VIX Option

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Abstract

Substantial progress has been made in developing more realistic option pricing models for S&P 500 index (SPX) options. Empirically, however, it is not known whether any by how much each generalization of SPX price dynamics improves VIX option pricing. This paper fills this gap by first deriving a VIX option model that allows simultaneous correlated and state-dependent jumps in stochastic volatility and SPX returns. Using both VIX options and VIX futures, this paper examines several alternative models from three perspectives: internal consistency of parameters with relevant time-series data, and out-of-sample pricing. Overall, incorporating stochastic volatility, price jumps and state-dependent correlated volatility jumps are important for pricing and internal consistency.

Keywords: VIX option, VIX futures, VIX, stochastic volatility, state-dependent jumps, internal consistency

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1. Introduction

In contrast to the implied volatility extracted from an option pricing model,[1] the VIX (volatility index) uses a model-free formula to derive expected volatility directly from the prices of a weighted strip of S&P 500 index (SPX) options over a wide range of strike prices, which incorporates information from the volatility skew. This VIX calculation supplies a script for replicating the VIX squared with a static portfolio of SPX options. This critical fact lays the foundation for tradable products based on the VIX, facilitating hedging and arbitrage of VIX derivatives.

The Chicago Board Options Exchange (CBOE) launched VIX futures on March 26, 2004 and VIX options on February 24, 2006. These were the first of an entire family of volatility products traded in exchanges. VIX futures (VIX options) are the futures (options) contracts on forward 30-day implied volatility. The current prices of both VIX futures and VIX options consequently reflect the market's expectation of the VIX level at expiration.[2] As a result of expectation VIX futures prices can swing from a premium to a discount to spot VIX, but converge to spot VIX at expiration. Hence, the current VIX futures price, rather than the spot VIX itself, is the underlying price of VIX options.[3]

Although the construction of VIX squared is model-free, the simple formula for the fair value of VIX futures given in the CBOE website is model-dependent. It involves the variance of the VIX futures price from current time to its expiry. Hence, this study examines the effect using alternate variance models to calculate the fair

[1] The idea of developing a volatility index was first suggested by Brenner and Galai (1989). In a follow-up paper, Brenner and Galai (1993) have introduced a volatility index based on implied volatilities from at-the-money options. The same idea is also described in Whaley (1993).
[2] The forward view offered by volatility implied by the market prices of VIX futures or VIX options is often regarded as bona fide investor fear gauge (Whaley, 2000).
[3] For example, on May 19, 2006 that was 98 days to the expiry of both VIX AUG Options and VIX AUG Futures, the spot VIX index was trading at 12.8 while VIX AUG Futures was 156.0 or 15.6 for comparison. The price for VIX AUG 15 Call was by 1.65 while VIX AUG 15 Put was traded at 1.10. This is the consequence that VIX options pricing is based on VIX futures rather than the spot VIX.

market value of VIX futures and thus the theoretical value of VIX options. In particular, the derivation for the theoretical VIX futures price and VIX option price is based on affine stochastic-volatility models with simultaneous jumps both in the asset price and variance processes. To the best of the author's knowledge, this process is one of the most general specifications for the SPX price in the literature (see Andersen, Benzoni, & Lund, 2002; Alizadeh, Brandt, & Diebuld, 2002; Duffie, Pan, & Singleton, 2000; Eraker, Johannes, & Polson, 2003; Eraker, 2004). For comparison, its nested models are also taken into account. The VIX futures formulas considered here are exactly the same as the one given in the CBOE website. The proof also answers the question of why this study uses a stochastic-volatility model as a benchmark for VIX futures pricing given that the CBOE uses a simple model.[4]

The literature on the price behavior of VIX spot and futures markets is growing fast. Carr and Wu (2006) present that the price of the VIX futures stays within a lower bound and an upper bound. The lower bound is the forward volatility swap rate and the upper bound is the forward variance swap rate. Dupire (2006) derives the concavity adjustment, which needs to be subtracted from the price of forward variance to arrive at the fair value of VIX futures. Zhang and Zhu (2006) posit a stochastic variance model of VIX time evolution and develop a numerical expression for the VIX futures price. Zhu and Zhang (2007) value VIX futures based on the term structure of forward variance. Dotsis, Psychoyios, and skiadopoulos (2007) empirically compare continuous-time models for implied volatility indices and point out the importance of the jump and mean reversion.

The literature on the developments in volatility options has emerged after 1987 crash. Brenner and Galai (1989, 1993) first suggested options written on a volatility

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[4] See "Additional Features of VIX Futures" at http://cfe.cboe.com/education/vixprimer/Features.aspx.

index. Various models to price volatility options written on the instantaneous volatility have also been developed (see Detemple & Osakwe, 2000; Grünbichler & Longstaff, 1996; Whaley, 1993). These models differ in the specification of the assumed stochastic process, and the assumptions made about the volatility risk premium. For the volatilities depending only on the instantaneous variance, any options can be hedged perfectly with a combination of any other option plus stock. The volatility skew, appropriately defined, is thus constant. The hedging effectiveness of those volatility options compared to that of plain-vanilla options has been studied by Psychoyios and Skiadopoulos (2006). Which dynamics are consistent with market prices? From historical developments in volatility derivatives pricing, the prior studies attempt to add factors other than one-factor stochastic volatility models. From Principle Component Analysis (PCA) of volatility surface time series (Jarrow, 2002), there are at least three important sources of fluctuation: level, term structure, and skew. It makes sense to add at least one more factor. Variance curves are more realistic in the two-factor case. For example, they can have humps. Historical attempts to add factors include Dupire's (1996) unified theory of local volatility; Derman et al.'s (1998) stochastic implied volatility which has, under diffusion, complex no-arbitrage condition, and is thus impossible to work with in practice; and variance curve models (Dupire, 1993; Bergomi, 2005; Balland, 2006; Buehler, 2006) that assume variances are tradable and allow for simple no-arbitrage condition. Rather than directly modeling volatility dynamics, this study reconciles the growing literature of SPX price processes to investigate how much each generalization of the SPX price dynamics improves VIX option pricing and hedging.

Since VIX futures and VIX options are tradable, and VIX futures prices are martingales under the risk-neutral measure, this study imposes consistent dynamics on forward VIX as a result of SPX price processes. This study evaluates the VIX options from four classes of the SPX price dynamics. The first class (SV model) consists of Heston's (1993) model with volatility parameterized as in Cox, Ingersoll, and Ross (1985). This class of models has the advantages of non-negative variance and capturing level-dependent volatility. The second class (SVJ model) consists of Bates (1996) and the Bakshi, Cao, and Chen (1997) models with asset returns containing price jumps. The third class (SVCJ model) indicates Duffie et al.'s (2000) model with correlated jumps both in the SPX return and volatility. The fourth class (SVSCJ model) extends the mean intensity of the jump frequency to be state-dependent as in Eraker (2004). Duffie et al. (2000) and Eraker (2004) illustrate that the SVSCJ model generalizes the models of SV model, SVJ model, and SVCJ model for the dynamics of the SPX price. It should be emphasized, however, that the main motivation here is to study how such underlying processes are applied to model forward VIX and thus the valuation of the VIX futures and VIX options. This study then demonstrates that the resultant formula for the prices of VIX futures and VIX options based on the SVSCJ model collapses to the ones under the SVCJ, SVJ, and SV models, respectively.

This study uses the most recent one-month joint VIX futures and VIX options to investigate the option pricing performance of the SV, SVJ, SVCJ and SVSCJ models. The whole sample period covers February 26, 2006 to November 30, 2007. The resultant parameter estimates are then adopted to investigate models' internal consistency and out-of-sample pricing. These two yardsticks judge the alternative models from different perspectives. First, the reasoning of the consistency test adopted by Bates (1996) and Bakshi et al. (1997) is that if an option model is correctly specified, its structural parameters implied by option prices will necessarily be consistent with those implicit in the observed time-series data. Second, out-of-sample pricing errors give a direct measure of model misspecification. In particular, while a more complex model will generally lead to better in-sample fit, it will not necessarily perform better out of sample as any overfitting may be penalized. Our results show that complex jump specifications add explanatory power in fitting options. But, incorporating price jumps mainly improves short-term out-of-sample pricing fits, whereas allowing for volatility jumps fares better in fitting the medium- and long-dated VIX options.

The rest of the paper is organized as follows: In the next section, we present the general model for the index price dynamics as formulated in Duffie et al. (2000) and Eraker (2004), and discuss implications for VIX option pricing. Section 3 presents the data. Section 4 discusses the econometric design and evaluates the in-sample fit of each model. Section 5 assesses the extent of each model's misspecification, including out-of-sample pricing results. Concluding remarks are offered in Section 6. Proof of pricing equations and most formulas are provided in the Appendix.

2. Models and Pricing

The stochastic volatility model accompanied with state-dependent and correlated jumps in both asset returns and volatility (denoted as SVSCJ) is the most general process in the equity derivatives literature (Eraker, 2004). This study first extends model-free implied volatility of Britten-Jones and Neuberger (2000) and Jiang and Tian (2005) to the asset price process under the SVSCJ specification. We then demonstrate that the VIX formula is still valid when the underlying asset price process includes volatility jumps. The second is to price all VIX futures of various maturities using the martingale pricing theory. The third is to price VIX options taking as given the market prices of the VIX futures. Overall, our methodology imposes stochastic structure directly on the evolution of the SPX price. It has a closed-form solution to the VIX options built on the resultant forward VIX term structure.

2.1 The S&P 500 Price Process

Although the construction of VIX is model-free, the fair values of VIX futures and VIX options are model-dependent. Different dynamics for the index price result in various expressions for VIX and thus different theoretical formulas for the VIX futures (Lin, 2007) and VIX option prices. A number of earlier studies (see Bates, 1996; Bakshi et al., 1997; Andersen et al., 2002; Chernov, Ghysels, Gallant, & Tauchen, 2003) point out the importance of stochastic volatility and price jumps to equity price models. Andersen, Bollerslev, Diebuld, and Ebens (2001), Alizadeh et al. (2002), and Eraker et al. (2003) further find the presence of an additional, rapidly moving factor driving conditional volatility, which, unlike jumps in returns, has a persistent component. Jumps in volatility provide such a factor. Together, this suggests a strong evidence for volatility driven by the diffusive and jump components. Jump models, however, typically specify jumps to arrive with constant intensity. This assumption poses problems in explaining the tendency of large movements to cluster over time. Bates (2000), Pan (2002), and Eraker (2004) use a linear specification $\lambda_0 + \lambda_1 v_t$, for instantaneous variance v and some nonnegative constants λ_0 and λ_1, of jump-arrival intensity to allow jumps to arrive more frequently in high-volatility regimes. The stochastic-volatility model with state-dependent and correlated jumps, both in the asset price and variance (SVSCJ model), is the most general process for the SPX price considered in this study.

Under the SVSCJ, there exists a forward measure *F* such that the forward asset price, denoted as $F_t(T) = S_t / B(t,T)$, is a martingale where S_t is the spot index price minus the present value of all expected future dividends to be paid prior to the option maturity, and $B(t,T)$ is time-*t* price of a zero-coupon bond that pays \$1 at time T. Note that the forward measure *F* is equivalent to the risk-neutral measure *Q* when interest rates are deterministic, which is the assumption of this study.

$$\frac{dF_t(T)}{F_t(T)} = \sqrt{v_t}\, d\omega_{S,t} + J_t dN_t - \lambda_t \kappa\, dt \tag{1}$$

$$d v_t = \kappa_v(\theta_v - v_t)\, dt + \sigma_v \sqrt{v_t} d\omega_{v,t} + z_v dN_t \tag{2}$$

where $J_t = \exp(z_S) - 1$ is the percentage price jump size with mean κ. Satisfying the no-arbitrage condition, $\kappa = \exp(\mu_j + \sigma_j^2/2)/(1-\rho_j \mu_v) - 1$. $\omega_{S,t}$ and $\omega_{v,t}$ are correlated Brownian motions with $\rho\, dt = \mathrm{corr}(d\omega_{S,t}, d\omega_{v,t})$. They are independent of the compound Poisson processes $z_S dN_t$ and $z_v dN_t$. The instantaneous variance v follows a mean-reverting square-root process with exponentially-distributed jump size z_v that is correlated to price jump size z_S by $z_S = \mu_j + \rho_j z_v$. Formally, jumps in volatility are assumed to have an exponential distribution, i.e., $z_v \sim \exp(\mu_v)$, whereas jumps in asset log-prices are normally distributed conditional on the realization of z_v, i.e., $z_S \mid z_v \sim N(\mu_j + \rho_j z_v, \sigma_j^2)$. Thus, z_S has mean $\mathrm{E}(z_S) = \mu_j + \rho_j \mu_v$, variance $\mathrm{var}(z_S) = \sigma_j^2 + \rho_j^2 \mu_v^2$, and is correlated with z_v by $\rho_j \mu_v / \sqrt{\sigma_j^2 + \rho_j^2 \mu_v^2}$. The underlying return and its volatility share the same jump arrival uncertainty followed by a Poisson process N_t with state-dependent intensity $\lambda_t = \lambda_0 + \lambda_1 v_t$. Further, κ_v, θ_v, and σ_v are the speed of adjustment, long-run mean, and variation of instantaneous variance v.

2.2 VIX and Expected Total Return Variance

Britten-Jones and Neuberger (2000) derived the model-free implied volatility under diffusion assumption, while Jiang and Tian (2005) generalized it to processes with price jumps. This study further generalizes the model-free implied volatility to the processes with correlated jumps in asset returns and volatility. We then demonstrate that model-free implied volatility is the VIX. This justifies the validness of the VIX formula under a general asset price process that includes both price jumps and volatility jumps.

Applying Itô's lemma to the SVSCJ process in equation (1), we have

$$d \ln F_t(T) = -\frac{1}{2} v_t dt + \sqrt{v_t}\, d\omega_{S,t} + \ln(1 + J_t) dN_t - \kappa \lambda_t\, dt \tag{3}$$

where $\ln(1+J_t) \approx J_t - \frac{1}{2} J_t^2$. Thus, we have the log profile for the SVSCJ,

$$\begin{aligned} d \ln F_t(T) &\approx -\frac{1}{2}\left(v_t dt + J_t^2 dN_t\right) + \sqrt{v_t}\, d\omega_{S,t} + J_t dN_t - \kappa \lambda_t\, dt \\ &\approx -\frac{1}{2}\left(v_t dt + J_t^2 dN_t\right) + \frac{dF_t(T)}{F_t(T)} \end{aligned} \tag{4}$$

Integrating both sides, the total variance of the instantaneous rate of return of the S&P 500 over a period is equal to twice the difference between the total variations of the continuously compounded and simple rates of return of the forward S&P 500 price ($F_t(T)$) over the period:

$$\int_t^T (\nu_u du + J_u^2 dN_u) \approx 2\int_t^T \left(\frac{dF_u(T)}{F_u(T)} - d\ln F_u(T) \right)$$
$$= \frac{2}{F_t(T)}[F_T(T) - F_t(T)] + 2[\ln F_t(T) - \ln F_T(T)] \tag{5}$$

A Taylor expansion of the logarithmic term $\ln(F_T(T)/F_t(T))$ shows that it can be replicated by trading stock index futures and a continuum of out-of-the-money (OTM) options:

$$\begin{aligned}
-\int_t^T d\ln F_u(T) = \ln F_t(T) - \ln F_T(T) &\approx \int_0^\infty \frac{(F_T(T)-K)^+ - (F_t(T)-K)^+}{K^2} dK \\
&= \int_0^\infty \frac{C_T(T,K) - (F_t(T)-K)^+}{K^2} dK \\
&= \int_0^\infty \frac{C_T(T,K)}{K^2} dK - \int_0^{F_t} \frac{(F_t(T)-K)}{K^2} dK \\
&\approx \int_0^{F_t} \frac{P_T(T,K)}{K^2} dK + \int_{F_t}^\infty \frac{C_T(T,K)}{K^2} dK + \frac{(F_T(T)-F_t(T))}{F_t}
\end{aligned} \tag{6}$$

Combining the two equations above, the total variance is

$$\begin{aligned}
&\int_t^T (\nu_u du + J_u^2 dN_u) = 2\int_t^T \left(\frac{dF_u(T)}{F_u(T)} - d\ln F_u \right) \\
&= 2\int_t^T \frac{dF_u(T)}{F_u(T)} + 2(\ln F_t(T) - \ln F_T(T)) \\
&\approx 2\int_t^T \frac{dF_u(T)}{F_u(T)} + 2\int_0^\infty \frac{(F_T(T)-K)^+ - (F_t(T)-K)^+}{K^2} dK \\
&= 2\int_t^T \frac{dF_u(T)}{F_u(T)} + 2\int_0^\infty \frac{C_T(T,K) - (F_t(T)-K)^+}{K^2} dK \\
&= 2\int_t^T \frac{dF_u(T)}{F_u(T)} + 2\frac{(F_T(T)-F_t(T))}{F_t(T)} + 2\int_0^{F_t} \frac{P_T(T,K)}{K^2} dK + 2\int_{F_t}^\infty \frac{C_T(T,K)}{K^2} dK
\end{aligned} \tag{7}$$

The first term of the total variance is the return to a position in index futures dynamically rebalanced to maintain a constant dollar exposure to the stock index, the second term is the return to a static position of $F_t(T)$ index futures held to expiration, and the last terms are the return to a static portfolio of OTM puts and calls, with moneyness defined relative to the forward price $F_t(T)$. This portfolio contains dK/K^2 puts with strike price *K* where *K* is the smaller than or equal to $F_t(T)$, and dK/K^2 calls with strike price *K*, where *K* is greater than or equal to $F_t(T)$. At expiration, the return to a put at strike *K* is $[K - F_T(T)]^+$ and the return to a call at strike *K* is $[F_T(T) - K]^+$. The replicating portfolio for total variance assumes continuous trading, a continuum of listed strike prices, and a listed strike at-the-money (ATM).

Two discrete approximations are needed to assemble and trade this portfolio. First, the dynamic component of the futures position is rebalanced at discrete intervals Δ_{t_i}; second, the strip of options consists of all listed puts with strikes at or below K_0, and all listed calls with strikes at or above K_0, where K_0 is the closest listed strike below $F_t(T)$. This leads to the discrete approximation:

$$\begin{aligned}\int_t^T (v_u du + J_u^2 dN_u) \approx\ & 2\sum_t^T \frac{\Delta F_i(T)}{F_i(T)} + 2\frac{[F_T(T) - F_t(T)]}{F_t(T)} \\ & + 2\sum_0^{K_0} \frac{P_T(T, K_i)}{K_i^2}\Delta K + 2\sum_{K_0}^{\infty} \frac{C_T(T, K_i)}{K_i^2}\Delta K - \left(\frac{F_t(T)}{K_0} - 1\right)^2\end{aligned} \tag{8}$$

$\Delta F_i(T)$ is the change in the futures position over Δt_i, ΔK is half the distance between the two strikes adjacent to *K*, i.e. $\Delta K_i = (K_{i+1} - K_{i-1})/2$, or the distance to the adjacent strike for the initial and final strikes in the put and call series. The last term of the approximation is an adjustment compensating for the fact that the strip is not centered around a strike exactly at-the-money. This term drops out if there is a listed strike at-the-money, i.e. $K_0 = F_t(T)$.

The forward price $P_t^{\text{var}}(T)$ of total variance is determined from the discrete approximation to this variance. Specifically, the forward price of total variance is the expected value of this approximation, with the expectation taken under the F-probability measure. Next taking into account the fact that the expected value of the futures positions is zero,[5] the forward price of total variance simplifies to the forward price of the strip of options plus the adjustment term:

$$\begin{aligned}
P_t^{\text{var}}(T) &= \mathrm{E}_t^F[\int_t^T (v_u du + J_u^2 dN_u)] = -2\mathrm{E}_t^F[\int_t^T d\ln F_u(T)] \\
&= 2\int_0^\infty \frac{e^{r(T-t)}C_t(T,K)}{K^2}dK - 2\int_0^{F_t}\frac{(F_t-K)}{K^2}dK \\
&= 2\int_0^{F_t}\frac{P_t^F(T,K)}{K^2}dK + 2\int_{F_t}^\infty \frac{C_t^F(T,K)}{K^2}dK \\
&\approx 2\sum_0^{K_0}\frac{\Delta K}{K_i^2}\frac{P_t(T,K_i)}{B(t,T)} + 2\sum_{K_0}^\infty \frac{\Delta K}{K_i^2}\frac{C_t(T,K_i)}{B(t,T)} - \left(\frac{F_t(T)}{K_0}-1\right)^2
\end{aligned} \tag{9}$$

The LHS of the above equation is equivalent to the expected integrated return variance, i.e. $\mathrm{E}_t^F\{var_t^F[\int_t^T dF_u(T)/F_u(T)]\} = \mathrm{E}_t^F\{\int_t^T [dF_u(T)/F_u(T)]^2\}$. Note that $F_t(T) = S_t / B(t,T)$ is the forward stock price. $C_t^F(\cdot) = C_t(\cdot)/B(t,T)$ and $P_t^F(\cdot) = P_t(\cdot)/B(t,T)$ are the forward prices of OTM European calls and puts, respectively, where $C_t(T,K_i)$ or $P_t(T,K_i)$ is the midpoint of the bid-ask spread for

[5] The current index futures price $F_t(T)$ is the expected value of future futures prices, and payment of futures contract is deferred. This implies that the expected value of $F_T(T)$–$F_t(T)$ and $\Delta F_i(T)$ are zero.

each option with strike K_i. This is consistent with Bakshi and Madan's (2000) results that any payoff function with bounded expectation can be spanned by a continuum of OTM European calls and puts.

The forward price of a 30-calender day variance is interpolated from the forward prices $P_t^{\text{var}}(T_1)$ and $P_t^{\text{var}}(T_2)$ of variances over the terms T_1 and T_2 of nearby and second-nearby listed options, with the nearby option at least a week from expiration. This price is annualized and VIX is the square root.[6]

$$\text{VIX}_t \equiv \sqrt{P_t^{\text{var}}(T)/\tau} \tag{10}$$

where $P_t^{\text{var}}(T) = wP_t^{\text{var}}(T_1) + (1-w)P_t^{\text{var}}(T_2)$, $w = (T_2 - 30)/(T_2 - T_1)$, $\tau = T - t = 30/365$.

Or equivalently, the VIX squared is the expected integrated return variance when the index process contains correlated jumps in returns and volatility.

$$\begin{aligned} &\frac{1}{\tau}\mathrm{E}_t^F[\int_t^{t+\tau}(v_u du + J_u^2 dN_u)] = -\frac{2}{\tau}\mathrm{E}_t^F[\int_t^{t+\tau} d\ln F_u(t+\tau)] \\ &\approx \frac{2}{\tau}\sum_0^{K_0}\frac{\Delta K}{K_i^2}\frac{P_t(t+\tau,K_i)}{B(t,t+\tau)} + \frac{2}{\tau}\sum_{K_0}^{\infty}\frac{\Delta K}{K_i^2}\frac{C_t(t+\tau,K_i)}{B(t,t+\tau)} - \left(\frac{F_t(t+\tau)}{K_0} - 1\right)^2 = \text{VIX}_t^2 \end{aligned} \tag{11}$$

Thus, under regularity conditions the VIX squared under the SVSCJ model is expressed by

$$\text{VIX}_t^2 \equiv \frac{1}{\tau}\mathrm{E}_t^F[\int_t^{t+\tau}(v_u du + J_u^2 dN_u)] = \frac{\zeta_1}{\tau}\mathrm{E}_t^F(\upsilon_{t,t+\tau}^c) + \zeta_2 = \frac{\zeta_1}{\tau}(a_\tau v_t + b_\tau) + \zeta_2 \tag{12}$$

where $\zeta_1 = 1 + 2\lambda_1[\kappa - (\mu_j + \rho_j\mu_v)]$, $\zeta_2 = 2\lambda_0[\kappa - (\mu_j + \rho_j\mu_v)]$, and $\tau = 30/365$. $\upsilon_{t,t+\tau}^c = \int_t^{t+\tau} v_u du$ is the integrated variance of the asset returns attributed to stochastic

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[6] More specifically, $\text{VIX}_t^2/10000 = P_t^{\text{var}}(t+\tau)/\tau = \mathrm{E}_t^F(\int_t^{t+\tau} v_u^{total} du)/\tau$ where $\tau = 30/365$ and v_t^{total} is annualized instantaneous variance.

volatility $d\omega_{v,t}$ and volatility jumps $z_v dN_t$. Under the SVSCJ, $a_\tau = \dfrac{1-e^{-(\kappa_v - \lambda_1\mu_v)\tau}}{\kappa_v - \lambda_1\mu_v}$ and $b_\tau = \left(\dfrac{\kappa_v\theta_v + \lambda_0\mu_v}{\kappa_v - \lambda_1\mu_v}\right)\left(\tau - \dfrac{1-e^{-(\kappa_v - \lambda_1\mu_v)\tau}}{\kappa_v - \lambda_1\mu_v}\right)$. The VIX squared is thus a linear function of the instantaneous variance v. From (12), we can also back out the instantaneous variance v from the current VIX level by

$$v_t \equiv \frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{\tau\zeta_2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau} \tag{13}$$

The theory behind the VIX calculation is that VIX is obtained as the square root of the price of variance, and this price is derived as the forward price of a particular strip of SPX options. The justification for this derivation is that variance is replicated by delta-hedging the options in the strip. An intuitive explanation of the mechanics of this replication based on Demeterfi, Derman, Kamal, and Zou (1999) is: (i) The price of a stock index option varies with the index level and with its total variance to expiration. This suggests using SPX options to design a portfolio that isolates the variance. (ii) The portfolio which isolates variances is centered around two strips of OTM SPX calls and puts. Its exposure to the risk of stock index variations is eliminated by delta hedging with a forward position in the S&P 500. (iii) A clean exposure to volatility risk independent of the value of the stock index is obtained by calibrating the options to yield a constant sensitivity to variance. If each option is weighted by the inverse of the square of its strike price times a small strike interval centered around its strike price, the sensitivity of the portfolio to total variance is equal to one. Holding the portfolio to expiration therefore replicates the total variance. (iv) Arbitrage implies that the forward price of variance must be equal to the forward price of the portfolio which replicates it. Observing that the S&P 500 forward positions in the portfolio contribute nothing to its value, the forward price of variance reduces to the forward price of the strips of options.

2.3 Fair Price of VIX Futures

The fair value of VIX futures corresponds to the VIX that one can contract for at time *t*, on VIX futures that takes an expectation on the implied volatility beginning on date *T* and returning 30 days later. Thus, the fair value of VIX futures is derived by pricing the forward 30-day volatility that underlines the settlement price of VIX futures. CBOE shows that the fair value of VIX futures (converted to our notation) is the square root of this expected variance less an adjustment factor, which reflects the concavity of the square root function used to extract volatility from variance. The fair value of VIX futures expiring at *T* is given by

$$\mathrm{F}_t^{\mathrm{VIX}}(T) = \mathrm{E}_t^F(\mathrm{VIX}_T) \tag{14}$$

Although the VIX squared is model-free which can be replicated by SPX option prices, the fair value of VIX futures is model-dependent. It involves the convexity adjustment relevant to the variance of the VIX futures price from current time *t* to its expiry *T*. From the approximation of Brockhaus and Long (2000) and Bates (2006), who use the second-order Taylor expansion for the square root of latent affine stochastic processes, Lin (2007) shows that the current VIX futures is worth theoretically:

$$\mathrm{F}_t^{\mathrm{VIX}}(T) = \mathrm{E}_t^F(\mathrm{VIX}_T) \approx \sqrt{\mathrm{E}_t^F(\mathrm{VIX}_T^2)} - \frac{\mathrm{var}_t^F(\mathrm{VIX}_T^2)}{8[\mathrm{E}_t^F(\mathrm{VIX}_T^2)]^{3/2}} \tag{15}$$

where $\mathrm{var}_t^F(\mathrm{VIX}_T^2)/\{8[\mathrm{E}_t^F(\mathrm{VIX}_T^2)]^{3/2}\}$ is the convexity adjustment relevant to the VIX futures. But, this study provides the closed-form solution to the fair value of the VIX futures using the characteristic function of the log VIX squared under probability measure *F* (see Appendix A for the details).

2.4 Fair Price of VIX Options

The VIX option is European-style exercise and AM settled. The opening value of VIX on the day after the last day of options trading determines the final settlement value of the VIX index. Thus, VIX options are priced on VIX futures. The VIX futures contract trades up to six near-term serial months and five months on the February quarterly cycle, whereas VIX options trade on two near-term month expirations plus one month in the February quarterly expiration cycle. The Final Settlement Date for both VIX futures and VIX options is the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the month in which the contract expires. Therefore the last day of options and futures trading is usually a Tuesday, and settlement is determined by Wednesday's open.

Now, consider a European call option written on the VIX with strike price K and expiry T and thus time-to-maturity τ_C, its time-t price $C(t,\tau_C)$ must, under the SVSCJ dynamics, solve

$$
\begin{aligned}
&\frac{1}{2}\nu\frac{\partial^2 C}{\partial L^2}+\left[r-\lambda_0\kappa-\left(\lambda_1\kappa+\frac{1}{2}\right)\nu\right]\frac{\partial C}{\partial L}+\rho\sigma_\nu\nu\frac{\partial^2 C}{\partial L\partial\nu}\\
&+\frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 C}{\partial\nu^2}+\kappa_\nu(\theta_\nu-\nu)\frac{\partial C}{\partial\nu}-\frac{\partial C}{\partial\tau_C}-rC\\
&+\mathrm{E}_t^F\{[\lambda_0+\lambda_1(\nu+z_\nu)]C(t,\tau_C;L+z_S,\nu+z_\nu)-(\lambda_0+\lambda_1\nu)C(t,\tau_C;L,\nu)]=0
\end{aligned}
\tag{16}
$$

subject to $C(t+\tau_C,0)=\max(\mathrm{VIX}_T-K,0)$ with $\mathrm{VIX}_T=\sqrt{\zeta_1 a_\tau\nu_T/\tau+\zeta_1 b_\tau/\tau+\zeta_2}$ and $L=\ln S$. In the Appendix A it is shown that

$$
C(t,\tau_C)=B(t,T)\,\mathrm{F}_t^{\mathrm{VIX}}(T)\,\Pi_1-KB(t,T)\,\Pi_2 \tag{17}
$$

where $B(t,T)=\exp(-r\tau_C)$ under the deterministic interest-rate assumption. The risk-adjusted probabilities, Π_1 and Π_2, are recovered from inverting the respective characteristic functions of the log VIX squared (see Bakshi et al., 1997; Bates, 1996;

Heston, 1993; Scott, 1997; Heston and Nandi, 2000 for similar treatments):

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi \ln K^2} f_2(t,\tau_C; i\phi + 1/2)}{i\phi\, f_2(t,\tau_C; 1/2)}\right] d\phi \tag{18}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi \ln K^2} f_2(t,\tau_C; i\phi)}{i\phi}\right] d\phi \tag{19}$$

where Re[] denotes the real part of a complex number. The characteristic function $f_2(t,\tau_C;i\phi)$ of $\ln \mathrm{VIX}_T^2$ is given in the Appendix B. The fair value of the VIX futures, $\mathrm{F}_t^{\mathrm{VIX}}(T)$, in equation (14) is given by setting $i\phi = 1/2$, i.e. $f_2(t,\tau_C;1/2) = \mathrm{E}_t^F(\mathrm{VIX}_T)$. The price of a European put on the same VIX can be determined from the put-call parity.[7]

The option valuation model in equation (17) has several distinctive features. First, it applies to economies with the index price process containing stochastic volatility and correlated jumps in returns and volatility. It reconciles most existing equity dynamics as special cases. For example, we obtain (i) the SV model by setting jump frequency equal to zero, i.e. $\lambda_0 = \lambda_1 = 0$; (ii) the SVJ model by setting volatility jump size equal to zero, i.e. $\mu_v = 0$; and (iii) the SVCJ model by setting a state-independent jump frequency, i.e. $\lambda_1 = 0$. The Appendix B provides the exact VIX option pricing formulas respectively for the SV, the SVJ, the SVCJ, and the SVSCJ models. Second, this general model allows for a flexible correlation structure between the index return and its volatility and the correlation between price jumps and volatility jumps. Third, the formula in equation (17) is parsimonious in the number of

[7] Grünbichler and Longstaff (1996, equation (14) at page 992) show that since volatility is not the price of a traded asset, the price of a volatility call can be below their intrinsic value. Hence, the traditional put-call parity relation does not hold for these options, and the pattern of time decay is perverse. However, the underlying of their volatility derivatives is "interpreted" as spot volatility, but actually they use the forward volatility as the underlying, which is exact the conception used in this study. To put more precisely, our volatility derivatives pricing framework is more analog to the forward interest rate dynamics of Heath, Jarrow and Morton (1990, 1992).

parameters; especially since it is given only as a function of identifiable variables such that all parameters can be estimated.

The closed-form option pricing formula in equation (17) makes it possible to derive comparative statistics and hedge ratios analytically. In the present context, there is one source of stochastic variation over time, i.e. price risk $\mathrm{F}_t^{\mathrm{VIX}}(T)$. Consequently, there is one delta:

$$\Delta_{\mathrm{F}_t^{\mathrm{VIX}}(T)}^{call} \equiv \frac{\partial C_t^{\mathrm{VIX}}(K,T)}{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)} = B(t,T)\Pi_1 \geq 0 \tag{20}$$

This analytical expression for the delta forms a convenient basis for constructing hedges.

3. Data Description

Since options written on the VIX are less actively traded than SPX options, this study uses both VIX call and put option prices for our empirical work. VIX options were launched on February 24, 2006. In April 2006, average daily volume for VIX options was 15,089 contracts, 55% above March-2006 volume, the first full month of trading in VIX options. Total volume in VIX options during April 2006 was 286,699 contracts. On May 10, 2006 VIX options were 150,493 contracts, beating the previous record of 62,461 contracts, set on May 1, 2006. Open interest stood at 561,207 contracts at the start of trading on May 10, 2006. On November 30, 2007, the trading volumes for VIX calls and puts were 44,277 and 12,852 contracts, in total 57,129 contracts; the open interests were 533,793 for VIX calls and 320,132 for VIX puts. Its underlying, VIX futures, had trading volume 130,278 and open interests 1,779,062 contracts on November 30, 2007. Figure 1 shows the trading volume and open interests of the VIX options and VIX futures across our trading months.

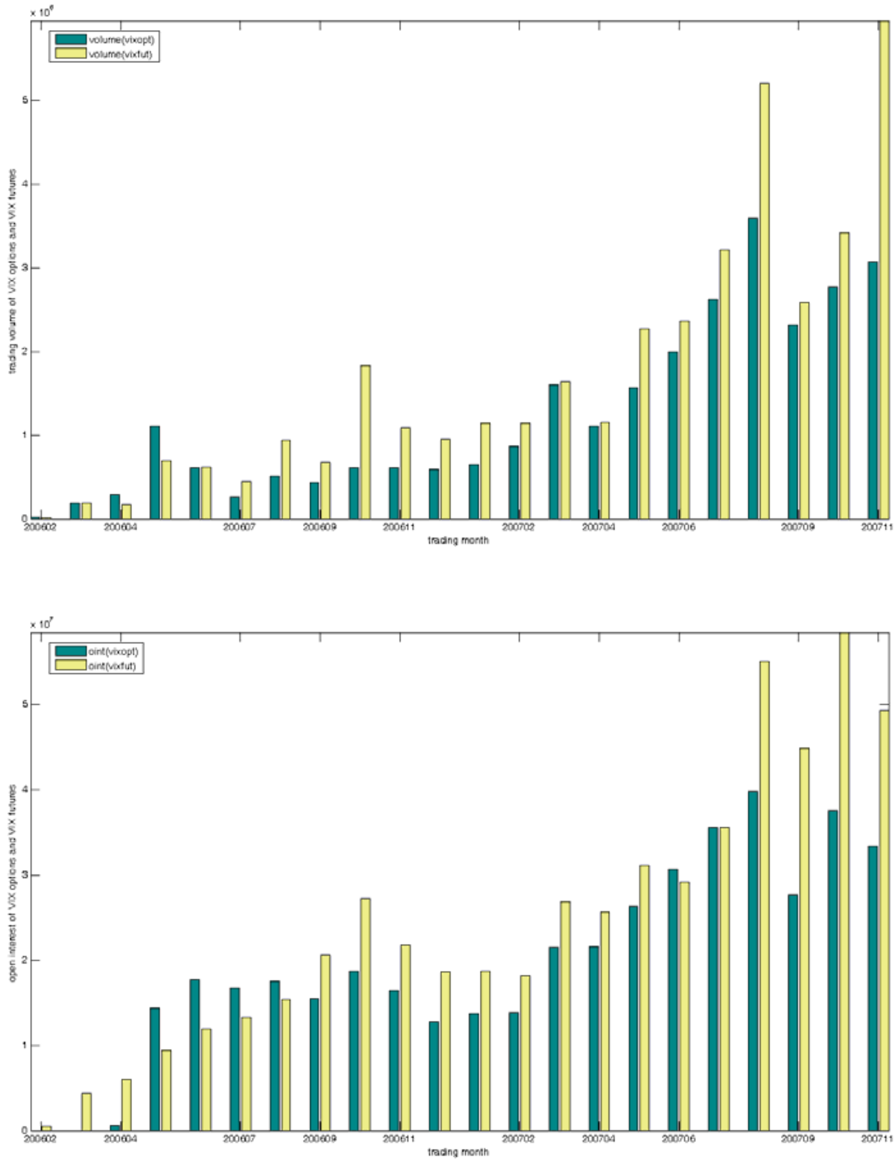

Figure 1. Trading volume and open interests. The figure shows the trading volume (volume) and open interest (oint) of the VIX options and VIX futures across trading month, February 2006 – November 2007.

The sample period extends from February 24, 2006 through November 30, 2007. Spot VIX, daily midpoints of the last bid and last ask quotations for VIX options, and daily settlement prices for VIX futures are obtained from CBOE.[8] The data on the daily U.S. Treasury-bill bid and ask discounts with maturities up to one year are

[8] Note that the recorded VIX futures prices are the daily settlement prices. They are not the corresponding futures levels at the moment when daily last bid and ask quotations of the VIX options are recorded. Thus, there is a non-synchronous price issue here. One solution to avoid non-synchronous prices between futures and options on VIX is to use the intraday data.

obtained from Datastream. By convention, the average of the bid and ask U.S. Treasury-bill discounts is used and converted to an annualized continuously compounded interest rate. Since Treasury bills mature on Thursdays while options and futures on VIX settle on Wednesday that is 30 days before the third Friday of the calendar month immediately following the month in which the contract expires, we utilize the two Treasury-bill rates straddling an option's expiration date to obtain the interest rate corresponding to the option's maturity. This is done for each contract and each day in the sample.

Several exclusion filters are applied to construct the option price data. First, as options with less than six days to expiration may induce liquidity-related biases, they are excluded from the sample. Second, to mitigate the impact of price discreteness on option valuation, price prices lower than \$3/8 are not included. Finally, the VIX call option prices not satisfying the arbitrage restrictions

$$\max\{0, B(t,T)[\mathrm{F}_t^{\mathrm{VIX}}(T)-K)]\} \le C(t,\tau_C) \le B(t,T)\mathrm{F}_t^{\mathrm{VIX}}(T) \tag{21}$$

, and the VIX put option prices violating the boundaries

$$\max\{0, B(t,T)[K-\mathrm{F}_t^{\mathrm{VIX}}(T)]\} \le P(t,\tau_C) \le B(t,T)K \tag{22}$$

are taken out of the sample. Based on these critera, 19,589 observations (approximately 29.43 percent of the original sample) are eliminated. A total of 46,969 records of joint futures and options prices on VIX are used for the estimation of the parameters of the SV, SVJ, SVCJ and SVSCJ models. Of these, 22,109 are calls and 24,860 are puts. Table I presents characteristics of the data sample across maturity and moneyness, where moneyness is defined as $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$. Average VIX option prices ranged from \$0.8214 for deep out-of-the-money (DOTM) short-term (SR) calls to \$9.0448 for deep in-the-money (DITM) medium-term (MR) puts. The average VIX futures price ranges from \$15.1965 corresponding to the SR at-the-money (ATM) calls to \$19.0767 within the SR DOTM puts.

Table I
Sample Characteristics of VIX options and VIX futures

The reported figures are respectively the average VIX option price, the average VIX futures price, and the total number of observations of VIX options for each moneyness-maturity category. The sample period extends from February 24, 2006 through November 30, 2007 for a total of 22,109 calls and 24,860 puts. Daily information from the midpoints of last quotations for each option contract and daily settlement futures prices are used to obtain the summary statistics. $\mathrm{F}_t^{\mathrm{VIX}}(T)$ indicates the time-*t* VIX futures price with expiry *T* and *K* is the option exercise price. Moneyness is defined as $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$.

		Days to Expiration					
		Calls			*Puts*		
Moneyness		<60	60–180	>180	<60	60–180	>180
<0.94	Option price	\$0.8214	\$1.0851	\$1.3180	\$8.1798	\$9.0448	\$8.5087
	Futures price	16.4892	16.9383	16.3520	15.3433	16.4938	16.2413
	Observations	2357	4476	4543	5257	6010	5008
0.94–0.97	Option price	\$1.2940	\$2.0751	\$2.4410	\$1.9336	\$2.8567	\$3.1758
	Futures price	15.5783	16.9965	16.5446	15.4194	16.9965	16.5448
	Observations	262	327	267	273	327	266
0.97–1.00	Option price	\$1.4981	\$2.1193	\$2.5023	\$1.7217	\$2.3559	\$2.7353
	Futures price	16.0649	16.5600	16.0585	16.0300	16.5600	16.0585
	Observations	253	286	309	255	286	309
1.00–1.03	Option price	\$1.5188	\$2.2664	\$2.7604	\$1.2824	\$2.0287	\$2.4883
	Futures price	15.1965	16.4529	16.3282	15.2760	16.4529	16.3282
	Observations	257	250	252	252	250	252
1.03–1.06	Option price	\$1.7835	\$2.5875	\$2.8483	\$1.1448	\$1.8476	\$2.1701
	Futures price	15.4616	17.1513	16.0491	15.6918	17.1513	16.0475
	Observations	256	285	269	240	285	268
>1.06	Option price	\$4.2986	\$4.0056	\$4.0630	\$0.8501	\$1.0239	\$1.2186
	Futures price	17.9294	17.8422	16.9714	19.0767	18.2766	17.0761
	Observations	2196	2833	2431	771	2138	2413

4. Structural Parameter Estimation and In-Sample Performance

A. Estimation Procedure

For the empirical work to follow, we concentrate on the four models: the SV, the SVJ, the SVCJ and the SVSCJ. As stated before, the analysis is intended to present a complete picture of how much each generalization of the SPX price dynamics can improves VIX option pricing and hedging. The vector of structural parameters for alternate models is $\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho\}$ for the SV,

$\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho, \lambda_J, \mu_j, \sigma_j\}$ for the SVJ, and $\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho, \lambda_J, \mu_j, \sigma_j, \mu_v, \rho_j\}$ for the SVCJ, and $\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho, \lambda_0, \lambda_1, \mu_j, \sigma_j, \mu_v, \rho_j\}$ for the SVSCJ. The vector of structural parameters is backed out by the minimization of the sum of the squared pricing errors between option model and market prices. Following the standard approach in the literature (Bates, 1991, 1996; Dumas, Fleming & Whaley, 1998; Longstaff, 1995; Madan, Carr, & Chang, 1998; Bakshi et al., 1997; Nandi, 1998), the minimization is given by

$$\min_{\Phi} \sum_{t=1}^{N_T} \sum_{n=1}^{N_t} [C_n - C_n^*(\Phi)]^2 \tag{23}$$

where N_T is the number of trading days in the estimation sample, N_t is the number of options on day *t*, and C_n and C_n^* are the observed and model option prices, respectively. The model is estimated separately each month and thus Φ is assumed to be constant over a month. The assumption that the structural parameters Φ are constant over a month is justified by an appeal to parameter stability (Bates, 1996; Eraker, 2004; Zhang and Zhu, 2006). Applying such an implied-parameter procedure to implement the candidate models should in some sense give each model an "equal" chance, and it is also consistent with the existing practice of judging a new option pricing model's performance among others.

B. Implied Parameters and In-Sample Pricing Fit

In implementing the above procedure, this study uses spot VIX and market prices of the options and futures on VIX available in each given month as inputs to estimate that month's structural parameters. This estimation is separately done for each model and for each month from March 2006 to October 2007. Table II reports the monthly average and *t*-statistic of each estimated parameter series as well as the monthly-averaged in-sample mean squared errors (*MSE*), respectively for the SV, SVJ, SVCJ, and SVSCJ models. The implied volatility of the VIX futures price changes is computed based on equation (25) below, using estimated parameters and spot VIX.

Table II
Implied Parameters and In-Sample Fit

The values of *MSE*, implied volatility and estimated structural parameters reported here are their averages over 20 non-overlapping estimation months from February 24, 2006 to October 16, 2007. The figures within the parentheses are the *t*-statistics of parameter estimates. The symbols of ** and * indicate significance of *t*-statistics at the 1% and 5% levels, respectively. Note that the variance of the VIX futures price changes $var(\mathrm{F}_t^{\mathrm{VIX}}(T))$ can be decomposed into the variations respectively attributed to stochastic volatility and volatility jumps, $v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)}$ and $v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)}$ (see equations (24)–(25) for the details). The implied volatility of the VIX futures price changes is computed from estimated parameters and spot VIX.

Parameters	SV	SVJ	SVCJ	SVSCJ
MSE	8.9036	7.511	3.4225	2.9443
κ_v	5.2028** (2.6744)	7.4837** (3.6125)	8.0532** (2.8168)	9.1425** (3.9566)
θ_v	0.0594** (5.6355)	0.0498** (4.7626)	0.0393** (2.9784)	0.0283* (3.1763)
σ_v	0.7262** (4.8042)	0.5391** (3.2701)	0.3672** (2.6242)	0.2725** (2.6385)
ρ	−0.8248** (−5.3758)	−0.6723** (−3.8882)	−0.5112** (−3.1853)	−0.3101** (−3.5935)
λ_0 or λ_J		1.3289* (3.1925)	1.2113** (2.9178)	1.2104** (3.3889)
λ_1				0.7658** (3.8457)
μ_j		−0.2673** (−3.6838)	−0.3142** (−3.0182)	−0.5482** (−2.8977)
σ_j		0.3236** (2.8845)	0.2513* (2.8146)	0.1412* (2.7286)
μ_v			1.6324** (3.2331)	0.8236** (4.1522)
ρ_j			−0.1658* (−3.7783)	−0.3175* (−3.6147)
Implied Volatility (%)				
$\sqrt{v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)}}$	0.7256	0.7243	0.4661	0.4218
$\sqrt{v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)}}$			0.5679	0.5935

These reported statistics are quite informative about the internal working of the models. As such, several observations are in order. First, the estimated implied variances of the VIX futures price changes, $v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)} + v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)}$, are generally very close for all models, where $v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)}$ is the variance per year attributed to volatility jumps. However, the sample path for spot variance estimated under the SVJ, the SVCJ, and the SVSCJ models involves a reflection off the minimum value of $v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)} = 2\lambda_0\mu_v^2(\partial\mathrm{F}_t^{\mathrm{VIX}}(T)/\partial\mathrm{VIX}_t^2)^2(\zeta_1 a_\tau/\tau)^2$, whereas the path estimated under the SV model never approaches the reflecting barrier at $v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)}$ =0. Second, the estimated structural parameters for the SPX price process generally differ across models. Recall that in the SV model the volatility is allowed to be mean-reverting and stochastic. The SVJ model relies on the same flexibility, with the additional feature of having price jumps to internalize more skewness and kurtosis without making other parameters unreasonable. In addition to inheriting all features of the SVJ model, the SVCJ model allows volatility jumps to occur. The SVSCJ model relies on the same flexibility, with the additional caveat of allowing volatility jumps to arrive more frequently in high-volatility regimes.

With this in mind, note that the speed-of-volatility adjustment is κ_v for the SV, SVJ, and SVCJ models and $\kappa_v - \lambda_1\mu_v$ for the SVSCJ model. The estimate of κ_v is highest for the SVSCJ model, partly attributed to the existence of volatility jumps. Estimates of the speed-of-volatility adjustment can be interpreted as approximately one minus autocorrelation of volatility. Hence, annualized estimates ranging from

5.2028 to 9.1425 imply daily volatility autocorrelations in the range 0.9637–0.9794 which is in line with the voluminous time-series literature on volatility models. The long-run mean variance is θ_v for the SV and SVJ models, $(\kappa_v\theta_v + \lambda_J\mu_v)/\kappa_v$ for the SVCJ model, and $(\kappa_v\theta_v + \lambda_0\mu_v)/(\kappa_v - \lambda_1\mu_v)$ for the SVSCJ model. The estimate of θ_v for the SV model is 0.0594 which is relatively high, whereas for the SVJ, SVCJ, and SVSCJ models the estimates of θ_v are much smaller, suggesting that the jump components are explaining a significant portion of the unconditional return variance.

The SV model has the strongest negative correlation between Brownian increments in volatility and index returns. More specifically, negative estimates of ρ are –0.8248, –0.6723, –0.5112, and –0.3101 respectively for the SV, SVJ, SVCJ and SVSCJ models.

Further, the SVJ, SVCJ and SVSCJ models attribute part of the negative skewness and excess kurtosis to the possibility of jumps. The jumps occur extremely rarely: The λ estimates in Table II indicate that one can expect about four to six jumps in a stretch of 1,000 trading days. The unconditional jump frequency is only marginally higher under the state-dependent SVSCJ model. Whenever spot volatility is high, say an annualized variance of 3%, the estimate of λ_1 is indicative of an instantaneous jump probability of about 0.023 – a 2% increase over the constant arrival intensity specifications. The possibility of price jumps occurs with an average price jump size μ_j of −0.2673, −0.3142, and −0.5482 (with the jump size uncertainty σ_j estimated at 0.3236, 0.2513, and 0.1412) for the SVJ, SVCJ, and SVSCJ models. Further, the volatility jump size under the SVCJ model is 1.6324 to be greater than its SVSCJ model counterpart, 0.8236, as one would expect the extra parameters (related to the state-dependent volatility jump process) to make the SVSCJ model fit the data better than the SVCJ model. The negative correlation between price jumps and volatility jumps (−0.1658 for the SVCJ and −0.3175 for the SVSCJ) indicates the asymmetry of volatility jumps across the index price level.

These estimates together present that, to the extent that the pricing structure of the options prices can be explained respectively by each model, the SVSCJ model or SVCJ model's demand on the $v(t)$ process is the most stringent as it requires both the highest variation and the greatest covariance (in magnitude) with underlying returns.

The parameter estimates in Table II are interesting in light of estimates obtained in the prior studies. Bakshi et al. (1997) estimate the jump frequency for the SVJ model is 0.59 annualized jump probability, and the jump-size parameter μ_j and σ_j are –5.37% and –7%, respectively. Their estimate of κ_v and σ_v are 2.03 and 0.38. Pan (2002) and Bates (2000) assume that the jump frequency, λ, depends on the spot volatility, and hence the jump size and jump frequency parameters are comparable to those reported in Eraker's (2004) SVSCJ model and here. The average jump intensity point estimates in Pan (2002) are in the range of [0.07%, 0.3%] across different model specifications, whereas Eraker (2004) indicates two to three jumps in a stretch of 1,000 trading days. Interestingly, Bates (2000) obtains quite different results, with an average jump intensity of 0.005.[9] Bates (2000) also reports a jump size mean ranging from –5.4% to –9.5% and standard deviations of about 10%–11%. Hence, Bates' estimates imply more frequent and more severe crashes than the parameter estimates reported in Bakshi et al. (1997) and Eraker (2004). Our estimates are in the same ballpark as those reported in Bates (2000). The practical implication of the difference is that Bates (2000) and our estimates will generate more skewness and kurtosis in the conditional index returns distributions.

Finally, the fact that allowing jumps to occur enhances the SV model's fit is illustrated by each model's *MSE* in an average month. The *MSE* is 8.90 for the SV model, 7.51 for the SVJ model, 3.42 for the SVCJ model, and 2.94 for the SVSCJ model. The in-sample mean squared errors are consistently smaller for a more complicated model. Table II also gives the *t* values for the parameter estimates of the index price process. These results indicate that all parameter estimates are significantly different from zero at the 95% confidence interval.

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[9] Pan (2002) and Bates (2000) model the jump frequencies as $\lambda = \lambda_1 v_t$ with v_t the spot volatility, whereas Eraker (2004) assumes the jump frequency as $\lambda = \lambda_0 + \lambda_1 v_t$. The average jump intensities reported here are obtained from Pan, Bates and Eraker's papers as the multiple of the long-run volatility mean, θ_v, and the proportionality parameter, λ_1, and plus λ_0.

C. Internal Consistency of Implicit Parameters

Another way to gauge model misspecification is to follow the approach taken by Bates (1996, 2000), Bakshi et al. (1997), Pan (2002) and Eraker (2004) and examine whether each model's implied parameters are consistent with those implicit in the time series of the VIX futures prices. That is, is the forward view offered by volatility implied by the market prices of VIX options similar in magnitude to those from the time series counterparts? The closer the implied parameters, the closer the implied time-series path to its observed counterpart and hence the less misspecified the model. The justification for the internal consistency test using time-series VIX futures prices is that VIX option pricing is based on VIX futures rather than the spot VIX.

Bates (1996, 2000), Bakshi et al. (1997), and Pan (2002) suggest that the volatility of volatility, σ_v implied by SPX option prices, cannot be reconciled with time-series estimates. In particular, Bates (1996, 2000) and Bakshi et al. (1997) show that their estimated volatility paths are too smooth to be consistent with the relatively high σ_v estimated from option prices. In contrast, Eraker (2004) reports that estimates of σ_v from the simulated values of the historical volatilities match almost exactly the option implied volatility of volatility. He points out the match partly attributed to their Markov Chain Monte Carlo (MCMC) method providing much more erratically behaving volatility paths than other methods based on Kalman filtering, and quasi-maximum likelihood methods.

This study re-examines the consistency of VIX option-implied volatility of the forward VIX, i.e. $var(d\mathrm{F}_t^{\mathrm{VIX}}(T))$, with the sample volatility of daily VIX futures price changes, i.e. $\mathrm{F}_{t+\Delta t}^{\mathrm{VIX}}(T) - \mathrm{F}_t^{\mathrm{VIX}}(T)$. By the law of iterated expectations, futures prices are martingale under the forward probability measure:

$$\mathrm{F}_t^{\mathrm{VIX}}(T) = \mathrm{E}_t^F(\mathrm{VIX}_T) = \mathrm{E}_t^F[\mathrm{E}_{t+1}^F(\mathrm{VIX}_T)] = \mathrm{E}_t^F[\mathrm{F}_{t+1}^{\mathrm{VIX}}(T)]$$

The futures price martingale property implies the drift term of $d\mathrm{F}_t^{\mathrm{VIX}}(T)$ equals zero. Further, the theoretical time-t value of T-expiry VIX futures, $\mathrm{F}_t^{\mathrm{VIX}}(T)$, equals the characteristic function of $\ln\mathrm{VIX}_T^2$ evaluated at $i\phi = 1/2$ under the probability measure F, i.e. $f_2(t,\tau_C;1/2) = \exp[C_2(\tau_C) + J_2(\tau_C) + D_2(\tau_C)\ln\mathrm{VIX}_t^2]$. Using Itô's lemma, the dynamics of the VIX futures price under the forward probability measure become

$$\begin{aligned} d\mathrm{F}_t^{\mathrm{VIX}}(T) = {} & \frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial \mathrm{VIX}_t^2}\frac{\zeta_1 a_\tau}{\tau}\sigma_v\sqrt{\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{\tau\zeta_2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau}}\, d\omega_{v,t} \\ & + \frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial \mathrm{VIX}_t^2}\frac{\zeta_1 a_\tau}{\tau}\left\{ z_v dN_t - \mu_v\left[\lambda_0 + \lambda_1\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{\tau\zeta_2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau}\right)\right]dt\right\} \end{aligned} \tag{24}$$

The variance of the VIX futures price change is expressed by

$$\begin{aligned} \sigma^2(t,T)dt &= \mathrm{var}[d\mathrm{F}_t^{\mathrm{VIX}}(T)] \\ &= \left(\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial \mathrm{VIX}_t^2}\frac{\zeta_1 a_\tau}{\tau}\right)^2\left[2\lambda_0\mu_v^2 + (\sigma_v^2 + 2\lambda_1\mu_v^2)\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{\tau\zeta_2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau}\right)\right]dt \\ &= v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)}dt + v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)}dt \end{aligned} \tag{25}$$

where $v_{t,\mathrm{F}_t^{\mathrm{VIX}}(T)} = \{[\partial \mathrm{F}_t^{\mathrm{VIX}}(T)/\partial \mathrm{VIX}_t^2](\zeta_1 a_\tau \sigma_v/\tau)\}^2(\tau\mathrm{VIX}_t^2/a_\tau\zeta_1 - \tau\zeta_2/a_\tau\zeta_1 - b_\tau/a_\tau)$ and

$v_{J,\mathrm{F}_t^{\mathrm{VIX}}(T)} = 2\mu_v^2\{[\partial \mathrm{F}_t^{\mathrm{VIX}}(T)/\partial \mathrm{VIX}_t^2](\zeta_1 a_\tau/\tau)\}^2[\lambda_0 + \lambda_1(\tau\mathrm{VIX}_t^2/a_\tau\zeta_1 - \tau\zeta_2/a_\tau\zeta_1 - b_\tau/a_\tau)]$

Two distinct approaches are adopted for estimating the volatility function $\sigma(t,T)$. Implicit volatility estimation uses the parameters recovered from the market prices of the VIX options (generated in the Table II), whereas historic volatility estimation uses time-series observations of the VIX futures price.

For the overall sample period from February 24, 2006 to November 30, 2007, the annualized daily VIX futures price changes have a mean of 0.03 percent, a volatility of 32.44 percent, a skewness of –1.90, and a kurtosis of 83.60. The historical volatility is indeed lower than its option-implied counterparts (see Table II). This departure between the average implied and the time-series estimated is remarkably similar across alternate models. Hence, each of the four models is hence equally misspecified. Figure 1 contains time series graphs of the VIX futures prices and the estimated forward VIX from the market prices of VIX options, i.e. $\mathrm{F}_t^{\mathrm{VIX}}(T) = f_2(t, \tau_C; 1/2)$, over the entire time period studied. There are 445 observations of daily means of VIX futures prices across maturities in this time period. In this graph, the forward VIX curves are piecewise constant. As seen, the forward VIX processes recovered from the market prices of both VIX futures and VIX options are remarkably similar in time-series patterns. However, the implied value of $var(d\mathrm{F}_t^{\mathrm{VIX}}(T))$ is, for each model, about two times its time-series estimate. The volatility process of the forward VIX implicit in option prices is therefore much too volatile, relative to each volatility time series implicit in futures prices. According to these yardsticks, each of the four models is hence significantly misspecified. On a relative scale, however, this departure between the average implied and the time-series estimates is the weakest for the SVJ, SVCJ and SVSCJ models, and the strongest for the SV model.

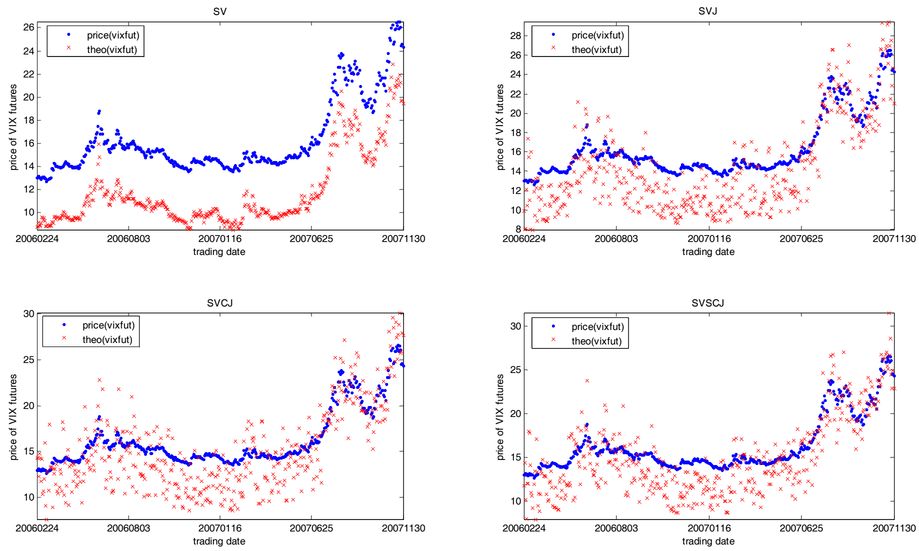

Figure 1 Forward VIX curve evolutions, 24 February 2006 – 30 November 2007. The dotted line is the average VIX futures prices and the line with cross symbols is the forward VIX recovered from VIX option prices.

Further, this study uses the time-series observations of VIX futures prices to examine the variance function in equation (24). For this analysis, we use daily changes in the market prices of the VIX futures of various maturities, i.e. $\mathrm{F}_{t+\Delta t}^{\mathrm{VIX}}(T)-\mathrm{F}_{t}^{\mathrm{VIX}}(T)$ for $\Delta t=1/252$ and adopt the PCA method to determine volatility function. The volatilities, therefore, depend on empirically specified factor loadings and these factors' volatilities. From these volatility functions, we can diagnose whether the volatility functions of the forward VIX posit the characteristics of the "term structure", "mean reversion", and "volatility of volatility".

Since CBOE may list for trading up to six near-term serial months and five months on the February quarterly cycle for the VIX futures contract, whereas the expiration months for the VIX options are generally up to three near-term months plus up to three additional months on the February quarterly cycle. For comparison, this study selects the VIX futures contracts with the same expiration months as the VIX options. Hence, there are at most seven different maturities of the VIX futures contracts in our sample period. For each of these maturities, we compute the VIX futures price changes over one month in the entire observation period (February 2006–November 2007). We then run the PCA on the covariance matrix across different maturities to estimate the volatility functions of the VIX futures price for each model.

The estimated volatility function (24) using PCA is graphed in Figure 2. We found three factors dominating the variation of VIX futures. On average, the first three factors on average explain about 99.32% of the variation in VIX futures. The first factor explains on average 85.91%, the second factor 10.68%, and the third factor 2.74%. The first three factors can be interpreted as the level, slope, and twist. The first factor of the principal component-based volatility function is roughly a parallel shift, except in the middle range. The second factor emphasizes the relation between the short and long forward VIX. Finally, the third factor accounts for isolated movements in short-term forward VIX. As we use a rolling horizon of one month, the estimated volatility functions change monthly, but the shapes of these volatility functions turn out to be quite robust over time in general.

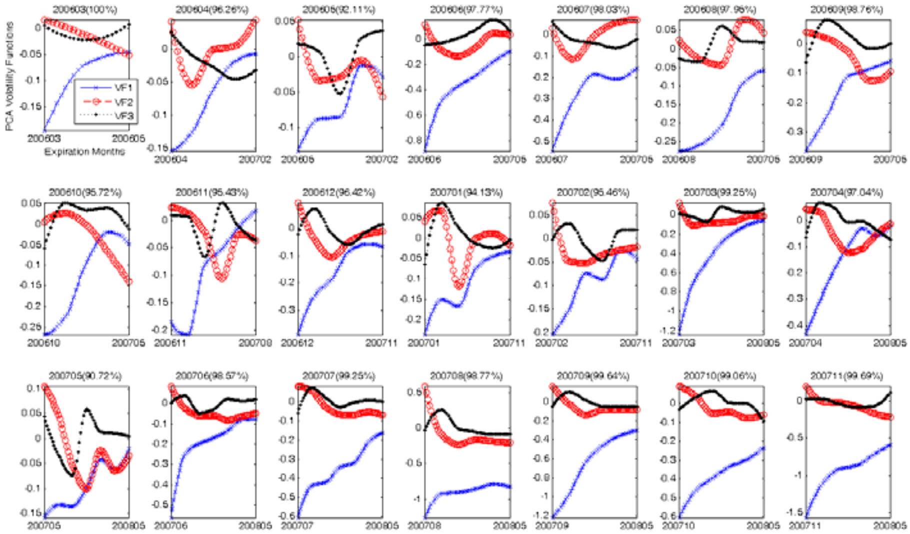

Figure 2 Graphs of volatility functions of forward VIX implicit in VIX futures prices. The volatility functions are estimated using a principal component analysis on a monthly basis. VF1 with cross markers, VF2 with cycle markers, and VF3 with dot markers respectively denote the first, the second and the third factors dominating the variation of VIX futures.

In summary, the four models rely on implausible levels of volatility variation of forward VIX to rationalize the observed option prices. This finding is similar to those of Bates (1996, 2000) using currency and SPX futures options and Bakshi et al. (1997) and Pan (2002) using SPX options. While the SV, the SVJ, the SVCJ, and the SVSCJ are clearly misspecified (though to a lesser degree compared to the SV and SVJ), how will they perform in pricing and hedging options? We answer this question in the sections to follow.

5. Assessment of Relative Model Misspecification

Although a more complicated model will generally lead to a better in-sample fit, it will not necessarily perform better in out-of-sample pricing because any overfitting will be penalized. To test whether the additional parameters of the volatility models are economically informative for VIX options pricing, this section provides a comparison of out-of-sample pricing.

Out-of-sample pricing is carried out with the *previous month's* structural parameters and *current day's* VIX and VIX futures prices to calculate *current day's* VIX option model price. Following Dumas, Fleming, and Whaley (1998), we define the pricing error outside the bid-ask spread as

$$\text{Pricing Error} = \begin{cases} \text{Model Price} - \text{Bid Price, if Model Price} < \text{Bid Price} \\ \text{Model Price} - \text{Ask Price, if Model Price} > \text{Ask Price} \end{cases} \tag{27}$$

Two measures of goodness of fit are then employed to assess the out-of-sample pricing performance of the VIX option pricing models on the SV, SVJ, SVCJ and SVSCJ specifications. These are the mean percentage pricing error (PE) and the mean absolute pricing error (MAE).

Table III reports MAE and PE values for several categories according to time to expiration and moneyness. Out of maturity combinations reported in Table III, RMSE and MAE are lowest for the SVJ (SVSCJ) model for the short-term (medium-term and long-term) options contracts. Thus, improvements are generated respectively for the short-dated VIX options under the SVJ model and for the medium- and long-dated VIX options under the SVSCJ model. From the panel of PE values, in contrast to the results of the SV and SVJ models, the SVCJ and SVSCJ models substantially overprice the short- and medium-dated VIX options. For the long-dated VIX options, all models are overpriced. The PE values show that the SVCJ and SVSCJ models pricing errors across maturities are less than zero, indicating an overpricing fit for VIX options.

Bakshi et al. (1997) point out that the price jump and the diffusive volatility features can in principle improve the pricing of, respectively, short-term and relatively long-term options. Therefore, the SVJ model enhances the flexibility of permissible return distributions and thus provides a better pricing fit for short-term and long-term options than the SV model (unless the *covariance* and *volatility variation* of the SV model are unreasonably high). Table III confirms this argument. The SVJ model provides a better out-of-sample pricing fit for VIX futures across maturities than the SV model.

Further, what is it that jumps in volatility provide that jumps in returns and diffusive stochastic volatility cannot? Eraker et al. (2003) point out that jumps in returns can generate large movements such as the crash of 1987, but the impact of a jump is transient: A jump in returns today has no impact on the future distribution of returns. On the other hand, diffusive volatility is highly persistent, but its dynamics are driven by a Brownian motion. For this reason, diffusive stochastic volatility can only increase gradually via a sequence of small normally distributed increments. Jumps in volatility fill the gap between jumps in returns and diffusive volatility by providing a rapidly moving but persistent factor that drives the conditional volatility of returns.

It is important to note that the presence of jumps in volatility does not eliminate the need for jumps in returns. With both types of jumps, jumps in returns occur less often (λ_J^{SVJ}=1.33> $\lambda_J^{\mathrm{SVCJ}}$=1.21), but they still play an important role, as they generate the large, though infrequently observed, crashlike movements. This indicates that jumps in volatility and returns play a greater role than diffusive stochastic volatility in generating the index dynamics. This suggests that jump components should command relatively larger risk premia than diffusive ones. When allowing both types of risk premia for diffusive volatility and volatility jumps to reconcile simultaneously the spot and option dynamics, Lin (2007) using the joint data of VIX futures prices and integrated volatility shows that the price jump-risk premium and volatility jump-risk premium dominate by far the diffusive volatility-risk premium.

In summary, like jumps in returns and unlike stochastic volatility, jumps in volatility are a rapidly moving factor driving returns. Like diffusive stochastic and unlike jumps in returns whose impact on returns is transient, a jump in volatility persists. Thus, jumps in volatility provide a rapidly moving but persistent factor driving volatility. Therefore, each factor (diffusive volatility, price jump and volatility jump) generates very different behavior. In particular, the persistent feature of volatility jumps enhances the long-dated derivatives valuation. This is because as maturity increases, the fat-tails and asymmetries in the conditional distribution are driven to a larger extent by diffusive volatility and the volatility jump through its persistence, rather than price jumps. Similarly, as maturity decreases, the fat-tails and asymmetries in the conditional distribution are driven to a larger extent by price jumps, rather than other two factors. Hence, according to the yardstick of out-of-sample pricing fits the SVJ performs best for our short-dated VIX options and the SVSCJ model outperforms for the medium- and long-dated VIX options.

In Table III, the dramatic pricing errors for the SV come mostly from OTM calls and OTM puts, whereas the large errors for the other three models are all associated with short-dated options. Outside of the OTM categories, the SV percentage pricing errors show no particular relation to maturity or moneyness. For medium- and long-dated options the pricing errors due to the SVJ, the SVCJ, or the SVSCJ are quite random across strike prices. Therefore, using moneyness-based parameter/volatility estimates for the SV and maturity-based parameter/volatility estimates for the other three models serves to correct for their respective weakness.

TABLE III
Out-of-Sample Pricing Errors

For a given model, we compute the price of each option using the *previous month*'s structural parameters and *current day*'s VIX and VIX futures prices. The reported mean absolute pricing error (MAE) is the sample average of the absolute difference between the market price and the model price for each option in a given moneyness-maturity category. The reported percentage pricing error (PE) is the sample average of the market price minus the model price, divided by the market price. The sample period is April 2006–November 2007, with a total of 22,109 call option prices and 24,860 put option prices. Moneyness is defined as $F_t^{VIX}(T)/K$.

		Absolute Pricing Errors			Percentage Pricing Errors		
Moneyness		Days to Expiration			Days to Expiration		
$(S_t–D_t)/X$	Model	<60	60–180	>180	<60	60–180	>180
< 0.94	SV	$2.53	$2.48	$2.76	–0.19	–0.50	–0.27
	SVJ	0.87	1.12	1.05	–0.09	–0.40	–0.22
	SVCJ	0.95	1.79	0.53	–0.12	–0.39	–0.17
	SVSCJ	1.02	0.58	0.32	–0.11	–0.25	–0.15
0.94–0.97	SV	1.38	1.79	2.79	–0.68	–0.31	–0.16
	SVJ	0.83	1.15	1.21	–0.33	–0.24	–0.15
	SVCJ	1.14	1.04	0.92	–0.61	–0.20	–0.12
	SVSCJ	1.10	0.51	0.42	–0.42	–0.15	–0.08
0.97–1.00	SV	1.69	2.14	2.10	–0.36	–0.17	–0.24
	SVJ	0.58	1.82	1.24	–0.08	–0.12	–0.22
	SVCJ	1.25	1.77	1.22	–0.26	–0.13	–0.21
	SVSCJ	0.85	0.84	0.95	–0.15	–0.07	–0.06
1.00–1.03	SV	1.22	1.18	2.36	–0.19	–0.26	–0.25
	SVJ	0.95	1.21	1.15	–0.09	–0.18	–0.20
	SVCJ	1.19	1.09	2.10	–0.15	–0.17	–0.16
	SVSCJ	1.02	0.86	1.01	–0.10	–0.54	–0.32
1.03–1.06	SV	0.91	1.66	1.72	–0.22	–0.23	–0.19
	SVJ	0.38	1.21	1.53	–0.65	–0.16	–0.17
	SVCJ	0.52	1.01	1.06	–0.20	–0.12	–0.17
	SVSCJ	0.45	0.97	0.93	–0.10	–0.08	–0.08
≥ 1.06	SV	1.70	2.67	2.27	–0.21	–0.24	–0.34
	SVJ	0.23	1.23	1.25	–0.09	–0.22	–0.25
	SVCJ	0.59	1.15	2.04	–0.15	–0.21	–0.22
	SVSCJ	0.43	0.71	0.89	–0.12	–0.10	–0.06

To further understand the structure of remaining pricing errors, we appeal to a regression analysis to study the association between the errors and factors that are either contract-specific or market condition-dependent. We first fix an option pricing model, and let $\varepsilon_n(t)$ denote the n^{th} option's percentage pricing error on day t. Then, we run the regression below for the entire sample:

$$\varepsilon_n(t) = \beta_0 + \beta_1 \frac{\mathrm{F}_t^{\mathrm{VIX}}(T)}{K_n} + \beta_2\, \tau_n + \beta_3\, BA_n(t) + \beta_4\, VJ(t) + u_n(t) \tag{28}$$

where K_n is the strike price of the option, τ_n the remaining time to expiration, and $BA_n(t)$ the percentage bid-ask spread at date t of the option (i.e., $(\mathrm{Ask} - \mathrm{Bid})/[0.5(\mathrm{Ask} + \mathrm{Bid})]$, all of which are contract-specific variables. The variable, $VJ(t)$, is the date-t (annualized) volatility jump of the SPX returns computed from the skewness premium introduced by Bates (1991) as a proxy for fears of a jump in volatility, and it is included in the regression to see whether the current day's volatility jump of the SPX returns may cause systematic pricing biases. Since volatility shocks tend to follow negative rather than positive shocks to the value of the underlying asset in the stock market, this study uses the skewness premium introduced by Bates (1991) as a proxy for fears of a jump in volatility:

$$skew_t(T^{\mathrm{SPX}}) \equiv \frac{C_t^{\mathrm{SPX}}(F_t, T^{\mathrm{SPX}}, X_C^{\mathrm{SPX}})}{P_t^{\mathrm{SPX}}(F_t, T^{\mathrm{SPX}}, X_P^{\mathrm{SPX}})} - 1 \tag{29}$$

where $skew_t(T^{\mathrm{SPX}})$ is the skewness premium at time t for SPX options maturating at T^{SPX} two days after the comparable maturity T of the VIX option, C_t^{SPX} and P_t^{SPX} are the prices of SPX call and put options as functions of F_t, the SPX's forward price at t, and the strike prices X_C^{SPX} and X_P^{SPX}, respectively. The strike prices of both options are defined to be x percent of out-of-the-money (x>0) and spaced geometrically around the forward price of the underlying SPX in the following way:

$$X_P^{\mathrm{SPX}} = \frac{F_t}{1+x} < F_t < F_t(1+x) = X_C^{\mathrm{SPX}} \tag{30}$$

As described in Bates (1991), the skewness premium can be used as a diagnostic of the symmetry or skewness in the risk neutral distribution implicit in option prices. Theoretically, negative skewness in the risk-neutral distribution reflects either the existence of crash fears or that volatility is expected to rise if the market falls. Since unexpected increases in volatility are much more often associated with negative shocks to the underlying in stock markets than with positive shocks (French, Schwert, & Stambaugh, 1987; Figlewski & Wang, 2000; Low, 2004; and Andersen, Bollerslev, Diebold, & Ebens, 2001), this study assumes this is a good proxy for the markets' expectations of a positive jump in volatility. The assumption of a strong negative relationship between rates of SPX returns and volatility in the stock market is confirmed by the negative estimates of ρ and $cov(d\ln S, dv)$ in Table II. This study uses SPX options prices that are 4% out-of-the-money to calculate the skewness premium. Since option exist only for specific strike prices, this study interpolates the relevant option prices fitting a constrained cubic spline through observed option price/forward price ratios as a function of observed strike price/forward price ratios (see Appendix A in Bates, 2000, for detailed information on the calculation of these constrained cubic splines). Similar to Bates (1991, 1997, 2000), this study requires that prices exist for at least four call strikes and four put strikes.

In some sense, the contract-specific variables help detect the existence of cross-sectional pricing biases, whereas $VJ(t)$ serves to indicate whether the pricing errors over time are related to the dynamically changing market conditions. Table IV reports the regression results based on the entire sample period, where the standard error for each coefficient estimate is adjusted according to the White's (1980) heteroskedasticity-consistent estimator and is given in the parentheses. Regardless of the model, each independent variable has statistically significant explanatory power of the remaining pricing errors. That is, the pricing errors from each model have some moneyness, maturity, volatility jump, and bid-ask spread related biases. The magnitude and sign of each such bias, however, differ among the models. The pricing errors due to the two models with volatility jumps are always biased in the same direction. To look at some point estimates, the SV and the SVJ percentage pricing errors will on average be 1.50 and 0.75 points higher when the bid-ask spread $BA(t)$ increases by one point, whereas the SVCJ and the SVSCJ percentage errors will only be, respectively, 3.23 and 1.23 points higher in response. Other noticeable patterns include the following. The SV and SVJ pricing errors are significantly, while the SVCJ and the SVSCJ pricing errors are only barely, decreasing in the SPX's volatility jump, which confirms that modeling volatility jumps is important. The deeper in-the-money the call, or equivalently the deeper out-of-the-money the put, the lower the SVJ, the SVCJ's, and the SVSCJ's mispricing. But for the SV model, its mispricing increases with moneyness. Even though all four models' pricing errors are, in most cases, statistically significantly related to each independent variable, the collective explanatory power of these variables is quite high only for the SV but not so for the others. The adjusted R^2 is 24 percent for the SV formula's pricing errors, 12 percent for the SV's, 5 percent for the SVCJ's, and 3 percent for the SVSCJ model's.

TABLE IV

Regression Analysis of Pricing Errors

The regression results below are based on the equation:

$$\varepsilon_n(t) = \beta_0 + \beta_1 \frac{\mathrm{F}_t^{\mathrm{VIX}}(T)}{K_n} + \beta_2\,\tau_n + \beta_3\,BA_n(t) + \beta_5\,VJ(t) + u_n(t)$$

where $\varepsilon_n(t)$ is the percentage pricing error of the *n*th call on date-*t*; $\mathrm{F}_t^{\mathrm{VIX}}(T)/K_n$ and τ_n respectively represent the moneyness and the term-to-expiration of the option contract; the variable $BA_n(t)$ is the percentage bid-ask spread; and $VJ(t)$ proxies the volatility jump computed from Bates' (1991) skewness premium using the ratio of deep out-of-the-money S&P 500 index puts and calls with comparable maturity to the VIX option. The standard errors, reported in parentheses, are White's (1980) heteroskedasticity consistent estimator. The percentage pricing errors are obtained using the parameters implied by all of the previous month's options. The sample period is March 2006 – November 2007 for a total of 46,969 observations. SV, SVJ, SVCJ, and SVSCJ, respectively, stand for the stochastic-volatility model, the stochastic-volatility model with random price jumps, the stochastic-volatility model with state-independent but correlated jumps in both S&P 500 index returns and their volatility, and the stochastic-volatility model with state-dependent and correlated jumps in both S&P 500 index returns and their volatility.

Coefficient	SV	SVJ	SVCJ	SVSCJ
Constant	-1.43^{**}	1.89^{**}	1.86^{**}	1.49^{**}
	(−6.95)	(2.96)	(7.37)	(2.59)
$\mathrm{F}_t^{\mathrm{VIX}}(T)/K$	3.08^{**}	-2.96^{**}	-2.13^{**}	-1.88^{**}
	(11.92)	(−11.42)	(−10.06)	(−6.69)
τ	0.07^{**}	0.04^{*}	-0.01^{*}	-0.02^{**}
	(3.53)	(1.74)	(−1.98)	(−5.95)
BA	1.50^{**}	0.75^{**}	3.23^{**}	1.23^{**}
	(6.47)	(3.23)	(8.38)	(6.92)
VJ	1.55^{**}	1.12^{**}	-0.75^{**}	-0.42^{**}
	(3.52)	(3.52)	(−2.90)	(−2.82)
Adj. R^2	0.24	0.12	0.05	0.03

6. Conclusion

This study has developed a parsimonious VIX option pricing model that reconciles the most general price processes of the SPX in the literature: stochastic volatility, random price jumps, and state-independent/state-dependent volatility jumps. It is shown that this closed-form pricing formula is practically implementable, leads to useful analytical hedge ratios, and contains many VIX option formulas as special cases. This last feature has made it relatively straightforward to study the relative empirical performance of several models of distinct interest.

Our empirical evidence indicates that regardless of performance yardstick, taking volatility jumps into account is of the first-order importance in improving upon the SV and SVJ formulas. In terms of internal consistency, the SV, the SVJ, the SVCJ, and the SVSCJ are still significantly misspecified. In particular, the four models rely on implausible levels of volatility variation of forward VIX to rationalize the observed option prices. But, such structural misspecifications do not necessarily preclude these models from performing better otherwise. According to the out-of-sample pricing measures, adding the random price jump feature to the SV model can further improve its performance, especially in pricing short-term VIX options; whereas modeling volatility jumps can enhance the fit of long-term VIX options. With the SVCJ, the SVSCJ and the SVJ, the remaining pricing errors show the least contract-specific or market-conditions-related biases. Overall, the two performance yardsticks employed in this article can rank a given set of models differently as they capture and reveal distinct aspects of a pricing model. Our results support the claim that a model with stochastic volatility and state-dependent correlated jumps in SPX returns and volatility (i.e. the SVSCJ) is a better alternative to the others in terms of pricing.

The contributions of this study are fivefold. First, it demonstrates that the VIX formula is still valid when the SPX price process contains volatility jumps. Second, it provides closed-form solutions to the fair value of the VIX futures and VIX options under alternate affine processes. Third, a methodology for an integrated analysis of futures and options on VIX is proposed. Finally, the models' internal consistency tests and out-of-sample pricing fits are assessed.

Appendix A: Closed-Form Solution to the VIX Call Option

One can calculate risk-adjusted probabilities, Π_1 and Π_2, from respective characteristic functions of the log VIX squared, following Feller (1971) or Kendall and Stuart (1977). Let $f_2(t,\tau_C;\phi)$ denote the moment generating function of the probability density under the measure *F*, $q_2(\ln \mathrm{VIX}_T^2)$. Let $q_1(\ln \mathrm{VIX}_T^2)$ be a probability density under measure *R* defined by

$$q_1(\ln \mathrm{VIX}_T^2) = \frac{\exp\left(\frac{1}{2}\ln \mathrm{VIX}_T^2\right)}{f_2\left(t,\tau_C;\frac{1}{2}\right)} q_2(\ln \mathrm{VIX}_T^2) = \frac{\mathrm{VIX}_T}{\mathrm{E}_t^F(\mathrm{VIX}_T)} q_2(\ln \mathrm{VIX}_T^2) \tag{A1}$$

It is easy to see that it is a valid probability density because it is non-negative and $f_2(t,\tau_C;1/2) = \mathrm{E}_t^F(\mathrm{VIX}_T)$. The moment generating function of logarithmic VIX squared for $q_1(\ln \mathrm{VIX}_T^2)$ is

$$\begin{aligned} f_1(t,\tau_C;\phi) &= \mathrm{E}_t^R\left(e^{\phi \ln \mathrm{VIX}_T^2}\right) = \int_{-\infty}^{\infty} e^{\phi \ln \mathrm{VIX}_T^2} q_1(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 \\ &= \frac{1}{f_2(t,\tau_C;1/2)} \int_{-\infty}^{\infty} e^{(\phi+1/2)\ln \mathrm{VIX}_T^2} q_2(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 \\ &= \frac{f_2(t,\tau_C;\phi+1/2)}{f_2(t,\tau_C;1/2)} \end{aligned} \tag{A2}$$

Since the terminal spot asset price is VIX_T, then the expectation of the payoff of a VIX call option separates into two terms with probability integrals.

$$\begin{aligned} &\mathrm{E}_t^F[C(t+\tau_C,0)] = \mathrm{E}_t^F[\max(\mathrm{VIX}_T - K,0)] \\ &= \int_{\ln K^2}^{\infty} \mathrm{VIX}_T\, q_2(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 - K\int_{\ln K^2}^{\infty} q_2(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 \\ &= f_2\left(t,\tau_C;\frac{1}{2}\right)\int_{\ln K^2}^{\infty} q_1(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 - K\int_{\ln K^2}^{\infty} q_2(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 \end{aligned} \tag{A3}$$

Note that $f_2(t,\tau_C;i\phi)$ is the characteristic function corresponding to $q_2(\ln \mathrm{VIX}_T^2)$ and $f_2(t,\tau_C;i\phi+1/2)/f_2(t,\tau_C;1/2)$ is the characteristic function corresponding to $q_1(\ln \mathrm{VIX}_T^2)$. Feller (1971) and Kendall and Stuart (1977) show how to recover the "probabilities" from the characteristic functions

$$\int_{\ln K^2}^{\infty} q_2(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi \ln K^2}\ f_2(t,\tau_C;i\phi)}{i\phi}\right] d\phi = \Pi_2 \tag{A4}$$

and similarly the other integral of $q_1(\ln \mathrm{VIX}_T^2)$.

$$\begin{aligned}\int_{\ln K^2}^{\infty} q_1(\ln \mathrm{VIX}_T^2)\, d\ln \mathrm{VIX}_T^2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi \ln K^2}\ f_1(t,\tau_C;i\phi)}{i\phi}\right] d\phi \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi \ln K^2}\ f_2(t,\tau_C;i\phi+1/2)}{i\phi\ f_2(t,\tau_C;1/2)}\right] d\phi = \Pi_1\end{aligned} \tag{A5}$$

Substituting (A.4) and (A.5) into the expression (A.3) proves the equations (17), (18) and (19) and noting that under the probability measure *F*, the time-*t* VIX futures price is $\mathrm{F}_t^{\mathrm{VIX}}(T) = \mathrm{E}_t^F(\mathrm{VIX}_T) = f_2(t,\tau_C;1/2)$.

Appendix B: Conditional Characteristic Function of the Log VIX Squared for the SVSCJ Model under the Probability Measure *F*

The valuation partial differential equation (PDE) in equation (16) can be rewritten as

$$
\begin{aligned}
&\left[r-\lambda_0\kappa-\left(\lambda_1\kappa+\frac{1}{2}\right)\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\frac{\partial C}{\partial L}+\frac{1}{2}\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial^2 C}{\partial L^2}\\
&+\rho\sigma_v\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial^2 C}{\partial L\partial(\ln\mathrm{VIX}_t^2)}\\
&+\frac{1}{2}\sigma_v^2\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)^2\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial^2 C}{\partial(\ln\mathrm{VIX}_t^2)^2}\\
&-\frac{1}{2}\sigma_v^2\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)^2\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial C}{\partial(\ln\mathrm{VIX}_t^2)}\\
&+\kappa_v\left[\theta_v-\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)\frac{\partial C}{\partial(\ln\mathrm{VIX}_t^2)}-\frac{\partial C}{\partial\tau}-rC\\
&+\left[\lambda_0+\lambda_1\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\mathrm{E}[C(t,\tau;L+z_S,\ln\mathrm{VIX}_{z_v}^2)-C(t,\tau;L,\ln\mathrm{VIX}^2)]\\
&+\lambda_1\times\mathrm{E}[z_vC(t,\tau;L+z_S,\ln\mathrm{VIX}_{z_v}^2)]=0
\end{aligned}
\qquad \text{(B1)}
$$

where we have applied the transformation $\ln\mathrm{VIX}_t^2=\ln(a_\tau\zeta_1 v_t/\tau+\zeta_1 b_\tau/\tau+\zeta_2)$. Assuming that jumps in volatility have an exponential distribution $z_v\sim\exp(1/\mu_v)$, conditional on volatility jump occurring the formula for VIX squared becomes

$$
\frac{1}{\tau}\mathrm{E}_t^F[\int_t^{t+\tau}(v_u+z_v)du]+\frac{1}{\tau}\mathrm{E}_t^F[\int_t^{t+\tau}J_u^2dN_u]=\mathrm{VIX}_t^2+\mu_v=\mathrm{VIX}_{t,z_v}^2 \qquad \text{(B2)}
$$

and the fair value of VIX futures is $\mathrm{F}_{t,z_v}^{\mathrm{VIX}}(T)=\mathrm{E}_t^F(\sqrt{\mathrm{VIX}_T^2+\mu_v})$. Inserting the conjectured solution in equation (17) into (B1) produces the PDEs for the risk-neutralized probabilities, Π_j for j=1, 2:

$$
\left\{\left[\frac{1}{2}\frac{\partial^2 \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L^2}-\left(\lambda_1\kappa+\frac{1}{2}\right)\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L}+\frac{1}{2}\sigma_\nu^2\left(\frac{\partial^2 \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial(\ln \mathrm{VIX}^2)^2}-\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial(\ln \mathrm{VIX}^2)}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)^2\right.\right.
$$

$$
\left.+\left(\rho\sigma_\nu\frac{\partial^2 \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L\partial(\ln \mathrm{VIX}^2)}-\kappa_\nu\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial(\ln \mathrm{VIX}^2)}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\right]\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)
$$

$$
\left.+\kappa_\nu\theta_\nu\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial(\ln \mathrm{VIX}^2)}+(r-\lambda_0\kappa)\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L}-\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial\tau_C}\right\}\Pi_1
$$

$$
+\left\{\left[\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L}-\left(\lambda_1\kappa+\frac{1}{2}\right)\mathrm{F}_t^{\mathrm{VIX}}(T)\right]\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)+(r-\lambda_0\kappa)\mathrm{F}_t^{\mathrm{VIX}}(T)\right\}\frac{\partial\Pi_1}{\partial L}
$$

$$
+\frac{1}{2}\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\mathrm{F}_t^{\mathrm{VIX}}(T)\frac{\partial^2\Pi_1}{\partial L^2}
$$

$$
+\left\{\left[\sigma_\nu^2\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial(\ln \mathrm{VIX}^2)}\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)-\frac{1}{2}\sigma_\nu^2\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\mathrm{F}_t^{\mathrm{VIX}}(T)+2\rho\sigma_\nu\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial L}-\kappa_\nu \mathrm{F}_t^{\mathrm{VIX}}(T)\right]\right.
$$

$$
\left.\times\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)+\kappa_\nu\theta_\nu \mathrm{F}_t^{\mathrm{VIX}}(T)\right\}\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\frac{\partial\Pi_1}{\partial(\ln \mathrm{VIX}^2)}
$$

$$
+\frac{1}{2}\sigma_\nu^2\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)^2\mathrm{F}_t^{\mathrm{VIX}}(T)\frac{\partial^2\Pi_1}{\partial(\ln \mathrm{VIX}^2)^2}
$$

$$
+\rho\sigma_\nu\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\mathrm{F}_t^{\mathrm{VIX}}(T)\frac{\partial^2\Pi_1}{\partial L\partial(\ln \mathrm{VIX}^2)}-\mathrm{F}_t^{\mathrm{VIX}}(T)\frac{\partial\Pi_1}{\partial\tau_{\mathrm{C}}}
$$

$$
+\left[\lambda_0+\lambda_1\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]
$$

$$
\times \mathrm{E}\left[\mathrm{F}_{t,z_\nu}^{\mathrm{VIX}}(T)\Pi_1(t,\tau_C;L+z_S,\ln \mathrm{VIX}_{z_\nu}^2)-\mathrm{F}_t^{\mathrm{VIX}}(T)\Pi_1(t,\tau_C;L,\ln \mathrm{VIX}^2)\right]
$$

$$
+\lambda_1\mathrm{E}[z_\nu \mathrm{F}_{t,z_\nu}^{\mathrm{VIX}}(T)\Pi_1(t,\tau_C;L+z_S,\ln \mathrm{VIX}_{z_\nu}^2)]=0 \tag{B3}
$$

and

$$
\begin{aligned}
&\left[r-\lambda_0\kappa-\left(\lambda_1\kappa+\frac{1}{2}\right)\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\frac{\partial\Pi_2(t,\tau;L,v)}{\partial L}\\
&+\frac{1}{2}\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial^2\Pi_2(t,\tau;L,v)}{\partial L^2}\\
&+\rho\sigma_v\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\frac{\partial^2\Pi_2(t,\tau;L,\ln \mathrm{VIX}^2)}{\partial L\partial(\ln \mathrm{VIX}^2)}\\
&+\left[\kappa_v\theta_v-\left(\kappa_v+\frac{1}{2}\sigma_v^2\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\frac{\partial\Pi_2(t,\tau_C;L,\ln \mathrm{VIX}^2)}{\partial(\ln \mathrm{VIX}^2)}\\
&+\frac{1}{2}\sigma_v^2\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)^2\frac{\partial^2\Pi_2(t,\tau_C;L,\ln \mathrm{VIX}^2)}{\partial(\ln \mathrm{VIX}^2)^2}\\
&-\frac{\partial\Pi_2(t,\tau_C;L,\ln \mathrm{VIX}^2)}{\partial\tau_C}\\
&+\left[\lambda_0+\lambda_1\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\mathrm{E}[\Pi_2(t,\tau_C;L+z_S,\ln \mathrm{VIX}_{z_v}^2)-\Pi_2(t,\tau_C;L,\ln \mathrm{VIX}^2)]\\
&+\lambda_1\mathrm{E}[z_v\Pi_2(t,\tau_C;L+z_S,\ln \mathrm{VIX}_{z_v}^2)]=0
\end{aligned}
\tag{B4}
$$

Observe that equation (B3) and (B4) are Fokker-Planck forward equations for probability functions. This implies that Π_1 and Π_2 must indeed be valid probability functions, with values bounded between 0 and 1. In Appendix A, we show that Π_1 can be calculated from the characteristic function $f_2(t,\tau_C;\phi)$ of Π_2. Hence, we focus on solving the PDE for Π_2 subject to the terminal condition:

$$
\Pi_2(t+\tau_C,0)=1_{\ln \mathrm{VIX}_T^2\geq\ln K^2}
\tag{B5}
$$

The corresponding characteristic functions for Π_2 will also satisfy the similar PDE:

$$\left[r-\lambda_0\kappa-\left(\lambda_1\kappa+\frac{1}{2}\right)\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\frac{\partial f_2(t,\tau;L,v)}{\partial L}$$
$$+\frac{1}{2}\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\frac{\partial^2 f_2(t,\tau;L,v)}{\partial L^2}$$
$$+\rho\sigma_v\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)\frac{\partial^2 f_2(t,\tau;L,\ln\mathrm{VIX}^2)}{\partial L\partial(\ln\mathrm{VIX}^2)}$$
$$+\left[\kappa_v\theta_v-\left(\kappa_v+\frac{1}{2}\sigma_v^2\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)\frac{\partial f_2(t,\tau_C;L,\ln\mathrm{VIX}^2)}{\partial(\ln\mathrm{VIX}^2)}$$
$$+\frac{1}{2}\sigma_v^2\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau\mathrm{VIX}_t^2}\right)^2\frac{\partial^2 f_2(t,\tau_C;L,\ln\mathrm{VIX}^2)}{\partial(\ln\mathrm{VIX}^2)^2}$$
$$-\frac{\partial f_2(t,\tau_C;L,\ln\mathrm{VIX}^2)}{\partial\tau_C}$$
$$+\left[\lambda_0+\lambda_1\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau\zeta_1}-\frac{b_\tau}{a_\tau}-\frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\mathrm{E}[f_2(t,\tau_C;L+z_S,\ln\mathrm{VIX}_{z_v}^2)-f_2(t,\tau_C;L,\ln\mathrm{VIX}^2)]$$
$$+\lambda_1\mathrm{E}[z_v f_2(t,\tau_C;L+z_S,\ln\mathrm{VIX}_{z_v}^2)]=0 \tag{B6}$$

with the boundary condition:

$$f_2(t+\tau_C,0;\phi)=e^{i\phi\ln\mathrm{VIX}_T^2} \tag{B7}$$

Conjecture that the solution to the PDE (B6) is given by

$$f_2(t,\tau_C;\phi)=e^{C_2(\tau_C)+J_2(\tau_C)+D_2(\tau_C)\ln\mathrm{VIX}_t^2+G_2(\tau_C)L_t} \tag{B8}$$

with $C_2(0)=J_2(0)=G_2(0)=0$ and $D_2(0)=i\phi$. Solving the resulting systems of differential equations respectively produce the desired characteristic functions.

The PDE (B6) contains an expression involving a characteristic function conditional on the occurrence of jumps. The jump-related characteristic function contains all the information required to describe the joint behavior of jumps in the asset price and volatility. Assuming that jumps in log-asset prices are normally distributed conditional on the realization of z_v, formally $z_S \mid z_v \sim N(\mu_j+\rho_j z_v,\sigma_j^2)$, the closed form of the jump-related characteristic function of the SVSCJ process is given by

$$f_2(t,\tau_C;L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})=\mathrm{E}^F_t[e^{i\phi\ln(\mathrm{VIX}^2_T+\mu_v)}]=\mathrm{E}^F_t[e^{i\phi\ln(\mathrm{VIX}^2_T)}e^{i\phi\ln(\mathrm{VIX}^2_T+\mu_v)-i\phi\ln(\mathrm{VIX}^2_T)}]$$
$$=\mathrm{E}_t\left[\mathrm{E}_t(e^{i\phi\ln(\mathrm{VIX}^2_T)}\mid \text{given jump occurring})\times e^{i\phi\ln\left(1+\frac{\mu_v}{\mathrm{VIX}^2_T}\right)}\right] \tag{B9}$$
$$=\mathrm{E}_t\left[f_2(t,\tau_C;L_t,\ln \mathrm{VIX}^2_t)\,e^{i\phi\ln\left(1+\frac{\mu_v}{\mathrm{VIX}^2_T}\right)}\right]=f_2(t,\tau_C;L_t,\ln \mathrm{VIX}^2_t)\,\mathrm{E}_t\left[e^{i\phi\ln\left(1+\frac{\mu_v}{\mathrm{VIX}^2_T}\right)}\right]$$

where $\ln \mathrm{VIX}^2_{T,z_v}=\ln(\mathrm{VIX}^2_T+\mu_v)$ obtained from (B2). The term in the expectation of (B9) can be re-written as

$$\ln\left(1+\frac{\mu_v}{\mathrm{VIX}^2_T}\right)=\ln\left(1+\mathrm{M}+\frac{\mu_v}{\mathrm{VIX}^2_T}-\mathrm{M}\right)=\ln(1+\mathrm{M})+\ln\left(1+\frac{\left(\frac{\mu_v}{\mathrm{VIX}^2_T}-\mathrm{M}\right)}{1+\mathrm{M}}\right) \tag{B10}$$

Take M largely enough to ensure the value of $(\mu_v/\mathrm{VIX}^2_T-\mathrm{M})/(1+\mathrm{M})$ is small. The Taylor's series expansion up to the first order should give a reasonable approximation to the log value, $\ln[1+(\mu_v/\mathrm{VIX}^2_T-\mathrm{M})/(1+\mathrm{M})]$, by $(\mu_v/\mathrm{VIX}^2_T-\mathrm{M})/(1+\mathrm{M})$. Since the maximum and minimum of the settlement VIX (converted to the notation of annualized standard deviation) in our sample are 0.2670 and 0.0995, respectively, with mean of 0.1493 and median of 0.1300, we choose M as $\mu_v/\min(\mathrm{VIX}^2_T)=\mu_v/0.0995^2$. Since $(\mu_v/\mathrm{VIX}^2_T-\mathrm{M})/(1+\mathrm{M})$ is small, we can approximate $\exp[i\phi\ln(1+\mu_v/\mathrm{VIX}^2_T)]$ in (B.9) at VIX^2_t using Taylor's expansion series up to the second order. The expectation term in (B9) is thus approximated by

$$\mathrm{E}_t\left[e^{i\phi\ln\left(1+\frac{\mu_v}{\mathrm{VIX}^2_T}\right)}\right]=\vartheta_1(\mathrm{VIX}^2_t,t)+\vartheta_2(\mathrm{VIX}^2_t,t) \tag{B11}$$

where

$$\vartheta_1(\mathrm{VIX}_t^2,t)=\frac{(i\phi)\mu_v}{(1+\mathrm{M})\mathrm{VIX}_t^4}\left(1+\frac{\mu_v}{\mathrm{VIX}_t^2}\right)^{i\phi}\Big[\ (1-\alpha_{T-t})\mathrm{VIX}_t^2+(\alpha_{T-t}-1)\zeta_2$$
$$+(\alpha_{T-t}b_\tau-a_\tau\beta_{T-t}-b_\tau)\frac{\zeta_1}{\tau}\Big]$$

$$\vartheta_2(\mathrm{VIX}_t^2,t)=\left[\frac{(i\phi)\mu_v}{(1+\mathrm{M})\mathrm{VIX}_t^6}+\frac{(i\phi)^2\mu_v^2}{2(1+\mathrm{M})^2\mathrm{VIX}_t^8}\right]\left(1+\frac{\mu_v}{\mathrm{VIX}_t^2}\right)^{i\phi}\Big\{(\alpha_{T-t}-1)^2\mathrm{VIX}_t^4$$
$$+\mathrm{VIX}_t^2\left[C_{T-t}\frac{a_\tau\zeta_1}{\tau}-2(\alpha_{T-t}-1)^2\zeta_2+2(\alpha_{T-t}-1)(a_\tau\beta_{T-t}+b_\tau-\alpha_{T-t}b_\tau)\frac{\zeta_1}{\tau}\right]-C_{T-t}\left(\frac{a_\tau\zeta_1}{\tau}\right)\zeta_2$$
$$-C_{T-t}\left(\frac{a_\tau\zeta_1}{\tau}\right)^2\frac{b_\tau}{a_\tau}+D_{T-t}\left(\frac{a_\tau\zeta_1}{\tau}\right)^2+\left[(1-\alpha_{T-t})\zeta_2+(a_\tau\beta_{T-t}+b_\tau-\alpha_{T-t}b_\tau)\frac{\zeta_1}{\tau}\right]^2\Big\}$$

$$\alpha_{T-t}=e^{-(\kappa_v-\lambda_1\mu_v)(T-t)},\quad \beta_{T-t}=\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)(1-\alpha_{T-t}),$$

$$C_{T-t}=\frac{\sigma_v^2+2\lambda_1\mu_v^2}{\kappa_v-\lambda_1\mu_v}(\alpha_{T-t}-\alpha_{T-t}^2),$$

$$D_{T-t}=\frac{[\sigma_v^2+2\lambda_1\mu_v^2]}{2(\kappa_v-\lambda_1\mu_v)}\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)(1-\alpha_{T-t})^2+\frac{\lambda_0\mu_v^2}{\kappa_v-\lambda_1\mu_v}(1-\alpha_{T-t}^2).$$

Therefore, the characteristic function $f_2(t,\tau_C;L_t+z_S,\ln\mathrm{VIX}_{t,z_v}^2)$ in (B9) can be written as

$$f_2(t,\tau_C;L_t+z_S,\ln\mathrm{VIX}_{t,z_v}^2)=f_2(t,\tau_C;L_t,\ln\mathrm{VIX}_t^2)[\vartheta_1(\mathrm{VIX}_t^2,t)+\vartheta_2(\mathrm{VIX}_t^2,t)] \qquad \text{(B12)}$$

Now, the expectation terms in the PDE (B6) become

$$\begin{aligned}&\mathrm{E}_t^F[\,f_2(t,\tau_C;L_t+z_S,\ln\mathrm{VIX}_{t,z_v}^2)-f_2(t,\tau_C;L_t,\ln\mathrm{VIX}_t^2)]\\&\quad=f_2(t,\tau_C;L_t,\ln\mathrm{VIX}_t^2)[\vartheta_1(\mathrm{VIX}_t^2,t)+\vartheta_2(\mathrm{VIX}_t^2,t)-1]\end{aligned} \qquad \text{(B13)}$$

and

$$\mathrm{E}_t^F[z_v f_2(t,\tau_C;L_t+z_S,\ln\mathrm{VIX}_{t,z_v}^2)]=\mu_v f_2(t,\tau_C;L_t+z_S,\ln\mathrm{VIX}_{t,z_v}^2) \qquad \text{(B14)}$$

Having specified the expectations term of the PDE (B6), we can proceed with finding a solution to it. The conjectured solution (B8) is inserted into the PDE (B6) and the terms in $\ln\mathrm{VIX}_t^2$, the ones related to the diffusion part of the process, and the ones related to jumps are grouped together to obtain four ordinary differential equations

(ODEs):

$$\frac{\partial G_2(\tau_C)}{\partial \tau_C} = 0$$

$$\frac{\partial C_2(\tau_C)}{\partial \tau_C} = -\kappa_v D_2(\tau_C) + (r-\delta) G_2(\tau_C)$$
$$+ \left(\rho \sigma_v \frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} D_2(\tau_C) + \frac{1}{2} G_2(\tau_C) - \frac{1}{2} \right) \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) G_2(\tau_C)$$

$$\frac{\partial J_2(\tau_C)}{\partial \tau_C} = -\kappa \left[\lambda_0 + \lambda_1 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \right] G_2(\tau_C) - \lambda_0 - \lambda_1 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right)$$
$$+ \left[\lambda_0 + \lambda_1 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) + \lambda_1 \mu_v \right] [\vartheta_1(\mathrm{VIX}_t^2, t) + \vartheta_2(\mathrm{VIX}_t^2, t)]$$

$$\frac{\partial D_2(\tau_C)}{\partial \tau_C} = \frac{1}{2} \sigma_v^2 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right)^2 \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right) D_2^2(\tau_C)$$
$$+ \left[\kappa_v \theta_v - \frac{\sigma_v^2 a_\tau \zeta_1}{2 \tau \mathrm{VIX}_t^2} \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) + \kappa_v \left(\frac{b_\tau}{a_\tau} + \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \right] \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right) \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right) D_2(\tau_C)$$

(B15)

Boundaries conditions are $C_2(0) = 0$, $J_2(0) = 0$, $G_2(0) = 0$, and $D_2(0) = i\phi$, so that $f_2(t + \tau_C, 0; \phi) = \exp(i\phi \ln \mathrm{VIX}_T^2)$.

The closed-form expression of the conditional characteristic function (B8) is obtained by finding solutions to ODEs (B15):

$$G_2(\tau_C) = 0$$

$$D_2(\tau_C) = i\phi + \frac{1}{2} \sigma_v^2 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right)^2 \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right) \int_0^{\tau_C} D_2^2(u) du$$
$$+ \left[\kappa_v \theta_v - \frac{1}{2} \sigma_v^2 \frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) + \kappa_v \left(\frac{b_\tau}{a_\tau} + \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \right]$$
$$\times \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right) \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right) \int_0^{\tau_C} D_2(u) du$$
$$= \frac{-B}{A} + \left\{ \frac{A}{B} + \left[\left(i\phi + \frac{B}{A} \right)^{-1} - \frac{A}{B} \right] \times e^{B \tau_C} \right\}^{-1}$$

$$
\begin{aligned}
C_2(\tau_C) &= -\kappa_\nu \int_0^{\tau_C} D_2(u)du \\
&= -\kappa_\nu \int_0^{\tau_C} \left\{ \frac{-B}{A} + \left\{ \frac{A}{B} + \left[\left(i\phi + \frac{B}{A} \right)^{-1} - \frac{A}{B} \right] \times e^{B\tau_C} \right\}^{-1} \right\} du \\
&= \frac{B}{A}\kappa_\nu \tau_C - \frac{\kappa_\nu}{A} \left\{ B\tau_C - \ln\left\{ \frac{A}{B} + \left[\left(i\phi + \frac{B}{A} \right)^{-1} - \frac{A}{B} \right] e^{B\tau_C} \right\} + \ln\left[\left(i\phi + \frac{B}{A} \right)^{-1} \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
J_2(\tau_C) &= -\lambda_0 \tau_C - \lambda_1 \tau_C \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \\
&\quad + \left[\lambda_0 + \lambda_1 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) + \lambda_1 \mu_\nu \right] \left[\vartheta_3(\mathrm{VIX}_t^2, t) + \vartheta_4(\mathrm{VIX}_t^2, t) \right]
\end{aligned}
$$

(B16)

where

$$
A = \frac{1}{2}\sigma_\nu^2 \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right)^2 \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right)
$$

$$
B = \left[\kappa_\nu \theta_\nu - \frac{1}{2}\sigma_\nu^2 \frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) + \kappa_\nu \left(\frac{b_\tau}{a_\tau} + \frac{\tau \zeta_2}{a_\tau \zeta_1} \right) \right] \left(\frac{a_\tau \zeta_1}{\tau \mathrm{VIX}_t^2} \right) \left(\frac{1}{\ln \mathrm{VIX}_t^2} \right)
$$

$$
\begin{aligned}
\vartheta_3(\mathrm{VIX}_t^2, t) &= \int_0^{\tau_C} \vartheta_1(\mathrm{VIX}_t^2, u)du = \frac{(i\phi)\mu_\nu}{(1+\mathrm{M})\mathrm{VIX}_t^4} \left(1 + \frac{\mu_\nu}{\mathrm{VIX}_t^2} \right)^{i\phi} \left[\left(\tau_C - \frac{(1-\alpha_{T-t})}{(\kappa_\nu - \lambda_1 \mu_\nu)} \right) \mathrm{VIX}_t^2 \right. \\
&\quad \left. - \tau_C a_\tau \beta_{T-t} \frac{\zeta_1}{\tau} - \left(\frac{\zeta_1}{\tau} b_\tau + \zeta_2 \right) \left(\tau_C - \frac{(1-\alpha_{T-t})}{(\kappa_\nu - \lambda_1 \mu_\nu)} \right) \right]
\end{aligned}
$$

$$
\vartheta_4(\mathrm{VIX}_t^2, t) = \int_0^{\tau_C} \vartheta_2(\mathrm{VIX}_t^2, u)du = \left[\frac{(i\phi)\mu_\nu}{(1+\mathrm{M})\mathrm{VIX}_t^6} + \frac{(i\phi)^2 \mu_\nu^2}{2(1+\mathrm{M})^2 \mathrm{VIX}_t^8} \right] \left(1 + \frac{\mu_\nu}{\mathrm{VIX}_t^2} \right)^{i\phi} \vartheta_5(\mathrm{VIX}_t^2, t)
$$

$$
\begin{aligned}
\vartheta_5(\mathrm{VIX}_t^2,t) &= \left[1+\frac{1}{2}\left(\frac{1-\alpha_{T-t}^2}{\kappa_v-\lambda_1\mu_v}\right)-2\left(\frac{1-\alpha_{T-t}}{\kappa_v-\lambda_1\mu_v}\right)\right]\mathrm{VIX}_t^4+\left[4\left(\frac{\zeta_1}{\tau}b_\tau+\zeta_2\right)\left(\frac{1-\alpha_{T-t}}{\kappa_v-\lambda_1\mu_v}\right)\right.\\
&-\left(\zeta_2+\frac{\zeta_1}{\tau}b_\tau\right)\left(\frac{1-\alpha_{T-t}^2}{\kappa_v-\lambda_1\mu_v}\right)+\frac{\zeta_1 a_\tau}{\tau}\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)\left(\frac{(1-\alpha_{T-t})(3-\alpha_{T-t})}{(\kappa_v-\lambda_1\mu_v)}-2\tau_C\right)\\
&\left.+\frac{a_\tau\zeta_1}{2\tau}\left(\frac{\sigma_v^2+2\lambda_1\mu_v^2}{\kappa_v-\lambda_1\mu_v}\right)\frac{(1-\alpha_{T-t})^2}{(\kappa_v-\lambda_1\mu_v)}-2\left(\zeta_2+\frac{\zeta_1}{\tau}b_\tau\right)\right]\mathrm{VIX}_t^2+\left(\frac{(1-\alpha_{T-t})(\alpha_{T-t}-3)}{2(\kappa_v-\lambda_1\mu_v)}\right)\\
&\times\left[\zeta_2^2+2\frac{\zeta_1\zeta_2}{\tau}b_\tau+\left(\frac{b_\tau\zeta_1}{\tau}\right)^2\right]+\frac{1}{2}\left(\frac{b_\tau\zeta_1}{\tau}\right)^2\left(\frac{1-\alpha_{T-t}^2}{\kappa_v-\lambda_1\mu_v}\right)+\left[\frac{a_\tau\zeta_1}{\tau}\zeta_2+\left(\frac{a_\tau\zeta_1}{\tau}\right)^2\frac{b_\tau}{a_\tau}\right]\\
&\times\left[\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)\left(2\tau_C-\frac{(1-\alpha_{T-t}^2)}{(\kappa_v-\lambda_1\mu_v)}\right)-\frac{1}{2}\left(\frac{\sigma_v^2+2\lambda_1\mu_v^2}{\kappa_v-\lambda_1\mu_v}\right)\frac{(1-\alpha_{T-t})^2}{(\kappa_v-\lambda_1\mu_v)}\right]\\
&+\left(\frac{a_\tau\zeta_1}{\tau}\right)^2\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)\left[\frac{(1-\alpha_{T-t})(\alpha_{T-t}-3)}{2(\kappa_v-\lambda_1\mu_v)}+\tau_C\right]\left[\left(\frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v}\right)+\frac{1}{2}\left(\frac{\sigma_v^2+2\lambda_1\mu_v^2}{\kappa_v-\lambda_1\mu_v}\right)\right]\\
&+\left(\frac{a_\tau\zeta_1}{\tau}\right)^2\left(\frac{\lambda_0\mu_v^2}{\kappa_v-\lambda_1\mu_v}\right)\left[\tau_C-\frac{(1-\alpha_{T-t}^2)}{2(\kappa_v-\lambda_1\mu_v)}\right]+\zeta_2^2+2\frac{\zeta_1\zeta_2}{\tau}b_\tau
\end{aligned}
$$

We now obtain the closed-form solution to the characteristic function $f_2(t,\tau_C;\phi)=e^{C_2(\tau_C)+J_2(\tau_C)+D_2(\tau_C)\ln\mathrm{VIX}_t^2}$ for Π_2. Further, equation (A2) produces the desired characteristic function for Π_1:

$$
\begin{aligned}
f_1(t,\tau_C;\phi) &= \frac{f_2(t,\tau_C;\phi+1/2)}{f_2(t,\tau_C;1/2)} \\
&= e^{C_2(\tau_C;\phi+1/2)-C_2(\tau_C;1/2)]+[J_2(\tau_C;\phi+1/2)-J_2(\tau_C;1/2)]+[D_2(\tau_C;\phi+1/2)-D_2(\tau_C;1/2)]\ln\mathrm{VIX}_t^2}
\end{aligned}
\tag{B17}
$$

The SV, the SVJ and the SVCJ models are all nested within the general formula in equation (17). In the SVCJ case, for instance, the state-dependent jump frequency vanishes in equation (B1). The general solution in equations (B16)–(B17) will still apply except that now $\lambda_1=0$. The final characteristic functions f_j for the SVCJ model are respectively given by setting $\lambda_1=0$ in equations (B8) and (B17). The characteristic functions for the SVJ model can be obtained by setting $\mu_v=0$ in (B8), (B16), and (B17). The characteristic functions for the SV model are obtained by further setting $\lambda_0=\lambda_1=0$ in (B8), (B16), and (B17).

Conditional on the occurrence of volatility jumps, the fair value of a VIX call option is re-written as

$$\begin{aligned}&C(t,\tau_C;L+z_S,\ln \mathrm{VIX}^2_{z_v})\\&=B(t,T)\,\mathrm{F}^{\mathrm{VIX}}_{t,z_v}(T)\,\Pi_1(L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})-KB(t,T)\,\Pi_2(L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})\end{aligned} \tag{B18}$$

where

$$\mathrm{F}^{\mathrm{VIX}}_{t,z_v}(T)=f_2(t,\tau_C;1/2;L_t,\ln \mathrm{VIX}^2_t)[\vartheta_1(\mathrm{VIX}^2_t,t;i\phi=1/2)+\vartheta_2(\mathrm{VIX}^2_t,t;i\phi=1/2)]$$

$$\Pi_1(L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K^2}}{i\phi}\frac{f_2(t,\tau_C;i\phi+1/2;L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})}{f_2(t,\tau_C;1/2;L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})}\right]d\phi$$

$$\Pi_2(L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K^2}}{i\phi}f_2(t,\tau_C;i\phi;L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})\right]d\phi$$

$$f_2(t,\tau_C;L_t+z_S,\ln \mathrm{VIX}^2_{t,z_v})=f_2(t,\tau_C;L_t,\ln \mathrm{VIX}^2_t)[\vartheta_1(\mathrm{VIX}^2_t,t)+\vartheta_2(\mathrm{VIX}^2_t,t)]$$

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