A stochastic volatility model for asset price estimation

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Abstract

In this paper, we propose a modification of the Heston (1993) model that can be used to effectively model any asset that follows a random process under stochastic volatility conditions in emerging markets like India. We use this idea to study the S&P CNX Nifty index process. We see that our simulation results match our analysis to a reasonable degree of accuracy. In the process we do an exact estimation of parameters for the volatility process, which we assume to follow the Cox-Ingersoll-Ross model. This, in turn, helps us to use this volatility prediction technique for derivative pricing and/or pricing of other nonlinear instruments.

KEY WORDS: Stochastic volatility, Cox-Ingersoll-Ross model, Gaussian log-likelihood estimator

1 Introduction

It has been observed that the celebrated GARCH model (and its variations like E-GARCH etc.) are highly inadequate for modelling volatilities in emerging markets in Asia like India (see for e.g., [3] and [2]). Hence we propose an alternative model for estimating asset price movements by suitably modifying the idea of Heston [1]. We indirectly model an asset price $\{N_t\}_{t\geq 0}$ by proposing the following SDEs for its log-return process $\{R_t\}_{t\geq 0}$:

$$\begin{aligned} dR_t &= \gamma_t dt + \sqrt{\sigma_t} dW_t, && (1) \\ d\sigma_t &= \alpha(\mu - \sigma_t)dt + \delta\sqrt{\sigma_t} dB_t, && (2) \end{aligned}$$

where $\{\sigma_t\}_{t\geq 0}$ is the return-volatility which follows the well-known Cox-Ingersoll-Ross (CIR) stochastic volatility model and $d[W,B]_t = \rho dt,\ -1 \leq \rho \leq 1$. The rest of the paper is organized as follows: Section 2 deals with the technique of estimating the parameters for CIR return-volatility model. Section 3 discussed the algorithms associated with equations (1) and (2). Section 4 discussed our co-simulation methodology and results while Section 5 gives a brief summary of future extensions and our ongoing research in this domain. We end the paper with an Appendix (Section 6) that gives our graphical outputs for the co-simulation.

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2 Approximation and parameter estimation

For time points $t,\ s,\ t > s$ and when they are close enough i.e., $\triangle s > 0$ is quite small where $\triangle s \equiv t - s$, we have σ_t distributed as $N(m(\sigma_s, \triangle s), v(\sigma_s, \triangle s))$ where:

$$m(\sigma_s, \triangle s) \equiv E[\sigma_t] = \sigma_s e^{-\alpha\triangle s} + \mu(1 - e^{-\alpha\triangle s}), \tag{3}$$

$$v(\sigma_s, \triangle s) \equiv Var[\sigma_t] = \sigma_s \frac{\delta^2}{\alpha}(e^{-\alpha\triangle s} - e^{-2\alpha\triangle s}) + \frac{\mu\delta^2}{2\alpha}(1 - e^{-\alpha\triangle s})^2. \tag{4}$$

With computation schemes in mind, we rewrite equations (3) and (4) in approximate form as follows:

$$m(\sigma_s, \triangle s) \approx \sigma_s + \alpha(\mu - \sigma_s) \triangle s, \tag{5}$$

$$v(\sigma_s, \triangle s) \approx \sigma_s \delta^2 \triangle s. \tag{6}$$

Given $n + 1$ equidistant observed instantaneous volatilities $\sigma_0, \sigma_1, \ldots, \sigma_n$ separated by $\triangle t$, we use as estimator the parameters' (μ, α, δ) value which maximizes the Gaussian log-likelihood function (given as $(\mu^*, \alpha^*, \delta^*)$):

$$l_n(\mu, \alpha, \delta) \stackrel{def}{=} \sum_{i=1}^{n} \ln \phi(\sigma_i; m(\sigma_{i-1}, \triangle t|(\mu, \alpha, \delta)), v(\sigma_{i-1}, \triangle t|(\mu, \alpha, \delta))), \tag{7}$$

where the dependence on the parameters has been made notationally explicit and $\phi(y; m, v)$ is the Normal density function given by:

$$\phi(y; m, v) \stackrel{def}{=} \frac{\exp\left(-\frac{(y-m)^2}{2v}\right)}{\sqrt{2\pi v}}. \tag{8}$$

Using equations (5) and (6) above, we write the equations for calculating $(\mu^*, \alpha^*, \delta^*)$ as follows:

$$\frac{\partial l_n(\mu, \alpha, \delta)}{\partial \mu} |_{\mu=\mu^*} \equiv \frac{\alpha^*}{\delta^{*2}} \sum_{i=1}^{n} \frac{(\sigma_i - \sigma_{i-1}) - \alpha^*(\mu^* - \sigma_{i-1}) \triangle t}{\sigma_{i-1}} = 0, \tag{9}$$

$$\frac{\partial l_n(\mu, \alpha, \delta)}{\partial \alpha} |_{\alpha=\alpha^*} \equiv \frac{1}{\delta^{*2}} \sum_{i=1}^{n} \frac{(\sigma_i - \sigma_{i-1}) - \alpha^*(\mu^* - \sigma_{i-1}) \triangle t}{\sigma_{i-1}}(\mu^* - \sigma_{i-1}) = 0, \tag{10}$$

$$\frac{\partial l_n(\mu, \alpha, \delta)}{\partial \delta} |_{\delta=\delta^*} \equiv -\frac{n}{\delta^*} + \frac{1}{\delta^{*3} \triangle t} \sum_{i=1}^{n} \frac{((\sigma_i - \sigma_{i-1}) - \alpha^*(\mu^* - \sigma_{i-1}) \triangle t)^2}{\sigma_{i-1}} = 0. \tag{11}$$

Solving the above equations (9), (10) and (11) for $(\mu^*, \alpha^*, \delta^*)$, we get

$$\mu^* = \frac{\left(\sum_{i=1}^{n} \frac{\sigma_i}{\sigma_{i-1}}\right) \sum_{i=1}^{n} \sigma_{i-1} - n \sum_{i=1}^{n} \sigma_i}{n(\sum_{i=1}^{n} \frac{\sigma_i}{\sigma_{i-1}} - n) - (\sigma_n - \sigma_0) \sum_{i=1}^{n} \frac{1}{\sigma_{i-1}}},$$

$$\alpha^* = \frac{n(\sum_{i=1}^{n} \frac{\sigma_i}{\sigma_{i-1}} - n) - (\sigma_n - \sigma_0) \sum_{i=1}^{n} \frac{1}{\sigma_{i-1}}}{\left(\sum_{i=1}^{n} \frac{1}{\sigma_{i-1}}\right) \sum_{i=1}^{n} \sigma_{i-1} - n^2},$$

$$\delta^* = \sqrt{\sum_{i=1}^{n} \frac{((\sigma_i - \sigma_{i-1}) - \alpha^*(\mu^* - \sigma_{i-1}) \triangle t)^2}{n\sigma_{i-1} \triangle t}}. \tag{12}$$

3 Algorithms

Now we discuss the algorithms to generate the simulated asset log-return $\{\hat{R}_t\}_{t\geq 0}$ and return-volatility $\{\hat{\sigma}_t\}_{t\geq 0}$. We propose the following algorithms for $\triangle t > 0$ small enough:

$$\hat{\sigma}_{t+\triangle t} = \hat{\sigma}_t + \alpha^*(\mu^* - \hat{\sigma}_t) \triangle t + \delta^* \sqrt{\hat{\sigma}_t}(B_{t+\triangle t} - B_t), \ \hat{\sigma}_0 = \sigma_0, \tag{13}$$

where $B_{t+\triangle t} - B_t$ is distributed as $N(0, \triangle t)$ and

$$\hat{R}_{t+\triangle t} = \hat{R}_t + \gamma_t \triangle t + \sqrt{\hat{\sigma}_t}(W_{t+\triangle t} - W_t), \ \hat{R}_0 = R_0, \tag{14}$$

where $W_{t+\triangle t} - W_t$ is distributed as $N(0, \triangle t)$, $Corr[W_{t+\triangle t} - W_t, B_{t+\triangle t} - B_t] = \rho$, $(\mu^*, \alpha^*, \delta^*)$ are the parameter values those maximize (7) and, σ_0 and R_0 are the corresponding obeserved quantitites at time 0.

4 Co-simulation results

We have co-simulated our algorithms (13) and (14) on a set of about 4000 data points of the Nifty index from 3rd July 1990 to 29th December 2006. We will use the *directional trend* measure for checking the accuracy of our model. This measure gives the percentage of time the observed and estimated movements are in the same direction i.e., both are increasing together or decreasing together. Note that we have used the estimated present day's log-return $\hat{R}_t$ and the previous day's observed index value N_{t-1} to get the estimate of the present day's index value $\hat{N}_t$ using the formula $\hat{N}_t = N_{t-1}\exp(\hat{R}_t)$. We observe the following results:

1. Index volatility prediction is about 66% accurate for in-sample forecasts.

2. The Nifty index prediction is about 60% accurate for the same.

3. For the actual out-of-sample forecasts, we get about **70**% accuracy for the stochastic volatility of the Nifty index.

4. We hit 60% accuracy for the Nifty index out-of-sample forecast.

For graphical output, please see the Appendix at the end.

5 Conclusions and Future Directions

We are in the process of developing a methodology for using this volatility estimation technique for the pricing of general nonlinear instruments. We are also testing equivalent alternative models for such emerging markets which captures the transient epochs of dependencies in the market data. We have seen that this modelling technique has strong implications for pricing of index options, portfolio selection, development of optimal hedging strategies as well as risk management.

References

[1] Heston S., 1993, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies 13 (3), pp. 585-625.

[2] Pandey A., *Modelling and forecasting volatility in Indian capital markets*, Indian Institute of Management, Ahmedabad 380015, India.

[3] Lim K.-P., Hinich M. J., and Liew K.-S. V., *Statistical inadequacy of GARCH models for Asian stock markets: evidence and implications*, Journal of Emerging Market Finance, 4:3, 2005, Sage Publ., London.

6 Appendix

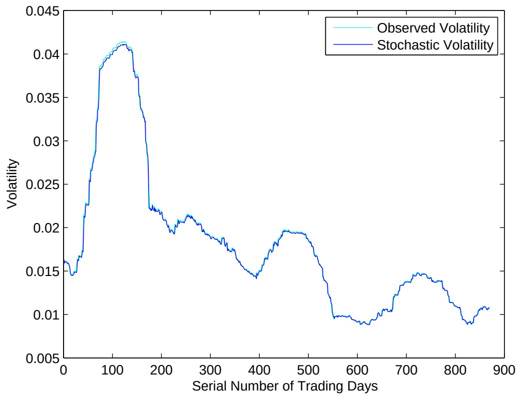

Figure 1: In-sample Volatility Comparison

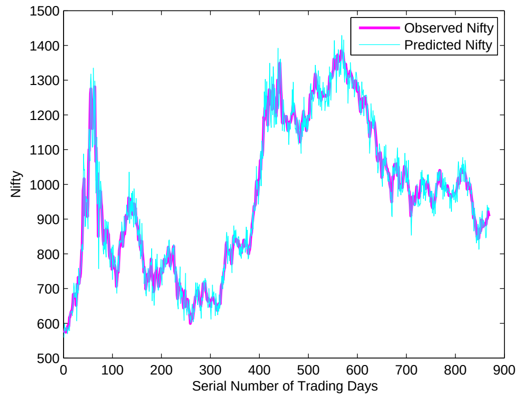

Figure 2: In-sample Nifty Index Comparison

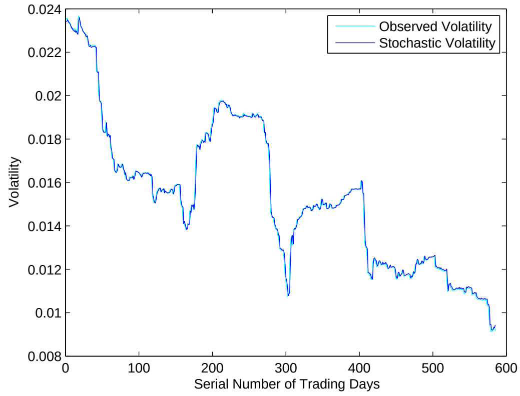

Figure 3: Out-of-sample Volatility Comparison

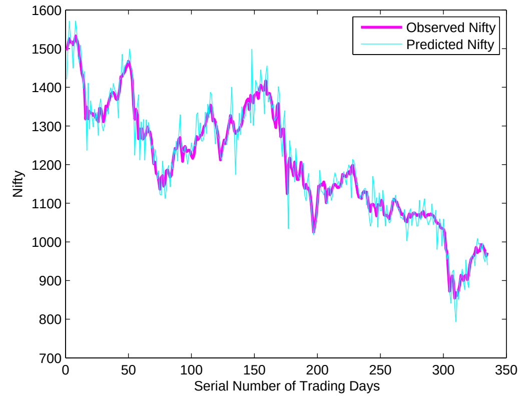

Figure 4: Out-of-sample Nifty Index Comparison