Risk Minimizing Option Pricing in a Markov Modulated Market *

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Abstract

We study option pricing in a regime switching market where the risk free interest rate, growth rate and the volatility of a stock depends on a finite state Markov chain. Using a minimal martingale measure we find explicit expressions for the risk minimizing option price and the corresponding hedging strategy.

1 Introduction

We consider option pricing in a regime switching market. We suppose that the state of the market is described by a finite state continuous time Markov chain $\{X_t,\ t \geq 0\}$ taking values in $\{1, 2, ..., M\}$. If $X_t = i$, the risk free interest rate is $r(i)$. The stock price process $\{S_t,\ t \geq 0\}$ is governed by a Markov modulated geometric Brownian motion, i.e., the drift and the volatility of S_t depends on X_t. The additional uncertainty arising due to the regime switching leads to incompleteness of the market. As a consequence there is no unique or fair price of an option on the stock S_t. At the same time the writer of the option cannot hedge himself perfectly. In other words every contingent claim in such a market will have an intrinsic risk. The option pricing in a regime switching framework has been studied by several authors using different approaches [2], [3], [7], [8], [10], [12] and [13]. In [5], Föllmer and Schweizer has addressed the option pricing in an incomplete market. By introducing a quadratic risk function they have obtained an abstract formula for the risk minimizing option price via the minimal martingale measure. In this paper we compute the minimal martingale measure P^* for the regime switching model and express the risk minimizing strategy under the minimal martingale measure P^*. We show that the risk minimizing option price satisfies a system of Black-Scholes partial differential equations with weak coupling; the coupling term representing the correction term arising due to regime switching. We also obtain the optimal

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mean self-financing strategy and the residual risk. Using a certain transformation we decouple the Black-Scholes system of equations into M number of decoupled Black-Scholes equations. We then obtain an explicit expression of the risk minimizing option price as the weighted average of the Black-Scholes options price in each regime.

Our paper is structured as follows. The model description is presented in Section 2. An risk minimizing strategy is described in Section 3. In Section 4, we obtain explicit expressions of option price, hedging strategies and other Greeks. We conclude our paper in Section 5 with a few remarks.

2 Model Description

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space. Let $\{X_t,\ t \geq 0\}$ be an irreducible Markov chain taking values in $\mathcal{X} := \{1, 2, ...M\}$ describing the state of the market. The evolution of X_t is given by

$$P(X_{t+\delta t} = j \mid X_t = i) = \lambda_{ij}\delta t + o(\delta t), \qquad i \neq j \tag{2.1}$$

where $\lambda_{ij} \geq 0,\ i \neq j;\ \lambda_{ii} = -\sum_{j=1}^{M} \lambda_{ij}$. Let $\Lambda = [\lambda_{ij}]$ denote the generating Q-matrix of the chain. We consider two assets: one (locally) risk free and the other risky. Let $r : \mathcal{X} \to [0, \infty)$ denote the (local) risk free interest rate; i.e., if the regime $X_t = i$, then the instantaneous interest rate is $r(i)$. Thus the interest rate process $r_t = r(X_t)$ is also an irreducible Markov chain taking values in $\mathcal{R} := \{r(1), r(2), ..., r(M)\}$ with the same generating matrix Λ. Let $\{B_t,\ t \geq 0\}$ denote the amount in the money market account at time t where the risk free interest rate is $r_t = r(X_t)$. If $B_0 = 1$, then

$$B_t = e^{\int_0^t r(X_s)ds}. \tag{2.2}$$

Thus

$$dB_t = r(X_t)B_t dt. \tag{2.3}$$

We assume that the risky asset is a stock whose price process $\{S_t,\ t \geq 0\}$ is governed by a Markov modulated geometric Brownian motion, i.e., the evolution of $\{S_t\}$ is given by

$$dS_t = \mu(X_t)S_t dt + \sigma(X_t)S_t dW_t \tag{2.4}$$

where $\{W_t,\ t \geq 0\}$ is a standard Wiener process independent of $\{X_t,\ t \geq 0\}$, $\mu : \mathcal{X} \to \mathbb{R}$ is the drift coefficient and $\sigma : \mathcal{X} \to (0, \infty)$ describes the volatility.

It would be convenient to write (2.1) in an equivalent way where $\{X_t\}$ is represented as a stochastic integral with respect to a Poisson random measure [6]. For $i, j \in \mathcal{X},\ \ i \neq j$, let $\triangle_{ij}$ be consecutive (w.r.t. to lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left closed right open intervals of the real line, each having length λ_{ij}. By embedding $\{1, 2, ..., M\}$ into $\mathbb{R}^M$, define a function $h : \mathcal{X} \times \mathbb{R} \longrightarrow \mathbb{R}^M$ by

$$h(i, z) = \begin{cases} j - i & \text{if} \quad z \in \triangle_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Then

$$dX_t = \int_{\mathbb{R}} h(X_{t-}, z) p(dt, dz) \tag{2.6}$$

where $p(dt, dz)$ is a Poisson random measure with intensity $dt \times m(dz)$, where $m(dz)$ is the Lebesgue measure on $\mathbb{R}$; $p(\cdot, \cdot)$ and $W(\cdot)$ are independent. Let $\tilde{p}(dt, dz)$ denote the corresponding compensated martingale measure. It is shown in [6] that $\{S_t, X_t\}$ is a Feller Markov process with infintesimal generator L whose action on smooth functions on $\mathbb{R} \times \mathcal{X}$ is given by

$$Lf(s, i) = \mu(i) s \frac{\partial f(s, i)}{\partial s} + \frac{1}{2} \sigma^2(i) s^2 \frac{\partial^2 f(s, i)}{\partial s^2} + \sum_j \lambda_{ij} f(s, j). \tag{2.7}$$

Let $\mathcal{F}_t = \sigma(S_t, X_t, t \geq 0)$. Without any loss of generality we assume that the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ is right continuous and P-complete. Let $T > 0$ be the planning horizon and H a European type contingent claim at time T. We wish to find the price of this contingent claim at any time $0 \leq t \leq T$. To this end we first find an equivalent martingale measure (EMM) P^* for this model. Set

$$\rho_T = \exp\{\int_0^T (\frac{r(X_s) - \mu(X_s)}{\sigma(X_s)}) dW_s - \frac{1}{2} \int_0^T (\frac{r(X_s) - \mu(X_s)}{\sigma(X_s)})^2 ds\}. \tag{2.8}$$

Let E denote the expectation under P. Then $E\rho_T = 1$ and $\{\rho_t,\ t \geq 0\}$ is an exponential martingale, where ρ_t is the expression given in (2.8) with t replacing T. Let P^* be defined by

$$\frac{dP^*}{dP} = \rho_T. \tag{2.9}$$

Then P^* is equivalent to P, and under P^*

$$\tilde{W}_t = W_t - \int_0^t (\frac{r(X_s) - \mu(X_s)}{\sigma(X_s)}) ds \tag{2.10}$$

is a standard Wiener process. Under P^* the dynamics of $\{S_t\}$ is given by

$$dS_t = r(X_t) S_t dt + \sigma(X_t) S_t d\tilde{W}_t. \tag{2.11}$$

Let $\{\tilde{S}_t\}$ denote the discounted stock price, i.e.

$$\tilde{S}_t = \frac{S_t}{B_t} = e^{-\int_0^t r(X_s) ds} S_t. \tag{2.12}$$

Then under P^* the dynamics of $\tilde{S}_t$ is given by

$$d\tilde{S}_t = \sigma(X_t)\tilde{S}_t d\tilde{W}_t. \tag{2.13}$$

Therefore $\{\tilde{S}_t\}$ is a martingale under P^*. This means that P^* is an EMM (equivalent martingale measure) for this model which implies that the model is arbitrage free. Hence an arbitrage free option price for the contingent claim H at time t is given by

$$B_t E^*[B_T^{-1} H \mid \mathcal{F}_t] = E^*[e^{-\int_t^T r(X_s)ds} H \mid \mathcal{F}_t] \tag{2.14}$$

where E^* denotes expectation under P^*. Let

$$\mathcal{M}(P) = \{\tilde{P} : \tilde{P} \equiv P\} \tag{2.15}$$

and $\{\tilde{S}_t\}$ is a martingale under $\tilde{P} \in \mathcal{M}(P)$. For a complete market $\mathcal{M}(P)$ is known to be a singleton [1], [11]. For an incomplete market with no arbitrage $\mathcal{M}(P)$ may have several elements. For each $\tilde{P} \in \mathcal{M}(P)$, the corresponding expression in (2.11) under $\tilde{P}$ is an arbitrage free price of H at t. Thus the option price is not unique. At the same time in an incomplete market, the writer of the option cannot hedge himself perfectly. Thus every contingent claim is associated with an intrinsic risk. In the next section we describe a risk minimizing option price in the framework of Fölmer and Schweizer [5].

3 Risk Minimizing Strategy

Let the contingent claim H at time T satisfy

$$H \in L^2(\Omega, \mathcal{F}, \mathcal{P}). \tag{3.1}$$

In order to replicate this claim we consider a strategy which involves the stock S_t and the money market account B_t, and which yields the terminal payoff H at time T. Let ξ_t and η_t denote the amounts invested in S_t and B_t respectively at time t; where $\xi = \{\xi_t,\ 0 \leq t \leq T\}$ is a predictable process satisfying

$$E\big[\int_0^T \xi_t^2 \sigma^2(X_t) S_t^2 dt + \big(\int_0^T \mid \xi_t \mid \mid \mu(X_t) \mid dt\big)^2\big] < \infty \tag{3.2}$$

and $\eta = \{\eta_t,\ 0 \leq t \leq T\}$ is an adapted process satisfying

$$E(\eta_t)^2 < \infty. \tag{3.3}$$

The value of the portfolio under the strategy $\pi = \{\pi_t,\ 0 \le t \le T\} = \{\xi_t, \eta_t,\ 0 \le t \le T\}$ at t is given by

$$V_t(\pi) = \xi_t S_t + \eta_t B_t. \tag{3.4}$$

The discounted value of the portfolio is given by

$$\tilde{V}_t(\pi) = \xi_t \tilde{S}_t + \eta_t. \tag{3.5}$$

The discounted cost accumulated upto time t is given by

$$\tilde{C}_t(\pi) = \tilde{V}_t(\pi) - \int_0^t \xi_u d\tilde{S}_u, \qquad 0 \le t \le T. \tag{3.6}$$

A strategy $\pi = \{\xi_t, \eta_t\}$ is said to be admissible

$$V_T(\pi) = H. \tag{3.7}$$

Note that for a self-financing strategy π, $\tilde{C}_t(\pi)$ is a constant. We look for an admissible strategy π which minimizes at each time t, the residual risk given by

$$R_t(\pi) := E[(\tilde{C_T}(\pi) - \tilde{C}_t(\pi))^2 \mid \mathcal{F}_t] \tag{3.8}$$

overall admissible strategies. We say that an admissible strategy π^* is risk minimizing if

$$R_t(\pi^*) \le R_t(\pi) \tag{3.9}$$

for any other admissible strategy π. In view of the results of [5] an admissible strategy π^* is optimal if the associated discounted cost process $\tilde{C}_t(\pi^*)$ is a square integrable martingale orthogonal to the martingale part $\{\tilde{S}_t\}$. We summarize this in the following Lemma.

LEMMA 3.1 An admissible strategy $\pi = \{\xi_t, \eta_t\}$ is optimal in the sense of (3.9) if the corresponding discounted cost $\tilde{C}_t(\pi)$ as in (3.6) is orthogonal to the martingale

$$M_t := \int_0^t \sigma(X_t)\tilde{S}_t dW_t. \tag{3.10}$$

Let $\tilde{H} = B_T^{-1}H$. It is shown in [5] that the existence of an optimal strategy is equivalent to the existence of a decomposition of $\tilde{H}$ in the form

$$\tilde{H} = \tilde{H}_0 + \int_0^T \xi_s^{\tilde{H}} d\tilde{S}_s \ + L_T^{\tilde{H}} \tag{3.11}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P})$, $\xi^{\tilde{H}} = \{\xi_t^{\tilde{H}}\}$ satisfies (3.2), and $L^{\tilde{H}} = \{L_t^{\tilde{H}}, 0 \le t \le T\}$ is a square integrable martingale orthogonal to the martingale $\{M_t,\ 0 \le t \le T\}$ (as in (3.10)). For the decomposition (3.11), the associated optimal strategy $\pi = (\xi_t, \eta_t)$ is given by

$$\xi_t = \xi_t^{\tilde{H}}, \quad \eta_t = \tilde{V}_t - \xi_t \tilde{S}_t. \tag{3.12}$$

with

$$\tilde{V}_t = \tilde{H}_0 + \int_0^t \xi_s^{\tilde{H}} d\tilde{S}_s \ + L_t^{\tilde{H}}, \qquad 0 \le t \le T. \tag{3.13}$$

Thus the discounted optimal cost $\tilde{C}_t(\pi)$ is given by

$$\tilde{C}_t(\pi) = \tilde{H}_0 + L_t^{\tilde{H}}. \tag{3.14}$$

We know define the minimal martingale measure for our model.

DEFINITION 3.1 An EMM $P' \equiv P$ is said to be minimal if $P' = P$ on $\mathcal{F}_0$, and if any square integrable P-martingale which is orthogonal to M (as in (3.6)) under P remains a martingale under P'.

In view of Theorem 3.5 in [5], it is easily seen that the unique minimal martingale measure in our case is given by

$$dP^* = \rho_T dP \tag{3.15}$$

where ρ_T is as in (2.7). In other words, the EMM P^* constructed in the previous section is the unique minimal martingale measure for our model. Note that the minimal martingale measure preserve orthogonality , i.e., for any square integrable martingale $\{L_t\}$ with $\langle L, M\rangle_t = 0$ under P satisfies

$$\langle L, M\rangle_t \ = 0 \qquad \text{under } P^*. \tag{3.16}$$

Also by Theorem 3.14 in [5], the optimal strategy, hence also the decomposition (3.11), is uniquely determined. In fact it can be determined in terms of the minimal martingale measure P^*. Note that $\{L_t^H\}$ is a square integrable martingale under P. Since P^* is the minimal martingale, $\{L_t^H\}$ is also a martingale under P^*. Thus $\tilde{V}_t$ as in (3.13) is a martingale under P^* which is the risk minimized discounted price of the H at t.

We now focus on a European call option on $\{S_t\}$ with strike price K and maturity time T. In this case the contingent claim H is given by

$$H = (S_T - K)^+. \tag{3.17}$$

For this case we now obtain the decomposition (3.11) so as to obtain the optimal strategy through (3.12) and (3.13). To this end consider the following system of partial differential equations

$$\frac{\partial\phi(t,s,i)}{\partial t}+\frac{1}{2}\sigma(i)^2s^2\frac{\partial^2\phi(t,s,i)}{\partial s^2}+r(i)s\frac{\partial\phi(t,s,i)}{\partial s}+\sum_{j=1}^{M}\lambda_{ij}\phi(t,s,j)=r(i)\phi(t,s,i) \tag{3.18}$$

for $i=1,2,...,M$, with the terminal condition

$$\phi(T,s,i)=(s-K)^+ \qquad \forall i. \tag{3.19}$$

The Cauchy problem (3.18)-(3.19) has a unique solution $\{\phi(t,s,i),\ \ i=1,2,...,M\}$ in the class of $C([0,T]\times\mathbb{R})\bigcap C^{1,2}((0,T)\times\mathbb{R})$ functions having at most polynomial growth [9]. Finally we have the following result.

THEOREM 3.1 Let $\{\phi(t,s,i),\ i=1,2,...,M\}$ denote the unique solution of the Cauchy problem (3.18), (3.19) in the above class of functions. Then
(i) $\phi(t,S_t,X_t)$ is the risk minimizing option price at time t;
(ii) An optimal strategy $\pi^*=\{\xi_t^*,\eta_t^*\}$ is given by

$$\xi_t^*=\frac{\partial\phi(t,S_t,X_{t-})}{\partial s} \tag{3.20}$$

$$\eta_t^*=\tilde{V}_t-\xi_t^*\tilde{S}_t \tag{3.21}$$

where

$$\begin{aligned}\tilde{V}_t &= \phi(0,X_0,S_0)+\int_0^t\frac{\partial\phi(u,S_u,X_{u-})}{\partial s}d\tilde{S}_u\\ &\quad+\int_0^t e^{-\int_0^u r(X_v)dv}\int_{\mathbb{R}}[\phi(u,S_u,X_{u-}+h(X_{u-},z))-\phi(u,S_u,X_{u-})]\tilde{p}(du,dz);\end{aligned} \tag{3.22}$$

(iii) The residual risk process is given by

$$R_t(\pi^*)=E[\int_t^T\sum_j\lambda_{X_{u_-j}}e^{-2\int_0^u r(X_v)dv}(\phi(u,S_u,j)-\phi(u,S_u,X_{u_-}))^2du\mid\mathcal{F}_t]. \tag{3.23}$$

Proof Let $0\le t\le T$. By applying Ito's formula to $e^{-\int_0^t r(X_u)du}\phi(t,S_t,X_t)$ under the measure P and using (2.4), (2.5), (2.6) and the PDE (3.18), we obtain after suitable rearrangement of terms

$$\begin{aligned}e^{-\int_0^t r(X_u)du}\phi(t,S_t,X_t) &= \phi(0,S_0,X_0)+\int_0^t\frac{\partial\phi(u,S_u,X_{u-})}{\partial s}d\tilde{S}_u\\ &\quad+\int_0^t e^{-\int_0^u r(X_v)dv}\int_{\mathbb{R}}[\phi(u,S_u,X_{u-}+h(X_{u-},z))-\phi(u,S_u,X_{u-})]\tilde{p}(du,dz).\end{aligned} \tag{3.24}$$

Letting $t \uparrow T$, we obtain

$$\begin{aligned} e^{-\int_0^T r(X_u)du}(S_T - K)^+ &= \phi(0, S_0, X_0) + \int_0^T \frac{\partial \phi(u, S_u, X_{u-})}{\partial s} d\tilde{S}_u \\ &+ \int_0^T e^{-\int_0^u r(X_v)dv} \int_{\mathbb{R}} [\phi(u, S_u, X_{u-} + h(X_{u-}, z)) - \phi(u, S_u, X_{u-})]\tilde{p}(du, dz). \end{aligned} \tag{3.25}$$

The desiring results (i) and (ii) now follow from (3.25). Finally the residual risk at time t is given by

$$\begin{aligned} R_t(\pi^*) &= E[\{\int_t^T \int_{\mathbb{R}} [e^{-\int_0^u r(X_v)dv}\{\phi(u, S_u, X_{u-} + h(X_u, z)) - \phi(u, S_u, X_{u-})\}]\tilde{p}(du, dz)\}^2 \mid \mathcal{F}_t] \\ &= E[\int_t^T \sum_j \lambda_{X_{u\_}j} e^{-2\int_0^u r(X_v)dv}(\phi(u, S_u, j) - \phi(u, S_u, X_{u\_}))^2 du \mid \mathcal{F}_t]. \end{aligned} \tag{3.26}$$

This completes the proof of the theorem. ■

4 Explicit Solutions and the Greeks

The equation (3.18) is a cooperative system of parabolic partial differential equations with weak coupling. It is cooperative in the sense that the Q-matrix Λ is irreducible. Thus the Markov Chain X_t does not have an absorbing state. This implies that each $\phi(t, s, i)$ depends on the other $\phi(t, s, j)$, $j = 1, 2, ..., M$. The coupling is weak in the sense that it occurs only through the zeroth order term. Using a transformation we decouple the system of equations (3.18) into M number of Black-Scholes PDE's. To this end we write (3.18) into a vector valued PDE. Set

$$\tilde{\phi}(t, s) = [\phi(t, s, 1), \phi(t, s, 2), ..., \phi(t, s, M)]'$$

where $'$ stands for transpose of a vector (or matrix).
Then the equation (3.18) can be written as

$$\frac{\partial \tilde{\phi}(t, s)}{\partial t} + \frac{1}{2}s^2\Sigma\frac{\partial^2 \tilde{\phi}(t, s)}{\partial s^2} + sR\frac{\partial \tilde{\phi}(t, s)}{\partial s} + \Lambda\tilde{\phi}(t, s) = R\tilde{\phi}(t, s) \tag{4.1}$$

where

$$\Sigma = diag[\sigma^2(1), \sigma^2(2), ..., \sigma^2(M)], \quad R = diag[r(1), r(2), ..., r(M)].$$

The terminal condition (3.19) becomes

$$\tilde{\phi}(T, s) = (s - K)^+\tilde{1} \tag{4.2}$$

where $\tilde{1} = [1, 1, ..., 1]'$. Let

$$\tilde{\phi}(t, s) = e^{\Lambda(T-t)}\tilde{\psi}(t, s) \tag{4.3}$$

where

$$\tilde{\psi}(t, s) = [\psi(t, s, 1), \psi(t, s, 2), ..., \psi(t, s, M)]'.$$

Substituting (4.3) in (4.1) and simplifying we obtain

$$\frac{\partial\tilde{\psi}(t, s)}{\partial t} + \frac{1}{2}s^2\Sigma\frac{\partial^2\tilde{\psi}(t, s)}{\partial s^2} + sR\frac{\partial\tilde{\psi}(t, s)}{\partial s} = R\tilde{\psi}(t, s). \tag{4.4}$$

The terminal condition (4.2) becomes

$$\tilde{\psi}(T, s) = \tilde{\phi}(T, s) = (s - K)^+\tilde{1}. \tag{4.5}$$

Now note that (4.4) is a system of M decoupled equations given by

$$\frac{\partial\psi(t, s, i)}{\partial t} + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2\psi(t, s, i)}{\partial s^2} + sr(i)\frac{\partial\psi(t, s, i)}{\partial s} = r(i)\psi(t, s, i), \quad i = 1, 2, ..., M. \tag{4.6}$$

The terminal condition (4.5) can be written as

$$\psi(T, s, i) = (s - K)^+, \quad i = 1, 2, ..., M. \tag{4.7}$$

For each i the equation (4.6) with the terminal condition (4.7) is a Black-Scholes PDE for the European call option with parameters $r(i), \sigma(i), K$. Thus the solution of (4.6), (4.7) is given by [11]

$$\psi(t, s, i) = s\Phi\Big(\frac{\log\frac{s}{K} + (r(i) + \frac{1}{2}\sigma^2(i))(T-t)}{\sigma(i)\sqrt{(T-t)}}\Big) - Ke^{-r(i)(T-t)}\Phi\Big(\frac{\log\frac{s}{K} + (r(i) - \frac{1}{2}\sigma^2(i))(T-t)}{\sigma(i)\sqrt{(T-t)}}\Big) \tag{4.8}$$

where as usual $\Phi(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{x} e^{\frac{-u^2}{2}}du$. Thus $\psi(t, s, i)$ is the price of a European call option with strike price K and terminal date T, where the interest rate is $r(i)$ and the volatility is $\sigma(i)$. We now obtain $\tilde{\phi}(t, s)$ from $\tilde{\psi}(t, s)$ using (4.3). Since Λ is a Q-Matrix, $e^{\Lambda t}$ is a probability transition matrix. Let

$$P(t) = [p_{ij}(t)] = e^{\Lambda t}. \tag{4.9}$$

Then

$$p_{ij}(t) = P(X_t = j \mid X_0 = i). \tag{4.10}$$

Therefore from (4.3), (4.9) and (4.10) it follows that

$$\phi(t, s, i) = \sum_{j=1}^{M} p_{ij}(T-t)\psi(t, s, j), \quad i = 1, ..., M, \tag{4.11}$$

where $\psi(t,s,j)$ is given in (4.8). Note that

$$p_{ij}(T-t) = P(X_{T-t} = j \mid X_0 = i) = P(X_T = j \mid X_t = i).$$

Hence when the stock price $S_t = s$, and the regime $X_t = i$, the risk minimizing option price $\phi(t,s,i)$ as given in (4.11) is the weighted average of the Black-Scholes prices in fixed regimes $j = 1, 2, \ldots, M$, with parameter $r(j), \sigma(j), K$ and with weights $p_{ij}(T-t) = P(X_T = j \mid X_t = i)$.

We now compute the risk minimizing hedging strategy. Let

$$\Delta^{\phi}(t,s,i) = \frac{\partial \phi(t,s,i)}{\partial s}, \quad i = 1, 2, ..., M \tag{4.12}$$

$$\Delta^{\psi}(t,s,i) = \frac{\partial \psi(t,s,i)}{\partial s}, \quad i = 1, 2, ..., M \tag{4.13}$$

and

$$\tilde{\Delta}^{\phi}(t,s) = [\Delta^{\phi}(t,s,1), ..., \Delta^{\phi}(t,s,M)]' \tag{4.14}$$

$$\tilde{\Delta}^{\psi}(t,s) = [\Delta^{\psi}(t,s,1), ..., \Delta^{\psi}(t,s,M)]'. \tag{4.15}$$

From (4.8) it follows that

$$\Delta^{\psi}(t,s,i) = \Phi\Big(\frac{\log \frac{s}{K} + (r(i) + \frac{1}{2}\sigma^2(i))(T-t)}{\sigma(i)\sqrt{(T-t)}}\Big). \tag{4.16}$$

Therefore from (4.3), (4.9) and (4.16) it follows that

$$\Delta^{\phi}(t,s,i) = \sum_{j=1}^{M} p_{ij}(T-t)\Phi\Big(\frac{\log \frac{s}{K} + (r(j) + \frac{1}{2}\sigma^2(j))(T-t)}{\sigma(j)\sqrt{(T-t)}}\Big). \tag{4.17}$$

Thus if at time t, the stock price $S_t = s$ and the regime $X_t = i$, then the risk minimizing hedging strategy $\xi_t^* = \Delta^{\phi}(t,s,i)$, as given in (4.17). Again we see that the risk minimizing hedging strategy is a weighted average of the hedging strategy in fixed regimes j with the weight given by $p_{ij}(T-t)$ which is the probability of regime switching from i at time t to j at the terminal time T. We now summarize these results in the following theorem.

Theorem 4.1 (i) The risk minimizing option price is a vector valued function given by

$$\tilde{\phi}(t,s) = [\phi(t,s,1), ..., \phi(t,s,M)]'$$

where $\phi(t,s,i)$ is the risk minimizing option price when the stock price $S_t = s$ and the regime $X_t = i$. This option price is a weighted average of the Black-Scholes option prices in fixed regimes; which is given explicitly in (4.11).

(ii) The risk minimizing hedging strategy is given by $\tilde{\Delta}(t,s) = [\Delta^\phi(t,s,1), ..., \Delta^\phi(t,s,M)]'$, where $\Delta^\phi(t,s,i)$ is the hedging strategy at t when $S_t = s$, $X_t = i$ (more precisely $X_{t-} = i$, so that the strategy is predictable); $\Delta^\phi(t,s,i)$ is the weighted average of the Black-Scholes hedging strategies in fixed regimes i. This strategy is given explicitly in (4.17).

Some comments are in order.

REMARK 4.1 (i) Since option price in a regime switching model has already been studied in the literature, it necessitates a comparison of our present work with the existing literature on this problem. We have addressed the risk minimizing option price in the framework of Föllmer and Schweizer [5]. To our knowledge this has not been done before. DiMasi et al [3] have studied the problem in the mean-variance set up. Guo [7] has addressed the problem by completing the market using a new security related to the cost of switching. The option price in [7] differs fundamentally from ours. Note that in [7] the option price formula depends on the drift parameters of the stock price whereas our option price formula has no explicit dependence on the drift process. The Δ-hedging and other Greeks are not addressed in [7], or for that matter in any paper in the existing literature. In [2], [10], the entire dynamics is described under a risk neutral measure. In particular in [2] the drift $\mu(X_t)$ of the stock process $\{S_t\}$ is different from the instantaneous interest $r(X_t)$ whereas in [10], it is assumed that $\mu(X_t) = r(X_t)$. Thus the option price formula in [2] has explicit dependence on μ whereas the option price formula in [10] is the same as that of ours. There is, however, a major difference in the interpretation of the option price formula in [10] and our option pricing formula (4.11). Our option pricing formula is valid under the real world market probability P, whereas the formula in [10] holds in an ideal risk neutral world. As a consequence, in our model the parameters $\lambda_{ij}, \sigma(i), r(i)$ etc. can be directly estimated from the market data, whereas the same quantities in [10] have to be estimated using specific risk neutral instruments such as federal bonds, treasury bills etc. To be more specific the $\sigma(X_t)$ in our model is the volatility of S_t as observed in the stock market whereas in [10] $\sigma(X_t)$ is the implied volatility. The same holds for all other parameters.

(ii) We can derive explicit expression of other greeks like gamma and theta. One can show that gamma retains the same characteristic as delta, but the nature of theta changes in the market modulated market.

5 Conclusions

We have studied the risk minimizing option price in the framework of Föllmer and Schweizer [5]. For our model we have obtained explicit expressions for the risk minimizing option pricing and the corresponding hedging strategy. Our method can be generalized to multi-dimensional case where there are n stocks which are correlated.

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