The Estimation of Local Volatility of Discount Bond with the Method of Dupire

Hajime Fujiwara[a], Masaaki Kijima[b], and Katsumasa Nishide[c*]

a *Mitsubishi UFJ Securities Co., Ltd. and Kyoto University*

b *Tokyo Metropolitan University and Kyoto University*

c *Kyoto University*

(Preliminry Version. Do not cite. Comments are welcome.)

Abstract. In order to price the derivatives written on a government bond, we have to consider the observable volatility structure, such as volatility smile, of the government bonds. This paper proposes a method based on Dupire (1993) to estimate the local volatility of discount bonds when the prices of coupon bond options are observed in the market. Numerical examples show that our method can construct the volatility structure consistent with the market data.

Keywords: Volatility estimation, the method of Dupire, volatility smile.

*Corresponding author. Daiwa Securities Chair, Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan.

TEL:+81-75-753-3411, FAX:+81-75-753-3492, E-mail: `knishide@econ.kyoto-u.ac.jp`

1 Introduction

This paper proposes a method based on Dupire (1993) to estimate the local volatilities of discount bonds when the prices of coupon bond options are observed. It is well known that the volatility smiles are observed not only in the stock options markets, but also in the bond options market. Therefore, in order to price the derivatives in the bond market with general payoffs , we should model the dynamics of the discount bonds that can capture the smile structure.

The volatility models that deal with volatility skews or smiles are roughly grouped in three categories. The first approach is stochastic volatility models. In the stochastic volatility models, the volatility is typically assumed to follow a mean reverting diffusion process. Depending on the correlation and parameters of the processes of the underlying asset and its volatility, a variety of volatility structure can be constructed. See for instance, Hull and White (1987) or Heston (1993). The second approach is adding jump process, originally proposed by Merton (1976). By modeling appropriately the jump intensity and the distribution of the jump size, the so-called jump-diffusion model can generate a volatility smile. The third approach is categorized into local volatility models. In this approach, the volatility is supposed to be a deterministic function of the time to maturity and the price of the underlying asset. This approach was first proposed by Dupire (1993), and is popular among stock market practitioners for its simplicity.

Our approach is based on the third approach. If the options of the discount bonds were traded, it would be straightforward to estimate the local volatilities of the discount bonds by the method of Dupire.

In the Japanese Government Bond (JGB) market, the options on the bond futures are actively traded and the volatility smiles are observed, but there is no market quotation for the discount bond options. Since the cheapest deliverable coupon bond of the bond futures in the JGB market can be specified uniquely before the maturity, the options on the 10-year bond future can be considered as those on the coupon bond.

Taking these circumstances into account, it is obvious that we need to estimate the local volatilities of the discount bonds from the observed volatility smiles of the coupon bond options. To do this, we use the following procedure.

1. Estimate the local volatility of the coupon bond by the method of Dupire.

2. Assume that the structure of the local volatility of the discount bond is the same as that of the coupon bond.

3. Estimate the parameters of the discount bonds to fit the observed prices of the coupon bond options.

Our local volatility model is quite simple, but is easy to fit the market data. In recent years, the Bank of Japan has been adopting a so-called "Zero-Interest Rate Policy." Therefore, it is observed that the discount coupon bonds with short maturity has a low volatility, and the volatility curve with respect to maturity is unimodal. Our numerical results show that our method can construct a volatility structure that is consistent with the market data as above.

The rest of the paper is organized as follows. In Section 2, we briefly review Dupire (1993), that our model is based on. We explain our model and the procedure for the parameters' estimation in Section 3. In Section 4, we apply our method for the options in the JGB market and estimate the local volatilities of the discount bonds. Our conclusion is presented in Section 5. We show the detailed calculation of our model in Appendix A.

2 Review of Dupire (1983)

As we mentioned above, our model is based on Dupire (1993). Here, we briefly review Dupire (1993) to make the following discussion clear.

Denote the price of a risky asset at time t by $S(t)$. Dupire (1993) assumes the SDE that $S(t)$ follows under the risk-neutral measure Q

$$\frac{\mathrm{d}S(t)}{S(t)} = (r-\delta)\mathrm{d}t + \sigma(t, S(t))\mathrm{d}W_t^Q, \tag{2.1}$$

where r is a risk-free interest rate, δ is an instantaneous dividend rate, and $\{W_t^Q\}$ is a Brownian motion under Q. Here, the volatility is assumed to be the function of time t and the price of the underlying asset S.

Now let us consider the European put option with maturity T, and strike price K. Denote the put price at time t by $P(T, P)$. Then, $P(T, P)$ can be expressed as

$$P(T,K) = \mathrm{e}^{-r(T-t)} \int_0^K (K-S)\phi(T,S)\mathrm{d}S \tag{2.2}$$

$$= \mathrm{e}^{-r(T-t)} \left(K \int_0^K \phi(T,S)\mathrm{d}S - \int_0^K S\phi(K,S)\mathrm{d}S \right), \tag{2.3}$$

where ϕ is a density function of $S(T)$. It follows from the forward Kolmogorov equation that ϕ satisfies

$$\frac{\partial \phi}{\partial T} + \frac{\partial}{\partial S}\{(r-\delta)S\phi(T,S)\} - \frac{1}{2}\frac{\partial^2}{\partial S^2}\{\sigma(T,S)^2 S^2 \phi(T,S)\} = 0. \tag{2.4}$$

Twice integrating the above equation with respect to S yields

$$\frac{1}{2}\sigma(T,S)^2S^2\phi(T,S) = \frac{\partial}{\partial T}\int_0^S\int_0^v \phi(T,u)\mathrm{d}u\mathrm{d}v + (r-\delta)\int_0^S v\phi(T,v)\mathrm{d}v. \tag{2.5}$$

On the other hand, by differentiating (2.2) with respect to K, we get

$$\frac{\partial P(T,K)}{\partial K} = \mathrm{e}^{-r(T-t)}\int_0^K \phi(T,u)\mathrm{d}u. \tag{2.6}$$

Therefore, from (2.3), we have an expression of $P(T,K)$ as

$$P(T,K) = K\frac{\partial P(T,K)}{\partial K} - \mathrm{e}^{-r(T-t)}\int_0^K S\phi(K,S)\mathrm{d}S, \tag{2.7}$$

or equilavently

$$\int_0^K v\phi(K,v)\mathrm{d}v = \mathrm{e}^{r(T-t)}\left[K\frac{\partial P(T,K)}{\partial K} - P(T,K)\right]. \tag{2.8}$$

Once again differentiating (2.6) with respect to K leads to

$$\frac{\partial^2 P(T,K)}{\partial K^2} = \mathrm{e}^{-r(T-t)}\phi(T,K). \tag{2.9}$$

Equation (2.9) indicates that the price of the European put option satisfies

$$\int_0^K\int_0^v \phi(T,u)\mathrm{d}u\mathrm{d}v = \int_0^K\int_0^v \mathrm{e}^{r(T-t)}\frac{\partial^2 P(T,u)}{\partial u^2}\mathrm{d}u = \mathrm{e}^{r(T-t)}P(T,K), \tag{2.10}$$

since $P(T,0)=0$. Then, setting $S=K$ and substituting (2.8) and (2.10) into (2.5), we have

$$\begin{aligned}&\frac{1}{2}\sigma(T,K)^2K^2\mathrm{e}^{r(T-t)}\frac{\partial^2 P(T,K)}{\partial K^2}\\ =&\mathrm{e}^{r(T-t)}\left[\frac{\partial P(T,K)}{\partial T} + rP(T,K) + (r-\delta)\left\{K\frac{\partial P(T,K)}{\partial K} - P(T,K)\right\}\right].\end{aligned} \tag{2.11}$$

Finally, solving (2.11) with respect to the volatility yields

$$\begin{aligned}\sigma(T,K) =&\sqrt{\frac{2}{K^2\frac{\partial^2 P(T,K)}{\partial K^2}}}\\ &\times\sqrt{\frac{\partial P(T,K)}{\partial T} + rP(T,K) - (r-\delta)P(T,K) + (r-\delta)K\frac{\partial P(T,K)}{\partial K}}.\end{aligned} \tag{2.12}$$

Equation (2.12) indicates that by estimating the functional form of P with respect to T and K, we obtain the volatility function $\sigma(T,S)$.

3 The Model

In this section, we provide our assumptions and construct our model to estimate the volatility of discount bonds.

We denote the price of discount bond with the maturity T_i by $P(t,T_i)$ and its time T forward price by $P(t,T_i;T) := \frac{P(t,T_i)}{P(t,T)}$. We assume that the dynamics of $P(t,T_i;T)$ is expressed as

$$\begin{aligned} \mathrm{d}P(t,T_i;T) &= P(t,T_i;T)\sigma_P(t,T,T_i)\eta_i\left(t,P(t,T_i;T)\right)\mathrm{d}W_t^T, \\ \sigma_P(t,T,T_i) &= \frac{\sigma}{a}\left(\mathrm{e}^{-a(T_i-t)}-\mathrm{e}^{-a(T-t)}\right), \end{aligned} \tag{3.1}$$

where $\eta_i(\cdot,\cdot):[0,T]\times R_+ \to R_+$ is a function which characterizes the local volatility of the discount bond with maturity T_i, W_t^T is a standard Brownian motion under T-forward measure, and both a and σ are constant. Our model (3.1) can be seen as an extension of Hull-White model to incorporate the volatility smile structure.

In the market, the options of the coupon bonds are traded, but those of the discount bonds are not. Thus, we can not estimate the local volatilities $\eta_i(\cdot,\cdot)$ directly from the market data by the Dupire method. In order to estimate the local volatility of the discount bond, we exploit the relation between the coupon bond price and the discount bond price.

Suppose that we have a coupon bond whose cash flow at time $T_i, i=1,\cdots,N$ is C_i. This coupon bond price can be represented by using the discount bond price as

$$bnd(t,T_N) = \sum_{i=1}^{N} C_i P(t,T_i),$$

and its time T forward price is expressed as

$$bnd(t,T_N;T) = \frac{bnd(t,T_N)}{P(t,T)} = \sum_{i=1}^{N} C_i P(t,T_i;T).$$

We assume that $bnd(t,T_N:T)$ follows

$$\frac{\mathrm{d}bnd(t,T_N;T)}{bnd(t,T_N;T)} = \sigma_B \xi(t,bnd(t,T_N;T))\mathrm{d}W_t^T,$$

where $\xi(\cdot,\cdot):[0,T]\times R_+ \to R_+$ is a function which characterizes the local volatility of the coupon bond, and σ_B is constant.

Let the time T forward price of the plain options (call option or put option) of the coupon bond with the strike K be $op(K,T)$, i.e.,

$$op(K,T) = \begin{cases} \mathrm{E}^T[(bnd(T,T_N;T)-K)^+] & \text{Call option} \\ \mathrm{E}^T[(K-bnd(T,T_N;T))^+] & \text{Put option} \end{cases},$$

where $\mathrm{E}^T[\cdot]$ is the expectation under T-forward measure. From the method of Dupire discussed in the previous section, we can calculate the local volatility of the coupon bond as

$$\xi(T,K)^2 = 2\frac{\frac{\partial op(K,T)}{\partial T} - \frac{\partial bnd(0,T_N;T)}{\partial T} 1_{bnd(0,T_N:T)\in ITM}}{\sigma_B^2 K^2 \frac{\partial^2 op(K,T)}{\partial K^2}}, \quad (3.2)$$

where "ITM" represents the region in which the option is "in the money".

To link the local volatility of the coupon bond $\xi(\cdot,\cdot)$ to the local volatilities of the discount bonds $\eta_i(\cdot,\cdot)$, the following assumption is made:

$$\eta_i(t, P(t,T_i;T)) = \alpha_i \xi\left(t, \frac{P(t,T_i;T)}{P(0,T_i;T)} bnd(0,T_N;T)\right), \quad i = 1, \cdots, N,$$

where $\{\alpha_i\}_{i=1}^N$ are constant. The parameters $\{\alpha_i\}$ in $\eta_i(\cdot,\cdot)$ are estimated with observed coupon bond option prices and Monte Carlo simulation as follows.

Let N_{sim} be the number of Monte Carlo simulation. Then, we have $\{P_k(T,T_m;T)\}_{k=1}^{N_{sim}}$, dicount bond prices at time T with maturity T_m by each simulation. Define the coupon bond price of k-th simulation to be

$$bnd_k(T,T_N;T) := \sum_{m=1}^{N} C_i P_k(T,T_m;T,\alpha_m). \quad (3.3)$$

We compute $\{\alpha_i\}$ as

$$\{\alpha_i\}_{i=1}^N = \arg\min_{\{\alpha_i\}} \sum_{j=1}^{M_{op}} \left(op_{obs}(T_j,K_j) - op_{sim}(T_j,K_j)\right)^2, \quad (3.4)$$

where

$$op_{sim}(T,K) := \frac{1}{N_{sim}} \sum_{k=1}^{N_{sim}} (K - bnd_k(T,T_N;T))^+$$

is a theoretical option price by simulation, and $\{op_{obs}(T_j,K_j)\}_{j=1}^{M_{op}}$ are option prices observed in the market, respectively.

To simulate the discount bond prices, we apply the Euler approximation for (3.1) as

$$P(t_{n+1},T_i;T) = P(t_n,T_i;T)\exp\left(-\frac{1}{2}\sigma_P^2(t_n,T,T_i)\eta^2(t_n,P(t_n,T_i;T)\Delta t_n + \sigma_P(t_n,T,T_i)\eta(t_n,P(t_n,T_i;T))\sqrt{\Delta t_n}\epsilon\right)$$

and

$$\eta(t_n,P(t_n,T_i;T)) = \alpha_i\xi\left(t_n, \frac{P(t_n,T_i;T)}{P(0,T_i;T)} bnd(0,T_N;T)\right),$$

where $\Delta t_n := t_{n+1} - t_n$, and ϵ is a random variable that follows a standard normal distribution.

4 Numerical Examples

In this section, we estimate the local volatility structure with the methodology presented in the previous section, and show that the estimation results are consistent with the data observed in the market.

4.1 Market data

In the Japanese bond market, the options on the 10y-bond futures are actively traded and the volatility smiles are observed. We provide the market quotations of the put options with the time to the maturity 0.104 year and 0.274 year on the 10y-bond futures as of 2006/5/23 in Table 1. For convenience, we express the options with the time to maturity 0.104 years as "option 1" and the options with the time to maturity 0.274 years as "option 2".

Table 1: Market quotations of the put options on the 10y-bond future as of 2006/5/23. Implied volatilities are presented in the column of IV.

option 1		option 2	
Time to Maturity	0.104 years	Time to Maturity	0.274 years
Strike	IV	Strike	IV
127.5	6.45 %	127	5.13 %
128	6.22 %	128	4.92 %
128.5	6.04 %	129	4.89 %
129	5.23 %	130	4.80 %
129.5	5.05 %	131	4.62 %
130	4.67 %	132	4.41 %
130.5	4.74 %	133	3.98 %
131	4.58 %	134	3.83 %
131.5	4.13 %		
132	3.86 %		
132.5	3.56 %		
133	3.39 %		

The cheapest deliverable bond of the underlying 10y-bond futures in Table 1 is the JGB 10y-bond series #253 (JGB # 253). Thus, the options on the 10y-bond future are considered as the options on the JGB #253 by using the conversion factor. We provide the details of the JGB # 253 in Table 2.

Table 2: The details of JGB # 253 as of 2006/5/23. This JGB is the cheapest bond, and so we can think that the bond option traded in the market is written on this bond.

Maturity date	2013/9/20
Time to Maturity	7.33 year
Coupon rate	1.6 %
Conversion Factor	0.751486
Market price	99.459

4.2 Local volatility of the coupon bond

To estimate the local volatility of the coupon bond $\xi(\cdot,\cdot)$, option prices for any strike and any maturity are necessary. Therefore, we model the implied volatility as following:

$$\sigma_{mkt}(K,T) = \sigma_B \exp\left(\sum_{n=1}^{M} a_n \left(\frac{K}{K_{ATM}(T)} - 1\right)^n\right) \tag{4.1}$$

where $K_{ATM}(T)$ is the *at-the-money* option strike at maturity T.

We estimate the parameters of the implied volatility model by the least square fitting to the observed implied volatilities in Table 1. We show the estimated parameters in Table 3 and the fitting results in Table 4.

Table 3: The estimated parameters of the implied volatility model. The number of parameters in the implied volatility model is two ($M = 2$).

	option 1	option 2
a_1	-17.19	-6.05
a_2	-25.25	-9.06
σ_B	0.036	0.042

From the implied volatility model (4.1), we can calculate the local volatility $\xi(\cdot,\cdot)$ explicitly (See Appendix).

Table 4: The estimated implied volatilities.

option 1	Time to Maturity	0.104	option 2	Time to Maturity	0.274
Strike	IV	model IV	Strike	IV	model IV
127.5	6.45 %	6.54 %	127	5.13 %	5.27 %
128	6.22 %	6.17 %	128	4.92 %	5.06 %
128.5	6.04 %	5.82 %	129	4.89 %	4.85 %
129	5.23 %	5.48 %	130	4.80 %	4.65 %
129.5	5.05 %	5.16 %	131	4.62 %	4.45 %
130	4.67 %	4.85 %	132	4.41 %	4.25 %
130.5	4.74 %	4.56 %	133	3.98 %	4.06 %
131	4.58 %	4.28 %	134	3.83 %	3.88 %
131.5	4.13 %	4.02 %			
132	3.86 %	3.77 %			
132.5	3.56 %	3.53 %			
133	3.39 %	3.31 %			

4.3 Local volatility of the discount bond

Once we have the local volatility of the coupon bond $\xi(\cdot,\cdot)$, we can estimate the parameters $\{\alpha_i\}_{i=1}^{N}$ by the procedure (3.4). For JGB #253, the number of coupon payments is 15 ($N = 15$). We use $a = 0.01508$ and $\sigma = 0.007$ for $\sigma_P(\cdot,\cdot,\cdot)$ in (3.1). Those values are obtained by fitting Hull-White model to the swaption volatilities. We employ the implied model parameters (a_1, a_2, σ_B) of option 1 for $t < 0.104$ and those of option 2 for $t > 0.104$. We show the estimated parameters $\{\alpha_i\}$ in Table 5 and the estimated coupon bond prices by the discount bonds in Table 6. The parameter α_i represents the magnitude of the volatility of the discount bond with the maturity T_i.

From Table 5, we can see that the discount bond with the shortest maturity ($T_i = 0.33$) has very low volatility. This result is consistent with the actual market where the movement of the short rate is tightly bounded under the "Zero-Interest rate" policy by the Bank of Japan. In Table 6, We have the better fitness for the longer maturity. This comes from that the option price of the longer maturity is higher and we use absolute option prices in the fitting procedure (3.4).

Table 5: The estimated parameters $\{\alpha_i\}_{i=1}^{N=15}$. We see that α has

i	T_i	α_i	i	T_i	α_i
1	0.33	0.100	9	4.33	1.352
2	0.83	0.336	10	4.83	1.226
3	1.33	2.511	11	5.33	1.135
4	1.83	2.459	12	5.83	1.052
5	2.33	2.256	13	6.33	0.975
6	2.83	1.950	14	6.83	0.915
7	3.33	1.689	15	7.33	0.852
8	3.83	1.496			

Table 6: The observed option prices and the estimated option prices

option 1	Time to Maturity	0.104	option 2	Time to Maturity	0.274
Strike	Observed price	Estimated price	Strike	Observed price	Estimated price
127.5	0.0318	0.0402	127	0.0793	0.0780
128	0.0375	0.0479	128	0.1276	0.1215
128.5	0.0455	0.0644	129	0.2039	0.1894
129	0.0569	0.0843	130	0.3219	0.2915
129.5	0.0733	0.1066	131	0.4995	0.4485
130	0.0969	0.1366	132	0.7569	0.6882
130.5	0.1312	0.1795	133	1.1124	1.0299
131	0.1823	0.2374	134	1.5766	1.4953
131.5	0.2564	0.3182			
132	0.3622	0.4249			
132.5	0.5113	0.5796			
133	0.7131	0.7774			

5 Conclusions

In this paper, we propose a methodology to estimate a local volatility function of the Japanese government bonds based on Dupire (1993). By including $\{\alpha_i\}$, the magnitude of the volatility at each maturity period, in our volatility model, a local volatility structure observed in the market can be captured. Our simulation results confirm that our estimation fits the market data. Therefore, our model enables us to price the JGB options in order that the prices are consistent with the observed volatility structure.

A Appendix

In Appendix A, we show the calculation of the local volatility of the coupon bond $\xi(\cdot,\cdot)$ in detail. We only demonstrate the calculation in the case of the put option. We can obtain the result for the call option in a similar way.

Using the implied volatility model (4.1), the plain option price is expressed as

$$\begin{aligned} op(K,T) &= E^T[(K - bnd(T,T_N;T))^+] \\ &= KN(\epsilon_K) - bnd(0,T_N;T)N(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}), \end{aligned}$$

and

$$\epsilon_K = \frac{\log(K) - \log(bnd(0,T_N;T))}{\sigma_{mkt}(K,T)\sqrt{T}} + \frac{1}{2}\sigma_{mkt}(K,T)\sqrt{T}.$$

From the above expression, the derivatives in the right-hand-side of (3.2) are calculated as

$$\begin{aligned} \frac{\partial op(K,T)}{\partial T} =& KN'(\epsilon_K)\frac{\partial \epsilon_K}{\partial T} - \frac{\partial bnd(0,T_N;T)}{\partial T}N(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}) \\ & - bnd(0,T_N;T)N'(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T}) \\ & \times \left(\frac{\partial \epsilon_K}{\partial T} - \frac{\partial \sigma_{mkt}(K,T)}{\partial T}\sqrt{T} - \frac{\sigma_{mkt}(K,T)}{2\sqrt{T}}\right) \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \frac{\partial^2 op(K,T)}{\partial K^2} =& 2N'(\epsilon_K)\frac{\partial \epsilon_K}{\partial K} + KN''(\epsilon_K)\left(\frac{\partial \epsilon_K}{\partial K}\right)^2 + KN'(\epsilon_K)\frac{\partial^2 \epsilon_K}{\partial K^2} \\ & - bnd(0,T_N;T)N''(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T})\left(\frac{\partial \epsilon_K}{\partial K} - \frac{\partial \sigma_{mkt}(K,T)}{\partial K}\sqrt{T}\right)^2 \\ & - bnd(0,T_N;T)N'(\epsilon_K - \sigma_{mkt}(K,T)\sqrt{T})\left(\frac{\partial^2 \epsilon_K}{\partial K^2} - \frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2}\sqrt{T}\right). \end{aligned} \tag{A.2}$$

Here, the function N in (A.1) and (A.2) is

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}s^2} \mathrm{d}s,$$

and its first and second derivatives are given by

$$N'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}, \text{ and } N''(x) = \frac{-x}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2},$$

respectively.

The partial derivatives in (A.1) and (A.2) are expressed as

$$\begin{aligned} \frac{\partial \epsilon_K}{\partial T} =& - \frac{\frac{\partial bnd(0,T_N;T)}{\partial T}}{\sigma_{mkt}(K,T)\sqrt{T} bnd(0,T_N;T)} \\ & - \frac{\log(K) - \log(bnd(0,T_N;T))}{\sigma_{mkt}^2(K,T)T} \left(\frac{\sigma_{mkt}(K,T)}{\partial T}\sqrt{T} + \frac{\sigma_{mkt}(K,T)}{2\sqrt{T}} \right) \\ & + \frac{1}{2}\frac{\partial \sigma_{mkt}(K,T)}{\partial T}\sqrt{T} + \frac{\sigma_{mkt}(K,T)}{4\sqrt{T}}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} \frac{\partial \epsilon_K}{\partial K} =& \frac{1}{\sigma_{mkt}(K,T)K\sqrt{T}} - \frac{\log(K) - \log(bnd(0,T_N;T))}{\sigma_{mkt}^2(K,T)\sqrt{T}} \frac{\partial \sigma_{mkt}(K,T)}{\partial K} \\ & + \frac{1}{2}\frac{\partial \sigma_{mkt}(K,T)}{\partial K}\sqrt{T}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} \frac{\partial^2 \epsilon_K}{\partial K^2} =& \frac{-1}{\sigma_{mkt}(K,T)^2 K^2 \sqrt{T}} \left[\sigma_{mkt}(K,T) + 2K \frac{\partial \sigma_{mkt}(K,T)}{\partial K} \right. \\ & \left. + K^2 \log\left(\frac{K}{bnd(0,T_N;T)} \right) \frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2} \right] \\ & + 2\frac{\log(K) - \log(bnd(0,T_N;T))}{\sigma_{mkt}^3(K,T)\sqrt{T}} \left(\frac{\partial \sigma_{mkt}(K,T)}{\partial K} \right)^2 \\ & + \frac{1}{2}\frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2}\sqrt{T} \end{aligned} \tag{A.5}$$

and

$$\frac{\partial bnd(0,T_N;T)}{\partial T} = R(T) bnd(0,T_N;T),$$

where $R(T) = -\frac{\log(P(0,T))}{T}$ is the spot rate.

From the implied volatility model (4.1), the derivatives appeared in (A.3), (A.4), and

(A.5) are calculated as

$$
\begin{aligned}
\frac{\partial \sigma_{mkt}(K,T)}{\partial T} =& -\sigma_{mkt}(K,T)\frac{K}{K_{ATM}^2(T)}\frac{\partial K_{ATM}(T)}{\partial T}\sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)}-1\right)^{n-1}, \\
\frac{\partial \sigma_{mkt}(K,T)}{\partial K} =& \frac{\sigma_{mkt}(K,T)}{K_{ATM}(T)}\sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)}-1\right)^{n-1}, \\
\frac{\partial^2 \sigma_{mkt}(K,T)}{\partial K^2} =& \frac{\sigma_{mkt}(K,T)}{K_{ATM}^2(T)}\left[\sum_{n=2}^{M} a_n n(n-1) \left(\frac{K}{K_{ATM}(T)}-1\right)^{n-2} \right. \\
& \left. + \left(\sum_{n=1}^{M} a_n n \left(\frac{K}{K_{ATM}(T)}-1\right)^{n-1}\right)^2\right],
\end{aligned}
$$

and

$$
K_{ATM}(T) = \frac{bnd(0,T_N)}{P(0,T)}.
$$

Thus, we can express ξ by substituting (A.3) - (A.5) into (3.2).

References

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