Measuring CDS rate with copula-dependent default intensity

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Abstract

In this paper, we consider a correlation between the default intensities to incorporate dependency between multivariate Cox process. Assuming that each obligor has its own default intensity process, we use multivariate shot noise intensity process where jumps (i.e. magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are correlated. A homogeneous Poisson process is used to describe simultaneous jumps in default intensities and the Farlie-Gumbel-Morgenstern (FGM) copulas are used to produce correlations between jump sizes. Using bivariate Cox process with exponential margins for FGM copulas, we derive joint survival/default probabilities, conditional default probabilities and linear default correlations. As an example of pricing of credit derivatives, we calculate the market credit default swaps (CDS) rates, assuming that a zero-coupon default-free bond price follows a generalised Cox-Ingersoll-Ross (CIR) model. Standard martingale theory is used to derive the joint Laplace transforms.

Keywords: Multivariate shot noise process; multivariate Cox process; joint survival/default probability; the Farlie-Gumbel-Morgenstern copulas; conditional default probabilities; linear default correlation; market credit default swaps (CDS) rate.

1. Introduction

Over the recent years, numerous papers have looked at the modelling for dependence of default intensities via the Cox process or a point process (Schönbucher and Schubert 2001; Jouanin et al. 2001; Yu 2005 and Giesecke 2005). Besides the construction of a point process, considerable attraction is given to the use of copulas to measure default dependence between the obligors (Li 2000; Schönbucher and Schubert 2001; Jouanin et al. 2001 and Giesecke 2004). The possiblilty to incorporate default dependence between multiple firms is to introduce correlation between thier intensity processes. The work by Duffie and Singleton (1998) considered joint jumps in the default intensity. Kijima (2000) and Jarrow & Yu (2001) developed it further considering the possibility of default-event triggers that cause joint default. Another approach is based on credit contagion which are from the previous research by Davis and Lo (2000, 2001). In practice, once one firm defaults, it causes increase of other firms default intensity accordingly due to business links or ties between firms. A couple of works for default intensities based on credit contagion can be found in Schönbucher (2003) Giesecke (2004) and Giesecke and Weber (2006).

This paper is mainly based on the first approach. We introduce separate intensity processes for each obligor. These intensities are triggered by primary events such as oil and commodity prices, the governments' fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the romours of mergers and acquisitions among firms, collapse and bankcrupcy of firms, September 11 WTC catastrophe and Hurricane Katrina etc. that will result in a positive jump *simultaneously* in intensity processes. As time passes, default intensity processes decrease respectively, as all firms in the market will do their best to avoid being in bankruptcy after the arrival of a primary event. This decrease continues until another event occurs which again will result in a positive jump simultaneously in intensity processes.

For the purpose of this paper, we concentrate on a very specific vector of intensity process, i.e multivariate shot noise process and show how within this model explicit calculation for market credit default swaps (CDS) rate can be performed. The multivariate default intensity model we consider has the following structure:

$$d\lambda_t^{(1)} = -\delta^{(1)}\lambda_t^{(1)}dt + dC_t^{(1)}, \quad C_t^{(1)} = \sum_{j=1}^{M_t} X_j^{(1)},$$

$$d\lambda_t^{(2)} = -\delta^{(2)}\lambda_t^{(2)}dt + dC_t^{(2)}, \quad C_t^{(2)} = \sum_{j=1}^{M_t} X_j^{(2)},$$

$$\vdots$$

$$d\lambda_t^{(n)} = -\delta^{(n)}\lambda_t^{(n)}dt + dC_t^{(n)}, \quad C_t^{(n)} = \sum_{j=1}^{M_t} X_j^{(n)}, \tag{1.1}$$

where:

· $\left\{X_j^{(i)}\right\}_{j=1,2,\cdots}$ is a sequence of independent and identically distributed random variables with distribution function $F(x)$ $(x > 0)$ and $i = 1, 2, \cdots, n$.

· M_t is the total number of events up to time t.

· $\delta^{(i)}$ is the rate of exponential decay for firm $i = 1, 2, \cdots, n$.

We also make the additional assumption that the point process M_t and the sequences $\left\{X_j^{(i)}\right\}_{j=1,2,\cdots}$ are independent of each other.

In this model, dependence between the intensities $\lambda_t^{(i)}$ comes from the common event arrival process M_t, together with dependence between the jumps $X_j^{(i)}$. The latter is modelled using the notion of copula (Nelson, 1998 and McNeil et al., 2005), i.e. the joint distribution of the vector $\left(X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)}\right)$ is assumed to be of the form $C(F_1, F_2, \cdots, F_n)$ with a given copula C. The uniqueness of this two stage construction goes back to Sklar's Theorem (Sklar, 1996).

We assume that event arrival process M_t follows a homogeneous Poisson process with frequency ρ. The multivariate Cox process (Cox 1955; Grandell, 1976 and Brémaud 1981) is used to model the joint default time. Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one used by Dassios and Jang (2003).

Definition 1.1 *Let (Ω, F, P) be a probability space with information structure given by $F = \{\Im_t, t \in [0, T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \quad almost\ surely\ \ (no\ explosions).$$

If for all $0 \le t_1 \le t_2$ and $u \in \Re$

$$\mathbb{E}\left\{e^{iu\left(N_{t_2}-N_{t_1}\right)}|\Im_{t_2}^\lambda\right\} = \exp\left\{\left(e^{iu}-1\right)\int_{t_1}^{t_2}\lambda_s ds\right\} \tag{1.2}$$

then N_t is call a $\Im_t$-doubly stochastic Poisson process with intensity λ_t where $\Im_t^\lambda = \sigma\{\lambda_s;\ s \le t\}$.

With the above model specification with $n = 2$, in Section 2 we derive the joint Laplace transform of the vector $\left(\Lambda_t^{(1)}, \Lambda_t^{(2)}\right)$ using standard martingale theory, i.e.

$$\mathbb{E}\left(e^{-\gamma\Lambda_t^{(1)}}e^{-\xi\Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right) \tag{1.3}$$

where $\gamma \geq 0$, $\xi \geq 0$ and $\Lambda_t^{(i)} = \int_0^t \lambda_s^{(i)} ds$. From the latter, we can then calculate the joint survival probability, i.e.

$$\Pr\left(\tau_1 > t, \ \tau_2 > t \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right) = \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}e^{-\Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right\}. \tag{1.4}$$

and relevant joint probabilities like:

$$\Pr\left(\tau_1 > t, \ \tau_2 \leq t \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right) = \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\left(1 - e^{-\Lambda_t^{(2)}}\right) \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right\}, \tag{1.5}$$

$$\Pr\left(\tau_1 \leq t, \ \tau_2 > t \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right) = \mathbb{E}\left\{\left(1 - e^{-\Lambda_t^{(1)}}\right)e^{-\Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right\} \tag{1.6}$$

and the joint default probability, i.e.

$$\Pr\left(\tau_1 \leq t, \ \tau_2 \leq t \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right) = \mathbb{E}\left\{\left(1 - e^{-\Lambda_t^{(1)}}\right)\left(1 - e^{-\Lambda_t^{(2)}}\right) \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right\}, \tag{1.7}$$

where $\tau^{(i)} \equiv \inf\left\{t: \ N_t^{(i)} = 1 \mid N_0^{(i)} = 0\right\}$ is the default arrival time for the firm i that is equivalent to the first jump time of the Cox process $N_t^{(i)}$.

As a specific example for C, we use the Farlie-Gumbel-Morgenstern copulas, which are given by

$$C(u, v) = uv + \theta uv(1 - u)(1 - v), \tag{1.8}$$

where $u \in [0, 1]$, $v \in [0, 1]$ and $\theta \in [-1, 1]$. In order to make latter calculation somewhat easier, we also assume that $F(x_1) = 1 - e^{-\alpha x_1}$ ($\alpha > 0$, $x_1 > 0$) and $F(x_2) = 1 - e^{-\beta x_2}$ ($\beta > 0$, $x_2 > 0$). The resulting joint distribution function $F(x_1, x_2)$ takes the form:

$$F(x_1, x_2) = 1 - e^{-\beta x_2} - e^{-\alpha x_1} + e^{-\alpha x_1 - \beta x_2} + \theta e^{-\alpha x_1 - \beta x_2} - \theta e^{-\alpha x_1 - 2\beta x_2} - \theta e^{-2\alpha x_1 - \beta x_2} + \theta e^{-2\alpha x_1 - 2\beta x_2}. \tag{1.9}$$

In Section 3, we illustrate the calculations of market credit default swaps (CDS) rate assuming that a zero-coupon default-free bond price follows a generalised Cox-Ingersoll-Ross (CIR) model. Section 4 contains some concluding remarks.

2. Joint Laplace transform of the vector $\left(\Lambda_t^{(1)}, \Lambda_t^{(2)}\right)$ and joint survival probability

In order to calculate the joint survival/default probability and relevant joint probabilities, we firstly consider using the joint Laplace transform of the vector $\left(\Lambda_t^{(1)}, \Lambda_t^{(2)}\right)$. Once its expression is derived we can easily calculate them by setting $\gamma = 1$ and $\xi = 1$ in the equation (1.3).

Using standard martingale theory, the joint Laplace transform of the vector $\left(\Lambda_t^{(1)}, \Lambda_t^{(2)}, \lambda_t^{(1)}, \lambda_t^{(2)}\right)$ is given by

$$\mathbb{E}\left\{e^{-\gamma\left(\Lambda_{t_2}^{(1)}-\Lambda_{t_1}^{(1)}\right)}e^{-\xi\left(\Lambda_{t_2}^{(2)}-\Lambda_{t_1}^{(2)}\right)}e^{-\kappa\lambda_{t_2}^{(1)}}e^{-\psi\lambda_{t_2}^{(2)}}\mid \lambda_{t_1}^{(1)},\lambda_{t_1}^{(2)}\right\}$$

$$=\ \exp\left[-\left\{\frac{\gamma}{\delta^{(1)}}+\left(\kappa-\frac{\gamma}{\delta^{(1)}}\right)e^{-\delta^{(1)}(t_2-t_1)}\right\}\lambda_{t_1}\right]$$

$$\times\exp\left[-\left\{\frac{\xi}{\delta^{(2)}}+\left(\psi-\frac{\xi}{\delta^{(2)}}\right)e^{-\delta^{(2)}(t_2-t_1)}\right\}\lambda_{t_2}\right]$$

$$\times\exp\left[-\rho\int_{t_1}^{t_2}\left[1-\hat{c}\left\{\frac{\gamma}{\delta^{(1)}}+\left(\kappa-\frac{\gamma}{\delta^{(1)}}\right)e^{-\delta^{(1)}s},\ \frac{\xi}{\delta^{(2)}}+\left(\psi-\frac{\xi}{\delta^{(2)}}\right)e^{-\delta^{(2)}s}\right\}\right]ds\right],\ (2.1)$$

where $t_2>t_1$, $\kappa\geq0$, $\psi\geq0$ and $\hat{c}(\zeta,\varphi)=\int_0^\infty\int_0^\infty e^{-\zeta x_1}e^{-\varphi x_2}dC(F(x_1),F(x_2))$. If we set $\kappa=\psi=0$ in (2.1), then the joint Laplace transform of the vector $\left(\Lambda_t^{(1)},\ \Lambda_t^{(2)}\right)$ is given by

$$\mathbb{E}\left\{e^{-\gamma\left(\Lambda_{t_2}^{(1)}-\Lambda_{t_1}^{(1)}\right)}e^{-\xi\left(\Lambda_{t_2}^{(2)}-\Lambda_{t_1}^{(2)}\right)}\mid \lambda_{t_1}^{(1)},\lambda_{t_1}^{(2)}\right\}$$

$$=\ \exp\left[-\frac{\gamma}{\delta^{(1)}}\left\{1-e^{-\delta^{(1)}(t_2-t_1)}\right\}\lambda_{t_1}\right]\times\exp\left[-\frac{\xi}{\delta^{(2)}}\left\{1-e^{-\delta^{(2)}(t_2-t_1)}\right\}\lambda_{t_2}\right]$$

$$\times\exp\left[-\rho\int_{t_1}^{t_2}\left[1-\hat{c}\left\{\frac{\gamma}{\delta^{(1)}}\left(1-e^{-\delta^{(1)}s}\right),\ \frac{\xi}{\delta^{(2)}}\left(1-e^{-\delta^{(2)}s}\right)\right\}\right]ds\right].\qquad(2.2)$$

If we use (1.9) as the joint distribution of the vector $\left(X_j^{(1)},\ X_j^{(2)}\right)$, the joint Laplace transform of the vector $\left(\Lambda_t^{(1)},\ \Lambda_t^{(2)}\right)$ is given by

$$\mathbb{E}\left\{e^{-\gamma\left(\Lambda_{t_2}^{(1)}-\Lambda_{t_1}^{(1)}\right)}e^{-\xi\left(\Lambda_{t_2}^{(2)}-\Lambda_{t_1}^{(2)}\right)}\mid \lambda_{t_1}^{(1)},\lambda_{t_1}^{(2)}\right\}$$

$$=\ \exp\left[-\frac{\gamma}{\delta^{(1)}}\left\{1-e^{-\delta^{(1)}(t_2-t_1)}\right\}\lambda_{t_1}^{(1)}\right]\times\exp\left[-\frac{\xi}{\delta^{(2)}}\left\{1-e^{-\delta^{(2)}(t_2-t_1)}\right\}\lambda_{t_1}^{(2)}\right]$$

$$\times\exp\left[-\rho\int_{t_1}^{t_2}\left[\frac{\{A(s,\xi)+B(s,\gamma)+C(s,\gamma,\xi)\}\,D(s,\gamma)F(s,\xi)-G(s,\gamma,\xi)}{H(s,\gamma)I(s,\xi)D(s,\gamma)F(s,\xi)}\right]ds\right],\quad(2.3)$$

where

$$A(s,\xi)=\frac{\alpha\xi\left(1-e^{-\delta^{(2)}s}\right)}{\delta^{(2)}},\ \ B(s,\gamma)=\frac{\beta\gamma\left(1-e^{-\delta^{(1)}s}\right)}{\delta^{(1)}},$$

$$C(s,\gamma,\xi)=\frac{\gamma\xi\left(1-e^{-\delta^{(1)}s}\right)\left(1-e^{-\delta^{(2)}s}\right)}{\delta^{(1)}\delta^{(2)}},$$

$$D(s,\gamma)=2\alpha+\frac{\gamma\left(1-e^{-\delta^{(1)}s}\right)}{\delta^{(1)}},\ \ F(s,\xi)=2\beta+\frac{\xi\left(1-e^{-\delta^{(2)}s}\right)}{\delta^{(2)}},$$

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$$G(s, \gamma, \xi) = \frac{\theta \alpha \beta \gamma \xi \left(1 - e^{-\delta^{(1)}s}\right)\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(1)}\delta^{(2)}},$$

$$H(s, \gamma) = \alpha + \frac{\gamma\left(1 - e^{-\delta^{(1)}s}\right)}{\delta^{(1)}}, \ \ I(s, \xi) = \beta + \frac{\xi\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(2)}}.$$

Assuming that $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ are stationary, it is given by

$$\mathbb{E}\left\{e^{-\gamma\left(\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)}\right)} e^{-\xi\left(\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)}\right)}\right\}$$

$$= \left\{\frac{\alpha}{\alpha + \frac{\gamma}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}(t_2 - t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(1)}}} \left\{\frac{\beta}{\beta + \frac{\xi}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}(t_2 - t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(2)}}}$$

$$\times \exp\left[-\rho \int_{t_1}^{t_2} \left[\frac{\{A(s, \xi) + B(s, \gamma) + C(s, \gamma, \xi)\} D(s, \gamma)F(s, \xi) - G(s, \gamma, \xi)}{H(s, \gamma)I(s, \xi)D(s, \gamma)F(s, \xi)}\right] ds\right] \quad (2.4)$$

and if we set $\gamma = \xi = 1$, we can easily obtain the joint survival probability, i.e.

$$\mathbb{E}\left\{e^{-\left(\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)}\right)} e^{-\left(\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)}\right)}\right\}$$

$$= \left\{\frac{\alpha}{\alpha + \frac{1}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}(t_2 - t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(1)}}} \left\{\frac{\beta}{\beta + \frac{1}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}(t_2 - t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(2)}}}$$

$$\times \exp\left[-\rho \int_{t_1}^{t_2} \left[\frac{\{A(s) + B(s) + C(s)\} D(s)F(s) - G(s)}{H(s)I(s)D(s)F(s)}\right] ds\right], \quad (2.5)$$

where

$$A(s) = \frac{\alpha\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(2)}}, \ \ B(s) = \frac{\beta\left(1 - e^{-\delta^{(1)}s}\right)}{\delta^{(1)}},$$

$$C(s) = \frac{\left(1 - e^{-\delta^{(1)}s}\right)\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(1)}\delta^{(2)}},$$

$$D(s) = 2\alpha + \frac{\left(1 - e^{-\delta^{(1)}s}\right)}{\delta^{(1)}}, \ \ F(s) = 2\beta + \frac{\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(2)}},$$

$$G(s) = \frac{\theta\alpha\beta\left(1 - e^{-\delta^{(1)}s}\right)\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(1)}\delta^{(2)}},$$

$$H(s) = \alpha + \frac{\left(1 - e^{-\delta^{(1)}s}\right)}{\delta^{(1)}}, \ \ I(s) = \beta + \frac{\left(1 - e^{-\delta^{(2)}s}\right)}{\delta^{(2)}}.$$

If we set $\theta = 0$ in (2.4), i.e. jump sizes in default intensities $\left\{X_j^{(1)}\right\}_{j=1,2,\cdots}$ and $\left\{X_j^{(2)}\right\}_{j=1,2,\cdots}$ are independent each other, the joint Laplace transform of the vector $\left(\Lambda_t^{(1)}, \Lambda_t^{(2)}\right)$ is given by

$$\mathbb{E}\left\{e^{-\gamma\left(\Lambda_{t_2}^{(1)}-\Lambda_{t_1}^{(1)}\right)}e^{-\xi\left(\Lambda_{t_2}^{(2)}-\Lambda_{t_1}^{(2)}\right)}\right\}$$

$$= \left\{\frac{\alpha}{\alpha + \frac{\gamma}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(1)}}} \times \left\{\frac{\beta}{\beta + \frac{\xi}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(2)}}}$$

$$\times \exp\left[-\rho\int_{t_1}^{t_2}\left[\frac{\frac{\alpha\xi}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}s}\right) + \frac{\beta\gamma}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}s}\right) + \frac{\gamma\xi}{\delta^{(1)}\delta^{(2)}}\left(1 - e^{-\delta^{(1)}s}\right)\left(1 - e^{-\delta^{(2)}s}\right)}{\left\{\alpha + \frac{\gamma}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}s}\right)\right\}\left\{\beta + \frac{\xi}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}s}\right)\right\}}\right]ds\right],$$

(2.6)

If we set $\gamma = 1$ and $\xi = 0$ either in (2.4) or in (2.6), the survival probability of firm 1 is given by

$$\mathbb{E}\left\{e^{-\left(\Lambda_{t_2}^{(1)}-\Lambda_{t_1}^{(1)}\right)}\right\}$$

$$= \left\{\frac{\alpha e^{-\delta^{(1)}(t_2-t_1)}}{\alpha + \frac{1}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(1)}}}\left\{\frac{\alpha + \frac{1}{\delta^{(1)}}\left(1 - e^{-\delta^{(1)}(t_2-t_1)}\right)}{\alpha e^{-\delta^{(1)}(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta^{(1)}\alpha+1}}$$

(2.7)

and if we set $\xi = 1$ and $\gamma = 0$ either in (2.4) or in (2.6), the survival probability of firm 2 is given by

$$\mathbb{E}\left\{e^{-\left(\Lambda_{t_2}^{(2)}-\Lambda_{t_1}^{(2)}\right)}\right\}$$

$$= \left\{\frac{\beta e^{-\delta^{(2)}(t_2-t_1)}}{\beta + \frac{1}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta^{(2)}}}\left\{\frac{\beta + \frac{1}{\delta^{(2)}}\left(1 - e^{-\delta^{(2)}(t_2-t_1)}\right)}{\beta e^{-\delta^{(2)}(t_2-t_1)}}\right\}^{\frac{\beta\rho}{\delta^{(2)}\beta+1}},$$

(2.8)

which also can be found in Dassios and Jang (2003) and Jang (2006).

3. Conditional default probabilities and linear default correlation.

Having dervied joint and marginal survival probability in the previous section, we can easily calculate the conditional default probabilities and the linear correlation coefficient between indicator random variables for two firms. Using Bayes' rule, the conditional default probabilities between firm 1 and 2, denoted by $p_{1|2}$ and $p_{2|1}$ are given by

$$p_{1|2} = \frac{p_{12}}{p_2} = \frac{\Pr\left(\tau_1 \leq t,\ \tau_2 \leq t\right)}{\Pr\left(\tau_2 \leq t\right)}$$

$$= \frac{1 - \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}e^{-\Lambda_t^{(2)}}\right\}}{1 - \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\}},$$

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$$
\begin{aligned}
p_{2|1} &= \frac{p_{12}}{p_1} = \frac{\Pr\left(\tau_1 \le t,\ \tau_2 \le t\right)}{\Pr\left(\tau_1 \le t\right)} \\
&= \frac{1 - \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_t^{(1)}} e^{-\Lambda_t^{(2)}}\right\}}{1 - \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\}}
\end{aligned}
$$

and the linear correlation coefficient between indicator random variables for two firms, denoted by $\rho\left\{1_{(\tau_1 \le t)}, 1_{(\tau_2 \le t)}\right\}$, is given by

$$
\begin{aligned}
\rho\left(1_{(\tau_1 \le t)}, 1_{(\tau_2 \le t)}\right) &= \frac{p_{12} - p_1 p_2}{\sqrt{p_1 \left(1 - p_1\right)} \sqrt{p_2 \left(1 - p_2\right)}} \\
&= \frac{\mathbb{E}\left\{e^{-\Lambda_t^{(1)}} e^{-\Lambda_t^{(2)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\} \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\}}{\sqrt{\left[1 - \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\}\right] \mathbb{E}\left\{e^{-\Lambda_t^{(1)}}\right\}} \sqrt{\left[1 - \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\}\right] \mathbb{E}\left\{e^{-\Lambda_t^{(2)}}\right\}}}.
\end{aligned}
$$

Now let us illustrate the calculations of the joint survival/default probabilities and relevant joint probabilities before we calculate the conditional default probabilities and the linear correlation coefficient.

Example 3.1

We assume that the magnitude of the contribution to the default intensity of the firm 1 from the primary events is smaller than that of the firm 2. We also assume that the decay rate for the firm 1, that measures how quick the firm gets out of the influence of primary events lowering their default intensity rate, is higher than that for the firm 2. So the parameter values used to calculate the joint probabilities are

$$
\alpha = 10,\ \beta = 5,\ \delta^{(1)} = 0.5,\ \delta^{(2)} = 0.3 \text{ and } \rho = 4.
$$

Setting $t_1 = 0$ and $t_2 = 1$, joint survival probability is given by

$$
\Pr\left(\tau_1 > 1,\ \tau_2 > 1\right) = \mathbb{E}\left\{e^{-\Lambda_1^{(1)}} e^{-\Lambda_1^{(2)}}\right\}
$$

and relevant joint probabilities are given by

$$
\Pr\left(\tau_1 > 1,\ \tau_2 \le 1\right) = \mathbb{E}\left\{e^{-\Lambda_1^{(1)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_1^{(1)}} e^{-\Lambda_1^{(2)}}\right\},
$$

$$
\Pr\left(\tau_1 \le 1,\ \tau_2 > 1\right) = \mathbb{E}\left\{e^{-\Lambda_1^{(2)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_1^{(1)}} e^{-\Lambda_1^{(2)}}\right\}
$$

and the joint default probability is given by

$$
\Pr\left(\tau_1 \le 1,\ \tau_2 \le 1\right) = 1 - \mathbb{E}\left\{e^{-\Lambda_1^{(1)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_1^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_1^{(1)}} e^{-\Lambda_1^{(2)}}\right\}.
$$

From the equation (2.5), (2.7) and (2.8), the calculations of the joint survival/default probabilities and relevant joint probabilities are shown in Table 3.1, 3.2, 3.3. and 3.4.

<div align="center">Table 3.1.</div>

θ	$\Pr(\tau_1 > 1,\ \tau_2 > 1)$
1	0.040875
0.5	0.040797
0	0.040720
-0.5	0.040643
-1	0.040565

<div align="center">Table 3.2.</div>

θ	$\Pr(\tau_1 > 1,\ \tau_2 \leq 1)$
1	0.42322
0.5	0.42330
0	0.42337
-0.5	0.42345
-1	0.42353

<div align="center">Table 3.3.</div>

θ	$\Pr(\tau_1 \leq 1,\ \tau_2 > 1)$
1	0.045414
0.5	0.045492
0	0.045570
-0.5	0.045647
-1	0.045724

<div align="center">Table 3.4.</div>

θ	$\Pr(\tau_1 \leq 1,\ \tau_2 \leq 1)$
1	0.49049
0.5	0.49041
0	0.49034
-0.5	0.49026
-1	0.49018

where

$$\Pr(\tau_1 > 1) = 0.46409 \quad \text{and} \quad \Pr(\tau_2 > 1) = 0.08629, \tag{3.1}$$

$$\Pr(\tau_1 \leq 1) = 0.53591 \quad \text{and} \quad \Pr(\tau_2 \leq 1) = 0.91371. \tag{3.2}$$

As we can see in (3.1), the survival probability of the firm 2 is very low. Hence the joint probabilities where the survivorship of the firm 2 is concerned are dominated by the survival probability of the firm 2 (see Table 3.1 and 3.3). On the contrary, the default probability of the firm 2 in (3.2) is very high. Hence joint probabilities where the defaultability of the firm 2 is concerned are dominated by the survival or default probability of the firm 1 (see Table 3.2 and 3.4).

Table 3.1 and 3.4 show that joint survival and default probability decrease as the values of copula parameter θ becomes -1 as time to default for each firm moves the same direction. On the other hand, Table 3.2 and 3.3 show that joint probabilities increase as the values of copula parameter θ becomes -1 as time to default for each firm moves the opposite direction. All joint probabilities in the tables above show that they are not very sensitive to the change of the values of the copula parameter θ.

Based on the same parameter values used in Example 3.1, let us illustrate the calculations of the conditional default probabilities and the linear correlation coefficient between indicator random variables for two firms.

Example 3.2

The calculation of the conditional default probabilities are shown in Table 3.5 and Table 3.6, respectively.

<div align="center">Table 3.5.</div>

θ	$p_{1|2}$
1	0.53682
0.5	0.53673
0	0.53665
-0.5	0.53656
-1	0.53648

<div align="center">Table 3.6.</div>

θ	$p_{2|1}$
1	0.91526
0.5	0.91511
0	0.91497
-0.5	0.91482
-1	0.91468

The conditional default probabilities in the tables above are rescaled to the joint default probabilities in Table 3.4 by the default probability of the firm1 and 2 in (3.2), respectively. Hence the default probabilities of the firm 1 (or 2) given that the firm 2 (or 1) defaults are very similar to the unconditional default probability of the firm 1 (or 2) in (3.2).

The calculations of the linear correlations between two firms are shown in Table 3.7.

Table 3.7.

θ	$\rho\left\{1_{(\tau_1\leq t)},1_{(\tau_2\leq t)}\right\}$
1	0.0059177
0.5	0.0053607
0	0.0048108
-0.5	0.0042609
-1	0.0037039

Table 3.7 shows that there exists very weak linear relationship between random variable $1_{(\tau_1\leq t)}$ and $1_{(\tau_2\leq t)}$. The complete specification of the default correlation is given by the joint survival/default probability and relevant joint probabilities in Example 3.1. Therefore, using $\rho\left\{1_{(\tau_1\leq t)},1_{(\tau_2\leq t)}\right\}$ as only dependence measure will misguide and fail us to capture all the dependence structure between two firms as they are covariance-based dependence measure. For details on inadequacy of linear correlation, we refer you McNeil, Frey and Embrechts (2005).

Next example shows that conditional default probabilities up to 1 can be achieved by changing the values of β (or α) and $\delta^{(2)}$ (or $\delta^{(1)}$) in the model specified.

Example 3.3

Using the same parameter values used in Example 3.1, the calculations of conditional default probabilities of $p_{2|1}$ at each value of β and $\delta^{(2)}$ are shown in Table 3.8 and 3.9 with $\theta = 1$ respectively

Table 3.8.   Table 3.9.

β	$p_{2|1}$
10	0.72357
5	0.91526
3	0.97995
1	0.99993
0.1	1.00000

$\delta^{(2)}$	$p_{2|1}$
0.5	0.77552
0.3	0.91526
0.2	0.97489
0.1	0.99935
0.01	1.00000

The bigger β and the higher $\delta^{(2)}$ is, i.e. the bigger the magnitude of the contribution to the default intensity of the firm from the primary events is and the slower the firm gets out of the influence of primary events, the conditional default probabilities are getting closer to 1.

4. **Measuring market credit default swaps (CDS) rate**

In order to calculate the market credit default swaps (CDS) rate, firstly let us assume that the interest rate process for a zero-coupon default-free bond, r_t follows a generalised Cox-Ingersoll-Ross (CIR) model (1985), i.e.

$$dr_t = c(b - ar_t)dt + \sigma\sqrt{r_t}dB_t, \tag{4.1}$$

where $a > 0$, $b > 0$ and $c > 0$. Then its price at time 0, paying 1 at time t is given by

$$B(0,t) = \mathbb{E}\left\{ \exp\left(-\int_0^t r_s ds \right) \mid r_0 \right\} = \mathbb{E}\left(e^{-R_t} \mid r_0 \right), \tag{4.2}$$

where $B(0,t)$ denotes the price of a default-free zero-coupon bond, $R_t = \int_0^t r_s ds$, $\Im_t^r = \sigma\{r_s; \ s \le t\}$. From CIR (1985) with $c = 1$, we can easily obtain the explicit expression for the equation (4.2):

$$
\begin{aligned}
&B(0,t)\\
=\ &\exp\left[-\left\{ \frac{2\left\{1 - \exp\left(-\sqrt{a^2 + 2\sigma^2}\,t\right)\right\}}{\left\{\left(\sqrt{a^2 + 2\sigma^2} + a\right) + \left(\sqrt{a^2 + 2\sigma^2} - a\right)\exp\left(-\sqrt{a^2 + 2\sigma^2}\,t\right)\right\}} \right\} r_0 \right]\\
&\times \left\{ \frac{2\sqrt{a^2 + 2\sigma^2}\exp\left(-\frac{\left(\sqrt{a^2+2\sigma^2}-a\right)}{2}t\right)}{\left(\sqrt{a^2 + 2\sigma^2} + a\right) + \left(\sqrt{a^2 + 2\sigma^2} - a\right)\exp\left(-\sqrt{a^2 + 2\sigma^2}\,t\right)} \right\}^{\frac{2b}{\sigma^2}}.
\end{aligned}
\tag{4.3}
$$

Now let us denote the default intensity process of the CDS buyer and seller by $\lambda_t^{(b)}$ and $\lambda_t^{(s)}$, respectively. We also specify the default intensity prcess of the reference credit by $\lambda_t^{(rc)}$. For simplicity, we assume a deterministic recovery rate π. Then the market credit default swaps (CDS) rate, denoted by $\overline{s}$, is given by

$$\overline{s} = (1 - \pi)\frac{\displaystyle\sum_{k=1}^{k_N} e^{rc,s}(0, t_{k-1}, t_k)}{\displaystyle\sum_{k=1}^{N}(t_{k_{n+1}} - t_{k_n})\overline{B}^b(0, t_{k_n})}, \tag{4.4}$$

where

$$
\begin{aligned}
&e^{rc,s}(0, t_{k-1}, t_k)\\
=\ &\mathbb{E}\left[\begin{array}{l} \exp\left(-\int_0^{t_k} r_s ds\right)\left\{\exp\left(-\int_0^{t_{k-1}} \lambda_s^{(rc)} ds\right) - \exp\left(-\int_0^{t_k} \lambda_s^{(rc)} ds\right)\right\}\\[2mm] \times\left\{\exp\left(-\int_0^{t_k} \lambda_s^{(s)} ds\right)\right\} \mid r_0, \lambda_0^{(rc)}, \lambda_0^{(s)} \end{array} \right],
\end{aligned}
\tag{4.5}
$$

$$\overline{B}^b(0, t_{k_n}) = \mathbb{E}\left[\exp\left\{ -\int_0^{t_{k_n}}\left(r_s + \lambda_s^{(b)}\right)ds \right\} \mid r_0, \lambda_0^{(b)} \right] \tag{4.6}$$

and $t_{k_1} < t_{k_2} < \cdots < t_{k_n}$. Assuming that r_t and $\lambda_t^{(i)}$ are independent each other and that $\lambda_t^{(i)}$ is stationary, the equation (4.5) and (4.6) can be expressed as

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$$e^{rc,s}(0, t_{k-1}, t_k) = B(0, t_k) \times \left[\mathbb{E}\left(e^{-\Lambda_{t_{k-1}}^{(rc)}} e^{-\Lambda_{t_{k-1}}^{(s)}} \right) \mathbb{E}\left\{ \exp\left(-\int_{t_{k-1}}^{t_k} \lambda_s^{(s)} ds \right) \right\} - \mathbb{E}\left(e^{-\Lambda_{t_k}^{(rc)}} e^{-\Lambda_{t_k}^{(s)}} \right) \right]$$

$$= B(0, t_k) \times \left[\mathbb{E}\left(e^{-\Lambda_{t_{k-1}}^{(rc)}} e^{-\Lambda_{t_{k-1}}^{(s)}} \right) \mathbb{E}\left\{ e^{-\left(\Lambda_{t_k}^{(s)} - \Lambda_{t_{k-1}}^{(s)} \right)} \right\} - \mathbb{E}\left(e^{-\Lambda_{t_k}^{(rc)}} e^{-\Lambda_{t_k}^{(s)}} \right) \right] \quad (4.7)$$

and

$$\overline{B}^b(0, t_{k_n}) = B(0, t_{k_n})\mathbb{E}\left(e^{-\Lambda_{t_{k_n}}^{(b)}} \right). \quad (4.8)$$

Using the equation (4.8), we can easily price defaultable bonds as well as credit spread between default-free bond and defaultable bond. For details we refer Jang (2006).

Let us illustrate the calculations of the market credit default swaps (CDS) rates using the expressions derived above.

Example 4.1

The parameter values used to calculate (4.4) are

$$r_0 = 0.05, \ a = 0.05, \ b = 0.025 \text{ and } \sigma = 0.8 \text{ for } r_t$$

and

$$\pi = 50\%, \ N = 2, \ t_{k_0} = 0, \ t_{k_1} = 0.5, \ t_{k_2} = 1.$$

We assume that the default intensity prcesses of CDS buyer and seller follows $\lambda_t^{(1)}$, i.e.

$$\lambda_t^{(1)} = \lambda_t^{(b)} = \lambda_t^{(s)}$$

and that the default intensity prcess of reference credit follows $\lambda_t^{(2)}$, i.e.

$$\lambda_t^{(2)} = \lambda_t^{(rc)}.$$

Using the same parameter values as in Example 3.1 for $\lambda_t^{(i)}$, the calculations of market credit default swaps (CDS) rates are shown in Table 4.1.

Table 4.1.

θ	$\overline{s}$
1	3647.7bp
0.5	3648.4bp
0	3649.1bp
-0.5	3649.7bp
-1	3650.4bp

Assuming that $\theta = 1$, we now examine the effect on market credit default swaps (CDS) rate caused by changes in the value of β and $\delta^{(rc)}$ for the reference credit and by changes in the value of $\alpha^{(s)}$ and $\delta^{(s)}$ for the CDS seller.

Example 4.2

Using the same parameter values used in Example 4.1, the calculations of market credit default swaps (CDS) rates caused by changes in the value of β and $\delta^{(rc)}$ for the reference credit and by changes in the value of $\alpha^{(s)}$ and $\delta^{(s)}$ for the CDS seller are shown in Table 4.2 and Table 4.3, respectively.

Table 4.2. Table 4.3.

	$\overline{s}$
$\beta = 0.1$	4000.1bp
$\beta = 1$	3999.7bp
$\beta = 3$	3914.8bp
$\delta^{(rc)} = 0.01$	4000.1bp
$\delta^{(rc)} = 0.1$	3997.3bp
$\delta^{(rc)} = 0.2$	3895.5bp

	$\overline{s}$
$\alpha^{(s)} = 0.1$	0bp
$\alpha^{(s)} = 0.5$	0.74161bp
$\alpha^{(s)} = 1$	24.42bp
$\delta^{(s)} = 0.01$	0bp
$\delta^{(s)} = 0.1$	172.64bp
$\delta^{(s)} = 0.2$	1161.6bp

Compared to CDS rates in Table 4.1, we can see more clear relationship between CDS rates and the parameter values of default intensity for the reference credit and for the CDS seller in Table 4.2 and 4.3, respectively. In Table 4.2, we can see that CDS rate is converging by lowering the values of β and $\delta^{(rc)}$ for the reference credit, respectively as the CDS seller's default intensity is not as bad as its counterpart for the reference credit. On the contrary, Table 4.3 shows that CDS rate is getting lower to 0 by decreasing the value of $\alpha^{(s)}$ and $\delta^{(s)}$ for the CDS seller. From the CDS buyers' point of view, it is better for them to purchase a CDS contract that the CDS seller is less likely to default. As long as the CDS seller's credit is strong enough, they can hedge against the default risk of the reference credit using a CDS contract. Hence the lower the CDS rate is, the more likely the CDS seller defaults. The worst case scenario for the CDS buyer is when both the reference credit and the CDS seller default.

5. **Conclusion**

Using the intensity-based framework, i.e. employing the bivariate Cox process with bivariate shot noise intensity, we examine how it can be used to calculate the market credit default swaps (CDS) rate. In this model, dependence between the two default intensities comes from the common primary event arrival process N_t, together with dependence between their jump sizes. The latter is modelled using the notion of copula. To be specific, we use a homogeneous Poisson process to describe simultaneous jumps in default intensities and the Farlie-Gumbel-Morgenstern copula with exponential margins for dependent jump sizes. Based on this copula-dependent default intensities of bivariate Cox process, we calculate joint survival/default probabilities, conditional default probabilities, linear default correlations and market CDS rates. We also examine the relationship between CDS rates and the parameter values of default intensity for the reference credit and for the CDS seller.

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