Preliminary and Incomplete— Comments Welcome

Asset Pricing with Skewed Payouts

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I examine how investors' preference for skewness impacts the price of limited liability equity. Using the actual probability density function of equity payouts, I obtain equity pricing relations that directly link investors' risk preferences to the skewness induced by the limited liability feature of equity. This approach differs from standard valuation techniques, which arrive at equity pricing relations using the risk-neutral density function of equity payouts.

I show expected equity returns may decline as leverage increases for firms with high asset systematic risk and low asset idiosyncratic risk. For these firms, as leverage increases, the value of the positive skewness induced by limited liability dominates compensation for systematic risk.

The results help explain the puzzling empirical relation between equity returns and financial distress. Equity returns for low book-to-market firms decline as financial distress and leverage increase (Dichev (Journal of Finance, 1998), Griffin and Lemmon (Journal of Finance, 2002), Vassalou and Xing (Journal of Finance, 2004)). Both Dichev (1998) and Griffin and Lemmon (2002) attribute the decline in average returns as leverage increases to investors' misperceptions of risk. In contrast, I show that when book-to-market ratios are negatively related to systematic risk, the declines are driven by value of the skewness induced by limited liability.

Keywords: Equity returns, skewness, limited liability, call option, Stein's lemma, skew normal distribution

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1. Introduction

Equity investments produce cash-flows that are asymmetrically distributed, or skewed. When a firms' assets are above its fixed obligations, equity holders own a proportion of the assets, and the firm remains a going concern. In the event that the firm's assets fall below the level, equity holders default, and receive nothing. These asymmetries may matter for investors.

This paper uses truncated distributions to derive equity pricing models that accommodate the skewness induced by limited liability and hold for general concave utility functions. I find that equity claim prices depend on how investors value limited losses. The value of limited liability increases with covariance risk and risk aversion. To obtain the equilibrium pricing relations, I derive decompositions of the covariance between marginal utility of wealth and equity payouts. The decompositions are generalizations of Stein's (1973) and Rubinstein's (1973) decomposition for normal random variables.

This approach is in contrast to much of the extant literature on asset pricing for skewed payouts. Two approaches are commonly used. The first relies on restricting preferences. This approach typically specifies the stochastic discount factor as a quadratic function. For example, Rubinstein (1973) and Kraus and Litzenberger (1976) use a Taylor's series expansion of the utility function, stopping with the cubic form.[1] My approach also differs from many pricing models that treat equity claims as a call option (Black and Scholes (1973) and Geske (1979)). Under certain restrictions, the claim may be valued under the risk-neutral payout density. My approach uses the actual payout density, and produces closed-form pricing relations without placing restrictions on utility.

The model has two important implications for expected equity returns. At the firm level, limited liability may cause expected equity returns to decline as leverage or the likelihood

[1] A quadratic approximation to the stochastic discount factor is only exact if investors have cubic utility over wealth (Levy (1969)).

of financial distress increases. As idiosyncratic cash-flow risk increases, investors are more likely to get a large positive payout, while having limited losses. When leverage is high, the benefits of having limited losses may offset the compensation for bearing the systematic risk of the firm's assets. This result is counter to pricing relations like the CAPM, where the expected equity returns increase monotonically with leverage.

This result helps explain the puzzling empirical relation between equity returns and financial distress. Equity returns for low book-to-market firms decline as financial distress and leverage increase, while the opposite occurs for high book-to-market firms (Vassalou and Xing (2001), Griffin and Lemmon (2002)). I show that when investors are risk averse and book-to-market ratios are positively related to systematic risk, expected equity returns decline as leverage and the likelihood of financial distress increase.

At the market level, relative risk aversion inferred from limited liability equity returns will decline. Limited liability induces a lower bound to realized wealth, so investors are able to bear greater risk. As a result, risk aversion estimated from equity prices under-estimates the risk aversion embedded in the underlying asset prices. This may explain why some empirical studies have difficulties finding a positive premium for bearing market-wide risk.

The paper proceeds with a derivation of the basic asset pricing relation, along with an application to the expected return of an equity claim on the market portfolio (Section 2). I then examine prices and returns for an equity claim to a single firm, when investors hold positions in many risky assets (Section 3). The first appendix presents a brief discussion of the statistical distribution theory and collects all relevant proofs, while proofs related to the economic model are in the second appendix.

2. The Price of Limited Liability Equity Claims

2.1. The Economy

Most modern asset pricing theories start with specifying a stochastic discount factor. When agents maximize expected utility of lifetime consumption, and no-arbitrage conditions hold, a non-negative stochastic discount factor exists that prices all traded assets (Harrison and Kreps (1979))

$$E(m_{t+1}K_j\left(x_{i,t+1}\right)|\Theta_t) = P_{j,i,t} \qquad \forall i, j. \tag{1}$$

m_{t+1} is the stochastic discount factor, while Θ_t is the information set of the economy. $K_j\left(x_{i,t+1}\right)$ is the j-th claim to cash-flow $x_{i,t+1}$. $P_{j,i,t}$ is the claim's corresponding price. This holds for any asset i, and any claim j. If we assume a one-period exchange economy, where a representative investor maximizes expected utility of next period wealth, m_{t+1} is proportional to the marginal utility of the aggregate wealth portfolio (Beja (1971)). Within a multiperiod setting, m_{t+1} is the intertemporal marginal rate of substitution of consumption (Rubenstein (1976)).

Using the properties of the covariance operator, and assuming the existence of a risk-free bond with return $R_{f,t+1}$, the fundamental pricing equation may also be expressed as

$$P_{j,i,t} = \frac{E_t(K_j\left(x_{i,t+1}\right))}{R_{f,t+1}} + \frac{Cov_t(m_{t+1}, K_j\left(x_{i,t+1}\right))}{R_{f,t+1}E_t(m_{t+1})}. \tag{2}$$

$E_t\left(\bullet\right)$ and $Cov_t(m_{t+1}, X_{j,t+1})$ are the expectation and covariance given the public information set at time t. The price of a claim depends on it's expected cashflow and a risk adjustment that decreases prices for payouts that do poorly when future consumption or wealth innovations are low.

In order to focus on the empirical implications of the limited liability feature of equity, I restrict investors to maximize over next period wealth. In this case, the stochastic discount factor is

$$m_{t+1} = \frac{u'(W_{t+1})}{E_t\left(u'(W_{t+1})\right) R_{f,t+1}}. \tag{3}$$

I assume that firm investment policy is independent of financing policy. I also abstract from managerial use of capital structure to signal future firm prospects. Firms can costlessly default, and have free access to external funds. The empirical content of Equation 2.2 comes from either specifying functional form of stochastic discount factor, or specifying the distribution of payouts. I specify the distribution of equity payouts in order to develop testable models that hold under weak restrictions on utility.

2.2. A Decomposition of Marginal Utility and Payouts

One way to give empirical content to Equation 2 is to assume that investment payouts are normally distributed. The covariance term may then be decomposed with Stein's lemma. As first discovered by Stein (1973) and Rubinstein (1973), Stein's lemma says that if the wealth portfolio, W, and a payout, x, have a joint normal distribution, and $u'(W)$ is a differentiable, lebegsue measurable function with $|E(u''(W))| < \infty$, then

$$Cov(u'(W), x) = \sigma_{Wx} E\left(u''(W)\right) \frac{\partial W}{\partial X_m}, \tag{4}$$

where $\frac{\partial W}{\partial X_m} = N_m$ is the investor's demand for the portfolio of risky assets.

Stein's Lemma decomposes the covariance into the average change in marginal utility over all future wealth outcomes, and the covariance of payouts with wealth. The greater the average change in marginal utility, the more averse investors are to risk. The larger the covariance between payouts and wealth, the more negative asset payouts are when overall wealth declines. A similar decomposition holds for the covariance between marginal utility of wealth and an equity claim, $K(x)$.

Lemma 1. If a payout, x, and the investor's final wealth portfolio, W, are normally distributed, the covariance between marginal utility of wealth and an equity claim that pays $K(x) = x - \lambda$ if $x > \lambda$, $\lambda \in \mathbb{R}^{+}$, and zero otherwise, is

$$\begin{aligned} Cov\left(u'\left(W\right), K\left(x\right)\right) &= \sigma_{Wx} E\left(u''\left(W\right)\right) \frac{\partial W}{\partial X_m} + \sigma_x \psi\left(\frac{\mu_x - \lambda}{\sigma_x}\right) \left(E_n(u'(W)) - E(u'(W))\right), \\ \psi\left(\frac{\mu_x - \lambda}{\sigma_x}\right) &= \frac{1}{\sqrt{2\pi\sigma_x^2}} Exp\left(-\frac{1}{2}\left(\frac{\mu_x - \lambda}{\sigma_x}\right)^2\right) P\left(A > \lambda\right)^{-1}. \end{aligned} \quad (5)$$

The expectation $E(\bullet)$ is formed under the conditional density $W|x > \lambda$ (Appendix A.1.2 derives the density). $E_n(\bullet)$ is formed under a normal distribution with mean $\mu_W - \frac{\sigma_{Wx}}{\sigma_x^2}(\mu_x - \lambda)$ and variance $\sigma_W^2\left(1 - \rho_{Wx}^2\right)$, where is ρ_{Wx} is the correlation coefficient between W and x.

Proof. See Appendix A.1.2.[2]

The first term is similar to the decomposition for normally distributed payouts The second term captures the benefits of limited liability. It consists of a weighted difference in expected marginal utilities. The first expectation is under a measure related to the wealth lost when the equity claim produces a zero payout. This marginal utility decreases as financial leverage, λ, increases or the covariance between payouts and wealth decreases. The second marginal utility has an expectation formed under the distribution of wealth conditional on equity claim producing a non-zero payout. The difference between the two captures how marginal utility differs between the worst outcome under limited liability (zero payout), and outcomes when the equity claim produces a positive payout. The convexity of marginal utility means that the difference between the two is positive and increases with investor risk aversion.

[2] The appendix derives the more general case where, essentially, λ is a random variable.

The truncated loss weight, $\psi\left(\frac{\mu_{Xm}-\lambda}{\sigma_{Xm}}\right)$, is a ratio of two functions.[3] The numerator is the a function the squared market payout, scaled by the inverse of the market variance. It reaches its maximum when the market payout net of λ is zero, and decreases as expected asset growth or financial leverage decreases. Assets with higher growth rates or lower debt levels have a diminished chance of producing a negative payout, so their impact on the covariance is smaller. The numerator also increases as the variance of the asset payout increases. Higher asset variance leads to a greater chance of gaining the benefits of limited liability. The denominator of the weight is the probability that the firm avoids bankruptcy. A lower probability of avoiding default is equivalent to an increased chance of gaining with limited liability, so it increases the covariance.

2.3 The Market Portfolio with Limited Liability

To see how limited liability will impact prices and expected returns, consider a representative investor who maximizes the expected utility of next period's wealth, W_{t+1}, by allocating between two claims to a single risky asset (the market) and a risk-free asset that pays unity, with price $P_{f,t}$ and return $R_{f,t+1}$. The first is an equity claim, with price $P_{E,t}$ and demand $N_{E,t}$. There is also a debt claim with price $P_{D,t}$ and demand $N_{D,t}$. The market cash-flow, $x_{m,t+1}$, is normally distributed with mean μ_{Xm} and variance σ^2_{Xm}. The investor is assumed to have a time-separable utility function with a strictly positive first derivative and strictly negative second derivative. Since no arbitrage is also assumed, there exists a possibly non-unique state price density function that prices the market asset. I assume that all agents use the same state price density function to price the market asset. The representative investor then reflects the aggregate preferences of the individual investors.

Applying Lemma 1 to the covariance term in Equation 2 gives the price of the equity

[3] The function $\psi(x)$ is the inverse Mill's ratio.

claim:

$$P_{E,t} = \frac{E_t\left(x_{E,t+1}\right)}{R_{f,t}} + \sigma_{Xm}^2 \frac{E_t\left(u''_{t+1}\left(W_{t+1}\right)\right)}{E_t(u'_{t+1}\left(W_{t+1}\right))R_{f,t}} N_{m,t} + \frac{\sigma_{Xm}}{R_{f,t}} \psi\left(\frac{\mu_{Xm} - \lambda}{\sigma_{Xm}}\right)\left(\eta_t - 1\right). \tag{6}$$

Again, $N_{m,t}$ is the demand for the risky asset. The expectation $E_t(\bullet)$ is under the distribution of wealth conditional on the equity claim producing a non-zero payout. The value of the equity claim is truncated at zero, so the total wealth of the investor, conditional on no default, is $N_{E,t}(x_{m,t+1} - \lambda) + N_{f,t} + N_{D,t}$. Under the equity payout distribution, the investor's wealth is truncated at $N_{f,t} + N_{D,t}$.

The pricing relation is similar to what is obtained when payouts are normally distributed. Price equals expected cashflow minus a covariance risk adjustment. In addition, there will be a truncated loss adjustment that reflects the benefits of limited liability. This adjustment consists of the maximum loss ratio, η_t, and the truncated loss weight $\psi(\bullet)$. The maximum loss ratio is

$$\eta_t = \frac{u'\left(W_t\left(1 - \alpha^*\right)R_f\right)}{E_t(u'(W_{t+1}))}, \tag{7}$$

and captures how investors value the benefits of limited liability; α^* is the optimal amount invested in the equity claim. The numerator is the marginal utility of wealth when the investor looses everything in the equity claim. At this point, the equity investor's marginal utility is highest. The marginal utility in the denominator is the expected marginal utility of the wealth portfolio conditional on no default. The ratio is greater than one for concave utility functions and increases as $-E_t(u''(W_{t+1}))$ increases.

We can re-express the price as an expected excess return, $r^e_{E,t+1}$, to investing in the market equity claim,

$$E_t(r^e_{E,t+1}) = \sigma^2_m \gamma_t + \sigma_m \psi \left(\frac{\mu_m - \lambda}{\sigma_m} \right) (1 - \eta_t) \,, \tag{8}$$

$$\gamma_t = -\frac{E_t(u''(W_{t+1}))}{E_t(u'(W_{t+1}))} W_t \alpha^*,$$

where γ_{t+1} is Rubinstein's (1973) coefficient of risk aversion, and $\sigma^2_m = \sigma^2_{Xm}/P^2_c$.

The positive skewness of the equity claim payout leads to a decrease in expected returns relative to an unlimited liability equity claim. The difference between the expected returns for the limited liability claim and an asset that pays $x_m - \lambda$ for all levels of x_m is

$$E_t(r^e_{E,t+1}) - E_{N,t}(r^e_{UN,t+1}) = \sigma^2_m \gamma_t - \sigma^2_{m,UN} \gamma_{UN,t} + \psi \left(\frac{\mu_m - \lambda}{\sigma_m} \right) \sigma_m (1 - \eta_t) \,, \tag{9}$$

where $\sigma^2_{m,UN}$ is σ^2_{Xm}/P^2_{UN}, and P_{UN} is the price for the equity claim with unlimited liability.

The difference in expected returns is positive. People are willing to pay to do better in bad times, so asset claims with limited liability protection command higher prices. If the investor has decreasing absolute risk aversion, the equity implied risk aversion is smaller than when the equity claim has unlimited liability. Compared with an unlimited liability claim, wealth is larger in some future states of the world. If investors become less risk averse as wealth increases, risk aversion measures implied by equity prices will also decrease.

The expected return relation generalizes Merton's (1973, 1980) dynamic CAPM, when investors have no intertemporal hedging demands. If trading occurs in continuous time, and assets values are log-normally distributed, expected returns are proportional to risk. Instantaneous returns are normally distributed, so there is no utility gained from having limited liability. The intertemporal CAPM model has met mixed empirical success. Some estimates of risk aversion using market level equity returns are positive (French et al (1987), Schwert (1989), Scruggs (1996)), while others are negative (Glosten et al (1993), Whitelaw (1994), Boudoukh, Richardson and Whitelaw (1997)). The impact of limited liability on the equity-implied risk aversion will lead to risk aversion coefficients are smaller than the risk aversion embedded in the price of the underlying market payout.

3 Expected Equity Returns With Multiple Risky Assets

3.1 Firms and Investors

Similar results hold when investors allocate between claims to many risky assets. Now the economy consists of a finite collection of firms, I; the terminal value of firm i is $A_{i,t+1}$ $(i \in I)$. At time t, firm value may be expressed as the sum the conditional expected value of the firm $E_t(A_{i,t+1})$ and a random cash-flow, $\epsilon_{A,i,t+1}$. The random component is assumed to be normally distributed with mean zero. Firms are subject to systematic shocks; the covariance matrix of cash-flows is Σ_A, the covariance between two assets σ_{A_i,A_j}.

The investor maximizes expected utility of wealth at $t+1$ by allocating initial wealth between equity and debt claims to the firm's cash-flows and a risk-free asset that pays unity next period.[4] The Ix1 equity and debt price vectors $P_{E,t}$ and $P_{D,t}$ and demand vectors, N_{Et} and $N_{D,t}$, are constrained so that $W_t = N'_{E,t}P_t + N'_{D,t}P_{D,t} + N_{f,t}P_{f,t}$. Wealth next period is $W_{t+1} = N_{m,t}A_{m,t+1} + N_{f,t} = N'_{E,t}x_{t+1} + N'_{D,t}D_t + N_{f,t}$, where x_{t+1} is the equity payout vector, and D_t is the vector of fixed claims. The equity claim to firm i's assets is $x_{i,t+1} = A_{i,t+1} - D_{i,t+1}$, conditional on the firm being able to meet it's debt payments, $D_{i,t+1}$.

I restrict the analysis to the case of an investor with constant absolute risk aversion utility; the absolute risk aversion coefficient is a. I re-normalize prices in terms of the risk-free asset. Since cash-flows are normally distributed, the expected return of the firm's assets and the wealth portfolio are

$$\begin{aligned} g_{i,t} &= 1 + \frac{\sigma_{i,m,t}}{\sigma^2_{m,t}}\left(g_{m,t} - 1\right), \qquad (10) \\ g_{m,t} &= 1 + \sigma^2_{m,t} RRA \alpha^*_{Am}. \end{aligned}$$

$\sigma_{i,m,t}$ and $\sigma^2_{m,t}$ are growth covariance and volatility, α^*_{Am} is the proportion of wealth invested

[4]As with the previous model, I assume a representative investor. I have also derived a similar model assuming a competitive equillibrium, and get analytically similar results.

in the risky wealth portfolio, and RRA is the relative risk aversion of the representative investor. The value of the firm is $P_{Ai,t} = E_t(A_{i,t+1})/g_{i,t}$.

The price of the equity claim is:[5]

$$
\begin{aligned}
P_{i,t} &= P_{i,t}^{UN} + \frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(P_{i,t}^{UN}/\sigma_{Ai}\right)^2\right) \Pr\left(A_{t+1} > D_t + \sigma_{Ai,Am} N_{m,t} a\right)^{-1}, \quad (11) \\
P_{i,t}^{UN} &= P_{Ai,t} - D.
\end{aligned}
$$

The first term, $P_{i,t}^{UN}$, is the price of equity with unlimited liability. Growth in the wealth portfolio is determined by the expected volatility of the market and the investor's risk aversion, while unlimited liability prices depend on covariance risk and firm cash-flows net of payouts to debt holders. The truncated loss adjustment is a ratio of the exponential of the squared Sharpe (1966) ratio and the probability that the firm's assets are bigger than debt and compensation for bearing covariance risk.

The value of limited liability increases as the covariance between firm assets and the wealth portfolio increases. Assets with higher covariance risk produce payouts that are worse when overall wealth falls. The decline in wealth is smaller with equity claims on these assets, so equity prices are higher. The value of limited liability also increases as overall economic uncertainty increases. Equity claims reduce the uncertainty about future wealth levels.

Table 1 presents the ratio of the truncated loss adjustment to equity price when risk and leverage vary. I use the asset's beta, along with market variance, to calculate asset covariance risk. The market variance is estimated using Federal Reserve Flow of Funds data. I calculate quarterly returns to a market portfolio consisting the value of non-farm, non-financial liabilities and the market value of outstanding equity.[6] The series spans 50

[5] See Appendix 2 for a derivation of the pricing function.

[6] The market value of equity series, FL103164003, includes corporate farm equity. For debt, I use non-

years, from Q1:1952 to Q4:2001. Wealth portfolio growth is determined by investor risk aversion; I fix absolute risk aversion to be consistent with relative risk aversion of 10, and assume that the investor places all funds in the risky market portfolio. The estimated market standard deviation, 4.80 percent, along with risk aversion of 10 produces an expected quarterly market return of 2.38 percent, which is close to the sample average of the market return (2.27 percent). I also calculate a five year rolling average of the market standard deviation. I use this to define high market volatility, which is two times the rolling standard deviation added to the average market standard deviation over the entire sample.

Total firm variance is fixed and equals that of an asset with no idiosyncratic risk and a beta of 2. Idiosyncratic risk decreases as beta increases from zero to two. I also vary leverage, which I define as the ratio of the outstanding debt and next period's expected asset level. When leverage is high, the truncated loss adjustment is a sizable component of equity price. As market volatility increases, the adjustment component increases. Even for relatively moderate leverage ratios, the value of limited liability contributes materially to the overall equity value. Figure 1 graphs the adjustment/price ratio for average and high market volatility and asset betas of 0.8 and 1.6. Again, when leverage is small, the adjustment term is a negligible component of the price. As leverage increases, the adjustment becomes a sizable component of the price of equity, especially for assets that have higher asset covariance.

Risk averse individuals prize investments that limit declines in their wealth. As risk aversion increases, the proportion of the truncated loss adjustment relative to total equity value increases. Figure 1 also shows the ratio of the adjustment term to equity price, but for different relative risk aversion coefficients. Asset beta is fixed at 1.2. For low levels of risk aversion, the value of the limited liability component is negligible, so I start the graphs

farm, non-financial liabilities (series FL104190005).

start at leverage of 0.5. As leverage increases, the limited liability value increases for all levels of risk aversion. Although it is larger for high risk aversion, from the lowest to highest risk aversion, the coefficient increases by an order of magnitude (2 to 20), while the value of the truncated loss adjustment merely doubles. Although more risk averse investors like limited liability, they also dislike cash flow risk. The aversion to uncertainty may dominate the value of limited liability.

3.2 Default Risk and Expected Returns

The value of the limited liability component may be used to help understand some of the puzzling empirical regularities on the relationship between leverage and debt.

1. Expected equity returns may decrease as leverage increases. The empirical relationship between average equity returns and financial leverage is complex. Portfolios returns for formed by degree of leverage may increase as leverage increases (Bhandari (1988), Fama and French (1992), Charoenrook (2001)). In addition, average returns for some equity portfolios may decline for very high leverage ratios, (Griffin and Lemmon (2002)). Individual equity returns also may decline as leverage increases (Rolph (2002)) or after the issuance of straight bonds (Affleck-Graves and Speiss (1999)).

Pronounced idiosyncratic cash-flow risk may lead to expected equity returns dropping as leverage increases. The risk of the equity cash-flow is positively related to leverage. The likelihood of getting a large positive payout, but having limited losses also increases. When idiosyncratic risk is large relative to systematic risk, the price impact of the limited loss component may dominate the impact of the higher equity cashflow risk.

Figure 2 graphs expected equity returns when idiosyncratic risk differs. Again, I fix absolute risk aversion to be consistent with relative risk aversion coefficient of 10. Firm systematic risk is held constant, under the assumption that asset beta is 1.5. Total asset volatility is determined by the correlation between firm and market payouts; a decline in asset correlation leads to an increase in idiosyncratic and total volatility. Low idiosyncratic risk corresponds to an asset correlation of 0.9, while high idiosyncratic risk corresponds to an asset correlation of 0.2. Equity returns for assets with little idiosyncratic risk increase as leverage increases. Assets with pronounced idiosyncratic volatility (and total volatility) have equity returns that increase over lower levels of leverage, but decline as leverage reaches higher levels.

There is empirical evidence that cash-flow covariance and idiosyncratic risk of firms partitioned by book-to-market ratios differ. Average firm returns are larger for high book-to-market firms than for low book-to-market firms when measured using either accounting values (Fama and French (1995)) or market equity and debt prices (Hecht (2000)). If underlying asset returns are well approximated by a linear function of the wealth portfolio, high book-to-market firms should have higher asset betas. In addition, Hecht (2000) reports that lower book-to-market firms also have higher total firm return variance. With a linear asset return model, higher firm variance and lower asset betas implies higher idiosyncratic risk.

Equity returns for low book-to-market firms decline after some leverage increasing transactions, while returns for high book-to-market firms increase. Speiss and Affleck-Graves (1999) report a significant negative equity return for low book-to-market firms, and negative, but smaller and statistically insignificant, returns for high book-to-market firms. Ikenberry, Lakonishok and Vermaelen (1991) also find average returns differ following leverage increasing transactions. They report that low book-to-market firms have negative stock price reactions after equity share repurchases; they also show positive, statistically significant equity returns for high book-to-market firms.

If higher book-to-market firms have higher asset betas, and lower book-to-market firms have lower asset betas and higher idiosyncratic risk, then the decline in expected returns for leverage increasing transactions have a risk-based explanation.

2. Expected return may decline for firms with high financial distress. Dichev (1998) and Griffin and Lemmon (2002) show that equity returns of firms likely to experience financial distress are sometimes smaller than returns of firms with greater likelihood of financial distress. Both studies estimate the likelihood of financial distress using accounting-based measures, such as Altman's (1968) Z-score or Olsen's (1980) O-score. These measures associate higher leverage and lower earnings with higher likelihood of default (Dichev (1998), Shumway (2001), Olsen (1968), Altman (1980)).

Again, we can likely properties of low and high book-to-market firms to evaluate the empirical evidence. Dichev (1998) reports that the likelihood of financial distress is positively associated with higher book-to-market ratios; these firms also have lower average monthly returns than firms with low likelihood of distress. Griffin and Lemmon (2002) find high book-to-market firms that are likely to enter into financial distress have greater one-year equity returns than firms with low likelihood of distress. The difference is statistically significant. They also report the converse occurs for low book-to-market firms.

The decrease in equity returns also occurs when the probability of default is inferred equity prices from risk-neutral valuation relations, such as Black and Scholes (1973) and Merton (1974). Vassalou and Xing (2001) estimate the risk-neutral probability of default. They find low book-to-market firms with high likelihood of default have lower one month returns than similar firms that have a low probability of default. The opposite occurs for high book-to-market firms.

As previously discussed, firms with pronounced idiosyncratic risk may have high probabilities of default, but low expected returns. Expected returns will also decline as asset beta increases. If low book-to-market firms have higher idiosyncratic risk and lower asset betas, then the empirical results have a risk-based explanation. The results reflect the interaction between the risk characteristics of the firm and limited liability.

4. Conclusion

This paper discusses how limited liability and risk aversion jointly impact expected equity returns. I derive covariance decompositions for truncated normal distributions. I use the decompositions to find explicit an pricing relation for an equity claim to investors' wealth portfolio that holds general concave utility functions. I discuss the impact of limited liability for firm-level equity returns when investors have CARA utility.

A key empirical prediction is that expected equity returns may decline as leverage increases. Compensation for systematic risk and limited liability jointly determine expected equity returns. When systematic asset risk is low, and total asset volatility is high, the limited liability component may dominate the systematic risk component, and expected equity returns decline as leverage.

The results may give insight into the risk characteristics of firms with high or low book-to-market ratios. One direction for future research is to use the model to estimate asset covariance and idiosyncratic risk for firms with differing book-to-market ratios. Existing empirical evidence suggests that low book-to-market firms have low systematic risk relative to idiosyncratic risk. To my knowledge, no study directly examines the asset risk characteristics of these firms. Griffin and Lemmon (2002) suggest that investors misperception of risk leads to the decline in average equity returns for low book-to-market firms. This paper indicates that investors preference for skewness, not mispricing, may explain Griffin and Lemmon's results.

Appendix 1

This appendix gives a brief discussion of the conditionally truncated normal distribution, in addition to providing relevant proofs. Appendix 1.1 provides some discussion of the distribution theory. Appendix 1.2 presents derivations of the distributions and lemmas used in the text. Appendix 1.3 derives conditional distributions, and moment generating functions, related to the conditionally truncated normal distribution.

Appendix 1.1

A.1.1 The Conditionally Truncated Normal Distribution

Consider a random cash flow x_{t+1} adapted to the public information set Θ_t at time t. When a claim $K(x_{t+1})$ to the cash flow x_{t+1} is nonzero only is the random variable z_{t+1} is above the threshold $\lambda \in \mathbb{R}^+$, the distribution of $K(x_{t+1})$ will be skewed.[7]

Theorem A.1.1. Assume that x and z are jointly normally distributed with arbitrary location parameters μ_x and μ_z, dispersion parameters σ_x^2 and σ_z^2 and correlation coefficient ρ. When $\lambda > 0$, the conditional distribution of $K(x) = x|\ z > \lambda$ is

$$\phi(K(x)) = \zeta\sigma_x^{-1}\phi(\frac{x-\mu_x}{\sigma_x})\Phi(\frac{\rho(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}} + \frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}}), \tag{12}$$

where $\zeta = P(z > \lambda)^{-1}$ is probability that $z > \lambda$, $\phi(x)$ represents the standard normal probability function and $\Phi(x)$ is the standard normal cumulative density function.

Proof. See Appendix 1.2.

The conditionally truncated normal distribution is skewed. When $\lambda = \mu_z$, and $\mu_x = 0$, the probability that $z > \mu_z$ is 0.5, and the skewness of the distribution of x depends only on ρ. For example, if $\rho = 0$, x and z are independent and $K(x)$ is not skewed. At the other extreme, if $\rho = 1$, $z > \mu_z$ is equivalent to $x > \mu_x$. In this case, the distribution of

[7] The distribution is a generalization of the skew-normal distribution (Azzalini (1985) and Azzalini and Della Valle (1996)). The skew-normal occurs when $\lambda = \mu_z$.

$K(x)$ is the half-normal distribution. In general, as the correlation coefficient between the x and z goes from zero to one, the distribution of $K(x)$ becomes more positively skewed. As the correlation coefficient goes from zero to negative one, the distribution becomes more negatively skewed.

In general, μ_z will not be equal to λ, so the skewness of $K(x)$ will be partially determined by the difference between μ_z and λ. As the difference becomes more positive, the probability that $K(x) = 0$ decreases, and the distribution of $K(x)$ becomes less skewed. The truncated normal distribution is a special case of Theorem A.1.1. Setting z equal to x produces the distribution of $K(x) = x|x > \lambda$.

The conditionally truncated normal distribution also extends to the bivariate case:

Theorem A.1.2. Suppose that the vector $x = (x_1, x_2)'$ and scalar z are jointly normally distributed. x has a positive definite covariance matrix Σ_{xx} and a 2x1 vector of means μ_x. The variance and mean of z are σ_z^2 and μ_z. The 1x2 vector of covariances between x and z is Σ_{xz}. $K(x) = x|z > \lambda$, $\lambda \in \mathbb{R}^+$ is a mapping from R^2 to R^2. The distribution of $K(x)$ is

$$\begin{aligned} \phi(K(x)) &= \zeta(2\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}} Exp\left(-\frac{1}{2}\left((x-\mu_x)'\,\Sigma_{xx}^{-1}(x-\mu_x)\right)\right) \\ &\quad \Phi\left(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}(x-\mu_x) - (\mu_z - \lambda)}{\sigma_{z|x}}\right), \end{aligned} \tag{13}$$

where $\sigma_{z|x} = \sigma_z\sqrt{1 - \Sigma_{xz}\Sigma_{xx}^{-1}\Sigma_{xz}'}$ and $\zeta = P(z > \lambda)^{-1}$ is probability that $z > \lambda$.

Proof. See Appendix 1.2.

The bivariate conditionally truncated normal distribution retains many of the same properties of the univariate distribution. Two factors influence the degree of skewness of the joint distribution. First are the covariances between x_i and z, $i = 1, 2$. As the covariance between x_i and z becomes more positive, the joint distribution becomes more skewed. In addition, when the covariances are positive, a greater positive difference between μ_z and

λ will lead to less skewed payouts. If $z = x_1$, we recover the bivariate truncated normal distribution of x_1 and x_2 given that x_1 is greater than zero.

Appendix 1.2

Proof of *Theorem A.1.1.*

Suppose that x and z are normally distributed with means μ_x and μ_z, variances σ_x^2 and σ_z^2 and correlation ρ. x is zero if $z > \lambda \in \mathbb{R}^+$. The joint distribution of x given $z > \lambda$ may be expressed using Bayes theorem

$$\phi(x, z|z > \lambda) = \zeta \frac{1}{2\pi\sqrt{\sigma_x^2 \sigma_z^2 \left(1-\rho^2\right)}} Exp\left(-\frac{1}{2}\left(\left(\epsilon_x^2 + \epsilon_z^2 - 2\rho\epsilon_x\epsilon_z\right) / \left(1-\rho^2\right)\right)\right), \tag{14}$$

where $\zeta = P(z > \lambda)^{-1}$, $\epsilon_x = \frac{x-\mu_x}{\sigma_x}$ and $\epsilon_z = \frac{z-\mu_z}{\sigma_z}$. The marginal distribution of x is obtained by integrating z out of the joint distribution,

$$\begin{aligned} \phi(x|z > \lambda) &= \frac{\zeta}{\sqrt{2\pi}\sigma_x} Exp\left(-\frac{1}{2}\left(\epsilon_x^2 / \left(1-\rho^2\right)\right)\right) \\ &\int_\lambda^\infty \frac{1}{\sqrt{2\pi\sigma_z^2\left(1-\rho^2\right)}} Exp\left(-\frac{1}{2}\left(\left(\epsilon_z^2 - 2\rho\epsilon_x\epsilon_z\right) / \left(1-\rho^2\right)\right)\right) dz. \end{aligned} \tag{15}$$

After a change in the variable $z = u + \lambda$, and algebraic manipulation, the integral may be written as

$$\begin{aligned} \phi(x|z > \lambda) &= \frac{\zeta}{\sqrt{2\pi}\sigma_x} Exp\left(-\frac{1}{2}\left(\epsilon_x^2 / \left(1-\rho^2\right)\right)\right) \\ &\int_0^\infty \frac{1}{\sqrt{2\pi\sigma_z^2\left(1-\rho^2\right)}} Exp\left(-\left(az^2 + 2bz + c\right)\right) dz, \end{aligned} \tag{16}$$

$$\begin{aligned} a &= \frac{1}{2\sigma_z^2(1-\rho^2)}, \\ b &= -\frac{1}{2}\left(\frac{(\mu_z - \lambda)}{\sigma_z^2(1-\rho^2)} + \frac{\epsilon_x\rho}{\sigma_z(1-\rho^2)}\right), \\ c &= \frac{(\mu_z - \lambda)^2}{2\sigma_z^2(1-\rho^2)} + \frac{(\mu_z - \lambda)\epsilon_x\rho}{\sigma_z(1-\rho^2)}. \end{aligned} \tag{17}$$

The resulting integral may be evaluated using the identity (Equation 7.4.2, Abramowitz and Stegun (1965))

$$\int_0^\infty e^{-(az^2+2bz+c)}dz = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{(\frac{b^2-ac}{a})}Erfc(\frac{b}{\sqrt{a}}). \tag{18}$$

$Erf(x)$ is the error function, and $Erfc(x) = 1 - Erf(x)$ is the complementary error function. Further algebraic manipulation yields the density

$$\begin{aligned}\phi(x|z > \lambda) = {} & \frac{\zeta}{2\sqrt{2\pi}\sigma_x}Exp\left(-\frac{1}{2}\left(\epsilon_x^2/\left(1-\rho^2\right) - \epsilon_x^2\rho^2/\left(1-\rho^2\right)\right)\right) \\ & Erfc(-\frac{(\mu_z - \lambda) + \sigma_z\epsilon_x\rho}{\sigma_z\sqrt{2(1-\rho^2)}}).\end{aligned} \tag{19}$$

$Erf(x)$ is an odd function, so $Erf(-x) = -Erf(x)$. In addition, it has been shown (Equation 26.2.29, Abromowitz and Steigum (1965)) that the cumulative density function for a standard normal random variable may be written as $\Phi(x) = \frac{1}{2}(1 + Erf(\frac{x}{\sqrt{2}}))$. Making the appropriate substitutions produces the density

$$\phi(x|z > \lambda) = \zeta\sigma_x^{-1}\phi(\frac{x-\mu_x}{\sigma_x})\Phi(\frac{\rho\,(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}} + \frac{(\mu_z - \lambda)}{\sigma_z\sqrt{(1-\rho^2)}}). \tag{20}$$

QED.

Theorem A.1.2. Suppose that the vector $x = (x_1, x_2)'$ and scalar z are jointly normally distributed. x has a positive definite covariance matrix Σ_{xx} and vector of means μ_x. The variance and mean of z are σ_z^2 and μ_z. The 2X1 vector of covariances between x and z is

Σ_{xz}. The conditional distribution of x given that $z > \lambda$ is

$$\phi(x_1, x_2 | z \; > \; \lambda) = \zeta (2\pi)^{-1} |\Sigma_{xx}|^{-\frac{1}{2}} Exp\left(-\frac{1}{2}\left((x-\mu_x)' \, \Sigma_{xx}^{-1} \, (x-\mu_x)\right)\right) \tag{21}$$
$$\Phi(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}\,(x-\mu_x)-(\mu_z-\lambda)}{\sigma_{z|x}})$$

where $\sigma_{z|x} = \sigma_z \sqrt{1 - \Sigma_{xz}\Sigma_{xx}^{-1}\Sigma_{xz}'}$.

Proof of *Theorem A.1.2*

The unconditional distribution of $(x', z)'$ is

$$\phi((x', z)') = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}Exp\left(-\frac{1}{2}\left((x', z)' - \left(\mu_x', \mu_z\right)'\right)' \Sigma^{-1}\left((x', z)' - \left(\mu_x', \mu_z\right)'\right)\right) \tag{22}$$

The covariance matrix may be partitioned as

$$\Sigma^{-1} = \Delta = \begin{bmatrix} \Delta_{xx} & \Delta_{xz}' \\ \Delta_{xz} & \Delta_{zz} \end{bmatrix} \tag{23}$$

Using Bayes theorem, the conditional distribution of $x|z > \lambda$ is

$$\begin{aligned}\phi(x_1, x_2|z \quad > \quad \lambda) &= \zeta(2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}Exp(-\frac{1}{2}\left(x - \mu_x\right)' \Delta_{xx}\left(x - \mu_x\right)) \\ &\int_{\lambda}^{\infty} Exp(-\frac{1}{2}((z - \mu_z)^2 \Delta_{zz} + 2\left(z - \mu_z\right)\Delta_{xz}\left(x - \mu_x\right)))dz \end{aligned} \tag{24}$$

where $\zeta = P(z > \lambda)^{-1}$. As in the proof of Theorem A.1.1, the density may be simplified by making a change in variable for $z = u + \lambda$, and applying Equation (18),

$$\begin{aligned}\phi(x_1, x_2|z \quad > \quad \lambda) &= \zeta(4\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}}Exp\left(-\frac{1}{2}\left(\left(x - \mu_x\right)' \Sigma_{xx}^{-1}\left(x - \mu_x\right)\right)\right) \\ &Erfc\left(\frac{\Delta_{zz}\left(\lambda - \mu_z\right) + \Delta_{xz}\left(x - \mu_x\right)}{\sqrt{2\Delta_{zz}}}\right). \end{aligned} \tag{25}$$

The density in Theorem A.1.2 is obtained taking the inverse of the partitioned matrix, Σ^{-1}, and the using the properties of the error function

$$\phi(x_1, x_2|z > \lambda) = \zeta(2\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}} Exp\left(-\frac{1}{2}\left((x-\mu_x)' \Sigma_{xx}^{-1}(x-\mu_x)\right)\right)$$
$$\Phi\left(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}(x-\mu_x) - (\mu_z - \lambda)}{\sigma_{z|x}}\right), \tag{26}$$

where $\sigma_{z|x} = \sigma_z\sqrt{1 - \Sigma_{xz}\Sigma_{xx}^{-1}\Sigma'_{xz}}$.

QED.

Proof of *Lemma 1*

To prove Lemma 1, consider a random variable that with the density function derived in Theorem A.1.1. Let $g(x)$ be a differentiable, lebesgue measurable function with $E|g'(x)| < \infty$. Lemma *1* states that if $\mathring{x} = x|z > \lambda$, then

$$Cov(g(\mathring{x}), \mathring{x}) = \sigma_x^2 E\left(g'(\mathring{x})\right) + \rho\sigma_x\psi\left(\frac{\mu_z - \lambda}{\sigma_z}\right)\left(E_n(g(\mathring{x})) - E(g(\mathring{x}))\right), \tag{27}$$

where $\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)$ is the inverse Mill's ratio. $E_n(g(\mathring{x}))$ is the expectation of $g(\mathring{x})$ formed under a normal distribution with mean $\mu_x - \frac{\sigma_{xz}}{\sigma_z^2}(\mu_z - \lambda)$ and variance $\sigma_x^2\sqrt{1-\rho^2}$. $E\left(g(\mathring{x})\right)$ is the expectation taken under the distribution in theorem A.1.1.

The covariance between $\mathring{x}$ and $g(\mathring{x})$ is

$$Cov(g(\mathring{x}), \mathring{x}) = E\left(g\left(\mathring{x}\right)\left(\mathring{x} - E\left(\mathring{x}\right)\right)\right), \tag{28}$$

while the first moment of $\mathring{x}$ is $E(\mathring{x}) = \mu_x + \psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x$ (see Appendix A.2, Eqn. A.2.2.2).

Substituting $E(\mathring{x})$ back into the covariance function leads to

$$Cov(g(\mathring{x}),\mathring{x}) = \int_{-\infty}^{\infty} g(x)\,(x-\mu_x)\,\zeta\frac{1}{\sqrt{2\pi}\sigma_x}Exp\left(-\frac{\epsilon_x^2}{2}\right)\Phi(\frac{\rho\,(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}}+\frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}})dx$$
$$-E(g(\mathring{x}))\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x \tag{29}$$

where $\epsilon_x = \frac{x-\mu_x}{\sigma_x}$. Since

$$-\sigma_x^2 dExp\left(-\frac{\epsilon_x^2}{2}\right) = (x-\mu_x)\,Exp\left(-\frac{\epsilon_x^2}{2}\right)dx, \tag{30}$$

the expression for the covariance is equivelent to

$$Cov(g(\mathring{x}),\mathring{x}) = \int_{-\infty}^{\infty} -\sigma_x^2 g(x)\zeta\frac{1}{\sqrt{2\pi}\sigma_x}\Phi(\frac{\rho\,(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}}+\frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}})dExp\left(-\frac{\epsilon_x^2}{2}\right)$$
$$-E(g(\mathring{x}))\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x \tag{31}$$

The first integral may be evaluated using integration by parts.[8]

$$Cov(g(\mathring{x}),\mathring{x}) = -\frac{\zeta\sigma_x}{\sqrt{2\pi}}\left[g(x)\left(\int_{-\infty}^{\delta(x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\right)e^{-\frac{\epsilon_x^2}{2}}\right]_{-\infty}^{\infty}$$
$$+\frac{\zeta\sigma_x}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g'(x)Exp\left(-\frac{\epsilon_x^2}{2}\right)\Phi\left(\frac{\rho\,(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}}+\frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}}\right)dx$$
$$+\frac{\zeta}{2\pi}\int_{-\infty}^{\infty}\frac{\rho}{\sqrt{(1-\rho^2)}}g(x)Exp\left(-\frac{1}{2}\left(\frac{\rho\,(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}}+\frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}}\right)^2\right)$$
$$Exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)dx - E(g(\mathring{x}))\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x. \tag{32}$$

[8] Integration by parts allows the following decomposition: $\int_D udv = uv|_D - \int_D vdu$

where $\delta(x) = \frac{\rho(x-\mu_x)}{\sigma_x\sqrt{(1-\rho^2)}} + \frac{(\mu_z-\lambda)}{\sigma_z\sqrt{(1-\rho^2)}}$. Since $E\left|g'(x)\right| < \infty$, the first term is zero. The second term is just $\sigma_x^2 E(g'(x))$. After completing the square, the third term may be rewritten as $\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x E_n\left(g(\mathring{x})\right)$, where $E_n\left(g(\mathring{x})\right)$ is the expectation of $g(x)$ taken under a normal density with expected value $\mu_x - \frac{\sigma_{xz}}{\sigma_z^2}(\mu_z-\lambda)$ and variance $\sigma_x^2\sqrt{1-\rho^2}$. Combining the three terms gives the result in Lemma 1:

$$Cov(g(\mathring{x}),\mathring{x}) = \sigma_x^2 E\left(g'(\mathring{x})\right) + \psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\rho\sigma_x\left(E_n(g(\mathring{x})) - E(g(\mathring{x}))\right). \tag{33}$$

QED.

Proof of *Lemma* 2.

Lemma 2 states that if the random 2X1 vector $(\mathring{x}_1,\mathring{x}_2)=(x_1,x_2)|z>\lambda$ has the joint distribution function defined in Theorem A.1.2, then

$$Cov(g\left(\mathring{x}_2\right),\mathring{x}_1) = \sigma_{12}E\left(g'\left(\mathring{x}_2\right)\right) + \psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z}\left(E_n\left(g\left(\mathring{x}_2\right)\right) - E\left(g\left(\mathring{x}_2\right)\right)\right). \tag{34}$$

$E_n\left(g\left(\mathring{x}_2\right)\right)$ is the expectation of x_2 taken under a normal density with mean $\mu_2 - \frac{\sigma_{2z}}{\sigma_z^2}\left(\mu_z-\lambda\right)$ and variance $\sigma_{2|z}^2$, the conditional variance of x_2 given z.

The bivariate version of Lemma 1 may be derived by expressing $Cov(g(x_2),x_1)$ as

$$Cov(g(x_2),x_1) = \int_{-\infty}^{\infty} g(x_2)(E(x_1|x_2) - E(x_1))\phi(x_2|z>\lambda)dx_2. \tag{35}$$

Appendix 1.3 establishes the following results. The marginal density function of $\mathring{x}_2$ is

$$\phi(x_2|z>\lambda) = \zeta\frac{1}{\sqrt{2\pi\sigma_2^2}}Exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}) \tag{36}$$

while the conditional expectation of $\mathring{x}_1$ is

$$E(\mathring{x}_1|\mathring{x}_2) = \mu_1 + \frac{\sigma_{12}}{\sigma_2^2}(x_2 - \mu_2) + \frac{\varsigma}{\sigma_{z|x_2}}\psi\left(\frac{E_n(z|x_2) - \lambda}{\sigma_{z|x_2}}\right) \tag{37}$$

with $\varsigma = \sigma_{1z} - \frac{\sigma_{2z}\sigma_{12}}{\sigma_2^2}$. The unconditional expectation of $\mathring{x}_1$ is

$$E(\mathring{x}_1) = \mu_1 + \psi\left(\frac{\mu_z - \lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z} \tag{38}$$

where $\psi(y)$ is the inverse Mill's ratio. The covariance may then be expressed as

$$\begin{aligned} Cov(g(\mathring{x}_2), \mathring{x}_1) &= \int_{-\infty}^{\infty} g(x_2)\zeta\frac{1}{\sqrt{2\pi\sigma_2^2}}Exp\left(-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\Phi(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}}) \\ &\left(\frac{\sigma_{12}}{\sigma_2^2}(x_2-\mu_2) + \frac{\varsigma}{\sigma_{z|x_2}}\psi\left(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}\right) - \psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z}\right)dx_2 \end{aligned} \tag{39}$$

The expression may be evaluated as three separate integrals. The first is

$$\begin{aligned} &-\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z}\int_{-\infty}^{\infty} g(x_2)\zeta\frac{1}{\sqrt{2\pi\sigma_2^2}}Exp\left(-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\Phi(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}})dx_2 \\ = &-\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z}E\left(u(\mathring{x}_2)\right). \end{aligned} \tag{40}$$

The second integral is

$$\frac{\sigma_{12}}{\sigma_2^2}\int_{-\infty}^{\infty}(x_2-\mu_2)\,g(x_2)\zeta\frac{1}{\sqrt{2\pi\sigma_2^2}}Exp\left(-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\Phi(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}})dx_2$$

$$= -\frac{\sigma_{12}\zeta}{\sqrt{2\pi\sigma_2^2}}\int_{-\infty}^{\infty}g(x_2)\Phi(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}})dExp\left(-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right).$$

$$= \frac{\sigma_{12}\sigma_{2z}\zeta}{\sigma_{z|x_2}\sigma_2^2\sqrt{2\pi\sigma_2^2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}g(x_2)Exp\left(-\frac{1}{2}\left(\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2+\left(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}}\right)^2\right)\right)dx_2$$

$$+\sigma_{12}E\left(u'(\mathring{x}_2)\right) \qquad (41)$$

The last line follows from integration by parts. The final integral is

$$\frac{\varsigma\zeta}{\sigma_{z|x_2}\sqrt{2\pi\sigma_2^2}}\int_{-\infty}^{\infty}g(x_2)\frac{1}{\sqrt{2\pi}}Exp\left(-\frac{1}{2}\left(\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2+\left(\frac{E(z|x_2)-\lambda}{\sigma_{z|x_2}}\right)^2\right)\right)dx_2. \qquad (42)$$

Further algebraic simplification gives the result in Lemma 2

$$Cov(g(\mathring{x}_2),\mathring{x}_1)=\sigma_{12}E\left(g'(\mathring{x}_2)\right)+\psi\left(\frac{\mu_z-\lambda}{\sigma_z}\right)\frac{\sigma_{1z}}{\sigma_z}\left(E_n\left(g(\mathring{x}_2)\right)-E\left(g(\mathring{x}_2)\right)\right), \qquad (43)$$

where the expectation $E_n\left(g\left(\mathring{x}_2\right)\right)$ is formed under a normal density with mean $\mu_2+\frac{\sigma_{2z}}{\sigma_z^2}\left(\lambda-\mu_z\right)$ and variance $\sigma_{2|z}^2=\sigma_{x_2}^2\left(1-\rho_{x_2,z}^2\right)$.

QED.

Appendix 1.3 Statistical Results Related to the Conditionally Truncated Normal.

Appendix 1.3.1 Univariate Moments of Conditionally Truncated Normal

Suppose that x is a random variable with a probability density function described in Theorem A.1.1. The moment generating function for x is

$$\begin{aligned} M(t) &= E\left(Exp\left(xt\right)\right) \\ &= \zeta \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\left(\frac{1}{2\sigma_x^2}(x-\mu)^2 - t(x-\mu) - t\mu\right)} \Phi\left(\frac{\epsilon_x \rho}{\sqrt{(1-\rho^2)}} + \frac{(\mu_z - \lambda)}{\sigma_z \sqrt{(1-\rho^2)}}\right) dx. \qquad (44) \end{aligned}$$

Completing the square of the argument of the exponential gives, and simplifying the moment generating function produces

$$M(t) = \zeta e^{t\mu + \frac{1}{2}t^2\sigma_x^2} E\left(\Phi\left(\frac{\rho}{\sqrt{(1-\rho^2)}} y + \frac{t\sigma_x\sigma_z\rho + \mu_z - \lambda}{\sigma_z\sqrt{(1-\rho^2)}}\right)\right), \qquad (45)$$

where y is a standard normal random variable. From Zacks (1981, Eq. 2.9.18), the expectation evaluates to

$$M(t) = \zeta e^{t\mu + \frac{1}{2}t^2\sigma_x^2} \Phi\left(\frac{t\sigma_z\sigma_x\rho + \mu_z - \lambda}{\sigma_z}\right). \qquad (46)$$

Evaluating $\frac{dM(t)}{dt}$ at $t = 0$ produces the first moment of x

$$\begin{aligned} E(x) &= \mu\zeta\Phi\left(\frac{\mu_z - \lambda}{\sigma_z}\right) + \zeta \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu_z - \lambda}{\sigma_z}\right)^2} \rho\sigma_x \\ &= \mu + \psi\left(\frac{\mu_z - \lambda}{\sigma_z}\right)\rho\sigma_x. \qquad (47) \end{aligned}$$

QED.

Appendix 1.3.2 Results Related To the Conditionally Truncated Normal

The results for Lemma 2 rely on certain results for the conditionally truncated normal distribution. Suppose that the random vector (x_1, x_2) follows the distribution in Theorem A.1.2.

1. The marginal distribution of x_2 is

$$\phi(x_2|z > \lambda) = \zeta \frac{1}{\sqrt{2\pi\sigma_2^2}} Exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}})$$

Proof. The marginal distribution for x_2 is found by integrating the density function in Theorem A.1.2 with respect to x_1. The joint density function may be rewritten as

$$\begin{aligned}
&\zeta(2\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}} \\
&\int_{-\infty}^{\infty} Exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_{1|2}^2} - 2\frac{\sigma_{12}}{\sigma_2^2\sigma_{1|2}^2}(x_1-\mu_1)(x_2-\mu_2) + \tilde{\Sigma}(x_2-\mu_2)^2\right)\right) \\
&\Phi(a(x_1-\mu_1) + b(x_2-\mu_2) - \frac{\lambda-\mu_z}{\sigma_{z|x}})dx_1
\end{aligned} \tag{48}$$

$$\begin{aligned}
\sigma_{1|2}^2 &= \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} \\
\tilde{\Sigma} &= \frac{1}{\sigma_2^2} + \frac{\sigma_{12}^2}{\sigma_2^4\sigma_{1|2}^2} \\
a &= \frac{1}{\sigma_{1|2}^2\sigma_{z|x}}\left(\sigma_{1z} - \frac{\sigma_{2z}\sigma_{12}}{\sigma_2^2}\right) \\
b &= \frac{1}{\sigma_{z|x}}\left(\sigma_{2z}\tilde{\Sigma} - \frac{\sigma_{1z}\sigma_{12}}{\sigma_2^2\sigma_{1|2}^2}\right)
\end{aligned}$$

Further algebraic simplification, along with the application of the fact that if y is standard normal, then $E(\Phi(\alpha y+\beta))=\Phi(\frac{\beta}{\sqrt{1+\alpha^2}})$ produces

$$\phi(x_2|z>\lambda)=\zeta\frac{1}{\sqrt{2\pi\sigma_2^2}}Exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\Phi\left(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}\right) \tag{49}$$

where $E_n(z|x_2)$ is the conditional expectation of z given x_2 given under the untruncated normal distribution, and $\sigma^2_{z|x_2}$ is its conditional variance.

2. The conditional expectation of x_1 given x_2 is

$$\begin{aligned} E(\mathring{x}_1|\mathring{x}_2) &= \mu_1+\frac{\sigma_{12}}{\sigma_2^2}(x_2-\mu_2)+\frac{\varsigma}{\sigma_{z|2}}\psi\left(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}\right), \\ \varsigma &= \sigma_{1z}-\frac{\sigma_{2z}\sigma_{12}}{\sigma_2^2}. \end{aligned} \tag{50}$$

This may be established by using the conditional density of x_1 given x_2 to derive the moment-generating function of x_1 given x_2. The conditional density function is

$$\phi(x_1|x_2,z>\lambda)= \tag{51}$$
$$\frac{1}{\sqrt{2\pi\sigma_{1|2}^2}}Exp\left(-\frac{1}{2\sigma_{1|2}^2}\left(x_1-\mu_1-\frac{\sigma_{12}}{\sigma_2^2}(x_2-\mu_2)\right)^2\right)\frac{\Phi(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}(x-\mu_x)-\lambda+\mu_z}{\sigma_{z|x}})}{\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}})}$$

The moment-generating function for $x1$ given $x2$ is

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_{1|2}^2}}Exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_{1|2}^2}\left(x_1-\mu_1-\frac{\sigma_{12}}{\sigma_2^2}(x_2-\mu_2)\right)^2-2\sigma_{1|2}^2tx_1\right)\right) \\ &\quad \frac{\Phi(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}(x-\mu_x)-\lambda}{\sigma_{z|x}})}{\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}})}dx_1. \end{aligned} \tag{52}$$

The evaluation of the integral is quite similar to that described in A.1.3.1, and is omitted.

The moment generating function, after simplification, is

$$M\left(t\right)=Exp\left(\left(\mu_1+\frac{\sigma_{12}}{\sigma_2^2}\left(x_2-\mu_2\right)\right)t+\frac{1}{2}\sigma_{1|2}^2t^2\right)\frac{\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}+\frac{a\sigma_{1|2}^2}{\sqrt{1+a^2\sigma_{1|2}^2}}t)}{\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}})} \tag{53}$$

The first moment is obtained by differentiating $M\left(t\right)$, and evaluating the derivative at $t=0$.

$$\begin{aligned} E(x_1|x_2,z &> \lambda)=\frac{\partial M(t)}{\partial t}|_{t=0} \\ &= \mu_1+\frac{\sigma_{12}}{\sigma_2^2}\left(x_2-\mu_2\right)+\frac{a\sigma_{1|2}^2}{\sqrt{1+a^2\sigma_{1|2}^2}}\frac{Exp\left(-\frac{1}{2}\left(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}\right)^2\right)}{\sqrt{2\pi}\Phi(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}})}. \end{aligned} \tag{54}$$

To further simplification produces

$$\begin{aligned} E(x_1|x_2) &= \mu_1+\frac{\sigma_{12}}{\sigma_2^2}\left(x_2-\mu_2\right)+\frac{\varsigma}{\sigma_{z|2}}\psi\left(\frac{E_n(z|x_2)-\lambda}{\sigma_{z|x_2}}\right) \\ \varsigma &= \sigma_{1z}-\frac{\sigma_{2z}\sigma_{12}}{\sigma_2^2} \end{aligned} \tag{55}$$

3. The first moment of x is derived for $(x_{1,}x_2)$ via the moment-generating function. The moment-generating function is

$$\begin{aligned} M\left(t\right) &= E(Exp(x't)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta(2\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}}Exp\left(-\frac{1}{2}\left(\left(x-\mu_x\right)'\Sigma_{xx}^{-1}\left(x-\mu_x\right)-2x't\right)\right) \\ &\quad \Phi(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}\left(x-\mu_x\right)-\lambda+\mu_z}{\sigma_{z|x}})dx_1dx_2 \end{aligned} \tag{56}$$

To evaluate the integral, complete the square in the exponential function using the rule that $x'Ax + 2x'b = (x + A^{-1}b)'\, A\,(x + A^{-1}b) - b'A^{-1}b$. The moment-generating function becomes

$$\begin{aligned} M(t) &= \zeta Exp\left(t'\mu_x + \frac{1}{2}t'\Sigma_{xx}^{-1}t\right) \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi)^{-1}|\Sigma_{xx}|^{-\frac{1}{2}}Exp\left(-\frac{1}{2}\left(x-\mu_x-\Sigma_{xx}t\right)'\Sigma_{xx}^{-1}\left(x-\mu_x-\Sigma_{xx}t\right)\right) \\ &\Phi(\frac{\Sigma_{xz}\Sigma_{xx}^{-1}\left(x-\mu_x\right)-\lambda+\mu_z}{\sigma_{z|x}})dx_1dx_2 \end{aligned} \tag{57}$$

Let $y = \Sigma_{xx}^{-\frac{1}{2}}\left(x - \mu_x - \Sigma_{xx}t\right)$, and use the following rule for the expectation of the multivariate cumulative normal probability integral: $E(\Phi(\alpha y + \beta)) = \Phi(\frac{\beta}{\sqrt{1+\alpha^2}})$ (Azzilani and De Valle (1993)). Moment generating function, after algebraic simplification, is

$$M(t) = \zeta Exp\left(t'\mu_x + \frac{1}{2}t'\Sigma_{xx}^{-1}t\right)\Phi(\frac{\Sigma_{xz}t - \lambda + \mu_z}{\sqrt{\sigma_z^2}}) \tag{58}$$

Differentiating the moment generating function, and evaluating at $t = 0$, produces

$$\frac{\partial E(Exp(x't))}{\partial t}|_{t=\mathbf{0}} = E(x) = \mu_x + \psi\left(\frac{\mu_z - \lambda}{\sigma_z}\right)\frac{\Sigma_{xz}}{\sigma_z}. \tag{59}$$

Appendix 2: The pricing relation when investors have negative exponential utility

There is an explicit closed form solution to the pricing relation in Section 2.6 when the representative investor has exponential utility. In this case, the investor's utility function is

$$u\left(W_{t+1}\right) = -Exp\left(-aW_{t+1}\right) \tag{60}$$

where a is the coefficient of absolute risk aversion. The pricing equation is

$$P_{i,t} = E\left(\frac{u'\left(W_{t+1}\right)}{E_t\left(u'\left(W_{t+1}\right)\right)R_{f,t+1}}K\left(x_{i,t+1}\right)\right), \tag{61}$$

After expanding the correlation using the properties of the covariance matrix, and applying Lemma 2 with the restriction that $z = x_{i,t+1}$, the equation may be also expressed as

$$P_{i,t} = E_t\left(\frac{K\left(x_{i,t+1}\right)}{R_{f,t+1}}\right) + \frac{E_t\left(u''\left(W_{t+1}\right)\right)}{E_t\left(u'\left(W_{t+1}\right)\right)R_{f,t+1}}N_{m,t}\sigma_{x_i,x_m} + \psi\left(\frac{\mu_{xi}-\lambda_i}{\sigma_{xi}}\right)\frac{\sigma_{x_i}}{R_{f,t}}\left(\eta_t - 1\right), \tag{62}$$

where $N_{m,t}P_{i,t} = N'P$. $N_{m,t}$ is the number of shares of the portfolio of equity claims held by the investor, and is defined relative to the percent of wealth invested in the risky asset $\alpha^* = \frac{N_{m,t}P_{m,t}}{W_t}$. Furthermore, I re-normalize the price of the risky asset in terms of the risk-free asset. The first and second derivative for the negative exponential function are

$$\begin{aligned} u'\left(W_{t+1}\right) &= aExp\left(-aW_{t+1}\right) \\ u''\left(W_{t+1}\right) &= -a^2Exp\left(-aW_{t+1}\right). \end{aligned} \tag{63}$$

Substituting the derivatives into the pricing relation gives

$$P_{i,t} = E_t\left(K\left(x_{i,t+1}\right)\right) - aN_{m,t}\sigma_{x_i,x_m} + \psi\left(\frac{\mu_{xi} - \lambda_i}{\sigma_{xi}}\right)\sigma_{x_i}\left(\eta_t - 1\right). \tag{64}$$

To evaluate η_t, the maximum loss ratio, recall that

$$\eta_t = \frac{E_{n,t}\left(u'\left(W_{t+1}\right)\right)}{E_t(u'(W_{t+1}))} \tag{65}$$

The expectation in the numerator is

$$\begin{aligned} E_{n,t}\left(u'\left(W_{t+1}\right)\right) &= \int_{-\infty}^{\infty} aExp\left(-aW_{t+1}\right)\frac{1}{\sqrt{2\pi}}Exp\left(-\frac{1}{2}y^2\right)dy \\ y &= \frac{x_m - \mu_m + \frac{\sigma_{xi,xm}}{\sigma_{xi}}\left(\mu_{xi} - \lambda_i\right)}{\sqrt{\sigma_{x_m}^2\left(1 - \rho_{x_m,x_i}^2\right)}} \end{aligned} \tag{66}$$

The investors budget constraint may be expressed as

$$W_{t+1} = W_t\left(1 - \alpha_t^*\right) + W_t\frac{\alpha_t^*}{P_{m,t}}x_m. \tag{67}$$

After some algebraic simplification, expected marginal utility may be expressed as

$$E_{n,t}\left(u'\left(W_{t+1}\right)\right) = aExp\left(-a\tilde{W}_{t+1}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}Exp\left(-\frac{1}{2}z^2\right)dy \tag{68}$$

$$\tilde{W}_{t+1} = W_t - N_{m,t}P_{m,t} + N_{m,t}\left(\mu_m - \frac{\sigma_{xm,xi}}{\sigma_{xi}^2}\left(\mu_{x_i} - \lambda_i\right)\right) - \frac{1}{2}aN_{m,t}^2\sigma_{m|x}^2$$

$$z = y + aW_t \frac{\alpha_t^*}{P_{m,t}} \sigma_{m|x}^2$$

where $\sigma_{m|x}^2$ is the conditional variance of the market payout given the asset payout $\sigma_{m|x}^2 = \sigma_m^2 \left(1 - \rho_{xm,xi}\right)$. The argument under the integral is that of a standard normal random variable, so the integral evaluates to one. The expected marginal utility is then

$$E_{n,t}\left(u'\left(W_{t+1}\right)\right) = aExp\left(a\tilde{W}_{t+1}\right) \tag{69}$$

The marginal utility in the denominator of the maximum loss ratio is

$$E\left(u'\left(W_{t+1}\right)\right) = \int_{-\infty}^{\infty} aExp\left(-aW_{t+1}\right) \frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(\frac{x_m - \mu_m}{\sigma_m}\right)^2\right) \Phi\left(\frac{E_n\left(x_i|x_m\right) - \lambda_i}{\sigma x_i|x_m}\right) dy \tag{70}$$

$E_n\left(x_i|x_m\right)$ is the linear projection of x_i onto x_m, while $\sigma_{x_i|x_m}$ is the square root of the conditional variance of x_i given x_m. We may combine the budget constraint and the argument in the exponential of the density. Algebraic manipulation yields

$$E\left(u'\left(W_{t+1}\right)\right) = \varphi aExp\left(-\left(1 + \alpha_t^*\left(\frac{\mu_{x_m}}{P_{m,t}} - 1\right) - \frac{1}{2}\left(aW_t \frac{\alpha_t^*}{P_{m,t}} \sigma_m\right)^2\right)\right) \tag{71}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_m^2}} Exp\left(-\frac{1}{2}\left(\frac{x_m - \mu_m + W_t \frac{\alpha_t^*}{P_{m,t}} \sigma_m^2}{\sigma_m}\right)^2\right) \Phi\left(\frac{E_n\left(x_i|x_m\right) - \lambda_i}{\sigma x_i|x_m}\right) dy \tag{72}$$

The integral may be evaluated using the fact that if y is normally distributed, $E\left(\Phi\left(by + c\right)\right) = \Phi\left(\frac{c}{\sqrt{1+b^2}}\right)$ (Zachs (1981)). Subsituting the evaluated marginal utility into the pricing relation, and simplifying gives the price of the equity claim

$$\begin{aligned} P_{i,t} &= P_{i,t}^{UN} + \frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(P_{i,t}^{UN}/\sigma_{Ai}\right)^2\right) \Pr\left(A_{t+1} > D_t + \sigma_{Ai,Am} N_{m,t} a\right)^{-1}, \quad (73) \\ P_{i,t}^{UN} &= P_{Ai,t} - D. \end{aligned}$$

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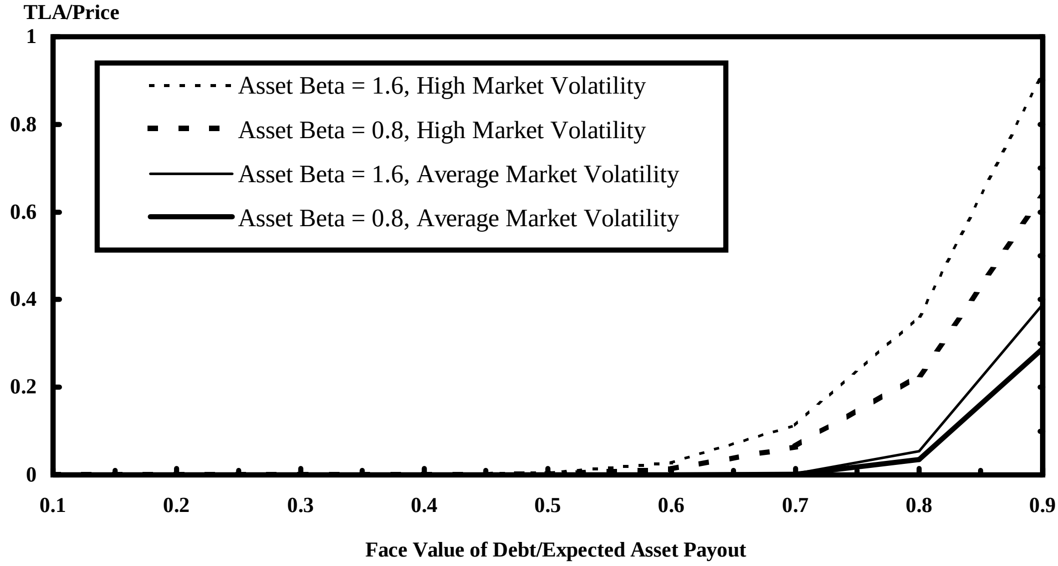
Table 1 Pricing Implications of Limited Liability

This table presents numerical results from the pricing relation described in Section 3.1, Equation 11. Market standard deviation is calculated using the returns to a quarterly index of aggregate US equity and debt. The index is constructed with Federal Reserve Flow of Funds data; the equity index is series FL103164003, and includes corporate farm equity. The debt index is series FL104190005, and includes nonfarm, non-financial liabilities. Market return is ARA*(Market Variance). ARA is the absolute risk aversion coefficient implied by a relative risk aversion coefficient of 10, when wealth equals one. High market volatility is the market standard deviation plus two times the five-year rolling average of the market standard deviation. Total asset volatility equals the volatility for a firm with an asset systematic risk, $\frac{\sigma_{i,m,t}}{\sigma^2_{m,t}}$, of two, and no idiosyncratic risk. E(r) is the expected equity return, and is calculated using the pricing relation (Equation 11). The expected equity payout is calculated using the mean of a truncated normal (Appendix 1.3 provides the equation.) Firm Beta is systematic risk. TLA/Price is the ratio of the truncated loss adjustment to the price of the equity claim. D/E is the debt/equity ratio. Debt is calculated by subtracting the equity price from the asset price, assuming the expected asset cash flow is one.

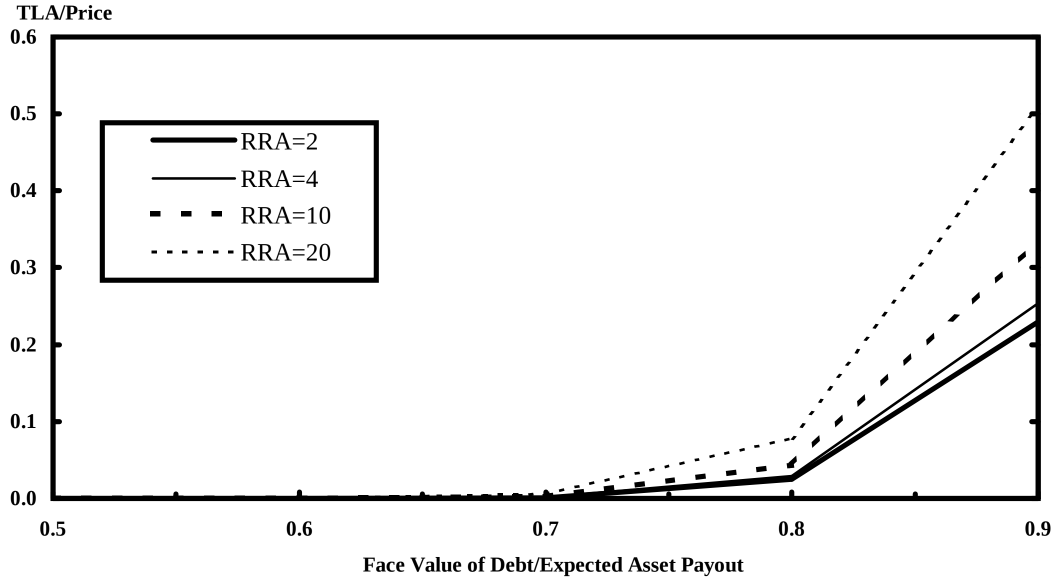
		Firm Beta	E(r)	TLA/Price	D/E
Panel A: Average Market Volatility, High Leverage					
Market Standard Dev.	0.048	0.4	0.048	0.246	7.23
Market Return	0.023	0.8	0.097	0.289	7.53
RRA Coef.	10	1.2	0.147	0.336	7.84
Debt/E(Asset)	0.9	1.6	0.119	0.388	8.16
Pr(A<D)	0.149	2	0.251	0.446	8.48

Table 1 continued

		Firm Beta	E(r)	TLA/Price	D/E
Panel B: High Market Volatility, Moderate Leverage					
Market Standard Dev.	0.078	0.4	0.049	0.001	1.05
Market Return	0.061	0.8	0.101	0.002	1.10
RRA Coef.	10	1.2	0.155	0.003	1.15
Debt/E(Asset)	0.5	1.6	0.211	0.005	1.21
Pr(A<D)	0.001	2	0.270	0.007	1.26
Panel C: High Market Volatility, High Leverage					
Market Standard Dev	0.078	0.4	0.077	0.513	5.23
Market Return	0.061	0.8	0.154	0.634	5.53
RRA Coef.	10	1.2	0.232	0.767	5.81
Debt/E(Asset)	0.9	1.6	0.310	0.913	6.08
Pr(A<D)	0.261	2	0.388	1.071	6.34



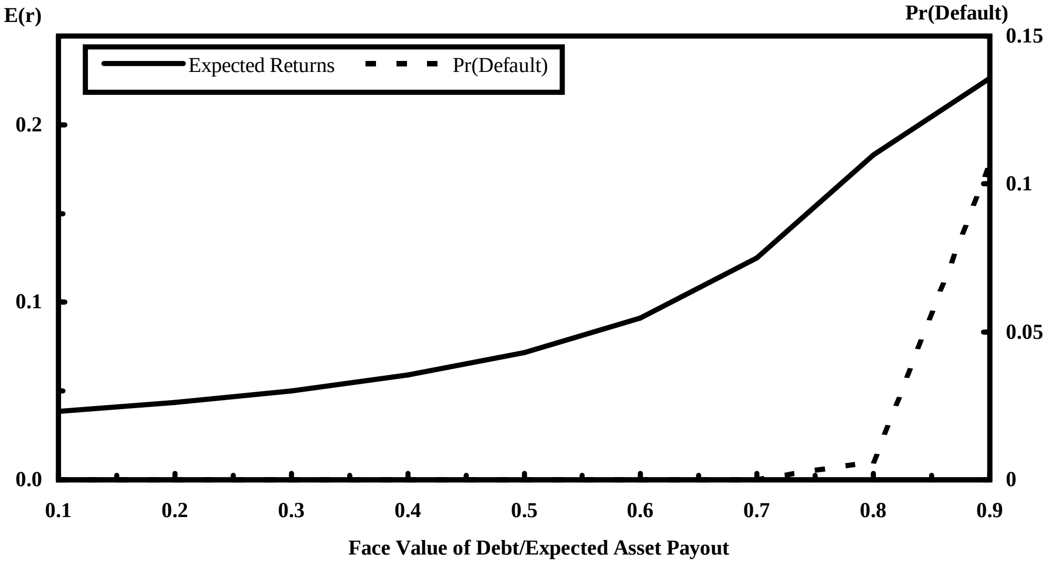
Differing Leverage Ratios and Asset Betas



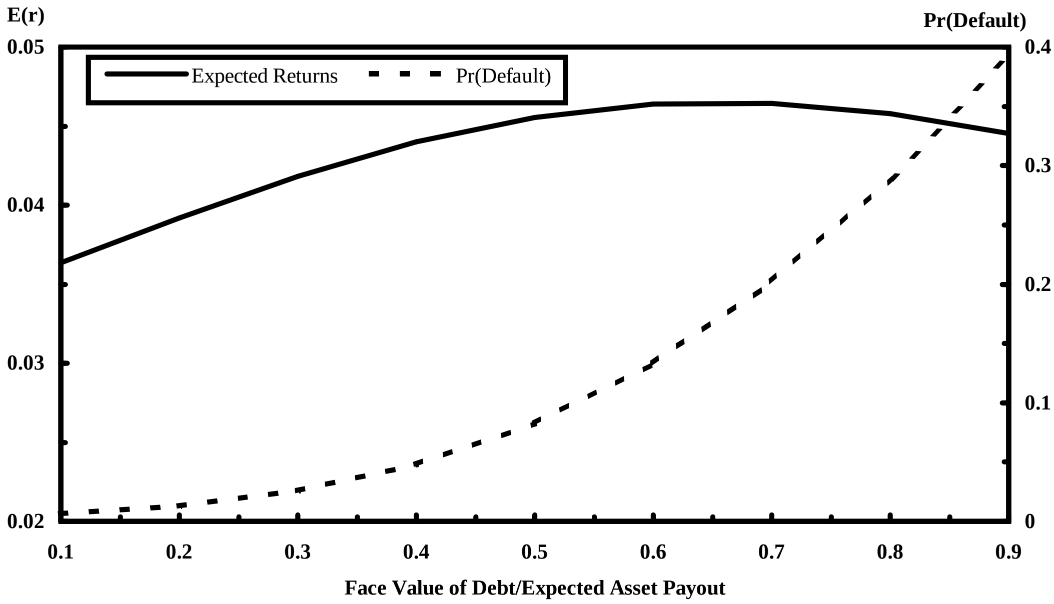
Differing Leverage Ratios and Risk Aversion

Figure 1 Truncated Loss Adjustment

This figure graphs the ratio of the truncated loss adjustment to the equity price using the pricing relation described in Section 3.1, Equation 11. TLA/Price is the ratio of the truncated loss adjustment to the price of the equity claim. The notes for Table 1 describe the method of calculating the ratio, and the definition of high market volatility.



Low Idiosyncratic Risk



High Idiosyncratic Risk

Figure 2 Expected Equity Returns and Financial Leverage

This figure graphs the expected equity returns using the pricing relation described in Section 3.1, Equation 11. The firm asset systematic risk equals 1.5. Firm idiosyncratic risk is calculated using the correlation between firm asset payouts and the market. High idiosyncratic risk corresponds to an asset correlation of 0.2, while low idiosyncratic risk corresponds to an asset correlation of 0.9. E(r) is the expected equity return; Pr(Default) is the probability of default.