A SIMPLE MODEL OF THE NOMINAL TERM STRUCTURE OF INTEREST RATES

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THIS VERSION: JANUARY 2007

ABSTRACT. This paper proposes a stylized two-factor model of the nominal term structure of interest rates, in which the log-price kernel has an autoregressive drift process and a nonlinear GARCH volatility process. With these two state variable processes, we derive closed form solutions for the zero-coupon bond prices as well as the yield-to-maturity for a given time to maturity.

1. INTRODUCTION

This paper proposes a stylized two-factor model of the term structure of interest rates with the logarithm of the nominal discount factor (plus its long-term mean) and its conditional variance being the state variables.[1] Motivated by the empirical evidence assembled by prior research on interest rate processes in the literature,[2] we model the dynamics of the state variables in the following way. The first state variable, which is the logarithm of the nominal discount factor, follows a first order autoregressive, AR(1), process. The second state variable, which is the conditional variance of the first state variable, has a nonlinear asymmetric generalized autoregressive conditional heteroskedasticity (NGARCH) process. With the above two state variable processes, we derive the closed form solutions for the zero-coupon bond price as well as the yield-to-maturity for a given time to maturity. The resulting yield to maturity is affine in the two state variables. Equivalently, the yield is a function of the spread of long-term rate and short rate, the difference between the conditional variance and its long-term mean, as well as time to maturity. An alternative representation of yield using another yield and its conditional volatility is also derived in this paper. Finally, a simple calibration

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[1] As a two-factor model, it provides two state variables in the squared-autoregressive-independent-variable nominal term structure (SAINTS) similar to the model proposed by Constantinides (1992).

[2] A partial list of the empirical studies includes Brennan and Schwartz (1982), Grossman, Melinio, and Shiller (1987), Longstaff (1989, 1990), and Chan, Karolyi, Longstaff, and Sanders (1992). See Chapman and Pearson (2001) for a critical survey on the empirical term structure literature.

exercise is performed on the model to show that the proposed model is capable of producing different shapes of yield curves and its volatility curves can be very different from those of the simple AR(1) model.

The remainder of the paper is organized as follows. In Section 2 we review the GARCH process, its continuous-time version, and a rearranged discrete-time version. The prices or yields of default-free bonds are derived and the functional dependences of the short-term rate and yields on the logarithm of the discount factor are illustrated in Section 3. The importance of the selection of moments for volatility curves as well as yield curves is illustrated through typical patterns of yield and volatility curves in Section 4. Section 5 contains concluding remarks. Finally detailed proofs for the prices and yields formula are relegated to the Appendix.

2. The Model

Let m_t be the nominal discount factor at time t. Let σ_t^2 be the conditional variance of the logarithm of discount factor between t and $t+\Delta$, where Δ is the length of time steps. This conditional variance is known from the information set at time t. Let $l_t = \ln m_t + \alpha$, where α is the mean of the logarithm of the nominal discount factor. Suppose that l_t and σ_t^2 have the following processes over Δ:

$$\begin{aligned} l_{t+\Delta} &= (1-\rho)l_t + \sigma_t v_{t+\Delta}, \qquad v_{t+\Delta} \overset{i.i.d}{\sim} N(0,1) && (1) \\ \sigma_{t+\Delta}^2 &= \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(v_{t+\Delta} - \gamma)^2, && (2) \end{aligned}$$

where $v_{t+\Delta}$, conditional on information at time t, is a standard normal random variable and $i.i.d$ means "identically and independently distributed as".

The state variable l_t in (1) follows an AR(1) process, while the conditional variance σ_t^2 in (2) follows a nonlinear asymmetric GARCH (NGARCH) process, that has been studied by Engle and Ng (1993) and Duan (1995). Note that this model is quite similar to but differs in some subtle ways from the affine discrete-time GARCH models proposed by Heston and Nandi (2000). More specifically, the Heston-Nandi model is designed specifically to produce closed-form option prices, while the specification in (1)-(2), like the Engle-Ng model, is designed foremost to provide a good fit to the interest-rate data.

The variance process, $\sigma_{t+\Delta}^2$, and the logarithm of the nominal discount factor, $\ln m_{t+\Delta}$, are assumed to be correlated, such that

$$\mathrm{Cov}_t\big(\sigma_{t+\Delta}^2, \ln m_{t+\Delta}\big) = -2\beta_2\gamma\sigma_t^3. \tag{3}$$

Given $\beta_2 > 0$, the negative parameter γ captures the positive correlation between discount factor and volatility innovations. That is, γ controls the skewness or the asymmetry of the distribution of the discount factor. Furthermore, the third power term on σ_t allows more variation over time in the leverage effect. This is likely to enhance the model's ability to fit the data to the extent that the leverage effect figures prominently in the term struture of interest rate. Thus our simple model accommodates two important stylized facts of interest-rate data: volatility clustering and leverage effect. Note that for $\gamma = 0$, the model simplifies to the popular GARCH model of Bollerslev (1986).

Since $v_{t+\Delta}$ and $v_{t+\Delta}^2 - 1$ are uncorrelated by construction, the variance equation can be rearranged in the following form:

$$\sigma_{t+\Delta}^2 - \sigma_t^2 = \beta_0 - \theta\sigma_t^2 - 2\beta_2\gamma\sigma_t^2 v_{t+\Delta} + \beta_2\sigma_t^2(v_{t+\Delta}^2 - 1),$$

where $\theta = 1 - \beta_1 - \beta_2(1+\gamma^2)$ with $1-\theta$ measuring the persistence of the variance. As the observation interval, Δ, shrinks to zero, a corresponding continuous-time system is obtained as:

$$\begin{aligned} dl_t &= -\rho\, l_t dt + \sigma_t dW_{1,t} \\ d\sigma_t^2 &= (\beta_0 - \theta\sigma_t^2)dt - 2\beta_2\gamma\sigma_t^2 dW_{1,t} + \beta_2\sigma_t^2 dW_{2,t}, \end{aligned} \tag{4}$$

where $(W_{1,t}, W_{2,t})$ is a bi-variate standard Brownian motion. From this continuous-time version, it is easy to see that the long-run variance of the logarithm of the discount factor is β_0/θ. That is, β_0/θ is the unconditional variance or, equivalently, the unconditional expectation of σ_t^2, which is $\mathrm{E}[\,\sigma_t^2\,] = \beta_0/\theta$. Also, we impose stationary such that $\mathrm{E}[\,\sigma_{t+\Delta}^2\,] = \mathrm{E}[\,\sigma_t^2\,]$ on the variance equation (2) by the following parameter restrictions:[3]

$$\beta_0 > 0 \qquad \text{and} \qquad \theta > 0,\ (\text{i.e.,}\ \beta_1 + \beta_2(1+\gamma^2) < 1).$$

Furthermore, we impose a stationary restriction such that $\mathrm{E}[\,l_{t+\Delta}\,] = \mathrm{E}[\,l_t\,]$ on the mean equation (1) by requiring that the speed of the mean-reversion equation obeys the additional restriction that $0 < \rho < 2$.

The continuous-time model in matrix form can be expressed as:

$$\begin{aligned} d\begin{pmatrix} l_t \\ \sigma_t^2 \end{pmatrix} &= \begin{pmatrix} -\rho\, l_t \\ \beta_0 - \theta\sigma_t^2 \end{pmatrix} dt + \begin{pmatrix} \sigma_t & 0 \\ -2\beta_2\gamma\sigma_t^2 & \beta_2\sigma_t^2 \end{pmatrix} \begin{pmatrix} dW_{1,t} \\ dW_{2,t} \end{pmatrix} \\ &= b\, dt + \begin{pmatrix} \sigma_{\cdot 1} & \sigma_{\cdot 2} \end{pmatrix} \begin{pmatrix} dW_{1,t} \\ dW_{2,t} \end{pmatrix} \end{aligned}$$

Next using Itô-Taylor formula, the Euler-Maruyama approximation scheme of the continuous-time version of (4) can be written as:

$$\begin{aligned} \begin{pmatrix} l_{t+\Delta} \\ \sigma_{t+\Delta}^2 \end{pmatrix} &= \begin{pmatrix} l_t \\ \sigma_t^2 \end{pmatrix} + b\Delta + \sum_{j=1}^{2} \sigma_{\cdot j}(W_{j,t+\Delta} - W_{j,t}) \\ &= \begin{pmatrix} l_t \\ \sigma_t^2 \end{pmatrix} + \begin{pmatrix} -\rho\, l_t \\ \beta_0 - \theta\sigma_t^2 \end{pmatrix} \Delta + \begin{pmatrix} \sigma_t \\ -2\beta_2\gamma\sigma_t^2 \end{pmatrix} \Delta_t W_1 + \begin{pmatrix} 0 \\ \beta_2\sigma_t^2 \end{pmatrix} \Delta_t W_2 \end{aligned}$$

[3] To ensure that the conditional variance is always positive further restrictions need to be imposed on β_1 and β_2 in (2). Alternatively, we can formulate the conditional variance by exponential GARCH (EGARCH) process (Nelson (1991)) instead of the NGARCH process (Equation (2)). The EGARCH process ensures positivity of the conditional variance and also allows for leverage effects and fat tails. However we choose the NGARCH process in this paper because it has been shown to improve the fit of empirical models substantially better than the GARCH process.

where $\Delta_t W_j = W_{j,t+\Delta} - W_{j,t}$ is an independent normal distribution with zero mean and variance Δ, i.e., $N(0,\Delta)$. Thus, another way of writing the equations in (1) and (2) is:

$$\begin{aligned} l_{t+\Delta} &= \rho_\Delta l_t + \sigma_t z_{t+\Delta}, \qquad z_{t+\Delta} \overset{i.i.d}{\sim} N(0,\Delta) &(5)\\ \sigma^2_{t+\Delta} &= \beta_0\Delta + \delta\sigma^2_t - 2\beta_2\gamma\sigma^2_t z_{t+\Delta} + \beta_2\sigma^2_t z^2_{t+\Delta}, &(6)\end{aligned}$$

where $\rho_\Delta = 1-\rho\Delta$ and $\delta = 1-(\theta+\beta_2)\Delta = 1-(1-\beta_1-\beta_2\gamma^2)\Delta$.

3. Zero-Coupon Bond Pricing and Yield-to-Maturity

It is well-known that the absence of arbitrage opportunities is characterized by the existence of an equivalent martingale measure $\mathbb{Q}$, so that the time-t price of a default-free, zero-coupon bond maturing at time $t+T$, $P_{t,T}$, is given by

$$P_{t,T} = \mathrm{E}^{\mathbb{Q}}_t\Big[\exp\Big(\int_t^{t+T} \ln m_s\, ds\Big)\Big]. \tag{7}$$

By [i] partitioning the time interval $[t, t+T]$ into subintervals of equal size; [ii] utilizing the Euler-Maruyama scheme in (5) and (6) and the tree property of conditional expectations; [iii] employing the trapezoidal rule to approximate the definite integral in the exponential function; and finally [iv] letting a subinterval size shrink to zero, we have an analytical approximation formula for the nominal price at time t of a default-free, zero-coupon bond maturing at time $t+T$. The result is stated in the following theorem.

Theorem 1. *If the yield factors follow the discrete stochastic differential equations in* (5) *and* (6)*, the nominal price at time t of a default-free, zero-coupon bond maturing at time $t+T$, $P_{t,T}$, is given by*

$$\ln(P_{t,T}) = -\alpha T + \frac{1-e^{-\rho T}}{\rho} l_t + \frac{\beta_0}{2\theta\rho^2} f(T) + \frac{\sigma^2_t - \beta_0/\theta}{2\rho^2} g(T), \tag{8}$$

where

$$\begin{aligned} f(T) &= f(T;\rho) = T - 2\frac{1-e^{-\rho T}}{\rho} + \frac{1-e^{-2\rho T}}{2\rho}\\ g(T) &= g(T;\rho,\theta) = \frac{1-e^{-\theta T}}{\theta} - 2\frac{e^{-\theta T}-e^{-\rho T}}{\rho-\theta} + \frac{e^{-\theta T}-e^{-2\rho T}}{2\rho-\theta}, \end{aligned} \tag{9}$$

where $\theta = 1-\beta_1-\beta_2(1+\gamma^2)$. Furthermore, $\mathrm{E}^{\mathbb{Q}}_t[\int_t^{t+T}\ln m_s\, ds]$ is given by the first two terms of $\ln P(t,T)$, $-\alpha T + \frac{1-e^{-\rho T}}{\rho} l_t$.

In a simple case of constant conditional variance, $\sigma^2 = \beta_0/\theta$, its corresponding nominal price, $P_{t,T}$, has the exact analytical formula of the following form:

$$\ln(P_{t,T}) = -\alpha T + \frac{1-e^{-\rho T}}{\rho} l_t + \frac{\sigma^2}{2\rho^2} f(T).$$

Note that the nominal yield-to-maturity is defined as

$$y^{(T)}_t = -T^{-1}\ln(P_{t,T}).$$

From Theorem 1, the yield-to-maturity can be written in terms of state variables, l_t and σ_t^2, as:[4]

$$y_t^{(T)} = \alpha - \frac{1-e^{-\rho T}}{\rho T} l_t - \frac{\beta_0}{2\theta\rho^2 T} f(T) - \frac{\sigma_t^2 - \beta_0/\theta}{2\rho^2 T} g(T), \tag{10}$$

where $f(T)$ and $g(T)$ are defined in (9).

As the time-to-maturity, T, tends to zero, the nominal (instantaneous) short rate is defined and calculated as

$$r_t = \lim_{T\to 0} y_t^{(T)} = \alpha - l_t = -\ln m_t, \tag{11}$$

where the second equality is obtained by applying the L'Hopital's rule into functions such as $f(T)/T$ and $g(T)/T$. Thus, it follows as a simple computation using (4) that the dynamics for the nominal short rate can be written as a stochastic process

$$dr_t = -dl_t = \rho(\alpha - r_t)dt - \sigma_t dW_{1,t}. \tag{12}$$

In a simple case of constant conditional variance, $\sigma^2 = \beta_0/\theta$, it becomes the well-known Vasicek (1977) model and if the nominal price, $P_{t,T-t}$, is represented by $A(t,T)e^{-r_t B(t,T)}$, where $\ln(A(t,T)) = -\alpha(T-t) + \alpha(1-e^{-\rho(T-t)})/\rho + \sigma^2 f(T-t)/(2\rho^2)$ and $B(t,T) = (1-e^{-\rho(T-t)})/\rho$, then $A(t,T)$ and $B(t,T)$ satisfy the following system of differential equations:

$$\begin{cases} \dfrac{\partial A}{\partial t} - \rho\alpha AB + \dfrac{1}{2}\sigma^2 AB^2 = 0 \\ \dfrac{\partial B}{\partial t} - \rho B + 1 = 0. \end{cases}$$

Similarly, as the time-to-maturity tends to infinity, the nominal long-term rate is defined as:

$$y_t^{(\infty)} = \lim_{T\to\infty} y_t^{(T)} = \alpha - \frac{\beta_0}{2\theta\rho^2}. \tag{13}$$

Note that it does not depend on the nominal short rate r_t. Thus, combining the short rate with the long-term rate, the nominal yield to maturity can be rearranged as:

$$y_t^{(T)} = y_t^{(\infty)} - \frac{1-e^{-\rho T}}{\rho T}(y_t^{(\infty)} - r_t) + \frac{g(T)}{2\rho^2 T}(\beta_0/\theta - \sigma_t^2) + \frac{\beta_0}{4\theta\rho^3 T}(1-e^{-\rho T})^2, \tag{14}$$

which implies that the yield-to-maturity is obtained by adjusting the long-term rate by the spread between the long-term rate and the short-term rate, the *difference between the current and long-run variances* (this is a new feature resulting from the GARCH effect on the variance equation), and the time to maturity.

Since the state variable, l_t or r_t[5], is unobservable, when we estimate the model parameters and calibrate the model, the shortest yield, for example, $y_t^{(\Delta)}$ for a Δ time-period,

[4]This shows that the yield to maturity is an affine function of the two state variables (logarithm of discount factor and conditional variance) defined in (1) and (2), in contrast to quadratic models (such as Ahn et al (2002)) in which the yield is a quadratic function of the state variables.

[5]From (11) and (13), it is clear that it does not matter whether we choose l_t or r_t.

is an alternative state variable instead of l_t or r_t. To this end, l_t can be rearranged in terms of one-period ahead state variables, $y_{t-\Delta}^{(\Delta)}$ and $\sigma_{t-\Delta}$, as:

$$l_t = \sqrt{\Delta}\sigma_{t-\Delta}\psi_t - \frac{\rho\Delta(1-\rho\Delta)}{1-e^{-\rho\Delta}}\left(y_{t-\Delta}^{(\Delta)} - \mathrm{E}[\,y^{(\Delta)}\,] + \frac{\sigma_{t-\Delta}^2 - \beta_0/\theta}{2\rho^2\Delta}g(\Delta)\right) \tag{15}$$

Substituting r_t with $\alpha - l_t$ in the equation (15), we have another form of (14) which is stated with its variance and kurtosis in the following theorem.

Theorem 2. *If the yield factors follow the discrete stochastic differential equations in* (5) *and* (6)*, the yield-to-maturity,* $y_t^{(T)}$*, can be written in terms of the shortest yield,* $y_{t-\Delta}^{(\Delta)}$*, and the volatility,* $\sigma_{t-\Delta}$*, as:*

$$\begin{aligned} y_t^{(T)} &= \frac{\Delta(1-\rho\Delta)}{T}\frac{1-e^{-\rho T}}{1-e^{-\rho\Delta}}\left(y_{t-\Delta}^{(\Delta)} - \mathrm{E}[\,y^{(\Delta)}\,] + \frac{\sigma_{t-\Delta}^2 - \beta_0/\theta}{2\rho^2\Delta}g(\Delta)\right) \\ &\quad + y_t^{(\infty)} - \frac{1-e^{-\rho T}}{\rho T}\sigma_{t-\Delta}\sqrt{\Delta}\psi_t + \frac{g(T)}{2\rho^2 T}(\beta_0/\theta - \sigma_t^2) + \frac{\beta_0}{2\theta\rho^2}\big(1 - \frac{f(T)}{T}\big), \end{aligned} \tag{16}$$

where $\sigma_t^2 = \beta_0\Delta + \delta\sigma_{t-\Delta}^2 - 2\beta_2\gamma\sigma_{t-\Delta}^2\sqrt{\Delta}\psi_t + \beta_2\sigma_{t-\Delta}^2\Delta\psi_t^2$ *and* $\delta = 1-(1-\beta_1-\beta_2\gamma^2)\Delta$ *and its variance and kurtosis per* Δ *time period are*

$$\begin{aligned} \mathrm{Var}_{t-\Delta}\big(y_t^{(T)}\big) &= \mathrm{E}_{t-\Delta}[\,(u_t)^2\,] = C^2 + 2D^2 \\ K\big(y_t^{(T)}\big) &= \frac{\mathrm{E}_{t-\Delta}\big[(u_t)^4\big]}{\big(\mathrm{Var}_{t-\Delta}(y_t^{(T)})\big)^2} = \frac{3C^4 + 2D^4}{(C^2+2D^2)^2} < 3, \end{aligned}$$

where $u_t = y_t^{(T)} - \mathrm{E}_{t-\Delta}[\,y_t^{(T)}\,]$ *and*

$$\begin{aligned} C &= \frac{\sigma_{t-\Delta}\sqrt{\Delta}}{\rho T}\big(\beta_2\gamma g(T)\sigma_{t-\Delta}/\rho - (1-e^{-\rho T})\big) \\ D &= -\frac{g(T)}{2\rho^2 T}\beta_2\sigma_{t-\Delta}^2\Delta. \end{aligned}$$

Note that Theorem 2 predicts that the excess kurtosis of nominal yields to maturity is negative even when the logarithm of the nominal discount factor is specified as an AR(1)-NGARCH(1,1) process.[6]

4. Yield and Volatility Curve of Yield-to-Maturity

In this section we illustrate the typical patterns of yield and volatility curves of the nominal yield-to-maturity of AR(1) (which is the Vasicek (1977) model) or AR(1)-NGARCH(1,1) process with the time-to-maturity and show the importance of moments for volatility curves as well as yield curves when we calibrate the model to match market data.

[6]This should be interpreted as short-term prediction only as negative excess kurtosis of nominal yields to maturity in the long run is hard to reconcile with the empirical data.

As we noted in (10) or (16), parameters affecting the shape of yield curves are the current volatility, σ_t, the long-run variance, β_0/θ, the AR(1) coefficient, ρ, the reproduced parameter from a stationary restriction on the conditional variance equation, θ, the current short-term yield, $y^{(T)}_{t-\Delta}$, the long-run short-term yield, $\mathrm{E}[\, y^{(\Delta)}\,]$, and the long-run long-term spread, α. Figure 1 – Figure 4 present the typical patterns for yield and volatility curves of AR(1) or AR(1)-NGARCH(1,1) processes and their related functions such as $f(T;\rho)$ and $g(T;\rho,\theta)$ defined in Theorem 1 and are calculated from parameters in Table 1. For the random innovation at $t-\Delta$, we assume that $\psi_t \sim N(0,1)$ have 7 possible values: $[\,-2,\,-1,\,-0.5,\,0,\,0.5,\,1,\,2\,]$.

Parameters	ρ	θ	σ_t	β_0/θ	$y^{(T)}_{t-\Delta}$	$\mathrm{E}[\, y^{(\Delta)}\,]$	α	γ
1(a)	0.12	0.08	0.012	0.02	0.02	0.05	0.07	-0.56
1(c)	0.25	0.02	0.02	0.015	0.04	0.05	0.08	-2
1(e)	0.05	0.03	0.01	0.015	0.035	0.055	0.07	-0.6
3(a)	0.2	0.25	0.01	0.02	0.03	0.05	0.06	-0.4
3(c)	0.07	0.1	0.02	0.022	0.05	0.04	0.07	-0.5
3(e)	0.16	0.01	0.02	0.01	0.05	0.035	0.07	-0.6

TABLE 1. **Parameter Values for Figure 1–Figure 4** This table provides parameter values for Figure 1–Figure 4. Since $\theta = 1-\beta_1-\beta_2(1+\gamma^2)$ is a function of β_1 and β_2, given by γ, there are infinite pairs (β_1, β_2) satisfying θ; thus, we impose the restriction that $\beta_1 = (1-\theta)/6$.

The left (right) panels of Figure 1 and Figure 3 present the yield curves for AR(1) (AR(1)-NGARCH(1,1)) processes and the left panels of Figure 2 and Figure 4 depict the volatility curves of AR(1) processes, AR(1)-NGARCH(1,1) processes, and market data for the yield volatility of On-the-Run Treasuries in 1987 (See Exhibit 22-10 in Fabozzi (1993)). Finally, the right panels of Figure 2 and Figure 4 present the function, $f(T;\rho)$, which appears in both processes and the function, $g(T;\rho,\theta)$, which appears only in the NGARCH processes. These panels illustrate the effects of the magnitude of function values in $f(T)$ and $g(T)$ on the volatility curve.

Since the random part in (16) at time $t-\Delta$ can be written as

$$C\psi_t + D\psi_t^2,$$

where $C = \frac{\sigma_{t-\Delta}\sqrt{\Delta}}{\rho T}\big(\beta_2\gamma g(T)\sigma_{t-\Delta}/\rho - (1-e^{-\rho T})\big)$ and $D = -\frac{g(T)}{2\rho^2 T}\beta_2\sigma^2_{t-\Delta}\Delta$, the volatility of yield curve, which we call a volatility curve, is calculated as:

$$\sqrt{\mathrm{Var}_{t-\Delta}\Big(\frac{y^{(T)}_t - y^{(T)}_{t-\Delta}}{y^{(T)}_{t-\Delta}}\Big)} = \frac{\sqrt{C^2+2D^2}}{y^{(T)}_{t-\Delta}}.$$

Figure 1a–1b, Figure 3a–3b, and Figure 2a and Figure 4a illustrate that although the yield curves are all upward-sloping, the volatility curves have different shapes depending on whether the corresponding model is an AR(1) process or an AR(1)-NGARCH(1,1) process. In addition they show that the yield curves retain their upward-sloping shape independent of the sign of a new innovation shock. However, Figure 1c–1d and Figure 2c illustrate that a yield curve can change its shape, provided that a new innovation shock is negative. Furthermore, Figure 1e–1f and Figure 2e show that although model parameter values are kept the same, except the parameter value of γ, the yield curves exhibit slightly different shapes. But volatility curves have even more strikingly different shapes. The AR(1)-NGARCH(1,1) process has a volatility curve. Recall that this process has been shown empirically to fit the observed volatility curve very well. Thus, we have demonstrated that the GARCH model for the conditional variance of interest rates is a necessary component of the model and it is important that we take into account of moments for the volatility curve as well as the yield curve when we calibrate the model to match the market data. Finally, Figure 3c–3d, Figure 3e–3f, and Figure 4c and 4e reproduce the results for the case of downward-sloping yield curves.

5. Conclusions

This paper presented a stylized model of the nominal term structure of interest rates. Our proposed model was derived with specific considerations for data availability and model tractability as discussed so eloquently in Dai and Singleton (2003). To achieve this objective, we established a linkage between a discrete-time version of the model and its continuous-time counterparts. This was done in two steps. First, the nominal discount factor was selected as a state variable of the model. Second, the logarithm of the discount factor was specified as an AR(1) process with its conditional variance following an NGARCH process. This particular modeling strategy has several distinct advantages: [i] the process has been shown to fit market data rather well; [ii] it allows tractability in deriving the formula for the prices or yields-to-maturity (yields) of default-free bonds; [iii] the functional dependence of the short-rate on this state variable can be easily obtained from the established linkage between the discrete-time model and its continuous-time counterpart. In other words, the model tractability, which is obtained from the result that the discrete-time GARCH models are linked to bivariate diffusion processes as limiting cases, can be exploited to show that the short-term rate is linearly dependent on the logarithm of this state variable as a limiting result; [iv] in a simple case of constant conditional variance, the discrete-time term-structure model can be shown to reduce to the well-known Vasicek (1977) model; and finally [v] the comparison between the yield volatility of On-the-Run Treasuries with the volatility curve of our model indicates that it is important in the modeling process to take into account moments for volatility curves and yield curves.

Thus, we attempted to take into considerations features known to be important to the empirical modeling of the term structure of interest rates, such as time-varying volatility, volatility clustering and, leverage effect, in the formulation of our model. With its closed form solution, which cannot be obtained by many other existing nonlinear, stochastic volatility models, our model can be potentially appealing for empirical purposes. Having said this, the paper, admittedly, is incomplete without actually establishing the empirical advantages of the proposed model based on actual data evaluation. To do, we first need to discuss identification and estimation issues of the model. For instance, what are the conditional densities of the state variables in this model? Which estimation technique is most appropriate to the model? What restrictions on the parameters of the model should be imposed to rule out arbitrage opportunities? What econometric issues may be involved when the state variables of the model are unobservable?

Further it is of interest to empirically implement the proposed model and assess its goodness-of-fit. In particular, it is useful to analyze the models ability to capture the historical movements in yields and volatilities for a full sample as well as for different subsamples, and perform across model comparison. Since the proposed model in this paper is a two-factor model, good candidates for model comparison include a two-factor affine Gaussian model (with contant volatility), and Longstaff and Schwartz (1992, using interest rate‘and its volatility as state variables). Other multifactor models known to have with empirical support, e.g. Ahn et al (2002), and Dai and Singleton (2000), may also serve as good candidates for model comparison. These empirical exercises would allow us to asses how well our model capture the historical movement in yields, whether it is able to produce different shapes of yield curves, whether the model's parameters can easily be identified, Which parameters of the model actually govern the dynamics of the state variables. All these empirical questions are undoubtedly important before the empirical appeal of the proposed model can be truly appreciated; therefore they will be researched in greater detail in the near future.

Appendix

Proof of Theorem 1:

By partitioning the interval $[t, t+T]$ into $t_0 = t < t_1 < \cdots < t_n = t+T$ with $t_j = t + j\Delta$ and $\Delta = \frac{T}{n}$, we compute the conditional expectation of $\exp\big(\int_t^{t+T} \ln m_s \, ds\big)$ under the equivalent martingale measure $\mathbb{Q}$. For simplicity, we denote l_{t_j}, σ_{t_j}, z_{t_j}, and $\mathrm{E}^{\mathbb{Q}}_{t_j}$ by l_j, σ_j, z_j, and E_j, respectively for $j = 0, 1, \ldots, n-1$ and assume that $t = 0$.

The trapezoidal rule is employed to approximate the definite integral in the exponent function of (7)), so that the integral part can be written in terms of l_t instead of $\ln m_t$

$$\int_t^{t+T} \ln m_s \, ds \approx -\alpha T + \Delta I_n,$$

where $I_n = l_0/2 + l_1 + \cdots + l_{n-1} + l_n/2$. It follows as an application of the tree property of conditional expectation that

$$\exp(\delta T) P_{t,T} \approx \mathrm{E}_0 \, \mathrm{E}_1 \cdots \mathrm{E}_{n-1} \left[e^{\Delta I_n} \right].$$

To compute the conditional expectation of $e^{\Delta I_n}$, it is necessary to represent l_j in terms of random variables $\{z_j\}_{j=1,\ldots,n}$ and obtain as an application of induction arguments on AR(1) process that

$$l_j = \rho_\Delta^j l_0 + \rho_\Delta^{j-1} \sigma_0 z_1 + \cdots + \rho_\Delta \sigma_{j-2} z_{j-1} + \sigma_{j-1} z_j$$

for $j = 1, \ldots, n$, where $\rho_\Delta = 1 - \rho\Delta$. A simple computation using the above formula yields the result that

$$I_n = a_{n+1} l_0 + \sum_{j=1}^{n} a_{n-j+1} \sigma_{j-1} z_j, \tag{17}$$

where $a_1 = 1/2$, $a_j = \frac{1-\rho_\Delta^{j-1}}{1-\rho_\Delta} + \frac{\rho_\Delta^{j-1}}{2}$ for $j = 2, \ldots, n$, and $a_{n+1} = \frac{1}{2} + \rho_\Delta \frac{1-\rho_\Delta^{n-1}}{1-\rho_\Delta} + \frac{\rho_\Delta^n}{2}$. To compute the conditional expectations, we stated a well-known result in the following lemma.

Lemma 1. *Suppose that ψ is a standard normal distribution, i.e., $\psi \sim N(0,1)$. Then we have*

$$\mathrm{E}[\, e^{b\sqrt{T}\psi} \,] = e^{\frac{1}{2} b^2 T}.$$

Furthermore, the moment generating function of $Q(\psi) = (\psi - w)^2$ is

$$\mathrm{E}[\, e^{vQ(w)} \,] = \exp\big(-\frac{w^2}{2}\big) \frac{1}{\sqrt{1-2v}} \exp\big(\frac{w^2}{2(1-2v)}\big) \tag{18}$$

First, applying the first equation of Lemma 1 into the case $a_1 \sigma_{n-1} z_n$ in I_n, we have

$$\mathrm{E}_{n-1} \left[e^{\Delta a_1 \sigma_{n-1} z_n} \right] = e^{\Delta^3 a_1^2 \sigma_{n-1}^2 / 2},$$

which implies that

$$\ln \mathrm{E}_{n-1}[\, e^{\Delta I_n} \,] = \Delta I_{n-1} + \Delta^3 a_1^2 \sigma_{n-1}^2 / 2$$

where

$$I_k = a_{n+1} l_0 + \sum_{j=1}^{k} a_{n-j+1} \sigma_{j-1} z_j$$

for $k = 1, 2, \ldots, n$. Using the recursive formula in (6) with $t = t_{n-2}$, the random part in $\ln \mathrm{E}_{n-1}[\, e^{\Delta I_n} \,]$ at time t_{n-2} is written as

$$\begin{aligned} I_{n-1,r} &= \Delta \left(a_2 \sigma_{n-2} z_{n-1} + \Delta^2 a_1^2 \sigma_{n-1}^2 / 2 \right) \\ &= v_{n-1} (\psi_{n-1} - w_{n-1})^2 - v_{n-1} w_{n-1}^2 + \frac{\Delta^3}{2} a_1^2 (\beta_0 \Delta + \delta \sigma_{n-2}^2), \end{aligned}$$

where $v_{n-1} = \beta_2 \Delta^4 a_1^2 \sigma_{n-2}^2 / 2$, $w_{n-1} = [\gamma - a_2 / (\beta_2 \Delta^2 a_1^2 \sigma_{n-2})] / \sqrt{\Delta}$, and $\psi_{n-1} = z_{n-1} / \sqrt{\Delta}$ is a standard normal distribution.

Second, applying (18) into $v_{n-1}(\psi_{n-1} - w_{n-1})^2$ in $I_{n-1,r}$ with v_{n-1} and w_{n-1} and the tree property of conditional expectation, we have

$$\begin{aligned} \ln \mathrm{E}_{n-2}[\, e^{\Delta I_n} \,] &= \Delta I_{n-2} + \frac{\Delta^3}{2} a_1^2 (\beta_0 \Delta + \delta \sigma_{n-2}^2) \\ &\quad + \frac{w_{n-1}^2}{2} \big(1/(1 - 2v_{n-1}) - 1 - 2v_{n-1} \big) - \frac{1}{2} \ln(1 - 2v_{n-1}). \end{aligned}$$

As a subinterval size, Δ, shrinks, v_{n-1} is sufficiently small. Thus we can approximate $\ln(1 - 2v_{n-1})$ by $-2v_{n-1}$ and $1/(1 - 2v_{n-1}) - 1 - 2v_{n-1} = 4v_{n-1}^2/(1 - 2v_{n-1})$ by $4v_{n-1}^2$, and then obtain

$$\begin{aligned} \ln \mathrm{E}_{n-2}[\, e^{\Delta I_n} \,] &\approx \Delta I_{n-2} + \frac{\Delta^3}{2} a_1^2 (\beta_0 \Delta + \delta \sigma_{n-2}^2) + 2 v_{n-1}^2 w_{n-1}^2 + v_{n-1} \\ &= \Delta I_{n-2} + \frac{\Delta^4}{2} \beta_0 a_1^2 + \frac{\Delta^3}{2} a_1^2 \sigma_{n-2}^2 \big(\delta + \beta_2 \Delta + (\beta_2 \gamma a_1 \sigma_{n-2} \Delta^2 - a_2/a_1)^2 \big) \\ &\approx \Delta I_{n-2} + \frac{\Delta^4}{2} \beta_0 a_1^2 + \frac{\Delta^3}{2} a_1^2 \sigma_{n-2}^2 \big(\delta + \beta_2 \Delta + (a_2/a_1)^2 \big) \\ &= \Delta I_{n-2} + \frac{\Delta^4}{2} \beta_0 a_1^2 + \frac{\Delta^3}{2} \sigma_{n-2}^2 b_2, \end{aligned}$$

where the second approximation comes from approximating $\beta_2 \gamma a_1 \sigma_{n-2} \Delta^2 - a_2/a_1$ by a_2/a_1, since $a_2 = 1 + \rho_\Delta/2$ and $a_1 = 1/2$ implies that $a_2/a_1 = O(1)$, and $b_2 = (\delta + \beta_2 \Delta) b_1 + a_2^2$ with the initial value $b_1 = a_1^2$.

Similarly, using the recursive formula in (6) with $t = t_{n-3}$, the random part in $\ln \mathrm{E}_{n-2}[\, e^{\Delta I_n} \,]$ at time t_{n-3} is written as

$$\begin{aligned} I_{n-2,r} &= \Delta \left(a_3 \sigma_{n-3} z_{n-2} + \Delta^2 b_2 \sigma_{n-2}^2 / 2 \right) \\ &= v_{n-2} (\psi_{n-2} - w_{n-2})^2 - v_{n-2} w_{n-2}^2 + \frac{\Delta^3}{2} b_2 (\beta_0 \Delta + \delta \sigma_{n-3}^2), \end{aligned}$$

where $v_{n-2} = \beta_2 \Delta^4 b_2 \sigma_{n-3}^2 / 2$, $w_{n-2} = [\gamma - a_3 / (\beta_2 \Delta^2 b_2 \sigma_{n-3})] / \sqrt{\Delta}$, and $\psi_{n-2} = z_{n-2} / \sqrt{\Delta}$ is a standard normal distribution. Thus, we have a similar computation problem as in the case of $t = t_{n-2}$. That is, we have

$$\ln \mathrm{E}_{n-3}[\, e^{\Delta I_n} \,] = \Delta I_{n-3} + \frac{\Delta^4}{2} \beta_0 (b_1 + b_2) + \frac{\Delta^3}{2} \sigma_{n-3}^2 b_3,$$

where $b_j = (\delta + \beta_2\Delta)b_{j-1} + a_j^2 = (1-\theta\Delta)b_{j-1} + a_j^2$ with the initial value $b_1 = a_1^2$ for $j = 2, \ldots, n$.

Continuing this procedure, we have

$$\ln \mathrm{E}_1[\, e^{\Delta I_n} \,] = \Delta I_1 + \frac{\Delta^4}{2}\beta_0(b_1 + b_2 + \cdots + b_{n-2}) + \frac{\Delta^3}{2}\sigma_1^2 b_{n-1}.$$

Finally,

$$\begin{aligned} \ln \mathrm{E}_0[\, e^{\Delta I_n} \,] &= a_{n+1} l_0 \Delta + \frac{\Delta^4}{2}\beta_0(b_1 + b_2 + \cdots + b_{n-2}) \\ &\quad + \ln \mathrm{E}_0[\, e^{\Delta(a_n\sigma_0 z_1 + \Delta^2 b_{n-1}\sigma_1^2/2)} \,] \\ &= a_{n+1} l_0 \Delta + \frac{\Delta^4}{2}\beta_0(b_1 + b_2 + \cdots + b_{n-1}) + \frac{\Delta^3}{2}\sigma_0^2 b_n. \end{aligned} \tag{19}$$

To obtain the convergence result as n approaches $+\infty$, we need several simple computations. Recall that the number e is defined as the limit of the sequence, i.e.,

$$\lim_{n\to\infty} \big(1 + \frac{1}{n}\big)^n = e,$$

which implies that for $\Delta = T/n$,

$$\lim_{n\to\infty} \big(1 - c\Delta\big)^n = e^{-cT}.$$

Applying this result to a sequence, we have the following lemma.

Lemma 2. *Given the sequence $\{a_j\}$, if a sequence $\{b_n\}$ is described by $b_1 = a_1^2$ and the recursive relationship $b_j = (1-\theta\Delta)b_{j-1} + a_j^2$ for $j = 2, \ldots, n$, then b_n can be explicitly written as*

$$b_n = \sum_{j=1}^{n} a_j^2 (1-\theta\Delta)^{n-j}.$$

Also, its partial sum is

$$\sum_{j=1}^{n-1} b_j = \left(\sum_{j=1}^{n} a_j^2 - b_n \right) / (\theta\Delta). \tag{20}$$

In particular, if $\{a_j\}$ is given by $a_1 = 1/2$, $a_j = \frac{1-\rho_\Delta^{j-1}}{1-\rho_\Delta} + \frac{\rho_\Delta^{j-1}}{2}$ for $j = 2, 3, 4, \ldots, n$,

$$\begin{aligned} \Delta^3 b_n &= \frac{1-(1-\theta\Delta)^{n-1}}{\rho^2\theta} + \big(\frac{\Delta^2}{4} - \frac{2c_\rho\Delta}{\rho} - c_\rho^2\Delta^2\big)(1-\theta\Delta)^{n-1}\Delta \\ &\quad + \frac{2c_\rho\Delta}{\rho}\frac{(1-\theta\Delta)^n - (1-\rho\Delta)^n}{\rho-\theta} + c_\rho^2\Delta^2\frac{(1-\theta\Delta)^n - (1-\rho\Delta)^{2n}}{2\rho-\theta-\rho^2\Delta} \end{aligned}$$

Thus, its limit, $\lim_{n\to\infty}\Delta^3 b_n$, is $g(T;\rho,\theta)/\rho^2$, where

$$g(T;\rho,\theta) = \frac{1-e^{-\theta T}}{\theta} - 2\frac{e^{-\theta T} - e^{-\rho T}}{\rho-\theta} + \frac{e^{-\theta T} - e^{-2\rho T}}{2\rho-\theta}$$

Also, we have

$$\Delta^3 \sum_{j=1}^{n} a_j^2 = \frac{\Delta n}{\rho^2} + \frac{2c_\rho\Delta}{\rho^2}(1-\rho_\Delta^n) + c_\rho^2\Delta^2\frac{1-\rho_\Delta^{2n}}{2\rho-\rho^2\Delta}$$

and its limit, $\lim_{n\to\infty} \Delta^3 \sum_{j=1}^{n} a_j^2$, *is* $f(T;\rho)/\rho^2$, *where*

$$f(T;\rho) = T - 2\frac{1-e^{-\rho T}}{\rho} + \frac{1-e^{-2\rho T}}{2\rho}$$

where $c_\rho = 1/2 - 1/(\rho\Delta)$, *hence,* $\lim_{n\to\infty} \Delta c_\rho = -1/\rho$. *Finally,* $a_{n+1}\Delta = (\frac{1}{2} + \rho_\Delta \frac{1-\rho_\Delta^{n-1}}{1-\rho_\Delta} + \frac{\rho_\Delta^n}{2})\Delta$ *has the limit*

$$\frac{1-e^{-\rho T}}{\rho}.$$

Substituting the relationship in (20) into (19) and rerranging terms give us the result that

$$\ln \mathrm{E}_0[\, e^{\Delta I_n} \,] = a_{n+1} l_0 \Delta + \frac{\beta_0}{2\theta}\Delta^3 \sum_{j=1}^{n} a_j^2 + \frac{\sigma_0^2 - \beta_0/\theta}{2}\Delta^3 b_n.$$

Thus, as the observation interval approaches zero, the desired result (8) is obtained as a simple application of Lemma 2.

Also, by the linear property of the expectation operator and the *i.i.d* property of $\{z_j\}$, it is easy to see that

$$\mathrm{E}_0\Big[\int_t^{t+T} \ln m_s \, ds\,\Big] \approx -\alpha T + a_{n+1} l_0 \Delta \to -\alpha T + \frac{1-e^{-\rho T}}{\rho} l_0$$

In a simple case of constant conditional variance, $\sigma^2 = \beta_0/\sigma$, the recursive formula of l_j can be written as

$$l_j = \rho_\Delta^j l_0 + \rho_\Delta^{j-1}\sigma\sqrt{\Delta}\psi_1 + \cdots + \rho_\Delta \sigma\sqrt{\Delta}\psi_{j-1} + \sigma\sqrt{\Delta}\psi_j$$

for $j = 1, \ldots, n$, where $\rho_\Delta = 1 - \rho\Delta$ and $\{\psi_j = z_j/\Delta\}$ are i.i.d standard normals. Thus, we have a similar form for (17) as follows

$$I_n = a_{n+1} l_0 + \sum_{j=1}^{n} a_{n-j+1}\sigma\sqrt{\Delta}\psi_j.$$

Using the i.i.d property of $\{\psi_j\}$, it is easy to obtain a similar form for (19) as follows

$$\ln \mathrm{E}_0[\, e^{\Delta I_n} \,] = a_{n+1} l_0 \Delta + \frac{\Delta^3}{2}\sigma^2 \sum_{j=1}^{n} a_j^2.$$

As the observation interval approaches zero, the limit results in Lemma 2 provides us the desired result.

Proof of Equation (15):

Recall that we have equation (10) with $T = \Delta$:

$$y_t^{(\Delta)} = \alpha - \frac{1-e^{-\rho\Delta}}{\rho\Delta} l_t - \frac{\beta_0}{2\theta\rho^2\Delta} f(\Delta) - \frac{\sigma_t^2 - \beta_0/\theta}{2\rho^2\Delta} g(\Delta)$$

Applying the stationary property such as $\mathrm{E}[\, l_t \,] = 0$ and $\mathrm{E}[\, \sigma_t^2 \,] = \beta_2/\theta$, this equation can be rearranged as

$$y_t^{(\Delta)} - \mathrm{E}[\, y^{(\Delta)} \,] = -\frac{1-e^{-\rho\Delta}}{\rho\Delta} l_t - \frac{\sigma_t^2 - \beta_0/\theta}{2\rho^2\Delta} g(\Delta) \tag{21}$$

Substituting l_t by (5) in terms of $l_{t-\Delta}$ and $\sigma_{t-\Delta}$ and replacing $l_{t-\Delta}$ by (21) in terms of $y_{t-\Delta}^{(\Delta)}$ and $\sigma_{t-\Delta}$, we have an another form of (21) as follows

$$\begin{aligned} y_t^{(\Delta)} - \mathrm{E}[\, y^{(\Delta)} \,] = (1-\rho\Delta) & \left[y_{t-\Delta}^{(\Delta)} - \mathrm{E}[\, y^{(\Delta)} \,] + \frac{\sigma_{t-\Delta}^2 - \beta_0/\theta}{2\rho^2\Delta} g(\Delta) \right] \\ & - \frac{1-e^{-\rho\Delta}}{\rho\Delta} \sigma_{t-\Delta}\sqrt{\Delta}\psi_t - \frac{\sigma_t^2 - \beta_0/\theta}{2\rho^2\Delta} g(\Delta) \end{aligned}$$

Subtracting the above equation from (21), we have

$$\frac{1-e^{-\rho\Delta}}{\rho\Delta} l_t = \frac{1-e^{-\rho\Delta}}{\rho\Delta} \sigma_{t-\Delta}\sqrt{\Delta}\psi_t - (1-\rho\Delta) \left[y_{t-\Delta}^{(\Delta)} - \mathrm{E}[\, y^{(\Delta)} \,] + \frac{\sigma_{t-\Delta}^2 - \beta_0/\theta}{2\rho^2\Delta} g(\Delta) \right],$$

which gives us the desired result in (15).

Proof of Theorem 2:

Combining (11) with (12), we have

$$y_t^{(\infty)} - r_t = l_t - \frac{\beta_0}{2\theta\rho^2}.$$

Substituting l_t in this above equation by (15) and then plugging this result into (14) give us the desired result in (16).

The random part in (16) at time $t-\Delta$ can be written as

$$C\psi_t + D\psi_t^2,$$

where $C = \frac{\sigma_{t-\Delta}\sqrt{\Delta}}{\rho T}\left(\beta_2\gamma g(T)\sigma_{t-\Delta}/\rho - (1-e^{-\rho T})\right)$ and $D = -\frac{g(T)}{2\rho^2 T}\beta_2\sigma_{t-\Delta}^2\Delta$. Thus, using the moments of the standard normal distribution ψ, that is, $\mathrm{E}[\, \psi^{2K} \,] = 1 \times 3 \times \cdots \times (2k-1)$ for $k = 1, 2, \ldots$ and the odd moments are zero, we have

$$y_t^{(T)} - \mathrm{E}_{t-\Delta}[\, y_t^{(T)} \,] = C\psi_t + D(\psi_t^2 - 1).$$

A simple calculation provides us the desired result about the variance and kurtosis of nominal yields to maturity.

References

[1] Ahn, D.-H., R.F. Dittmar, and A.R. Gallant, 2002, Quadratic term structure models: theory and evidence, *Review of Financial Studies,* 15, 243-288.

[2] Bollerslev, T., 1986, Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 307-327

[3] Brennan, Michael J. and Eduardo S. Schwartz, 1982, An Equilibrium Model of Bond Pricing and a Test of Market Efficiency, *Journal of Financial and Quantitative Analysis* 17, 75-100

[4] Chan, K. C., G. Andrew Karolyi, Francis A. Longstaff, and Anthony B. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-term Interest Rate, *Journal of Finance*, Vol. XLVII, 1209-1227

[5] Chapman, D. and N. Pearson, 2001, What Can be Learned From Recent Advances in Estimating Models of the Term Structure ? *Financial Analysts Journal July/August,* 77-95

[6] Constantinides, G, 1992, A Theory of the Nominal Term Structure of Interest Rates, *Review of Financial Studies 5,* 531-552

[7] Dai, Q. and K. Singleton, 2000, Specification Analysis of Affine Term Structure Models, *Journal of Finance* LV , 1943-1978

[8] Dai, Q. and K. Singleton, 2003, Fixed Income Pricing *Handbook of Economics and Finance*, Chapter 20, C. Constantinides, M. Harris, and R. Stulz, eds. North Holland

[9] Duan, J., 1995, The GARCH Option Pricing Model, *Mathematical Finance* 5, 13-32

[10] Fabozzi J., *Fixed Income Mathematics*, Prentice Hall, New Jersey, 1993

[11] Grossman, Sanford J., Angelo Melino, and Robert J. Shiller, 1987, Estimating the Continuous-time Consumption-based Asset Pricing Model, *Journal of Business and Economic Statistics* 5, 315-327

[12] Heston, S. and S. Nandi, 2000, A Closed-form GARCH Option Valuation Model, *Review of Financial Studies* 13, 585-625

[13] Longstaff, Francis A., 1989, Temporal Aggregation and the Continuous-time Capital Asset Pricing Model, *Journal of Finance* 44, 871-887

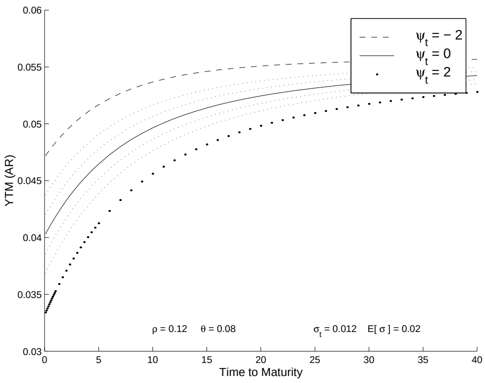
[14] Longstaff, Francis A., 1990, Time Varying Term Premia and Traditional Hypotheses about the Term Structure, *Journal of Finance* 45, 1307-1314

[15] Longstaff, F. and E. Schwartz, 1992, Interest Rate Volatility and the Term Structure : A Two-Factor General Equilibrium Model, *Journal of Finance* 47, 1259-1282

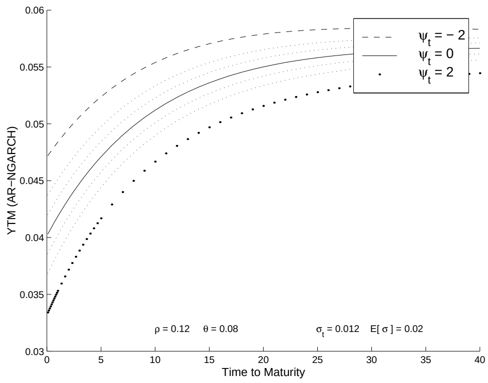
[16] Longstaff, F., 1993, The Valuation of Option on Coupon Bonds, *Journal of Banking and Finance,* 27-42

[17] Nelson, D.B., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica* 59, 347-370.

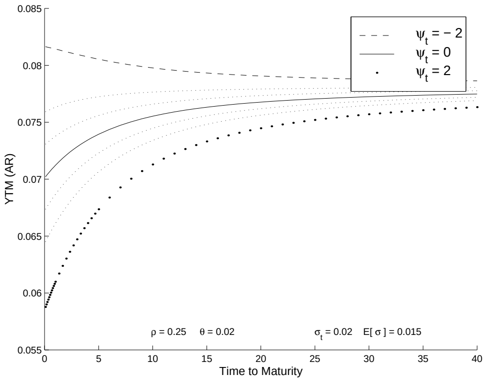
[18] Vasicek, Oldrich, 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177-188



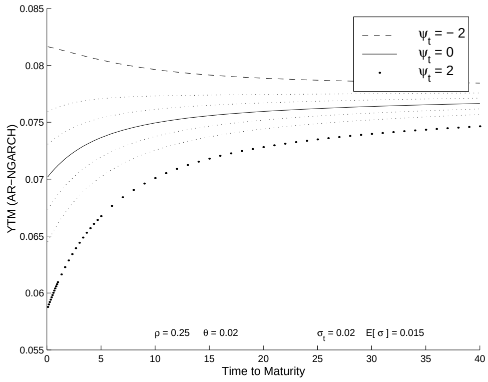
(a) $\rho = 0.12$, $\theta = 0.08$, $\sigma_t = 0.012$, $\beta_0/\theta = 0.02$



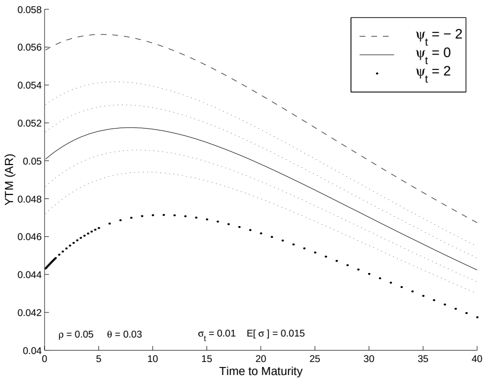
(b) $\rho = 0.12$, $\theta = 0.08$, $\sigma_t = 0.012$, $\beta_0/\theta = 0.02$



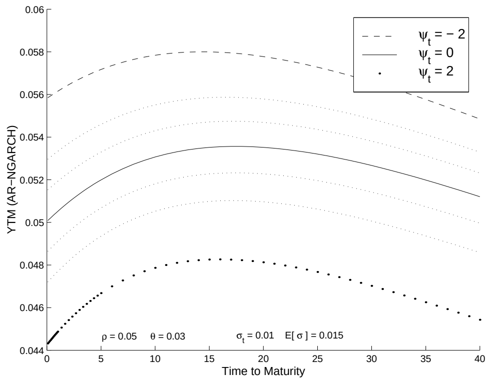
(c) $\rho = 0.25$, $\theta = 0.02$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.015$



(d) $\rho = 0.25$, $\theta = 0.02$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.015$

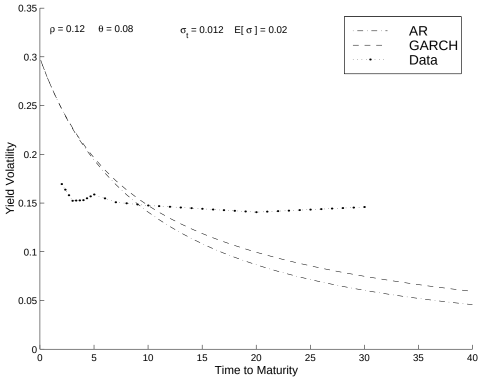


(e) $\rho = 0.05$, $\theta = 0.03$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.015$

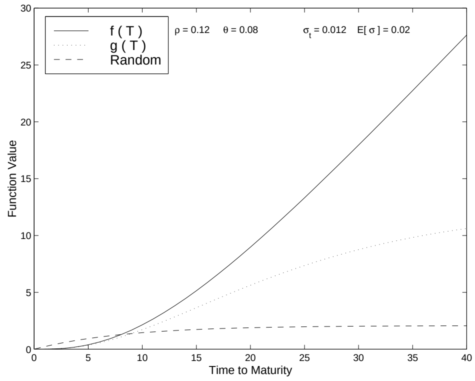


(f) $\rho = 0.05$, $\theta = 0.03$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.015$

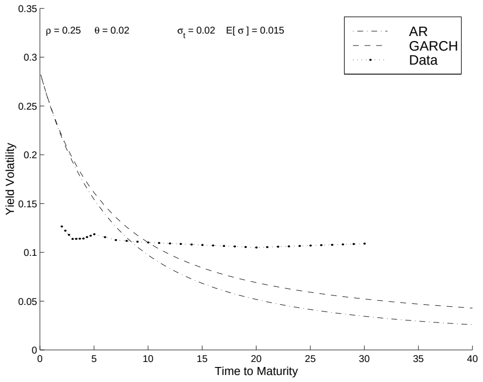
FIGURE 1. Typical patterns of yield curves for the case of AR(1) process or AR(1)-NGARCH(1,1) process and the desired result (8) is obtained-with the time to maturity.



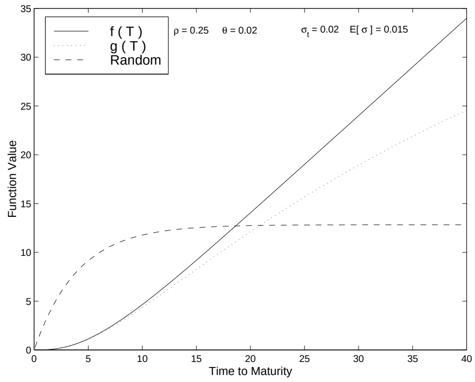
(a) $\rho = 0.12$, $\theta = 0.08$, $\sigma_t = 0.012$, $\beta_0/\theta = 0.02$



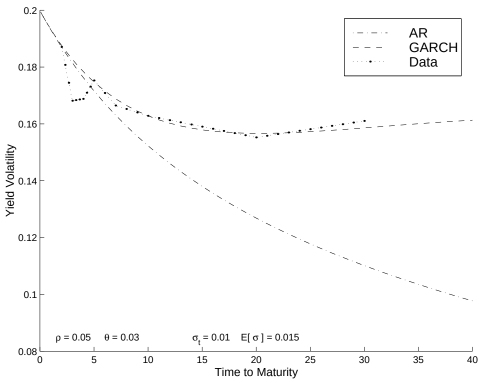
(b) $\rho = 0.12$, $\theta = 0.08$, $\sigma_t = 0.012$, $\beta_0/\theta = 0.02$



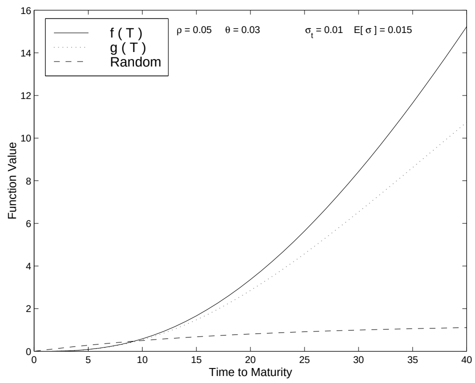
(c) $\rho = 0.25$, $\theta = 0.02$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.015$



(d) $\rho = 0.25$, $\theta = 0.02$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.015$

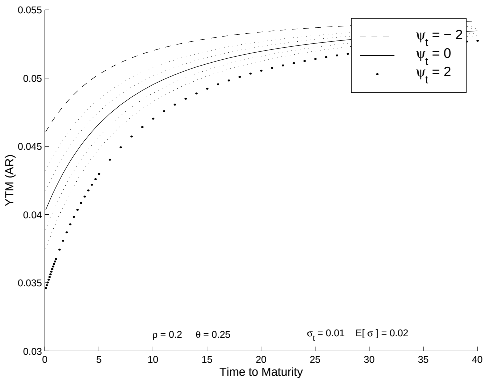


(e) $\rho = 0.05$, $\theta = 0.03$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.015$

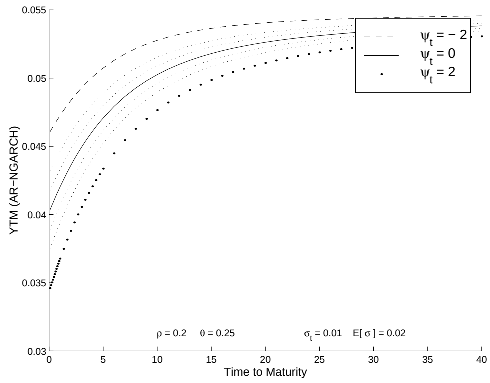


(f) $\rho = 0.05$, $\theta = 0.03$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.015$

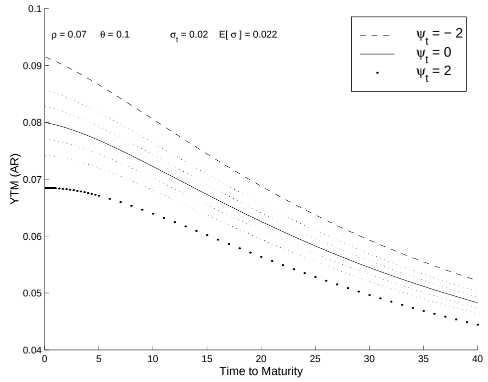
FIGURE 2. Typical patterns of volatility curves for the case of AR(1) process or AR(1)-NGARCH(1,1) process when function values appear in the yield curve and with the time to maturity.



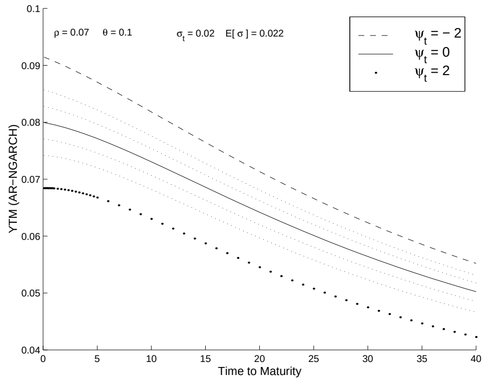
(a) $\rho = 0.20$, $\theta = 0.25$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.02$



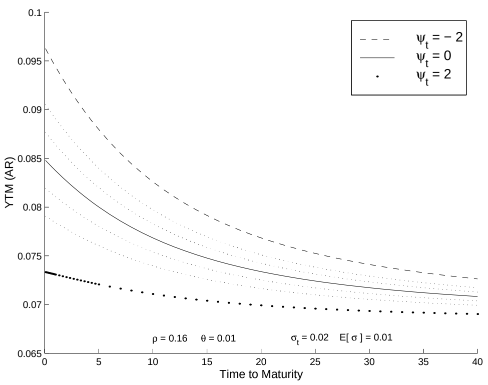
(b) $\rho = 0.20$, $\theta = 0.25$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.02$



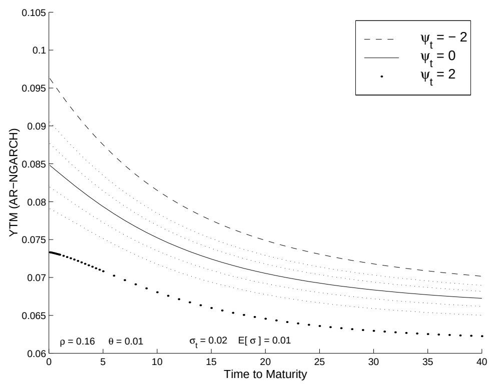
(c) $\rho = 0.07$, $\theta = 0.1$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.022$



(d) $\rho = 0.07$, $\theta = 0.1$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.022$

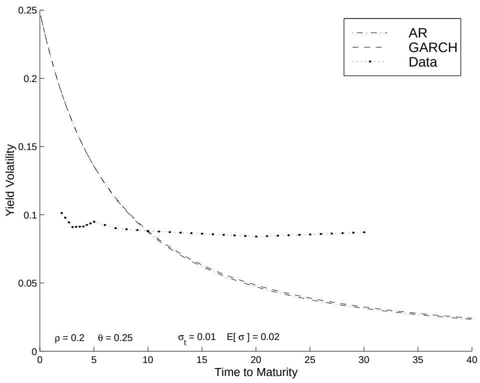


(e) $\rho = 0.16$, $\theta = 0.01$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.01$

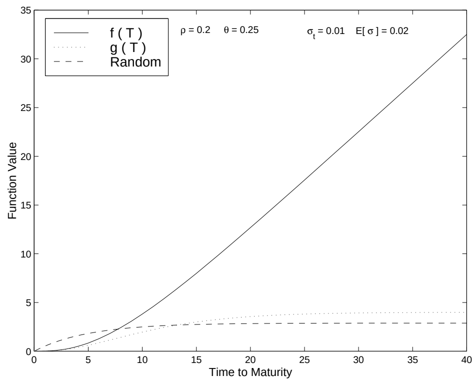


(f) $\rho = 0.16$, $\theta = 0.01$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.01$

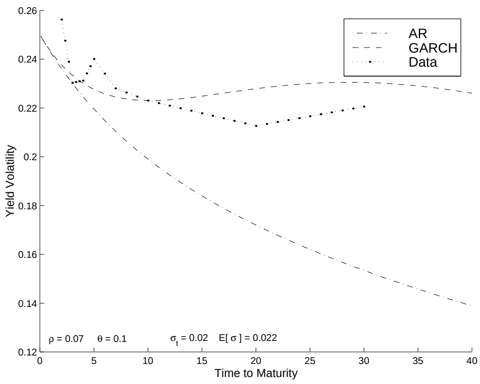
FIGURE 3. Typical patterns for yield curves for the case of AR(1) process or AR(1)-NGARCH(1,1) process and with the time to maturity.



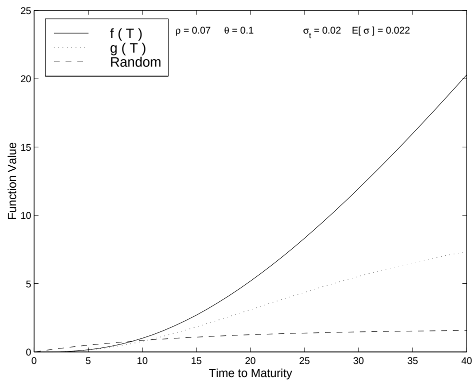
(a) $\rho = 0.20$, $\theta = 0.25$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.02$



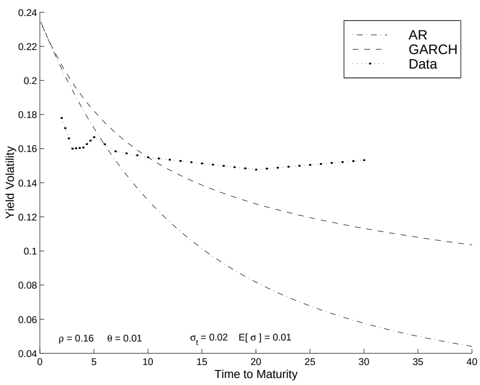
(b) $\rho = 0.20$, $\theta = 0.25$, $\sigma_t = 0.01$, $\beta_0/\theta = 0.02$



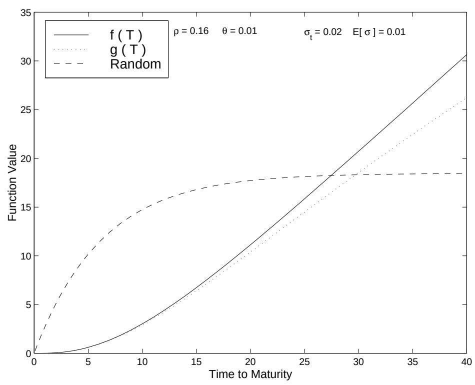
(c) $\rho = 0.07$, $\theta = 0.1$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.022$



(d) $\rho = 0.07$, $\theta = 0.1$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.022$



(e) $\rho = 0.16$, $\theta = 0.01$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.01$



(f) $\rho = 0.16$, $\theta = 0.01$, $\sigma_t = 0.02$, $\beta_0/\theta = 0.01$

FIGURE 4. Typical patterns for volatility curves for the case of AR(1) process or AR(1)-NGARCH(1,1) process where function values appear in the yield curve and with the time to maturity.