Forward Volatility Hedge

Yueh-Neng Lin*
Department of Finance, National Chung Hsing University
e-mail: ynlin@dragon.nchu.edu.tw
tel:+(886) 4 2285-7043; fax:+(886) 4 2285-6015

Abstract

VIX futures are launched on 26 March 2004, and their underlying VIX obviously becomes an indicator of investors' confidence towards the U.S. equity market. Since VIX futures settle to the forward 30-day implied volatility of the S&P 500, they are natural to hedge the "forward vega" risk of S&P 500 options. Empirically, however, it is not known whether and by how much VIX futures improves hedging. This study fills this gap by first deriving a futures option model that allows volatility and jumps to be stochastic. Using VIX futures, S&P 500 futures and S&P 500 futures options, this study examines several alternative models within two types of hedging strategy, including (i) the minimum-variance hedge of option contracts that rely on the underlying futures as the single hedging instrument, and (ii) a delta-neutral hedge, in which as many hedging instruments as there are risk sources are used to make the net position completely risk-immunized (locally). For comparison, a forward-start strangle portfolio is constructed using S&P 500 options. Our findings show that VIX futures outperform forward-strangle strategy at most moneyness-maturity categories and the results are robust across models and hedging strategy. Overall, VIX futures are a better hedging instrument than standard options if the target option is a *futures option*, or equivalently if the risk exposure is the *forward* volatility risk. But for VIX futures, incorporating stochastic volatility and jumps yields the best hedging performance.

Keywords: VIX futures, forward-start strangle, forward volatility risk, stochastic volatility, price jumps, futures option

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*Corresponding author: Yueh-Neng Lin, Department of Finance, National Chung Hsing University, 250 Kuo-Kuan Road, Taichung, Taiwan; Tel: +(886) 4 2285-7043; Email: ynlin@dragon.nchu.edu.tw. The work of Lin was supported by a grant from the National Science Council in Taiwan (grant number: *NSC 94-2416-H-005-019*).

1. Introduction

Brenner et al. (2004) argue that option market makers' and portfolio managers' performance may be affected by unexpected changes in volatility and thus suggest a standard straddle using standard call and put options to hedge volatility risk. However, this option portfolio will need to hedge the changes in delta and volatility simultaneously. One can refer to Carr and Madan's (2002) dynamic trade to isolate the volatility risk, but it will incur a higher cost. Carr and Madan (2002) also provide another opinion to hedge volatility risk—options on a straddle (STO). They use a numerical method to calculate the value of STO, and find the value of STO is highly sensitive to the volatility, especially for high volatility variation and shorter maturities. Therefore, they conclude that STO is an efficient tool to hedge volatility risk. Other than straddle, Rebonato (1999) proposes another idea to hedge volatility risk. The target instrument is a long position on the forward-start option. It is considered as an at-the-money (ATM) option because its strike price is set in the future. Therefore, there are no delta and gamma risk exposures, but only volatility risk remained. Rebonato (1999) constructs two wide strangles with different maturities so that the changes of underlying stock price will not affect the payoff of the portfolio. In other words, this strangle portfolio can hedge forward volatility risk, without exposure to delta and gamma risk. Since volatility is time-dependent and state-dependent, however, this strategy must rebalance whatever time passes by or the volatility risk exposure exists. Further, using Eurodollar options Chaput and Ederington (2005) examine the trading and structure of alternate volatility trades. They find that straddles are the most popular volatility trade since volatility traders seek volatility trading designs with low deltas, low transaction costs and high gammas and vega.

Briefly, the risks of an option writer can be partitioned into the price risk and volatility risk. For the futures option writer, however, the volatility risk he faces is the so-called ***forward volatility risk***. The forward volatility risk refers to the risk exposure induced by volatility randomness between the futures option's expiry and the futures' expiry. In contrast, the traditional straddle or strangle strategy mainly hedges the volatility risk between current day and option's expiry, so-called ***spot volatility risk***.

The Chicago Board Options Exchange (CBOE) introduces the VIX (volatility index) in 2003, which is the sentiment volatility implicit in S&P 500 index (SPX) option prices.[1] For offering more instruments to manage volatility risk, CBOE launches the VIX futures in March 2004 and the VIX option in February 2006. Before CBOE launches VIX and derivatives on VIX, Brenner and Galai (1989) have introduced the

[1] Since 1990, VIX has moved opposite the SPX 88% of the time, with a negative daily return correlation of –0.67. On average, VIX has risen 16.8% on days when SPX fell 3% or more. Simply buying VIX call options, therefore, could be considered a hedge to protect against sharply falling stock prices.

concept of volatility index and options on volatility index. They believe that different volatility measures that are highly correlated to the volatility exposures of all market participants are different. Hence, the volatility index may be based on standard deviation of historical data, implied volatility of options, or the combination of historical and option data. Then, the futures or options on volatility index may satisfy the hedging demand of different participants. However, it is still an empirical question of whether the hedging efficiency of derivatives on volatility is as perfect as the theoretical result. This paper fills this gap by examining whether and by how much VIX futures improves hedging. Since VIX futures settle to the forward 30-day implied volatility of the S&P 500, they are natural to hedge the "forward vega" risk of S&P 500 options. A short position on S&P 500 futures call options is chosen as our target instrument, since it consists of the volatility risk between option's expiry and its underlying futures' expiry, or equivalently the "forward vega" risk. But in reality most futures option contracts are American in nature. This paper derives a model for American futures options in the following manner. For options with early exercise potential, compute the Barone-Adesi and Whaley (1987) early-exercise premium, treating it as if the stock volatility. Adding this early-exercise adjustment component to the European futures option price in the pricing formula should result in a reasonable approximation of the corresponding American futures option price (e.g., Bates, 1996; Broadie et al., 2007). Alternatively, one can follow such a nonparametric approach as in Aït-Sahalia and Lo (1998) and Broadie et al. (2000) to price American options. The closed-form option pricing formula makes it possible to derive hedge ratios analytically. The analytical expressions for the hedge ratios form a convenient basis for constructing hedges.

In the present literature, there are at least three sources of stochastic variations over time: price risk, volatility risk and interest rate risk. Bakshi et al. (1997) relax the Black-Scholes-Merton (1973) (BSM) model to incorporate stochastic volatility, stochastic interest rates and random price jumps and study what the most important element to improve hedging effectiveness. They use a short position on a SPX call option as the target instrument and investigate the performance of two hedging strategies, the single-instrument hedge[2] and the delta-neutral hedge[3]. For both strategies, the stochastic-volatility model is the best performer, followed by the random jumps and stochastic-volatility model. This illustrates that once the stochastic volatility is modeled, the hedging performance may be improved by incorporating neither price jumps, nor

[2] The single-instrument hedging strategy only consists of underlying asset, this strategy mainly hedges price risk.

[3] Under the delta-neutral hedge, BSM only uses the underlying stock, the SV and SVJ models consist of the underlying stock and another SPX option with different strikes or maturities, and the SVSI model further adds a discount bond.

stochastic interest rate into the option pricing framework. Nevertheless, the delta-neutral strategy provides a better hedging performance than a single-instrument strategy. Bakshi and Kapadia (2003) use Heston's (1993) stochastic-volatility option pricing model to construct a delta-hedged strategy for a long position on SPX call options.[4] They find that the volatility risk is priced and the price jump affects the hedging efficiency. Vishnevskaya (2004) follows the structure of Bakshi and Kapadia (2003) and constructs a delta-vega-hedged portfolio for a long position on the SPX call option, consisting of the underlying stock, another option and the money-market fund. His result suggests the existence of some other sources of risk. Further, Mwanga and Ndogmo (2005) describe the characteristics of BSM greeks and their hedging functions for the target position on the option, futures and forward contracts. By assuming deterministic interest rates, their hedge gets a lot of exposure to delta and vega risks. Other than using a traditional option to hedge vega risk, Neuberger (1994) adopts the log contract to hedge volatility. Finally, Psychoyios and Skiadopoulos (2006) use the short position on a standard European-style call option as the target instrument and apply a joint Monte Carlo simulation to generate data. They then couple two hedging schemes with two traditional option models and three volatility option models in order to investigate which

[4] A delta-hedged portfolio is a replicating portfolio constructed to hedge a long position in option by shorting delta units of the underlying stock, such that the net investment earns the risk-free rate.

combination gets the best hedging performance. Psychoyios and Skiadopoulos (2006) further investigate the hedging performance under different rebalance frequencies, maturity and moneyness. Consistent with Boyle and Emanuel's (1980) work, the delta hedging errors for the short- and medium-term options under daily rebalance are less than the hedging errors under weekly rebalance. In addition, the short-term volatility options are more sensitive to the changes of volatility, and using short-term volatility options with rolling-over strategy will thus get a better hedging performance. In conclusion, the traditional option, on the one hand, is a more efficient instrument than the volatility option for hedging a tradition option. On the other hand, the hedging performance using volatility option can be improved by either increasing the rebalance frequency or with a rolling-over strategy.

Guided by previous studies, the price risk, stochastic-volatility risk and price-jump risk apparently become the key factors when constructing a hedging strategy for option writers. Hence, this paper derives a futures option model that allows volatility and jumps to be stochastic, abbreviated as the SVJ model. The setup is rich enough to contain competing futures option formulas as special cases, including the constant-volatility (CONST) model, and the stochastic-volatility (SV) model. For comparison, a forward-start strangle portfolio is constructed using SPX options. For comparison, a forward-start strangle portfolio using SPX options is also constructed. The forward-start strangle is a combination of long-term and short-term strangles and is aimed to hedge the forward volatility risk. This paper then uses these models and two instruments, VIX futures and the forward strangle portfolio, to construct the hedged portfolio.

The contributions of this paper are threefold. First, closed-form solutions to the target American futures options under alternate underlying processes are provided. Second, the concept of *forward* volatility risk applied to VIX futures and the construction of a forward strangle portfolio are introduced. Third, we derive the hedging weights of VIX futures and the forward strangle portfolio that will be convenient to practical participants for risk management purposes.

Our finding reveals that the VIX futures generally outperform the forward strangle portfolio over the hedging period 20 October 2004 – 30 June 2005. In particular, based on the absolute hedging errors, the SVJ model is the best overall performer, followed by the SV model, and then by the CONST for a short position on the SPX futures call option. Our findings are in sharp contrast with that obtained by Psychoyios and Skiadopoulos (2006). They find that volatility options are not better hedging instruments than plain-vanilla options, and that the most naïve volatility option-pricing model can be reliably used for pricing and hedging purposes. Further, regardless of hedge rebalancing frequency, the real significant improvement by the stochastic volatility models over the CONST occurs only when OTM calls are being hedged. The hedging-based ranking of the models is in contrast with that obtained in Bakshi et al. (1997). Bakshi et al. (1997) find that the SVJ does not improve over the SV's hedging performance for a short position in a SPX call option. Based on our results, we conclude that the VIX futures is a better hedging instrument than standard options if the target option is a traditional *futures option*, or equivalently if the risk exposure is the *forward* volatility risk. Hedging performance can also be improved further by incorporating price jumps into the American-style futures option pricing framework.

The rest of this paper proceeds as follows. Section 2 develops the hedging models. Section 3 provides a description of the data for the empirical work. The empirical procedure is presented in Section 4. Finally, Section 5 concludes.

2. Empirical Model

2.1 The Futures Options

To investigate whether VIX futures hedge the forward volatility risk better than the forward strangle portfolio composed by standard plain-vanilla options, this study first develops in closed form the fair prices of index futures options under alternate SPX price processes. The jump-diffusion and stochastic volatility (SVJ) process of Bates (1996) and Bakshi et al. (1997) for the SPX price contains constant volatility (CONST) of BSM (1973) and stochastic volatility (SV) of Heston (1993) as special cases. Consequently, we concentrate our efforts on the SVJ and the two processes just described. Given a constant volatility (BSM) structure, the price process of the SPX futures under a risk-neutral probability measure Q becomes

$$dF_t = \sigma F_t \, d\omega_{S,t} \tag{1}$$

where F_t is the futures price at time t, σ is the volatility of the underlying SPX price, and $\omega_{S,t}$ is Brownian motion under Q. The time-t price of a futures call option ($C_t(F)$) with option expiry T_1 and futures expiry T_2, for $T_2 \geq T_1$, is given by,

$$C_t(F) = e^{-r(T_1-t)}[F_t N(d_1^*) - XN(d_2^*)] \tag{2}$$

where X is the strike price of the futures option, $N(\cdot)$ is the cumulative probability function of a standard normal distribution, $d_1^* = \ln(F_t / X) + \sigma^2(T_1 - t)/2]/\sigma\sqrt{T_1 - t}$ and $d_2^* = \ln(F_t / X) - \sigma^2(T_1 - t)/2]/\sigma\sqrt{T_1 - t}$. The accurate derivation is given in the Appendix A.

Under the SV, Heston (1993) assumes that the underlying stock of futures follows a geometric Brownian motion and the instantaneous variance v_t of the underlying stock follows a mean-reverting square root process; that is,

$$dS_t = rS_t + \sqrt{\nu_t} S_t \, d\omega_{S,t} \tag{3}$$

$$d\nu_t = (\theta_\nu - \kappa_\nu \nu_t)dt + \sigma_\nu \sqrt{\nu_t} \, d\omega_{\nu,t} \tag{4}$$

where S_t is the price of the underlying stock; r is the annualized continuously compound interest rate; κ_ν is the speed of mean-reverting adjustment of ν_t; θ_ν / κ_ν is the long-run mean of ν_t; σ_ν is the variation coefficient of ν_t; and $\omega_{S,t}$ and $\omega_{\nu,t}$ are two correlated risk-neutral Brownian processes with the correlation coefficient $\rho \, dt = corr(d\omega_{S,t}, d\omega_{\nu,t})$. Referred to Bates (1996), the price of American-style futures option ($C^A(F)$) is given as follows:

$$C(F) \equiv \begin{cases} C^E(F) + KA_2\left(\dfrac{F/K}{y_c^*}\right)^{q_2} & \text{for } F/K < y_c^* \\ F - K & \text{for } F/K \ge y_c^* \end{cases} \tag{5}$$

where $C^E(F) = e^{r(T_2 - T_1)} S_t \Pi_1' - K e^{-r(T_1 - t)} \Pi_2'$ is the price of European-style futures options. The q_2 is the positive root of equation (B14) of Appendix B for the SV model. Π_1' and Π_2' are risk-neutral probabilities recovered from inverting the characteristic functions f_1 and f_2, respectively, for *j* = 1 and 2:

$$\Pi_j'(t, T_1 - t; S_t, r, \nu_t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln[Ke^{-r(T_2 - T_1)}]} f_j(t, T_1 - t; S_t, r, \nu_t; \phi)}{i\phi}\right] \tag{6}$$

where T_2 is the expiration date of SPX futures, and $T_2 > T_1$. The accurate derivation is given in the Appendix B. Under the SVJ model, the process of the underlying stock is given as

$$dS_t = (r - \lambda_J \mu_J) S_t + \sqrt{v_t} S_t \, d\omega_{S,t} + J_t S_t dN_t \tag{7}$$

where J_t is the percentage jump size with mean κ^*. The jumps in asset log-prices are assumed to be normally distributed, i.e., $\ln(1+J_t) \sim N(\mu_J, \sigma_J^2)$. Satisfying the no-arbitrage condition, $\kappa^* = \exp(\mu_J + \sigma_J^2/2) - 1$. Further, dN_t is the jump frequency following a Poisson process with mean λ_J. Finally, v_t follows the equation (4). The price of a futures option under the SVJ model is similar with the one under the SV model. The differences between the SV and the SVJ are the characteristic functions as shown in equations (C1) and (C2), and the function of q_2 is given in equation (C6) of Appendix C.

2.2 Hedging Strategies

This study uses a short position on the futures option as the target portfolio at time *t*,[5] i.e. $TAR_t = -C_t(F)$, where the expiration dates of SPX futures and SPX futures options are respectively T_2 and T_1 and $t < T_1 < T_2$. Furthermore, this study constructs two hedging schemes.

[5] Because the last trading date of the VIX futures is usually the third Tuesday and the last trading date of SPX futures options is the third Friday of the expiration month.

Hedging Scheme 1 (HS1): The instrument portfolio consists of $N_{1,t}$ shares of underlying futures, and $N_{2,t}$ shares of forward-start strangle portfolios. Following Rebonato (1999), the forward-start strangle portfolio consists of a short position of short-term strangle (the expiration date of the standard call and standard put is T_1; the strikes of the call and put are K_2 and K_1 with $K_2 > K_1$) and a long position of long-term strangle (the expiration date of the standard call and standard put are T_2; the strikes of the standard call and put are K_2 and K_1 with $K_2 > K_1$), i.e. $INST_t = -C_t(T_1, K_2) - P_t(T_1, K_1) + C_t(T_2, K_2) + P_t(T_2, K_1)$.

Hedging Scheme 2 (HS2): The instrument portfolio consists of $N_{1,t}$ shares of underlying futures, and $N_{2,t}$ shares of the VIX futures, i.e. $INST_t = \mathrm{F}_t^{\mathrm{VIX}}(T)$.

Next, this study incorporates two hedging schemes with the BSM, the SV and the SVJ option models to construct six hedging strategies, respectively.

1. HS1−BSM hedging strategy: Under the hedging scheme 1, the BSM model is used to calculate the greeks of the target and the instrument portfolio. Therefore, the profit of this hedging portfolio is defined as

$$\pi_t = N_{1,t}F_t + N_{2,t}INST_t - C_t(F) \tag{8}$$

Further add the constraints of delta-neutral and vega-neutral by

$$\frac{\partial \pi_t}{\partial F_t} = N_{1,t} + N_{2,t}\frac{\partial INST_t}{\partial F_t} - \frac{\partial C_t(F)}{\partial F_t} = 0 \tag{9}$$

$$\frac{\partial \pi_t}{\partial \nu_t} = N_{2,t} \frac{\partial INST_t}{\partial \nu_t} - \frac{\partial C_t(F)}{\partial \nu_t} = 0 \tag{10}$$

We will get the shares of instrument assets

$$\begin{aligned} N_{1,t} &= \frac{\partial C_t(F)}{\partial F_t} - N_{2,t} \frac{\partial INST_t}{\partial F_t} \\ &= \frac{\partial C_t(F)}{\partial F_t} - e^{-r(T_2-t)} N_{2,t} (-\Delta_{C(T_1,K_2)} - \Delta_{P(T_1,K_1)} + \Delta_{C(T_2,K_2)} + \Delta_{P(T_2,K_1)}) \\ &= e^{-r(T_1-t)} N(d_1^*) - e^{-r(T_2-t)} N_{2,t} (-\Delta_{C(T_1,K_2)} - \Delta_{P(T_1,K_1)} + \Delta_{C(T_2,K_2)} + \Delta_{P(T_2,K_1)}) \end{aligned} \tag{11}$$

$$N_{2,t} = \left(\frac{\partial C_t(F)}{\partial \nu_t} \right) / \left(\frac{\partial INST_t}{\partial \nu_t} \right) = \frac{F_t e^{-r(T_1-t)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_1^*)^2}{2} \sqrt{T_1 - t} \right)}{-\upsilon_{C(T_1,K_2)} - \upsilon_{P(T_1,K_1)} + \upsilon_{C(T_2,K_2)} + \upsilon_{P(T_2,K_1)})} \tag{12}$$

where $\Delta_{C(T,K)}$ and $\Delta_{P(T,K)}$ are BSM's deltas that respectively equal $N(d_1)$ for a call and $\Delta_{P(T,K)} - 1$ for a put with $d_1 = [ln(S_t / K) + 0.5\sigma^2 (T-t)] / \sigma\sqrt{T-t}$; $\upsilon_{C(T,K)}$ and $\upsilon_{P(T,K)}$ are BSM's vegas for call and put options, respectively, equal to $S_t \sqrt{T-t} N(d1)$. The hedging ratios are derived in equations (A7)–(A10) of the Appendix A. Assuming that there are no arbitrage opportunities, this portfolio should earn the risk-free rate. The change of the value of this portfolio is defined as

$$\Delta\pi_{t+\Delta t} = \pi_{t+\Delta t} - \pi_t \tag{13}$$

Finally, the hedging error is defined as the additional profit (loss) over the risk-free return and it can be written as

$$\begin{aligned} HE_t(t+\Delta t) &= \Delta\pi_{t+\Delta t} - \pi_t (e^{r\Delta t} - 1) \\ &= (C_{t+\Delta t} - C_t) + N_{1,t}(F_{t+\Delta t} - F_t) + N_{2,t}(INST_{t+\Delta t} - INST_t) \\ &\quad - [-C_t + N_{1,t} F_t + N_{2,t} INST_t](e^{r\Delta t} - 1) \end{aligned} \tag{14}$$

And the absolute hedging error through a hedging period ($T_1 - t$) is calculated as

$$TDHE(t,T_1)=\sum_{l=1}^{M}|\ HE_{t+(l-1)\Delta t}(t+l\Delta t)\ e^{r\Delta t(M-l)}\ | \tag{15}$$

where $M=(T_1-t)/\Delta t$.

2. HS1−SV hedging strategy: Under the hedging scheme 1, the SV model is used to calculate the greeks of the target and the instrument portfolio. This study further addes the conditions of delta-neutral and vega-neutral by

$$\frac{\partial \pi_t}{\partial F_t}=N_{1,t}+N_{2,t}\frac{\partial INST_t}{\partial F_t}-\frac{\partial C_t^A(F)}{\partial F_t}=0 \tag{16}$$

$$\frac{\partial \pi_t}{\partial \nu_t}=N_{2,t}\frac{\partial INST_t}{\partial \nu_t}-\frac{\partial C_t^A(F)}{\partial \nu_t}=0 \tag{17}$$

The shares of instrument assets are given by

$$\begin{aligned}
N_{1,t}&=\frac{\partial C_t^A(F)}{\partial F_t}-N_{2,t}\frac{\partial INST_t}{\partial F_t}\\
&=\frac{\partial C_t^A(F)}{\partial F_t}-e^{-r(T_2-t)}N_{2,t}[-\Pi_1(T_1-t,K_2)-\Pi_1(T_1-t,K_1)\\
&\quad+\Pi_1(T_2-t,K_2)+\Pi_1(T_2-t,K_1)]\\
&=e^{-r(T_1-t)}\Pi_1'+KA_2\left(\frac{1}{Ky_c^*}\right)^{q_2}q_2F^{(q_2-1)}-e^{-r(T_2-t)}N_{2,t}[-\Pi_1(T_1-t,K_2)\\
&\quad-\Pi_1(T_1-t,K_1)+\Pi_1(T_2-t,K_2)+\Pi_1(T_2-t,K_1)]
\end{aligned} \tag{18}$$

$$N_{2,t}=\left(\frac{\partial C_t^A(F)}{\partial \nu_t}\right)\Big/\left(\frac{\partial INST_t}{\partial \nu_t}\right) \tag{19}$$

where $\dfrac{\partial C_t^A(F)}{\partial \nu_t}=e^{r(T_2-T_1)}S_t\dfrac{\partial \Pi_1'}{\partial \nu_t}-e^{-r(T_1-t)}K\dfrac{\partial \Pi_2'}{\partial \nu_t}$

$$\begin{aligned}\frac{\partial INST_t}{\partial v_t} &= -S_t \frac{\partial \Pi_1(T_1-t,K_2)}{\partial v_t} + K_2 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_1-t,K_2)}{\partial v_t} \\ &- S_t \frac{\partial \Pi_1(T_1-t,K_1)}{\partial v_t} + K_1 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_1-t,K_1)}{\partial v_t} \\ &+ S_t \frac{\partial \Pi_1(T_2-t,K_2)}{\partial v_t} - K_2 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_2-t,K_2)}{\partial v_t} \\ &+ S_t \frac{\partial \Pi_1(T_2-t,K_1)}{\partial v_t} - K_1 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_2-t,K_1)}{\partial v_t}\end{aligned}$$

The hedging ratios for the hedging instrument and the target portfolio are given in the Appendix B.

3. HS1−SVJ hedging strategy: Under the hedging scheme 1, the SVJ model is used to calculate the greeks of the target and the instrument portfolio. This study further adds the constraints of delta-neutral and vega-neutral as given in equations (16) and (17). The formulas for the shares of instrument assets, $N_{1,t}$ and $N_{2,t}$, are derived in the Appendix C which exactly change Π_1' and Π_2' in equations (18) and (19) into their counterparts under the SVJ model.

4. HS2−BSM hedging strategy: Under the hedging scheme 2, the BSM model is used to calculate the greeks of the target and the instrument portfolio. Therefore, the profit of the hedged portfolio is defined as equation (8). By further adding the constraints of delta-neutral and vega-neutral as shown in equations (9) and (10), we can obtain the share of instrument assets:

$$N_{1,t} = \frac{\partial C_t(F)}{\partial F_t} - N_{2,t} \frac{\partial INST_t}{\partial F_t} = \frac{\partial C_t(F)}{\partial F_t} = e^{-r(T_1-t)} N(d_1^*) \tag{20}$$

$$N_{2,t} = \left(\frac{\partial C_t(F)}{\partial \nu_t}\right) / \left(\frac{\partial INST_t}{\partial \nu_t}\right) = F_t e^{-r(T_1-t)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_1^*)^2}{2}\sqrt{T_1-t}\right) \tag{21}$$

The hedging ratio of the VIX futures is derived in the Appendix D.

5. HS2−SV hedging strategy: Under the hedging scheme 2, the SV model is used to calculate the greeks of the target and the instrument portfolio. By imposing delta-neutral and vega-neutral conditions given in equations (16) and (17), the shares of instrument assets are computed as

$$N_{1,t} = e^{-r(T_1-t)}\Pi_1' + KA_2\left(\frac{1}{Ky_c^*}\right)^{q_2} q_2 F^{(q_2-1)} \tag{22}$$

$$N_{2,t} = \left(\frac{\partial C_t^A(F)}{\partial \nu_t}\right) / \left(\frac{\partial INST_t}{\partial \nu_t}\right) \tag{23}$$

where the related symbols are

$$\frac{\partial C_t^A(F)}{\partial \nu_t} = e^{r(T_2-T_1)} S_t \frac{\partial \Pi_1'}{\partial \nu_t} - e^{-r(T_1-t)} K \frac{\partial \Pi_2'}{\partial \nu_t}$$

$$\frac{\partial INST_t}{\partial \nu_t} = \frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T_1)}{\partial \nu_t}$$

$$\equiv \frac{1}{2}\frac{a_\tau \zeta_1}{\tau\sqrt{\mathrm{E}_t^Q(\mathrm{VIX}_{T_1}^2)}}\left\{\alpha_{T_1-t} + \frac{1}{4\mathrm{E}_t^Q(\mathrm{VIX}_{T_1}^2)}\left[\frac{3}{2}\alpha_{T_1-t}\frac{\mathrm{var}_t^Q(\mathrm{VIX}_{T_1}^2)}{\mathrm{E}_t^Q(\mathrm{VIX}_{T_1}^2)} - \left(\frac{\zeta_1 a_\tau}{\tau}\right)^2 C_{T_1-t}\right]\right\}$$

$$\mathrm{E}_t^Q(\mathrm{VIX}_{T_1}^2) \equiv \frac{a_\tau\zeta_1}{\tau}\alpha_{T_1-t}\nu_t + \frac{a_\tau\zeta_1}{\tau}\beta_{T_1-t} + \frac{b_\tau\zeta_1}{\tau} + \zeta_2$$

$$\mathrm{var}_t^Q(\mathrm{VIX}_{T_1}^2) \equiv \left(\frac{\zeta_1 a_\tau}{\tau}\right)^2 C_{T_1-t}\nu_t + \left(\frac{\zeta_1 a_\tau}{\tau}\right)^2 D_{T_1-t}$$

$\nu_t \equiv \frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{\tau\zeta_2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau}$ with the spot VIX, VIX_t, and $\tau = 30/365$

$a_\tau = (1-e^{-\kappa_\nu \tau})/\kappa_\nu$, $b_\tau = \theta_\nu[\tau - (1-e^{-\kappa_\nu \tau})/\kappa_\nu]$

$\alpha_{T_1-t} = e^{-\kappa_v (T_1-t)}$, $\beta_{T_1-t} = \theta_v [1-e^{-\kappa_v (T_1-t)}]$

$C_{T_1-t} = \sigma_v^2 (\alpha_{T_1-t} - \alpha_{T_1-t}^2)/\kappa_v$, $D_{T_1-t} = \sigma_v^2\theta_v (1-\alpha_{T_1-t})^2/(2\kappa_v)$, $\zeta_1 = 1$, and $\zeta_2 = 0$.

6. HS2–SVJ hedging strategy: Under hedging scheme 2, the SVJ model is used to calculate the greeks of the target and the instrument portfolio. Satisfying the conditions of delta-neutral and vega-neutral, we have the shares of instrument assets:

$$N_{1,t} = e^{-r(T_1-t)}\Pi_1' + KA_2\left(\frac{1}{Ky_c^*}\right)^{q_2} q_2 F^{(q_2-1)} \tag{24}$$

$$N_{2,t} = \left(\frac{\partial C_t^A(F)}{\partial v_t}\right) \Big/ \left(\frac{\partial INST_t}{\partial v_t}\right) \tag{25}$$

where the values for $\partial C_t^A(F)/\partial v_t$ and $\partial INST_t/\partial v_t$ are obtained by replacing $\zeta_2 = 0$ in equations (22) and (23) with $\zeta_2 = 2\lambda_J(\kappa^* - \mu_J)$. The details of the hedging ratio of the VIX futures are given in the Appendix D.

3 Data Description

3.1 Contracts introduction

The target of the hedging portfolio is the S&P 500 index (SPX) futures option traded in Chicago Mercantile Exchange (CME). It lists four months in the March quarterly cycle and two serial months. For options that expire in the March quarterly cycle, options trading shall terminate at the same date with the underlying futures contract. For options that expire in months other than those in the March quarterly cycle, options trading shall terminate on the third Friday of the contract month. Every one point of futures option equals $250. The underlying SPX futures are traded in CME. It lists eight months in the March quarterly cycle. The last trading day for futures will be the Thursday prior to the third Friday of the contract month. Every one point of futures equals $250. The SPX options are traded in CBOE. Its expiration months are three near-term months followed by three additional months from the March quarterly cycle. The last trading date is the Thursday before the expiration date (the third Friday) of the expiration month. Every one point of the SPX options equals $100. Finally, the VIX futures are traded in CBOE. It lists two near-term months plus two months in February quarterly cycle (February, May, August and November). The settlement date for the VIX futures is the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the month in which the contract expires. And the last trading day of the VIX futures is the day before the settlement date. Every one point of the VIX futures equals $100.

3.2 Empirical data

In order to assess the hedging performance of forward volatility risk, we use the

SPX futures, the SPX options and the VIX futures to hedge a short position on the SPX futures call option. The sample period for hedging is from October 20, 2004 to June 30, 2005. The intraday prices for the SPX futures and the SPX futures option are obtained from the CME. The daily prices for the SPX options are obtained from CBOE, and the daily prices for the VIX futures are captured from CBOE. Further, for our empirical work, the contracts that we select are as follows: First, the SPX futures contracts employed expire on March, June, September, and December. Second, the SPX futures call options that expire on February, May, August, and November are selected as our target instrument. Third, the forward-start strangle portfolio involves in two strangles. We use the SPX options contracts expire on February, May, August, and November to construct a strangle, and May, June, September, and December for another one. Finally, The VIX futures expire on February, May, August, and November that are selected as the hedging instrument. The interest rate data are daily annualized Treasury-bill rates obtained from DataStream Database. The daily dividend-yield ratio data are obtained from the S&P Corporation.

The raw data of futures options from March 26, 2004 to May 4, 2006, are in total 99,875 observations.[6] For our empirical work, we use only that options expire on

[6] The listing date of the VIX futures is March 26, 2004.

February, May, August, and November. After we filter out the raw data, the remainder observations are 29,804. Next, we employ the last reported quote of each contract for each day.[7] Hence, there are 6,507 traded prices, 174 middle points of bid-ask quotes, 218 bid quotes, and 332 ask quotes, in total 7,231 observations. The time gap between the last ask and the last bid quotes is about one hour and five minutes. After we couple with VIX futures expiring on February, May, August, and November, there are 7,003 observations in our sample.[8] Because the SPX options available for constructing the forward-start strangle in only October 20, 2004 to June 30, 2005, we filter out the 4,521 and 2,482 SPX futures options observations remains. We define the moneyness of the SPX futures options as the point of the underlying futures divided by the strike of the futures option (F/K). According to the moneyness, we classify these observations as deep out-of-the-money (DOTM) if $F/K \leqq 0.94$; out-of-the-money (OTM) if $F/K \in [0.94,0.97)$; at-the-money 1 (ATM1) if $F/K \in [0.97,1)$; at-the-money 2 (ATM2) if $F/K \in [1,1.03)$; in-the-money (ITM) if $F/K \in [1.03,1.06)$; and deep in-the-money (DITM) if $F/K > 1.06$. By the term to maturity, these observations are classified as short-term (<30 days), medium-term (30–60 days), and long-term ($\geqq$60 days). Table 1 describes

[7] If two call option data with the same trading day, expiration month, and strike price. Then, these two data are the same contract.

[8] The last trading date of the VIX futures is usually the third Tuesday and the last trading date of SPX futures options is the third Friday of expiration month.

some properties of the SPX futures call options. It reports the average point of the SPX futures call option, its corresponding futures price and the observations for each moneyness-maturity category. There are in total 2,482 futures call option observations, with OTM and ATM options about 56 and 40 percent, respectively. The average futures call price range is from 0.1827 points for short-term DOTM call options to 117.7 points for medium-term DITM call options.

For the forward-start strangle strategy, we use SPX options expiring in February-quarterly cycles as T_1 and expiring in March-quarterly cycles as T_2, where the pair of (T_1, T_2) must be February–March, May–June, August–September and November–December. The maximum and the minimum strikes of SPX options available for each pair of (T_1, T_2) on each day are selected as the two strike prices. Hence, there are four pairs of SPX options with strikes (K_1, K_2) corresponding to SPX options expiring on the four pairs of (T_1, T_2). In summary, there are 692 SPX option observations selected. Hence, the K_1 is the minimum strike that is available in the options expiring on the February quarterly cycle and the March quarterly cycle, simultaneously, and traded for each working date. The result shows that the K_1 of the options are all 700 index points. The K_2 of the options expires on November 2004 and February, May, August 2005 are 1,250, 1,250, 1,300 and 1,350, respectively.

Those SPX options expiring on the pairs of (T_1, T_2) from October 20, 2004 to June 30, 2005 are reported in Table 2. We define the moneyness of the SPX options as the point of the underlying index divided by the strike of the SPX option (*S/K*). According to the moneyness, we classify these observations as deep out-of-the-money (DOTM) if *S/K*≦0.94; out-of-the-money (OTM) if *S/K*∈[0.94,0.97); at-the-money 1 (ATM1) if *S/K*∈[0.97,1); at-the-money 2 (ATM2) if *S/K*∈[1,1.03); in-the-money (ITM) if *S/K*∈[1.03,1.06); and deep in-the-money (DITM) if *S/K*>1.06. By the term to maturity, these observations are classified as short-term (<30 days), medium-term (30-60 days), and long-term (≧60 days). It reports the average point of the SPX option and the observations for each moneyness-maturity category. There are totally 30,166 option observations, 15,083 observations for call and put. The average call price range is from 0.1628 points for short-term DOTM call options to 301.1628 points for long-term DITM call options. The average put price range is from 0.3092 points for short-term DITM call options to 176.2546 points for long-term DOTM call options.

Based on Lin (2007), the parameters of the SV and the SVJ are estimated using the joint VIX and 30-day realized volatility calculated from five-minute index returns over the period, April 21, 2004 to October 19, 2004. The risk-neutral parameters κ_v, θ_v, σ_v and ρ of the SV model are on average 5.6269, 0.6866, 0.5320 and –0.5012, respectively. The risk-neutral parameters κ_V, θ_V, σ_V, ρ, λ_J, μ_J and σ_J of the SVJ model are 8.7697, 0.6517, 0.4366, –0.4182, 2.1651, –0.3503, 0.3123, respectively. Next, we use the traded SPX futures options data from September 15, 2004 (Wednesday) to October 19, 2004 (Tuesday) to estimate the moneyness for immediate exercise y_c^* by assessing the estimation error of these parameters for the SV and the SVJ, respectively. It is because the settlement day of the VIX futures is the third Wednesday, and the last trading day is Tuesday. Hence, the month is defined as the period from the third Wednesday of prior calendar month to the third Tuesday of this calendar month. At last, the hedging period of this study is from October 20, 2004 to June 30, 2005, of SPX options for constructing a forward-start strangle portfolio.

Table 1

Sample Properties of SPX Futures Call Options

The three rows of each moneyness denote the average points of futures options ($250 per point), underlying futures ($250 per point) and observations, respectively. The futures option contracts listed at CME are four quarterly and two serial, and the longest maturity for our observations is 120 days. Therefore, by the term to maturity, we classified these observations as short-term (<30 days), medium-term (30-60 days), and long-term (≧60 days). By the moneyness, we classified these observations as deep out-of-the-money (DOTM) if $F/K \leqq 0.94$; out-of-the-money (OTM) if $F/K \in [0.94,0.97)$; at-the-money 1 (ATM1) if $F/K \in [0.97,1)$; at-the-money 2 (ATM2) if $F/K \in [1,1.03)$; in-the-money (ITM) if $F/K \in [1.03,1.06)$; and deep in-the-money (DITM) if $F/K > 1.06$. The data period is from 20 October 2004 to 30 June 2005. F is the price of the SPX futures and K is the strike price of S&P500 futures options.

	Moneyness	Maturity			
	F/K	<30	30–60	≧60	Subtotal
DOTM	≦0.94	0.1827 1156.245 231	0.5007 1175.539 238	1.3969 1200.946 113	582
OTM	0.94–0.97	0.7433 1162.678 345	2.5717 1184.388 289	4.7168 1203.839 184	818
ATM1	0.97–1	4.9073 1168.642 424	10.5415 1181.149 196	12.7566 1205.591 61	681
ATM2	1–1.03	19.4346 1171.62 257	24.2600 1176.276 50	29.6692 1211.154 13	320
ITM	1.03–1.06	48.7425 1171.179 60	52.0571 1166.5 7	NA NA NA	67
DITM	>1.06	81.6333 1171.55 12	117.7 1216 1	74.2 1198.8 1	14
Subtotal		1329	781	372	2482

Table 2

Sample Properties of SPX Options

The first row of each moneyness denotes the average points of the SPX options ($100 per point). The second row of each moneyness denote observations. The S&P 500 index options are traded in CBOE. Its expiration months are three near-term months followed by three additional months from the March quarterly cycle., and the longest maturity for our observations is 723 days. Therefore, by the term to maturity, we classified these observations as short-term (<30 days), medium-term (30-60 days), and long-term (≧60 days). By the moneyness, we classified these observations as deep out-of-the-money (DOTM) if $S/K \leqq 0.94$; out-of-the-money (OTM) if $S/K \in [0.94,0.97)$; at-the-money 1 (ATM1) if $S/K \in [0.97,1)$; at-the-money 2 (ATM2) if $S/K \in [1,1.03)$; in-the-money (ITM) if $S/K \in [1.03,1.06)$; and deep in-the-money (DITM) if $S/K > 1.06$. The data period is from 20 October 2004 to 30 June 2005. S is the price of the SPX and K is the strike price of SPX options.

		All				Call				Put			
	Moneyness	Maturity				Maturity				Maturity			
	S/K	<30	30-60	≧60	Subtotal	<30	30-60	≧60	Subtotal	<30	30-60	≧60	Subtotal
DOTM	≦0.94	65.1301	83.5214	88.5615		0.1628	0.4002	0.8683		130.0973	166.6426	176.2546	
		668	1852	1992	4512	334	926	996	2256	334	926	996	2256
OTM	0.94-0.97	27.4918	29.8519	33.4532		0.6334	2.9607	7.2748		54.3503	56.7430	59.6316	
		782	1290	842	2914	391	645	421	1457	391	645	421	1457
ATM1	0.97-1	13.6321	20.3251	27.0631		4.5939	11.5286	19.8819		22.6703	29.1216	34.2442	
		940	1294	1110	3344	470	647	555	1672	470	647	555	1672
ATM2	1-1.03	14.2759	21.7983	29.6033		22.8446	30.6625	39.0762		5.7072	12.9341	20.1303	
		876	1086	842	2804	438	543	421	1402	438	543	421	1402
ITM	1.03-1.06	26.9440	31.2917	38.2495		52.1228	56.4900	65.0060		1.7651	6.0934	11.4929	
		806	888	552	2246	403	444	276	1123	403	444	276	1123
DITM	>1.06	109.4576	131.7505	151.3518		218.6059	262.5436	301.1628		0.3092	0.9574	1.5408	
		3346	4992	6008	14346	1673	2496	3004	7173	1673	2496	3004	7173
Subtotal		7418	11402	11346	30166	3709	5701	5673	15083	3709	5701	5673	15083

4 Empirical Results

This study follows two steps to assess the hedging performance of writing a SPX futures call option using two hedging schemes under three S&P 500 index processes. First, based on the parameters above, this study uses the S&P 500 index, U.S. Treasury-bill rates, the SPX futures, the SPX options and the VIX futures of day t to construct the hedging portfolio. Next, we calculate the hedging error of day $t + n$, where n is the available trading dates till SPX futures call option's expiry, and also rebalance the hedging portfolio. Since the quotes of each futures option are not all available for each day until its expiry, we only take rebalance on the day with available quote data after day t. These steps are repeated for each futures option contract expiring in February quarterly cycle on every trading date with quote data available in our sample.

The hedging performance is shown in Tables 3 and 4. We define the absolute hedging error as the mean of absolute hedging errors and define the average hedging error as the mean of hedging errors. We illustrate the hedging errors in points and each point represents $250. In Table 3, under the BSM model, the range of hedging errors of

HS1 is from 1.63 points (DOTM short-term) to 42.42 points (DITM long-term) and the hedging errors of HS2 is from 8.26 points (OTM short-term) to 115.71 points (DITM long-term). For all moneyness-maturity categories, HS1 is better than HS2 and short-term SPX futures calls have smaller errors.

For the SV model, the range of hedging errors is from 0.12 points (DOTM short-term) to 49.58 points (DITM long-term) for HS1 and it is from 7.08 points (OTM short-term) to 77.92 points (ATM1 medium-term) for HS2. We can also find that HS1 of SV outperforms HS2 of SV and the short-term hedging errors of HS1 are smaller.

The range of hedging errors of HS1 in the SVJ model is from 0.13 points (DOTM short-term) to 45.29 points (DITM long-term). The range of hedging errors of HS2 in the SVJ model is from 6.11 points (OTM short-term) to 71.65 points (ATM1 medium-term). Therefore, the absolute hedging errors of HS1 are less than HS2.

Hence, the results indicate that the forward-start strangle portfolio is a more efficient instrument to hedge forward volatility risk than the VIX futures. Next, we can find out that the absolute hedging errors of the SVJ model are less than that of the SV model. It represents the random price jump feature commonly exists in the SPX futures option. However, this result seems not to be consist with those of Bakshi et al. (1997) and Bakshi and Kapadia (2003). The parameter of jump-frequency intensity λ_J in Bakshi et al. (1997) is 0.59, i.e. one year and half for a price jump to occur. Their hedging portfolio is rebalanced daily or every five days. Hence, they think the reason for the SV model dominates the SVJ model is the chance for a price jump to occur is small in the daily or five-day rebalancing period. Other than the uncertain rebalance frequency in our empirical work,[9] the parameter λ_J of our empirical work is 2.1651 and larger than that of Bakshi et al. (1997). Our empirical result also shows that the SVJ has better hedging performance than SV. Therefore, their reason does not hold for our empirical result. We think it is because the SPX futures options are American-style in this study, while SPX options are European-style for the prior research. Since the traders with American-style options positions have early-exercise choice and thus can take caution to prevent any loss from the potential jump events than the ones with European-style

[9] There are in total 169 unique SPX futures options contracts over our hedging period, 20 October 2004–30 June 2005. Among data, there are 18 unique contracts can be daily rebalanced. The maximum rebalancing period is 26 days for only one unique contract (with May-2005 maturity at its first trading date, 8 March 2005). On average, the rebalancing period is 4.44 days.

options. Thus, American-style option buyers (sellers) even favor (hate) volatility risk than the ones with European-style options. In addition, given the possibility of price jumps, the specification of SVJ can provide more accurate parameter estimates than SV. Thus, the delta-vega-neutral strategy can be constructed in a more effective way under SVJ than SV. Thus, it is not surprising for our results showing that SVJ outperforms SV in terms of hedging efficiency.

About the effect of the maturity, by comparing with the medium- and long-term options, most of the absolute hedging errors of short-term options are smaller. Except for the HS2 strategy under ATM1 and ATM2, the absolute hedging errors of medium-term are smaller than long-term, i.e. the absolute hedging errors commonly increase with maturity. This result consists with Psychoyios and Skiadopoulos (2006) in the case of ITM and OTM target options. They also examine the relationship between the maturity and the difference between hedging schemes. The difference decreases with maturity in case of ITM and OTM, and increases with maturity in case of ATM. In our result, except for the HS2 scheme in the case of ATM1 and ATM2 target options, the absolute hedging error increases dramatically from short-term to medium- and long-term, and the difference between HS1 and HS2 increases.

In terms of the moneyness effect, Psychoyios and Skiadopoulos (2006) think the options perform best for ATM and worse for ITM, the difference is minimized for ATM and maximized for ITM. Our empirical result shows that the absolute hedging errors of HS1 increase with moneyness, except for the category of ATM2 long-term. However, the relationship between the absolute hedging errors of HS2 and the moneyness is uncertain. Therefore, the difference between HS1 and HS2 across moneyness is uncertain.

Table 3
Absolute Hedging Errors

The numbers in this table denote the points of absolute hedging errors ($250 per point): $\frac{M-t}{\Delta t}\sum_{l=1}^{M}|HE_{t+(l-1)\Delta t}(t+l\Delta t)\times e^{r\Delta t(M-l)}|$. The hedging error between time t and time $t+\Delta t$ is defined as $HE_t(t+\Delta t)$. The hedging period is from 20 October 2004 to 30 June 2005. The futures option contracts listed at CME are 4 quarterly and 2 serial, and the longest maturity for our observations is 120 days. Therefore, by the term to maturity, we classified these observations as short-term (<30 days), medium-term (30-60 days), and long-term (≧60 days). By the moneyness, we classified these observations as deep out-of-the-money (DOTM) if *F/K*≦0.94; out-of-the-money (OTM) if *F/K*∈[0.94,0.97); at-the-money 1 (ATM1) if *F/K*∈[0.97,1); at-the-money 2 (ATM2) if *F/K*∈[1,1.03); in-the-money (ITM) if *F/K*∈[1.03,1.06); and deep in-the-money (DITM) if *F/K*>1.06. *F* is the price of the SPX futures and *K* is the strike price of S&P500 futures options.

Moneyness *F/K*			Maturity <30	30–60	≧60
DOTM	HS1	BSM	1.63	1.91	2.83
		SV	0.12	0.77	3.47
		SVJ	0.13	0.80	3.51
	HS2	BSM	23.95	54.29	58.53
		SV	18.45	23.95	29.01
		SVJ	11.22	19.13	22.33
OTM	HS1	BSM	2.31	3.54	3.21
		SV	0.68	7.21	7.88
		SVJ	0.68	6.96	7.63
	HS2	BSM	8.26	64.31	66.83
		SV	7.08	23.91	26.33
		SVJ	6.11	21.92	24.65
ATM1	HS1	BSM	4.39	15.87	17.24
		SV	3.44	7.18	8.16
		SVJ	3.37	6.15	7.75
	HS2	BSM	40.96	92.75	104.61
		SV	19.71	77.92	58.54
		SVJ	15.59	71.65	29.69
ATM2	HS1	BSM	10.52	10.38	10.65
		SV	9.41	20.33	11.217
		SVJ	5.04	6.95	8.616
	HS2	BSM	14.91	32.42	108.54
		SV	13.32	25.21	18.925
		SVJ	12.01	22.38	16.641
ITM	HS1	BSM	12.40	24.89	NA
		SV	13.95	27.33	NA
		SVJ	13.58	26.20	NA
	HS2	BSM	29.48	63.54	NA
		SV	18.06	44.53	NA
		SVJ	16.46	28.92	NA
DITM	HS1	BSM	26.42	NA	42.42
		SV	29.01	NA	49.58
		SVJ	28.87	NA	45.29
	HS2	BSM	38.51	NA	115.71
		SV	24.40	NA	71.13
		SVJ	23.80	NA	50.61

Theoretically, if a portfolio is perfectly hedged, it should earn the risk-free rate, and the average hedging errors should be close to zero. In this study, the hedging error is defined as the changes of the hedged portfolio minus the risk-free return. Table 4 reports the average hedging errors. If the number is greater (less) than zero, it means that the strategy gets more (less) profits than risk-free rate. Oppositely, the hedging performance through all hedging periods is less than risk-free rate. The average hedging error of HS1−SVJ strategy is the smallest in most moneyness-maturity categories. Noticeably, HS1 scheme is superior to HS2 scheme for the BSM model. The BSM model assumes the volatility of underlying asset is constant. As a result, the HS2−BSM strategy only considers price risk and the weights of this strategy is the vega of the futures option. The price changes of the VIX futures will not be explained. However, there are not only price risks of the SPX futures option but also risk exposure of the VIX futures. That will let this strategy incurs additional risk exposure and incurs losses.

Compared with SV model, most hedging performance of the SVJ model is smaller. It is consistent with the results in Table 3 and represents the existence of the random price jump feature for SPX futures options. Still, the hedging performance of HS1 is better than HS2, i.e. the forward-start strangle portfolio is more efficient than the VIX futures. The average hedging error of HS2 is negative under short-term. However, it is uncertain for the medium- and long-term categories. Under the short-term categories, the HS1 gets reverse results. Psychoyios and Skiadopoulos (2006) think that the volatility is more stable in long-term than short-term. According to BSM, the options have greater vega if the moneyness close to 1. Therefore we think that HS1 still dominates HS2, i.e. the forward-start strangle portfolio can hedge the forward vega risk more efficient than VIX futures. This conclusion is consistent with Psychoyios and Skiadopoulos (2006), stating that volatility options are not better hedging instrument than standard options if the target options are standard options. They argue that volatility options may be a useful hedging tool for other type of target options or exotic options. Similarly, compared to the forward-start strangle portfolio, the VIX futures get worse hedging performance for our target instrument, i.e. the American-style futures call option, may not necessarily indicate it is not a good instrument for hedging forward volatility risk.

Table 4
Average Hedging Errors

The numbers in this table denote the points of absolute hedging errors ($250 per point):

$$\frac{M-t}{\Delta t}\sum_{l=1}^{M} HE_{t+(l-1)\Delta t}(t+l\Delta t)\times e^{r\Delta t(M-l)}$$

The hedging error between time t and time $t+\Delta t$ is defined as $HE_t(t+\Delta t)$. The hedging period is from 20 October 2004 to 30 June 2005. The futures option contracts listed at CME are 4 quarterly and 2 serial, and the longest maturity for our observations is 120 days. Therefore, by the term to maturity, we classified these observations as short-term (<30 days), medium-term (30-60 days), and long-term (≧60 days). By the moneyness, we classified these observations as deep out-of-the-money (DOTM) if *F/K*≦0.94; out-of-the-money (OTM) if *F/K*∈[0.94,0.97); at-the-money 1 (ATM1) if *F/K*∈[0.97,1); at-the-money 2 (ATM2) if *F/K*∈[1,1.03); in-the-money (ITM) if *F/K*∈[1.03,1.06); and deep in-the-money (DITM) if *F/K*>1.06. *F* is the price of the SPX futures and *K* is the strike price of S&P500 futures options.

Moneyness			Maturity		
F/K			<30	30–60	≧60
DOTM	HS1	BSM	0.88	-0.40	-1.89
		SV	0.01	0.01	1.65
		SVJ	0.01	0.00	1.66
	HS2	BSM	-23.23	-32.48	-57.88
		SV	2.70	-2.36	5.34
		SVJ	-1.52	-1.85	2.85
OTM	HS1	BSM	1.06	-3.18	-3.58
		SV	0.21	-3.00	2.84
		SVJ	0.23	-3.03	2.77
	HS2	BSM	-70.419	-13.72	-43.17
		SV	-1.64	3.39	-5.32
		SVJ	-1.09	3.98	-2.59
ATM1	HS1	BSM	1.46	-5.41	-5.60
		SV	1.02	-6.85	-6.48
		SVJ	1.03	-6.71	-5.21
	HS2	BSM	-33.58	-19.66	-68.35
		SV	-4.45	-14.80	11.30
		SVJ	-2.03	-13.98	9.44
ATM2	HS1	BSM	4.21	-15.32	-7.51
		SV	4.42	-15.60	-8.46
		SVJ	4.31	-12.49	-8.50
	HS2	BSM	-14.30	-30.01	-108.44
		SV	-8.84	-16.50	19.93
		SVJ	-6.72	-3.82	12.67
ITM	HS1	BSM	9.79	-17.80	NA
		SV	11.09	-20.40	NA
		SVJ	10.87	-19.38	NA
	HS2	BSM	-29..37	-62.76	NA
		SV	-20.51	28.01	NA
		SVJ	-18.41	19.87	NA
DITM	HS1	BSM	26.42	NA	-42.42
		SV	29.01	NA	-49.58
		SVJ	28.87	NA	-45.29
	HS2	BSM	-38.51	NA	-57.20
		SV	24.40	NA	-51.34
		SVJ	23.80	NA	-50.61

5 Conclusions

This study intends to compare the hedging efficiency of the forward-start strangle portfolio and the VIX futures. Since the vega risk of the SPX futures options is related to forward volatility between option's expiry and underlying futures' expiry, this study chooses this option as our target asset. However, the short position on the SPX futures option still has delta risk. For our empirical work, we construct a delta-vega-neutral hedging strategy and use strangle and VIX futures as instruments. In terms of forward volatility risk, the SPX futures and the forward-start strangle portfolio are used to construct three hedging strategies (HS1−BSM, HS1−SV and HS1−SVJ), the SPX futures and the VIX futures are used to construct the other three hedging strategies (HS2−BSM, HS2−SV and HS2−SVJ). The empirical results show that the SVJ model gets better hedging performance, i.e. incorporating random price jump feature may help improve the hedging performance of the SV model. The result shows that the random price jump feature exists in futures options. In terms of the effect of maturity to the hedging performance, if the target futures options are short-term (the term to maturity is less than 30 days), most of these strategies will get better hedging performance and the HS1 all gets more profits than risk-free return. That indicates that both forward-start strangle portfolio and VIX futures are volatility risk hedging instrument for the short-term options. The moneyness is negatively related to the hedging performance, i.e. the hedging error increases with the moneyness.

HS1 dominates HS2 in all moneyness-maturity categories. That is, the hedging errors of the forward-start strangle portfolio are mostly less than those of the VIX futures strategy. Hence, comparing with the VIX futures strategies, we think that the forward-start strangle portfolio is more efficient to hedge forward volatility risk. The option writers of the futures options can use the forward-start strangle portfolio as the instrument to hedge the forward volatility of short-term options. Besides, if the option writes use the SVJ model to determine the weights of hedging instruments, they will get better hedging performance.

Appendix A. Pricing SPX Futures Options under Constant Volatility

Following Black-Scholes-Merton, we assume that the underlying stock of futures follow a geometric Brownian motion process; that is,

$$dS_t = rS_t dt + \sigma S_t d\omega_{S,t} \tag{A1}$$

where σ is the volatility of the underlying stock, and $\omega_{S,t}$ is a Brownian motion under the risk-neutral measure *Q*. The process of the futures on this stock is given by

$$dF_t = \sigma F_t d\omega_{S,t} \tag{A2}$$

Using the martingale theory, the time-*t* price of a European-style futures option with strike K and expiry T_1 is derived as

$$\begin{aligned} C_t(F) &= e^{-r(T_1-t)}\mathrm{E}_t^Q[C_{T_1}(F)] = e^{-r(T_1-t)}\mathrm{E}_t^Q(F_{T_1}1_A) - Ke^{-r(T_1-t)}\mathrm{E}_t^Q(1_A) \\ &= e^{-r(T_1-t)}[F_t N(d_1^*) - KN(d_2^*)] \end{aligned} \tag{A3}$$

where 1_A is an indicator function having the value of 1 if $F_{T_1} > K$ and 0 otherwise; $d_1^* = [\ln F_t - \ln K + \sigma^2(T_1 - t)/2]/\sigma\sqrt{T_1 - t}$ and $d_2^* = d_1^* - \sigma\sqrt{T_1 - t}$. The delta and vega of the futures options are

$$\frac{\partial C_t(F)}{\partial F_t} = e^{-r(T_1-t)}N(d_1^*) \tag{A4}$$

$$\frac{\partial C_t(F)}{\partial \nu_t} = \frac{\sqrt{T_1 - t}}{2\sigma^2}F_t e^{r(T_1-t)}\phi(d_1^*) \tag{A5}$$

where $\phi(\cdot)$ and $N(\cdot)$ are the probability density function and cumulative density function of a standardized normal distributed random variable, respectively.

Further, the delta of the forward-start strangle portfolio is

$$\begin{aligned}\frac{\partial INST_t}{\partial F_t} &= -\frac{\partial C(T_1,K_2)}{\partial F_t} - \frac{\partial P(T_1,K_1)}{\partial F_t} + \frac{\partial C(T_2,K_2)}{\partial F_t} + \frac{\partial P(T_2,K_1)}{\partial F_t} \\ &= -\frac{\partial C(T_1,K_2)}{\partial S_t}\frac{\partial S_t}{\partial F_t} - \frac{\partial P(T_1,K_1)}{\partial S_t}\frac{\partial S_t}{\partial F_t} + \frac{\partial C(T_2,K_2)}{\partial S_t}\frac{\partial S_t}{\partial F_t} + \frac{\partial P(T_2,K_1)}{\partial s_t}\frac{\partial S_t}{\partial F_t} \\ &= \frac{\partial S_t}{\partial F_t}\left[-\frac{\partial C(T_1,K_2)}{\partial S_t} - \frac{\partial P(T_1,K_1)}{\partial S_t} + \frac{\partial C(T_2,K_2)}{\partial S_t} + \frac{\partial P(T_2,K_1)}{\partial s_t}\right] \\ &= e^{-r(T_2-t)}\left(-\Delta_{C(T_1,K_2)} - \Delta_{P(T_1,K_1)} + \Delta_{C(T_2,K_2)} + \Delta_{P(T_2,K_1)}\right)\end{aligned} \tag{A6}$$

where $\Delta_{(T,K)}$ is the BSM delta that equals $N(d_1)$ for call and $N(d_1)-1$ for put with $d_1 = [\ln S_t - \ln K + \sigma^2(T-t)/2]/\sigma\sqrt{T-t}$. The vega of the forward-start strangle portfolio is

$$\begin{aligned}\frac{\partial INST_t}{\partial \nu_t} &= \frac{\partial INST_t}{\partial \sigma}\frac{\partial \sigma}{\partial \nu_t} \\ &= \frac{1}{2\sigma^2}\left[-\frac{\partial C(T_1,K_2)}{\partial \sigma} - \frac{\partial P(T_1,K_1)}{\partial \sigma} + \frac{\partial C(T_2,K_2)}{\partial \sigma} + \frac{\partial P(T_2,K_1)}{\partial \sigma}\right] \\ &= \frac{1}{2\sigma^2}\left[-\upsilon_{C(T_1,K_2)} - \upsilon_{P(T_1,K_1)} + \upsilon_{C(T_2,K_2)} + \upsilon_{P(T_2,K_1)}\right]\end{aligned} \tag{A7}$$

where $\upsilon_{C(T,K)}$ is the BSM vega, equal to $S_t\sqrt{T-t}N(d1)$.

Appendix B. Pricing SPX Futures Options under Stochastic-Volatility

Following Heston (1993), the underlying stock of futures follows a geometric Brownian motion and the instantaneous variance ν_t of the underlying stock is driven by a mean reverting squared root process; that is,

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t d\omega_{S,t} \tag{B1}$$

$$d\nu_t = (\theta_\nu - \kappa_\nu \nu_t)dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t} \tag{B2}$$

where $\omega_{S,t}$ and $\omega_{\nu,t}$ are risk-neutral Brownian motions correlated by $\rho\, dt = corr(d\omega_{S,t}, d\omega_{\nu,t})$. Using the martingale method, the time-*t* price of a European-style futures option with strike K and expiry T_1 is given by

$$C_t(F) = e^{-r(T_1-t)}(F_t\Pi_1' - K\Pi_2') = e^{r(T_2-T_1)}S_t\Pi_1' - e^{-r(T_1-t)}K\Pi_2' \tag{B3}$$

where 1_A is an indicator function having the value of 1 if $F_{T_1} > K$, or equivalently $S_{T_1} > Ke^{-r(T_2-T_1)}$, and 0 otherwise. Π_1' and Π_2' are risk-neutral probabilities that are recovered from inverting the characteristic functions f_1 and f_2, respectively:

$$\Pi_j'(t, T_1 - t; S_t, r, \nu_t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi \ln[Ke^{-r(T_2-T_1)}]} f_j(t, T_1 - t; S_t, r, \nu_t; \varphi}{i\varphi}\right] d\varphi \quad \text{for } j = 1,\ 2$$

(B4)

The characteristic functions for the SV model are given as follows

$$f_1 = \exp\left\{ i\varphi\, r(T_1 - t) - \frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1 - \frac{[\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v][1 - e^{-\xi_v (T_1 - t)}]}{2\xi_v} \right)\right] \right.$$
$$- \frac{\theta_v}{\sigma_v^2}[\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v](T_1 - t) + i\varphi \ln S_t \tag{B5}$$
$$\left. + \frac{i\varphi\,(i\varphi+1)[1 - e^{-\xi_v (T_1 - t)}]}{2\xi_v - [\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v][1 - e^{-\xi_v (T_1 - t)}]} \nu_t \right\}$$

$$f_2 = \exp\left\{ i\varphi\, r(T_1 - t) - \frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1 - \frac{[\xi_v^* - \kappa_v + i\varphi\rho\sigma_v][1 - e^{-\xi_v^* (T_1 - t)}]}{2\xi_v^*} \right)\right] \right.$$
$$- \frac{\theta_v}{\sigma_v^2}[\xi_v^* - \kappa_v + i\varphi\rho\sigma_v](T_1 - t) + i\varphi \ln S_t \tag{B6}$$
$$\left. + \frac{i\varphi\,(i\varphi-1)[1 - e^{-\xi_v^* (T_1 - t)}]}{2\xi_v^* - (\xi_v^* - \kappa_v + i\varphi\rho\sigma_v)[1 - e^{-\xi_v^* (T_1 - t)}]} \nu_t \right\}$$

$$\xi_v = \sqrt{[\kappa_v - (1+i\varphi)\rho\sigma_v]^2 - i\varphi\,(i\varphi+1)\sigma_v^2}$$

$$\xi_v^* = \sqrt{(\kappa_v - i\varphi\rho\sigma_v)^2 - i\varphi\,(i\varphi-1)\sigma_v^2}$$

Next, the delta of the futures options is

$$\begin{aligned}
\frac{\partial C_t(F)}{\partial F_t} &= \frac{\partial C_t(F)}{\partial S_t e^{r(T_2 - t)}} = \frac{\partial C_t(F)}{\partial (S_t e^{r(T_2 - T_1)} e^{r(T_1 - t)})} \\
&= \frac{\partial C_t(F)}{\partial S_t'} \frac{\partial S_t'}{\partial (S_t e^{r(T_2 - T_1)} e^{r(T_1 - t)})} \quad \text{with} \quad S_t' = S_t e^{r(T_2 - T_1)} \\
&= e^{-r(T_1 - t)} \frac{\partial C_t(F)}{\partial S_t'} = e^{-r(T_1 - t)} \frac{\partial [e^{r(T_2 - T_1)} S_t \Pi_1' - K e^{-r(T_1 - t)} \Pi_2']}{\partial S_t'} \\
&= e^{-r(T_1 - t)} \left[e^{r(T_2 - T_1)} \Pi_1' + e^{r(T_2 - T_1)} S_t \frac{\partial \Pi_1'}{\partial S_t'} - K e^{-r(T_1 - t)} \frac{\partial \Pi_2'}{\partial S_t'} \right] \\
&= e^{-r(T_1 - t)} \Pi_1'
\end{aligned} \tag{B7}$$

The vega of the futures options is

$$\frac{\partial C_t(F)}{\partial \nu_t} = e^{r(T_2-T_1)} S_t \frac{\partial \Pi_1'}{\partial \nu_t} - K e^{-r(T_1-t)} \frac{\partial \Pi_2'}{\partial \nu_t} \tag{B8}$$

where

$$\frac{\partial \Pi_1'}{\partial \nu_t} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi \ln[Ke^{-r(T_2-T_1)}]}}{i\varphi} \frac{i\varphi(i\varphi+1)[1-e^{-\xi_\nu (T_1-t)}]}{2\xi_\nu - [\xi_\nu - \kappa_\nu + (1+i\varphi)\rho\sigma_\nu][1-e^{-\xi_\nu (T_1-t)}]} f_1\right] d\varphi$$

$$\frac{\partial \Pi_2'}{\partial \nu_t} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi \ln[Ke^{-r(T_2-T_1)}]}}{i\varphi} \frac{i\varphi(i\varphi-1)[1-e^{-\xi_\nu^* (T_1-t)}]}{2\xi_\nu^* - [\xi_\nu^* - \kappa_\nu + i\varphi\rho\sigma_\nu][1-e^{-\xi_\nu^* (T_1-t)}]} f_2\right] d\varphi$$

Further, according to put-call parity the delta of the forward-start strangle portfolio is

$$\begin{aligned}
\frac{\partial INST_t}{\partial F_t} &= -\frac{\partial C(T_1,K_2)}{\partial F_t} - \frac{\partial P(T_1,K_1)}{\partial F_t} + \frac{\partial C(T_2,K_2)}{\partial F_t} + \frac{\partial P(T_2,K_1)}{\partial F_t} \\
&= -\frac{\partial C(T_1,K_2)}{\partial S_t}\frac{\partial S_t}{\partial F_t} - \frac{\partial P(T_1,K_1)}{\partial S_t}\frac{\partial S_t}{\partial F_t} + \frac{\partial C(T_2,K_2)}{\partial S_t}\frac{\partial S_t}{\partial F_t} + \frac{\partial P(T_2,K_1)}{\partial S_t}\frac{\partial S_t}{\partial F_t} \\
&= e^{-r(T_2-t)}\left[-\frac{\partial C(T_1,K_2)}{\partial S_t} - \frac{\partial P(T_1,K_1)}{\partial S_t} + \frac{\partial C(T_2,K_2)}{\partial S_t} + \frac{\partial P(T_2,K_1)}{\partial S_t}\right] \\
&= e^{-r(T_2-t)}[-\Pi_1(T_1-t,K_2) - \Pi_1(T_1-t,K_1) + \Pi_1(T_2-t,K_2) + \Pi_1(T_2-t,K_1)]
\end{aligned} \tag{B9}$$

where $\Pi_j(T-t,K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi \ln K}}{i\varphi} f_j(t,T-t;S_t,r,\nu_t;\varphi)\right] d\varphi$ for $j = 1, 2$.

The vega of the forward-start strangle portfolio is

$$\begin{aligned}
\frac{\partial INST_t}{\partial \nu_t} &= -\frac{\partial C(T_1,K_2)}{\partial \nu_t} - \frac{\partial P(T_1,K_1)}{\partial \nu_t} + \frac{\partial C(T_2,K_2)}{\partial \nu_t} + \frac{\partial P(T_2,K_1)}{\partial \nu_t} \\
&= -S_t \frac{\partial \Pi_1(T_1-t,K_2)}{\partial \nu_t} + K_2 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_1-t,K_2)}{\partial \nu_t} \\
&\quad - S_t \frac{\partial \Pi_1(T_1-t,K_1)}{\partial \nu_t} + K_1 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_1-t,K_1)}{\partial \nu_t} \\
&\quad + S_t \frac{\partial \Pi_1(T_2-t,K_2)}{\partial \nu_t} - K_2 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_2-t,K_2)}{\partial \nu_t} \\
&\quad + S_t \frac{\partial \Pi_1(T_2-t,K_1)}{\partial \nu_t} - K_1 e^{-r(T_1-t)} \frac{\partial \Pi_2(T_2-t,K_1)}{\partial \nu_t}
\end{aligned} \tag{B10}$$

However, the SPX futures option is American-style option. Therefore, referred to Bates

(1996), the European-style futures option is converted to American-style one:

$$C^A(F) \equiv \begin{cases} C(F) + KA_2\left(\dfrac{F/K}{y_c^*}\right)^{q_2} & \text{for } \; F/K < y_c^* \\ F - K & \text{for } \; F/K \geq y_c^* \end{cases} \tag{B11}$$

where $A_2 = (y_c^* - 1) - c(y_c^*, V, T; 1)$, and the q_2 is the positive root to

$$\frac{1}{2}\overline{v}q^2 + \left(r - \frac{1}{2}\overline{v}\right)q - \frac{r}{1 - e^{-r(T_1 - t)}} = 0 \tag{B12}$$

$\overline{v}$ is the expected average variance over the lifetime of the option conditional on no jumps:

$$\overline{v} \equiv \frac{1}{T_1 - t}\mathrm{E}_t^Q\left(\int_t^{T_1} v_u du\right) = \frac{\theta_v}{\kappa_v} + \left(v_0 - \frac{\theta_v}{\kappa_v}\right)\frac{[1 - e^{-\kappa_v(T_1 - t)}]}{\kappa_v(T_1 - t)} \tag{B13}$$

y_c^*, the critical spot price/strike price ratio for immediate exercise of calls, is given implicitly by

$$y_c^* - 1 = C(y_c^*, v, T; 1) + \left(\frac{y_c^*}{q_2}\right)[1 + C_F(y_c^*, v, T; 1)] \tag{B14}$$

where $C_F(y_c^*, v, T; 1) = \partial C(y_c^*, v, T; 1)/\partial F$. The delta and the vega of the American-style futures option are given as follows:

$$\frac{\partial C_t^A(F)}{\partial F} \equiv \begin{cases} e^{-r(T_1 - t)}\Pi_1' + KA_2\left(\dfrac{1}{Ky_c^*}\right)^{q_2} q_2 F^{q_2 - 1} & \text{for } \; F/K < y_c^* \\ 1 & \text{for } \; F/K \geq y_c^* \end{cases} \tag{B15}$$

$$\frac{\partial C_t^A(F)}{\partial v_t} \equiv \begin{cases} \dfrac{\partial C_t(F)}{\partial v_t} & \text{for } \; F/K < y_c^* \\ 0 & \text{for } \; F/K \geq y_c^* \end{cases} \tag{B16}$$

Appendix C. Pricing SPX Futures Options under Diffusion-Jump and Stochastic Volatility (SVJ)

The SVJ model of Bates (1996) and Bakshi et al. (1997) assumes that the underlying asset price of futures follows a diffusion-jump process with a mean-reverting squared root stochastic volatility, as specified in equations (4) and (7). The pricing equation of European-style futures option and risk-neutral probabilities under the SVJ model are the same to (B3) and (B4), respectively. The only difference is the characteristic functions. The characteristic functions for the SVJ are given as follows:

$$\begin{aligned} f_1 = \exp\Bigg\{ & i\varphi\, r(T_1 - t) - \frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1 - \frac{[\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v][1 - e^{-\xi_v (T_1 - t)}]}{2\xi_v} \right) \right] \\ & - \frac{\theta_v}{\sigma_v^2}[\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v](T_1 - t) + i\varphi \ln S_t \\ & + \lambda_J (1+\kappa^*)(T_1 - t)[(1+\kappa^*)^{i\varphi} e^{(i\varphi/2)(1+i\varphi)\sigma_J^2} - 1] - \lambda_J i\varphi\, \kappa^* (T_1 - t) \\ & + \frac{i\varphi\,(i\varphi + 1)[1 - e^{-\xi_v (T_1 - t)}]}{2\xi_v - [\xi_v - \kappa_v + (1+i\varphi)\rho\sigma_v][1 - e^{-\xi_v (T_1 - t)}]} v_t \Bigg\} \end{aligned} \tag{C1}$$

$$f_2 = \exp\left\{ i\varphi\, r(T_1 - t) - \frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1 - \frac{[\xi_v^* - \kappa_v + i\varphi\rho\sigma_v][1 - e^{-\xi_v^*(T_1 - t)}]}{2\xi_v^*} \right)\right] \right.$$
$$- \frac{\theta_v}{\sigma_v^2}[\xi_v^* - \kappa_v + i\varphi\rho\sigma_v](T_1 - t) + i\varphi \ln S_t \tag{C2}$$
$$+ \lambda_J (T_1 - t)[(1+\kappa^*)^{i\varphi} e^{(i\varphi/2)(i\varphi-1)\sigma_J^2} - 1] - \lambda_J i\varphi\, \kappa^*(T_1 - t)$$
$$\left. + \frac{i\varphi(i\varphi - 1)[1 - e^{-\xi_v^*(T_1 - t)}]}{2\xi_v^* - (\xi_v^* - \kappa_v + i\varphi\rho\sigma_v)[1 - e^{-\xi_v^*(T_1 - t)}]} v_t \right\}$$

where λ_J is the frequency of the jumps per year and κ^* is percentage mean jump size.

Then, the delta of the American-style futures options is

$$\frac{\partial C_t^A(F)}{\partial F} \equiv \begin{cases} e^{-r(T_1 - t)}\Pi_1' + KA_2\left(\dfrac{1}{Ky_c^*}\right)^{q_2} q_2 F^{q_2 - 1} & \text{for } F/K < y_c^* \\ 1 & \text{for } F/K \geq y_c^* \end{cases} \tag{C3}$$

The vega of the American-style futures options is

$$\frac{\partial C_t^A(F)}{\partial v_t} \equiv \begin{cases} \dfrac{\partial C_t(F)}{\partial v_t} & \text{for } F/K < y_c^* \\ 0 & \text{for } F/K \geq y_c^* \end{cases} \tag{C4}$$

Further, the greeks of forward-start strangle portfolio in the SVJ are equivalent to (B9) and (B10). The only different is the characteristic functions. For the same reason, we turn the European-style option to American-style one. The pricing equations of American call and put options are the same to (B11), and the only difference is the equation of *q* to follow

$$\frac{1}{2}\overline{v}q^2+\left(r-\lambda_J\overline{\kappa}^*-\frac{1}{2}\overline{v}\right)q-\frac{r}{1-e^{-r(T_1-t)}}+\lambda_J[(1+\overline{\kappa}^*)^q e^{q(q-1)\delta^2/2}-1]=0 \qquad \text{(C5)}$$

where $\overline{\kappa}^*=\exp(\mu_J+\sigma_J^2/2)-1$ and σ_J is the standard deviation of $\ln(1+J_t)$.

Appendix D. Pricing VIX Futures under the Diffusion-Jump and Stochastic Volatility (SVJ) SPX Price Process

The time-*t* fair value of VIX futures expiring at *T*, denoted as $\mathrm{F}_t^{\mathrm{VIX}}(T)$, is given in the CBOE website by

$$\mathrm{F}_t^{\mathrm{VIX}}(T)=\sqrt{\frac{1}{\tau}\{\mathrm{E}_t^Q[P_T^{\mathrm{var}}(T+\tau)]-\mathrm{var}_t^Q[\mathrm{F}_T^{\mathrm{VIX,de\text{-}annualized}}(T)]\}} \qquad \text{(D1)}$$

where $\tau=30/365$ and Q is the risk-neutral probability measure. $\mathrm{E}_t^Q[P_T^{\mathrm{var}}(T+\tau)]$ is the forward price of de-annualized variance in the 30 days after the futures expiration where $P_T^{\mathrm{var}}(T+\tau)=\tau\mathrm{VIX}_T^2$, and $\mathrm{var}_t^Q[F_T^{\mathrm{VIX,de\text{-}annualized}}(T)]$ is the concavity adjustment. The adjustment subtracts the variance of the futures price at expiration, which can also be expressed as the cumulative daily variance of VIX futures from the current date to expiration. Since $F_T^{\mathrm{VIX,de\text{-}annualized}}(T)$ is the de-annualized price of the VIX futures at expiry *T*, one has $F_T^{\mathrm{VIX,de\text{-}annualized}}(T)=\sqrt{P_T^{\mathrm{var}}(T+\tau)}=\mathrm{VIX}_t\sqrt{\tau}$. Thus the variance of $F_T^{\mathrm{VIX,de\text{-}annualized}}(T)$ is equivalent to

$$\begin{aligned} \mathrm{var}_t^Q(F_T^{\text{VIX,de-annualized}}) &= \mathrm{E}_t^Q[P_T^{\text{var}}(T+\tau)] - [\mathrm{E}_t^Q\sqrt{P_T^{\text{var}}(T+\tau)}]^2 \\ &= \mathrm{E}_t^Q[\tau^2\mathrm{VIX}_T^2] - \tau[\mathrm{E}_t^Q(\mathrm{VIX}_T)]^2 \end{aligned} \quad \text{(D2)}$$

The fair value of the VIX futures provided by the CBOE becomes,

$$\begin{aligned} \mathrm{F}_t^{\text{VIX}}(T) &= \sqrt{\frac{1}{\tau}\{\mathrm{E}_t^Q[P_T^{\text{var}}(T+\tau)] - \mathrm{E}_t^Q[P_T^{\text{var}}(T+\tau)] + [\mathrm{E}_t^Q\sqrt{P_T^{\text{var}}(T+\tau)}]^2\}} \\ &= \frac{1}{\sqrt{\tau}}\mathrm{E}_t^Q[\sqrt{P_T^{\text{var}}(T+\tau)}] \\ &= \mathrm{E}_t^Q(\mathrm{VIX}_T) \end{aligned} \quad \text{(D3)}$$

From the approximation of Brockhaus and Long (2000), Bates (2006) and Lin (2007), who use the second-order Taylor expansion for the square root of latent affine stochastic processes, the current VIX futures is worth theoretically

$$\mathrm{F}_t^{\text{VIX}}(T) = \mathrm{E}_t^Q(\mathrm{VIX}_T) \equiv \sqrt{\mathrm{E}_t^Q(\mathrm{VIX}_T^2)} - \frac{\mathrm{var}_t^Q(\mathrm{VIX}_T^2)}{8\times[\mathrm{E}_t^Q(\mathrm{VIX}_T^2)]^{3/2}} \quad \text{(D4)}$$

where $\mathrm{var}_t^Q(\mathrm{VIX}_T^2)/\{8\times[\mathrm{E}_t^Q(\mathrm{VIX}_T^2)]^{3/2}\}$ is the convexity adjustment relevant to the VIX futures. Thus, to calculate the VIX futures one needs both $\mathrm{E}_t^Q(\mathrm{VIX}_T^2)$ and $\mathrm{var}_t^Q(\mathrm{VIX}_T^2)$.

Hence, different dynamics for the SPX index price S will result in various expressions for VIX squared and thus different theoretical formulas for the VIX futures price. The stochastic volatility model with price jumps (SVJ) is the most general model considered in this paper. Its risk-neutral processes of $(\ln S, v)$ are given in equations (4) and (7). Referred to Lin (2007), the VIX squared under the SVJ model is expressed by,

$$\mathrm{VIX}_t^2 \equiv \zeta_2 + \frac{\zeta_1}{\tau}(a_\tau v_t + b_\tau) \tag{D5}$$

where $\zeta_1 = 1$, $\zeta_2 = 2\lambda_J(\kappa^* - \mu_J)$, $\tau = 30/365$, $a_\tau = (1 - e^{-\kappa_v \tau})/\kappa_v$, and $b_\tau = \theta_v(\tau - a_\tau)/\kappa_v$. Under the SVJ model, the mean and variance of VIX_T^2 conditional on VIX_t^2 are computed as

$$\mathrm{E}_t^Q(\mathrm{VIX}_T^2) \equiv \zeta_2 + \frac{\zeta_1}{\tau}(a_\tau \alpha_{T-t} v_t + a_\tau \beta_{T-t} + b_\tau) \tag{D6}$$

$$\mathrm{var}_t^Q(\mathrm{VIX}_T^2) \equiv \left(\frac{a_\tau \zeta_1}{\tau}\right)^2 (C_{T-t} v_t + D_{T-t}) \tag{D7}$$

where $v_t \equiv \frac{\tau}{a_\tau \zeta_1}\mathrm{VIX}_t^2 - \frac{\tau \zeta_2}{a_\tau \zeta_1} - \frac{b_\tau}{a_\tau}$, $\alpha_{T-t} = e^{-\kappa_v (T-t)}$, $\beta_{T-t} = \frac{\theta_v}{\kappa_v}(1 - \alpha_{T-t})$, $C_{T-t} = \frac{\sigma_v^2}{\kappa_v}(\alpha_{T-t} - \alpha_{T-t}^2)$, and $D_{T-t} = \frac{\sigma_v^2 \theta_v}{2\kappa_v^2}(1 - \alpha_{T-t})^2$.

By substituting $\mathrm{E}_t^Q(\mathrm{VIX}_T^2)$ and $\mathrm{var}_t^Q(\mathrm{VIX}_T^2)$ in equation (D4) with the ones above, the fair price of the VIX futures expiring at *T* under the SVJ model is obtained, and the parameter vector is $\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho, \lambda_J, \mu_j, \sigma_j\}$. The vega of the VIX futures under the SVJ model is thus given by

$$\begin{aligned}\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial v} \equiv & \left\{\frac{1}{2[\mathrm{E}_t^Q(\mathrm{VIX}_T^2)]^{1/2}} + \frac{3\,\mathrm{var}_t^Q(\mathrm{VIX}_T^2)}{16[\mathrm{E}_t^Q(\mathrm{VIX}_T^2)]^{5/2}}\right\}\frac{\partial \mathrm{E}_t^Q(\mathrm{VIX}_T^2)}{\partial v} \\ & - \frac{1}{8[\mathrm{E}_t^Q(\mathrm{VIX}_T^2)]^{3/2}}\frac{\partial\,\mathrm{var}_t^Q(\mathrm{VIX}_T^2)}{\partial v}\end{aligned} \tag{D8}$$

where $\partial \mathrm{E}_t^Q(\mathrm{VIX}_T^2)/\partial v = \zeta_1 a_\tau \alpha_{T-t}/\tau$ and $\partial \mathrm{var}_t^Q(\mathrm{VIX}_T^2)/\partial v = C_{T-t}(a_\tau \zeta_1/\tau)^2$.

The volatility specification of the SV model, introduced by Heston (1993), captures important stylized features of the stock return dynamics: mean-reversion, stochastic volatility and volatility asymmetry. The SV model obtains as a special case of the general model with jumps restricted to zero ($J_t dN_t = 0$), and thus $\zeta_2 = 0$. Hence the fair value of the VIX futures under the SV model is given by equation (D4) with the parameter vector $\Phi = \{\kappa_v, \theta_v, \sigma_v, \rho\}$. The vega of the VIX futures for the SV model is obtained by replacing ζ_2 with zero in equation (D8). Further, under the constant volatility of BSM, the VIX squared is the constant volatility:

$$\mathrm{VIX}_T^2 = \frac{1}{\tau}\left(\int_T^{t+\tau} v_u du\right) = \sigma^2 \tag{D9}$$

Therefore, the price of the VIX futures under BSM is

$$\mathrm{F}_t^{\mathrm{VIX}}(T) = \mathrm{E}_t^Q(\mathrm{VIX}_T) = \sigma \tag{D10}$$

, and the vega of the VIX futures under BSM is

$$\frac{\partial \mathrm{F}_t^{\mathrm{VIX}}(T)}{\partial v} = \frac{\partial \sigma}{\partial v} = \frac{1}{2\sigma^2} \tag{D11}$$

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