampling interval to a quarter or a half-year did not alter the result in a notable way. So, we chose to stay with the montly observations to maximize the sample size. The estimated covariance matrix is shown in Table 1. In the table, the upper triangular part contains covariances normalized by the sample mean, and the lower triangular part contains corresponding correlation coefficients. Sectors are very highly correlated, from which we imply that survival of small and mid-size firms are very sensitive to the macro economic trend.

3.1 Comparison of CreditRisk+ models

The covariance matrix in Table 1 was calibrated by the hidden gamma model (HG), the compound gamma model (CG), and the common factor model (CF) and the results are shown in Table 2. The estimates of parameters of the hidden gamma model and the compound gamma model were bounded, respectively by the largest and the smallest sector variances. Comparing Table 2 to Table 1, you see that the common factor model well calibrates a rather heterogeneous covariance matrix, significantly outperforming the other

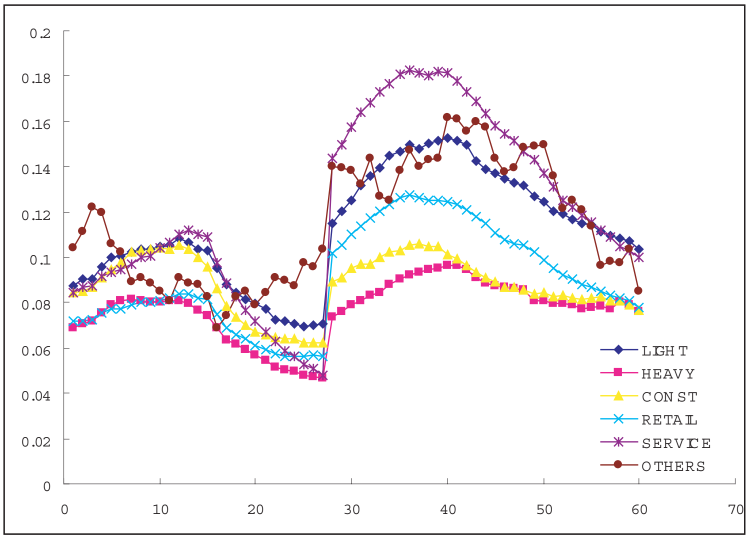


Fig. 1. Sector default rate time series. Dafault rates were monthly observed for five years from January 2001 to December 2005.

	LIGHT	HEAVY	CONST	RETAIL	SERVICE	OTHERS
LIGHT	0.0489	0.0350	0.0232	0.0542	0.0729	0.0424
HEAVY	0.9282	0.0291	0.0213	0.0373	0.0504	0.0265
CONST	0.7038	0.8371	0.0222	0.0257	0.0347	0.0127
RETAIL	0.9870	0.8806	0.6936	0.0617	0.0827	0.0489
SERVICE	0.9867	0.8858	0.6959	0.9966	0.1116	0.0643
OTHERS	0.8183	0.6644	0.3627	0.8401	0.8221	0.0549

Table 1. Sector covariance matrix estimated from 60 default rate time series observed for five years from January 2001 to December 2005. Default rates of each sector were normalized by their sample mean. The upper triangular part of the table contains covariances and the lower triangular part contains correlation coefficients.

two models. Calculated residual sum of squares were 0.0000951, 0.00484, and 0.0116, respectively.

With the estimates of each model's parameters, we computed the loss distribution of the portfolio (Table 3). As expected, the classical model with independency assumption returns the smallest estimate of the unexpected loss and the other two models also more or less underestimate the unexpected loss compared with the common factor model: The hidden gamma model with the second largest loss estimates, still underestimates the 99% loss by 222,611 million won (about 8% of the loss) compared to the common factor model. This result reflects how using the CreditRisk+ methodology without

	LIGHT	HEAVY	CONST	RETAIL	SERVICE	OTHERS
LIGHT	0.0489	0.0127	0.0098	0.0270	0.0489	0.0241
HEAVY	0.3377	0.0291	0.0058	0.0161	0.0291	0.0143
CONST	0.2956	0.2278	0.0222	0.0123	0.0222	0.0109
RETAIL	0.4922	0.3794	0.3320	0.0617	0.0617	0.0304
SERVICE	0.6620	0.5102	0.4465	0.7436	0.1116	0.0549
OTHERS	0.4643	0.3578	0.3131	0.5215	0.7013	0.0549

(a) Hidden gamma model

	LIGHT	HEAVY	CONST	RETAIL	SERVICE	OTHERS
LIGHT	0.0489	0.0222	0.0222	0.0222	0.0222	0.0222
HEAVY	0.5902	0.0291	0.0222	0.0222	0.0222	0.0222
CONST	0.6745	0.8751	0.0222	0.0222	0.0222	0.0222
RETAIL	0.4050	0.5255	0.6005	0.0617	0.0222	0.0222
SERVICE	0.3011	0.3907	0.4465	0.2681	0.1116	0.0222
OTHERS	0.4294	0.5571	0.6366	0.3823	0.2842	0.0549

(b) Compound gamma model

	LIGHT	HEAVY	CONST	RETAIL	SERVICE	OTHERS
LIGHT	0.0490	0.0341	0.0227	0.0552	0.0730	0.0419
HEAVY	0.9057	0.0289	0.0158	0.0384	0.0508	0.0291
CONST	0.6814	0.6176	0.0226	0.0255	0.0338	0.0194
RETAIL	0.9995	0.9058	0.6815	0.0621	0.0821	0.0472
SERVICE	0.9909	0.8981	0.6757	0.9911	0.1106	0.0624
OTHERS	0.8064	0.7309	0.5499	0.8065	0.7997	0.0550

(c) Common factor model

Table 2. Calibrated covariance matrices. The upper triangular part of the table contains covariances and the lower triangular part contains correlation coefficients.

the knowledge of sector correlation can be dangerous underestimating the credit risk.

3.2 Performance of the Simulation based approach

Table 4 compares percentile losses obtained from the simulation to those from the analytical procedure. 10000 samples were drawn for the simulation. As shown in the last column, simulation error is very small around the region of interest, less than 0.1%. Though not presented here, simulation with only 2000 iterations increased the error in a trivial way; less than 1% for a wide percentile range. This is very impressive result for the number of obligors.

	INDEP	HG	CG	CF
EL	1,757,152	1,757,152	1,757,152	1,757,152
50.00%	1,747,171	1,731,366	1,740,626	1,716,367
75.00%	1,888,368	1,947,578	1,952,889	1,984,383
90.00%	2,023,641	2,163,954	2,157,046	2,259,301
95.00%	2,108,487	2,303,922	2,285,262	2,439,720
99.00%	2,275,860	2,588,760	2,538,149	2,811,371
99.90%	2,476,821	2,944,298	2,841,059	3,281,133

Table 3. Percentile losses of different CreditRisk+ models. Losses are in millions. INDEP, HG, CG, and CF respectively stand for the classical model, the hidden gamma model, the compound gamma model, and the common factor model.

	Anlytic	Simulation	Error(%)
EL	1,757,152	1,752,533	-0.263
50.00%	1,716,367	1,733,579	1.003
75.00%	1,984,383	1,990,321	0.299
90.00%	2,259,301	2,253,257	-0.268
95.00%	2,439,720	2,439,275	-0.018
99.00%	2,811,371	2,813,805	0.087
99.90%	3,281,133	3,279,864	-0.039

Table 4. Analytic vs. simulation results of the common factor model. The pre-specified loss x in Equation (37) was set to 99% percentile loss computed analytically. Simulation consisted of 10000 iterations. Losses are in millions.

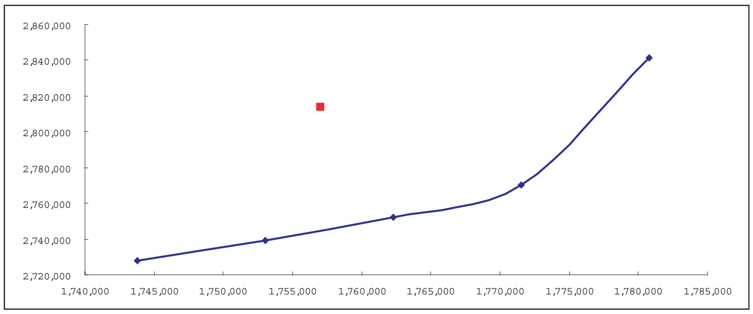
3.3 Minimum CVaR Portfolio

The objective of KCGF, as a public institution, is not to maximize its profit. It rather aims to control the expected loss to a level determined by the policy. So the usual practice of portfolio optimization; minimize risk given a required expected return, is not appropriate to the fund. The main source of profit is the gurantee fee which is received at the time of gurantee, but it does not correctly reflect the risk associated with the contract. So we minimized CVaR equating the expected loss to a pre-determined expected loss.

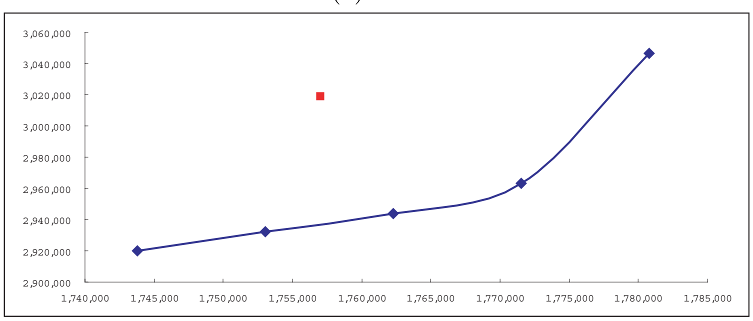
The obligors were grouped by six sectors and the rebalancing of each sector was constrained by the bound $[70\%, 130\%]$. Instead of specifying an expected loss level, we found the minimum and maximum values of the expected loss achievable by the portfolio subject to the rebalancing bounds, and uniformly chose five EL points in the range. For each of these five expected losses, we obtained the minimum CVaR portfolio allocation. Connecting the five $(EL, CVaR)$ points, we constructed a *efficient frontier-like* curve as shown in panel (b) of Figure 2. The current portfolio lies well above the efficient line, providing room for improvement of the portfolio's risk profile. Even though we minimized CVaR, EL-VaR curve in panel (a) also has very similar pattern.

Table 5 and Figure 3 show the optimal portfolio weights. It is expected that a sector with high default rate volatility relative to the expectation will be weighted less since it will increase the CVaR for the same level of EL. As expected, SERVICE sector whose VOL/PD ratio is the greatest, is bounded by the lower bound in most EL values. Similarly, the optimal weights of RETAIL sector whose VOL/PD ratio is the second greatest, are also smaller than the current weight. Many sectors were bounded by the rebalancing constraints.

We grouped assets into only six sectors for simplicity. Adding more classification criteria such as ratings will give more control for asset rebalancing and the resulting portfolio allocation would improve the risk profile even more.



(a) VaR



(b) CVaR

Fig. 2. Minimum CVaR given a required expected loss level. The point above curve in each panel represents the current portfolio.

4 Concluding Remarks

In this paper, we proposed a simple and intuitive way of incorporating sector correlation in the CreditRisk+ framework. Our model, referred to as the

	VOL/PD	W0	W1	W2	W3	W4	W5
LIGHT	0.746	14.660	10.262	13.128	15.994	18.519	19.058
HEAVY	0.507	26.783	34.818	34.818	34.818	34.818	34.818
CONST	0.429	10.948	14.233	14.233	14.233	13.320	7.664
RETAIL	1.120	37.381	33.528	30.661	27.795	26.167	26.167
SERVICE	1.983	10.198	7.139	7.139	7.139	7.139	12.256
OTHERS	0.596	0.029	0.020	0.020	0.020	0.038	0.038

Table 5. Optimal portfolio weights. Values in column VOL/PD are the volatilities of default probability divided by its expectation. W0 is the current portfolio weights and W1 to W5 are optimal portfolio weights for five EL points in ascending order.

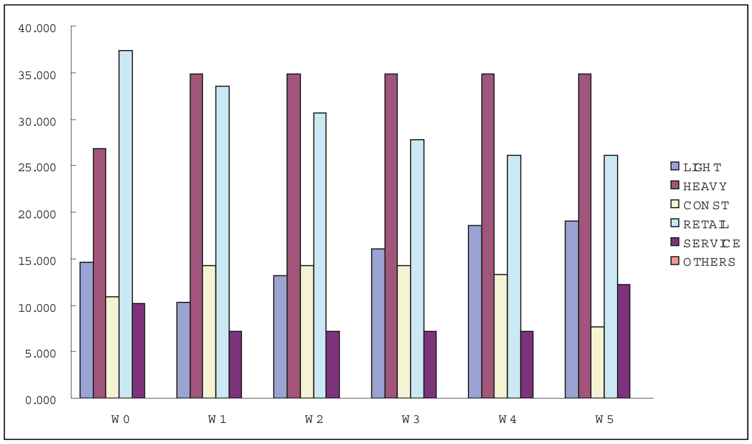


Fig. 3. Optimal portfolio weights. W0 is the current portfolio weights and W1 to W5 are optimal portfolio weights for five EL points in ascending order.

common factor model, is very flexible in that it can accomodate various types of heterogeneous covariance structures. Case studies show that ignoring or falsely estimating sector correlation can significantly underestimate the credit risk.

Measuring risk is not everything of risk management. Once risk is measured, the results have to be reflected to portfolio management. We found optimal portfolio allocations by using the methods in [Andersson *et al.*, 2001]. For this we had to draw random samples based on the model assumptions. Importance sampling deployed in the simulation procedure notably reduces the simulation error. The results suggest that rebalancing the current portfolio by minimizing the CVaR significantly improves the portfolio's risk profile.

We, by simulating losses, relied on a linear programming approach for the portfolio optimization. if we could minimize an analytic form of the CVaR directly, a more accurate result would be expected. In [Kurth and Tasche, 2002], analytical forms of the CVaR and its contribution under the CreditRisk+ framework are derived. These formulas provide a basis for nonlinear optimization: they can be used as the objective function and its first derivatives. One difficulty of applying nonlinear programming is that the losses in the CreditRisk+ model have only integer values. We hope to find a nonlinear programming algorithm for minimizing the CVaR within the CreditRisk+ framework.

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A Numerical Procedure for Loss Distribution

Consider the generalized form of the CreditRisk+ model

$$p_i^s = p_i \left(w_{0i} + \sum_{k=1}^{K} w_{ki} S_k \right) \tag{18}$$

where

$$S_k \sim \quad Gamma\ (\alpha_k, \beta_k)\,, \quad k = 1, \ldots, K. \tag{19}$$

The probability generating function for loss distribution has the form

$$\begin{aligned} &G_X(z) \\ &= \exp\left(\sum_{i=1}^{N} w_{0i} p_i (z^{\nu_i} - 1) - \sum_{k=1}^{K} \alpha_k \ln \left[1 - \beta_k \sum_{i=1}^{N} w_{ki} p_i \left(z^{\nu_i} - 1 \right) \right) \right] \quad (20) \\ &= \exp\left[-Q_0(1) + Q_0(z) - \sum_{k=1}^{K} \alpha_k \ln\left(1 + \beta_k Q_k(1) - \beta_k Q_k(z)\right) \right] \quad (21) \end{aligned}$$

where

$$Q_k(z) = \sum_{i=1}^{N} w_{ki} p_i z^{\nu_i}$$

and ν_i is the net exposure of obligor i, $i = 1, \ldots, N$. Then the numerical procedure in [Haaf *et al.*, 2003] can be rewritten as follows:

- First, define δ_{ij} as follows

$$\delta_{ij} = \begin{cases} 1, \, \nu_i = j \\ 0, \, \text{otherwise} \end{cases} . \tag{22}$$

  and determine the order of polynomial M sufficiently large,

$$M = \sum_{i=1}^{N} \nu_j,$$

  for example.
- Define $a_j^{(k)}$ such that

$$a_0^{(k)} = 1 + \beta_k Q_k(1) \tag{23}$$

$$a_j^{(k)} = \beta_k \sum_{i=1}^{N} w_{ki} p_i \delta_{ij}, \quad j = 1, \ldots, M \tag{24}$$

- Define $b_j^{(k)}$ such that

$$b_0^{(k)} = -\log(a_0^{(k)}) \tag{25}$$

$$b_j^{(k)} = \frac{1}{a_0^{(k)}}\left[a_j^{(k)} + \frac{1}{j}\sum_{q=1}^{j-1} q b_q^{(k)} a_{j-q}^{(k)}\right], \quad j = 1, \ldots, M \tag{26}$$

- Next, define

$$c_0 = -Q_0(1) + \sum_{k=1}^{K} \alpha_k b_0^{(k)} \tag{27}$$

$$c_j = \sum_{i=1}^{N} w_{0i} p_i \delta_{ij} + \sum_{k=1}^{K} \alpha_k b_j^{(k)} \tag{28}$$

- Finally, define

$$d_0 = \exp(c_0) \tag{29}$$

$$d_j = \sum_{q=1}^{j} \frac{q}{j} d_{j-q} c_q \quad \text{for } j \geq 1. \tag{30}$$

d_j is the probability of loss j, $\Pr[X = j]$.

For the common factor model, specify the parameters as follows:

- Let
$$K = K + 1,$$
i.e., increase the number of sectors by one to include the common sector.
- Let, for $k = 1, \ldots, K$
$$\alpha_k = \theta_k$$
$$\beta_k = \delta_k$$
- And let, for $k = K + 1$,
$$\alpha_{K+1} = \hat{\theta}$$
$$\beta_{K+1} = 1$$
$$w_{(K+1)i} = \sum_{k=1}^{K} w_{ki} \gamma_k$$

In the above, the right hand side parameters are as defined in the main text of the paper.

B Importance Sampling for Loss Simulation

In this section, we summarize the importance sampling procedure in [Glasserman, 2003].

Define θ as the exponential twisting parameter for importance sampling. Given the conditional probability of default as in Equation (18), the cumulant generating function of the portfolio loss, $X = \sum X_i$ is given by

$$\psi(\theta) = \psi^{(1)}(\theta) + \psi^{(2)}(\theta) \tag{31}$$

where

$$\psi^{(1)}(\theta) = \sum_{i=1}^{N} p_i w_{0i} \left(e^{\nu_i \theta} - 1\right), \tag{32}$$

$$\psi^{(2)}(\theta) = -\sum_{k=1}^{K} \alpha_k \log\left(1 - \beta_k \sum_{i=1}^{N} p_i w_{ki} \left(e^{\nu_i \theta} - 1\right)\right). \tag{33}$$

And its first order derivative with respect to θ is

$$\psi'(\theta) = {\psi^{(1)}}'(\theta) + {\psi^{(2)}}'(\theta) \tag{34}$$

where

$${\psi^{(1)}}'(\theta) = \sum_{i=1}^{N} p_i w_{0i} \nu_i \, e^{\nu_i \theta}, \tag{35}$$

$${\psi^{(2)}}'(\theta) = \sum_{k=1}^{K} \left\{ \frac{\alpha_k \beta_k \sum_{i=1}^{N} p_i w_{ki} \nu_i \, e^{\nu_i \theta}}{1 - \beta_k \sum_{i=1}^{N} p_i w_{ki} \left(e^{\nu_i \theta} - 1\right)} \right\}. \tag{36}$$

Portfolio loss simulation is performed by following the steps below.

- Solve

$$\psi'(\theta) = x \tag{37}$$
$$\theta = \max(0, \theta) \tag{38}$$

  for θ. x is a pre-determined portfolio loss, *e.g.*, VaR, around which samples are drawn. For the CreditRisk+, x can be obtained analytically.
- Compute $\tau_k,\ k = 1, \ldots, K$ from

$$\tau_k = \sum_{i=1}^{N} p_i w_{ki} \left(e^{\nu_i \theta} - 1\right)$$

- Draw samples

$$S_k \sim Gamma\left(\alpha_k, \frac{\beta_k}{1-\beta_k \tau_k}\right), \ k = 1, \ldots, K.$$

- Compute the conditional default probabilities, $p_i^s, \ i = 1, \ldots, N$.

$$p_i^s = p_i \left(w_{i0} + \sum_{k=1}^{K} w_{ki} S_k\right)$$

- Draw samples

$$D_i \sim Poisson\left(p_i^s \, e^{\nu_i \theta}\right), \ i = 1, \ldots, N.$$

- Portfolio loss is given by

$$X = \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} \nu_i \cdot D_i$$

- Repeat from the third step until you reach the desired number of iterations.

For the common factor model, α_k, β_k, and $w_{(K+1)i}$ shoud be defined as described in Appendix A.