Bond Valuation for log-normal interest rate processes using a conditioning factor approach

Sankarshan Basu[1]

Abstract

This paper looks at a different approach to pricing of bonds. The underlying interest rate process is taken to be log-normal, to ensure positivity (Rogers, 1995). However, the approach is different from the usual bond pricing techniques in the sense that the pricing is not based on the solution to a set of partial differential equations, but based on a more analytical approach with only one numerical integration being involved at the end.

The first part of the paper deals with the methods to calculate the bounds for the prices of zero coupon bonds; note that instead of finding the exact prices, we get the bounds to the prices and if for a large number of cases, the bounds are identical implying that the bounds and the price match exactly. Thus, the lower bounds obtained are so accurate that they are essentially the true prices. The method can also be easily extended to value contingent payments on the interest rates. The approach here is by using a conditioning factor similar to the ones used by Rogers and Shi (1995) and Basu (1999). The simple bond pricing using conditioning factors is compared with an alternative method - that of using a direct expansion.

The instantaneous rate of interest r_s (interest rate process) is defined by

$$r_s = be^{X_s}, \quad \text{and} \quad X_s = \mu_s + Y_s$$

where b is a scaling constant, $\{Y_s; 0 \leq s \leq T\}$ is a Gaussian process with zero mean and μ_s is the drift of X_s. The price of a zero coupon bond is given by

$$E(e^{-b\int_0^1 e^{X_s} ds}), \tag{1}$$

where X_s is as defined earlier. The exponential nature of the model ensures that interest rates do not go negative since negative interest rates are unrealistic and could lead to undesirable consequences, as outlined by Rogers (1995). This can be put in the framework of Heath, Jarrow and Morton (1992) and is also an extension of Black and Karasinski (1991) and Black, Derman and Toy (1990).

[1]Indian Institute of Management, Bangalore, India. Email: sankarshanb@iimb.ernet.in

The approach is then extended to a few other bond pricing scenarios :

- **Pricing of non-defaultable (sovereign) bonds:** Here, we want to calculate,

$$E\left[C\int_0^T e^{-\int_0^s r_u du}ds + e^{-\int_0^T r_u du}\right] = E\left[C\int_0^T e^{-\int_0^s r_u du}ds\right] + E\left[e^{-\int_0^T r_u du}\right], \qquad (2)$$

  where, the first term $E\left[C\int_0^T e^{-\int_0^s r_u du}ds\right]$ is the value of the coupon and the second term $E\left[e^{-\int_0^T r_u du}\right]$ is the value of the principal.

- **Pricing of defaultable (non-sovereign or corporate) bonds:** We are interested in calculating

$$E\left[D\int_0^T (e^{-rs-\int_0^s \lambda_u du}\lambda_s)ds + (e^{-r}e^{-b\int_0^T \lambda_s ds}) + C\int_0^T (\int_0^s (e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds)\right.$$
$$\left. + C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right], \qquad (3)$$

$$\text{where} \qquad \lambda_t = be^{\sigma Y_t} \qquad \text{and} \qquad Y_t = \int_0^t e^{-a(t-s)}dB_s.$$

  Here, λ_t is the rate of default and Y_t is a non-stationary Ornstein - Uhlenbeck process. r is the interest rate which is assumed to be constant, σ is the instantaneous variance. Further, D is the percentage paid out in case default occurs, C is the rate of coupon payments during the life of the bond and b is a scaling factor, representing the discount rate. The terms in equation (3) represent the following:

  $E\left[D\int_0^T e^{-rs-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Payment at default.

  $E\left[e^{-r}e^{-b\int_0^T \lambda_s ds}\right]$ = Final payment on maturity, when no default takes place.

  $E\left[C\int_0^T (\int_0^s e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Coupon payments in case of default.

  $E\left[C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right]$ = Coupon payments in case no default occurs.

- **Bonds with multiple drivers - here we further discuss two cases:** Here, in general, r_t is governed by n stochastic processes - say $\{Y_t^{(i)}, 0 \leq t \leq 1, i = 1, 2, ..., n\}$. Further, the stochastic processes $\{Y_t^{(i)}, 0 \leq t \leq 1\}$ and $\{Y_t^{(j)}, 0 \leq t \leq 1\}$ could be correlated amongst themselves with a correlation coefficient ρ. Now, we can have two situations.

  1. **Interest rate process log-normally distributed:** This is the first case - here r_t is just the sum of the stochastic processes. That is, we have

$$r_t = exp(\sum_{i=1}^n \beta_i Y_t^{(i)}), \qquad (4)$$

where $\beta_i, i = 1, 2, ..., n$ is a constant. In this situation, r_t is still a log-normal process and hence cannot go negative.

2. **Interest rate process not log-normally distributed but necessarily giving a positive output:** A slightly different model which we will look at is when r_t is based on n drivers directly. This is given by

$$r_t = \sum_{i=1}^{n} \gamma_i e^{\beta_i Y_t^{(i)}}, \tag{5}$$

where γ_i and β_i are constants for $i = 1, 2, ..., n$. Here r_t is a sum of n log-normal processes.

In all these cases, we shall make use of the conditioning factor approach as discussed by Basu (1999) and Rogers and Shi (1995) to value them.

References

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