Credit Analysis of Corporate Credit Portfolios---

A Cash Flow Based Conditional Independent Default Approach

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Abstract

Most existing studies on portfolio credit analysis are reduced from models. Few of them can consider dynamics of risk structure and none can estimate portfolio loss rate endogenously. As a cash flow based structural form credit model, we suggest combining a factor modeled cash flow model and conditional independent default approaches to estimate the multi-period credit risk of a corporate credit portfolio. The proposed approach includes a dynamic risk structure and is able to endogenously estimate the portfolio loss rate. We have exemplified how the approach is applied to credit tranching and pricing of a cash funded CBO.

Keywords: Conditional independent default, Cash flow, Credit portfolio tranching and pricing, Risk dynamics

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Abstract

Most existing studies on portfolio credit analysis are reduced from models. Few of them can consider dynamics of risk structure and none can estimate portfolio loss rate endogenously. As a cash flow based structural form credit model, we suggest combining a factor modeled cash flow model and conditional independent default approaches to estimate the multi-period credit risk of a corporate credit portfolio. The proposed approach includes a dynamic risk structure and is able to endogenously estimate the portfolio loss rate. We have exemplified how the approach is applied to credit tranching and pricing of a cash funded CBO.

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Due to the implementation of New Basel Accord (or Basel II) and the fast development of collateralized debt obligations (CDO), portfolio credit analysis has become an important research area in recent years. Most existing studies are reduced from models and focus on handling the default correlation issue between component assets comprising a credit portfolio. To deal with the issue, several approaches are developed in the literature such as conditional independent default (later denoted as C.I.D.) approach, contagion models, and some other varieties.[1] Few of them take into considerations of the dynamics of risk structure and are able to endogenously estimate the portfolio recovery (loss) rate. Alternatively, within the framework of structural form credit models, this paper suggests combining a cash flow based credit model that has a factor structure and a conditional independent default approach, such as the factor copula or the Fourier transform methods, to analyze the credit risk of a corporate credit portfolio.[2] Since the approach is based upon a firm's future free cash flows, it can evade the controversies stemming from most Merton type option-based structural models that employ a market-based valuation approach.[3]

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1 The major conditionally independent defaults (CID) studies include Duffee (1999), Zhou (2001), Schonbucher (2003), Driessen (2005), Bakshi, Madam and Zhang (2004), Janosi, Jarrow and Yildirim (2002), and Zhang (2003). The contagion models include the infectious default model (Davis and Lo, 1999) and the propensity model (Jarrow and Yu, 2001). The other varieties are for example, Giesecke and Weber (2004) and Hull and White(2005).

2 Because a firm's free cash flow is mainly affected by both the firm's management policies and macroeconomic economic cycle, its free cash flow dynamics include both systematic factors and a firm specific effect.

3 Most structural form models use an option-based theory to convert a firm's equity market value into its asset value, assuming that stock prices are log-normally distributed and therefore stock returns are normally distributed. They also assume the existence of an efficient market. However, the literature has shown that stock return distribution is asymmetric, fat tailed, and volatility smiled. In addition, according to Merton (1974), most traditional option-based structural models assume that the value of a firm is not affected by its capital structure.

The approach employs a state-dependent free cash flow process to generate each component firm's multi-period asset value distributions and therefore its multi-period default probabilities and recovery rates endogenously. [4] Multi-period loss distributions of the credit portfolio can then be obtained through conditional default approaches such as the factor copula or the Fourier transform methods. The multi-period credit information is useful in the portfolio credit tranching and the tranche pricing by employing the method suggested by Geske (1977) and Jarrow and Turnbull (1995).

Conforming to a common understanding that the growth rates of most economic indicators are weakly stationary, this study suggests a mean-reverting Gaussian process to model the common state factors underlying the cash flow models of portfolio component firms. The estimated forward-looking state factor information is then used to adjust the parameters of the cash flow models. Each component firm's multi-period unconditional asset value distributions can be spawned by its free cash flow process. Thereofere, with proper default boundary information, we are able to estimate each component firm's unconditional multi-period PDs and RRs endogenously and concurrently. Because all component firms' cash flow process are affected by the same common state factors, their PDs are independent conditioning on a specific set of state factors. As the conditional independent default approach (later denoted as CID approach) is the common methodology underlying the two recently developed techniques for credit

[4] Free cash flow to firm is a firm's operating free cash flow prior to the payment of interests to the debt holders and after deducting the funds required to maintain the firm's productivity (i.e. non-discretionary capital expenditures). Free cash flow to firm is a measure to estimate the value of the total firm. On the other hand, free cash flow to equity is used to estimate the value of a firm's equity and is equal to the free cash flow to firm minus debt repayments.

portfolio analysis, the factor copula and the Fourier transform methods (later denoted as FTM), we are able to develop a cash flow based portfolio credit model by combining the cash flow based credit model and the CID-based techniques. To demonstrate the application of our approach, we provide an example of tranching and valuation of a cash funded CBO comprising fifteen corporate bonds.

The rest of the paper is divided into five sections: First, we present the setting of the state-dependent corporate free cash flow model; Second, we introduce two CID-based techniques, the factor copula and the FTM methods; Third, we develop the cash flow based corporate portfolio credit model; Fourth, we demonstrate the examples of the model's application; and in the last, we conclude this study.

I. The State-dependent Corporate Free Cash Flow Model

As a structural form credit model, we use free cash flow to "firm" (instead of free cash flow to "equity") to estimate a firm's asset value distribution. In this section, we first introduce the definition of corporate free cash flows and then its modeling.

A. Definition of Free Cash Flow to Firm

The definition of free cash flow to firm (later denoted as C_t) adopted in this study is as (1).[5] Its most significant characteristic is that it distinguishes capital expenditures into the discretionary and

[5] This definition is also adopted by the COMPUSTAT database. This definition is used later in the examples of model application. It is to be noted that the database deducts pre-specified cash dividend in C_t calculation because it is a non-discretionary item.

non-discretionary ones.[6] It deducts only non-discretionary capital expenditures in the C_t calculation.

$$C_t = C_t^o - E_t^c \tag{1}$$

Where, at time *t*, C_t denotes a firm's free cash flow to firm, C_t^o denotes a firm's operating cash flow and E_t^C denotes non-discretionary capital expenditure.

Specifically, the non-discretionary capital expenditure is defined in the following

$$E_t^C = E_t^\ell + F_t^c + ppe_t^r \tag{2}$$

Where, at time *t*, E_t^ℓ denotes expenditures for capital leases, F_t^c denotes an increase in funds for construction, and ppe_t^r denotes reclassification of inventory to property, plant, and equipment.

From (1), we know that C_t is essentially influenced by a firm's operating cash flows and the non-discretionary capital expenditures. Because sales revenue is a major component of a firm's operating cash flows and because it is primarily affected by the macroeconomic cycle and corporate policies, there must be a close relationship between C_t and macroeconomic environment. Regarding a firm's management policies, those with short-term objectives might be formulated to conform to economic cycle and therefore are more related to recent market conditions.[7] On the other hand, a firm's long-term policies are less likely to consider short-term fluctuations in the state of the economy and are more firm

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[6] The non-discretionary capital expenditures indicate the capital expenditures necessary to maintain its productivity or sustainable growth.

[7] For example, in a recessionary period, a firm may reduce its operating activities by layoff its employees and lower the level of inventories. Conversely, a firm may adopt an opposite action in a boom periods.

specific. In sum, a firm's C_t is mainly affected by two forces: macro-industrial cyclicality and a firm specific effect. To eliminate the scale effect, the free cash flow to firm C_t used in the rest of the paper is de-scaled by dividing the firm's total asset. That is, C_t indicates free cash flow to firm per unit asset.

B. The State-dependent Free Cash Flow Model

According to previous discussion, we establish the relationship between the i^{th} firm's C_{it} and the state of the economy as (3). In (3) the i^{th} firm's C_{it} is affected by both a set of k systematic factors and an idiosyncratic (firm specific) effect. In addition, F_{jt} indicates the unobservable state factors; α_{ij} indicates the sensitivities of the i^{th} firm's C_{it} to the j^{th} state factor; and ξ_{it} indicates the i^{th} firm's idiosyncratic factor representing the part the variations of the i^{th} firm's C_{it} that can not be explained by the state factors and is normally distributed with mean zero and variance equal to residual variance not explained by the systematic factors, that is $1-h_i$, where h_i indicates the variance explained by the systematic factors. According to Liao and Chen (2004), in most cases, a firm's free cash flow to firm follows a mean-reverting (weakly stationary) process, the number of factors (k) and the factor loading α_{ij} can therefore be estimated by a factor analysis that extracts the unobservable common factors underlying the free cash flows of the component firms comprising the credit portfolio.

$$C_{it} = E(C_{it}) + \sum_{j=1}^{k} \alpha_{ij} F_{jt} + \xi_{it} \qquad \xi_{it} \sim N(0, \sqrt{1-h}_{it}) \tag{3}$$

To take into consideration of the changes in risk structure, we employ a mean-reverting Gaussian process to model each state factor process as (4)

$$dF_{jt} = a_{F_j}[b_{F_j} - F_{j,t-1}]dt + \sigma_{F_j} dz_j \tag{4}$$

Where, F_{jt} indicates the j^{th} state factor value in the time t; a_{F_j} indicates the mean-reverting speed of F_{jt}; b_{F_j} is the long-term average level of F_{jt}; σ_{F_j} indicates the standard deviation of the term variation of F_{jt}, and dz_j is a wiener process. Assuming that the stochastic characteristics of the economy will not structurally change in foreseeable future, the parameters of each state factor's process are set constant. Combining (3) and (4), we can obtain many probable free cash flow paths and therefore their value distributions for each component firm by simulation.

II. The Cash Flow Based Multi-Period Credit Model

A. Single-Firm Credit Model

To construct a structural-form type of multi-period credit model, the first and most important task is to derive a firm's multi-period value distributions. To achieve this, we first employ a firm's the state-dependent cash flow model to simulate many probable C_t paths. Then, many asset value paths corresponding to each simulated C_t path can be generated with an appropriately estimated cost of capital. From a cross-sectional perspective of each future time *t*, we are able to obtain a firm's multi-period unconditional value distributions. The firm's multi-period credit risk (i.e., the unconditional multi-period *PDs* and *RRs*) can then be assessed from the relationship between the asset value distributions and debt boundary (or appropriately designated multi-period default boundaries).

Employing a common practice in firm valuation, we assume that a firm has a two-stage growth pattern in its C_t. In the first stage, C_t follows a state-dependent process before time T (a future time point). In the second stage, C_t grows at a constant rate g after time T. When T is large, the present value of the cash flows after T is not a significant portion of the firm's value. The insignificance is more manifest when the cost of capital increases. With this assumption, we can generate one C_t path (to the future) of a firm by just one simulation. For each C_t path, we can obtain a firm's present value at any time t according to (11):

$$V_{it} = \left[\sum_{\tau=t+1}^{T} \frac{C_{i\tau}}{(1+\gamma_A)^{\tau-t}} \right] + \frac{C_{iT}(1+g)}{(1+\gamma_A)^{T-t}(\gamma_A - g)} \tag{5}$$

In (5), V_{it} is a firm's present value for the i^{th} free cash flow path at time t (the end of period t), $C_{i\tau}$ is the firm's C_t for the i^{th} cash flow path at time τ, T is the beginning time of constant growth, γ_A is the firm's weighted average cost of capital, and g is the firm's constant growth rate after time T.[8] Since all the information employed in the model is derived from historical corporate finance and industrial economic figures, the valuation is under real measure and is subject to systematic risk. The common way to estimate the γ_A is to employ an asset pricing models for equity required return and an appropriate method to estimate the cost of debt(s). Regarding constant growth rate after T, there are several ways to do the estimation in literature such as an appropriate average of historical growth rates, internal growth

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8 Later in the model application example, we set T as 10 years since this is long enough to reduce the portion of the value of the cash flows after T to mitigate any possible errors in the estimation of a constant growth period.

rate,[9] or an adjustment of industrial or macroeconomic growth rates. Because the firm value is very sensitivity to the estimates of the growth rate, it is rather controversial in its estimation. Alternatively, we estimate the growth rate by calculating the market implied growth rate given the firm's cost of capital γ_A and current market value of the firm's asset. That is, we calculate the g that equates the average value of the simulated cash flow paths to the current market value of the firm' asset.[10] The market value of the asset can be estimated by either employing the sum of the firm's equity market value and book value of the debt or converting from equity market value through the option theory (Merton (1974)). This market implied constant growth rate after T incorporates all information regarding the firm available in the market without the errors involved the estimation methodology.

Given the default boundary $\overline{L}_t$, the probability of default for the i^{th} firm at time t (denoted as PD_{it}) is defined as (6). In (6), $f_{it}(V)$ indicates the unconditional distribution of the i^{th} firm's asset value at time t.

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9 Internal growth is defined as $g = \gamma_I \kappa$; where γ_I stands for the long-term average return rate of invested capital and κ represents the long-term average re-investment rate.

10 The mathematical expression is to calculate the g that makes the following equation exist:

$$V_0 = \frac{1}{N}\sum_{i=1}^{N} V_{i,0}$$

Where there is N simulated cash flow paths and V_t represents the current market value of the firm's asset.

When the researchers are more confident in the estimation of the constant growth g than the weighted average cost of capital γ_A, they can calculated the implied γ_A in a similar way given the g and market value of the firm is known, that is to calculate the γ_A that makes the above equation exist.

$$PD_{it} = \int_{\infty}^{\overline{L}_t} f_{it}(V)dV \tag{6}$$

When a default occurs, the creditor's recovery rate at time *t* (later denoted as RR_{it}) can be written as

$$RR_{it} = \frac{1}{\overline{L}_{it}(PD_{it})} \int_{0}^{\overline{L}_{it}} V f_{it}(V) \cdot dV \, . \tag{7}$$

In the current study, we define a new variable: the expected recovery rate (later denoted as ERR_{it}), which indicates how much the creditors expect to recover from their loans. The ERR_{it} can be written as follows:

$$ERR_{it} = 1(1 - PD_{it}) + (RR_{it})(PD_{it}) \, . \tag{8}$$

In the current model, the two main credit risk indicators, PD_t and RR_t, are both endogenously determined. Besides, we know from (6) and (7) that PD_t and RR_t are inversely related and PD_t and ERR_t are negatively related.[11]

B. Portfolio Corporate Credit Model

In this study, we employ the conditional independent default approach (the CID approach) to handle the issues of default correlation and to some extent default contagion between and among the component firms in a credit portfolio. Two different CID methods are introduced in

[11] Equation (21) can be rearranged as $ERR_t = 1(1 - PD_t) + (RR_t)(PD_t) = 1 + PD_t(RR_t - 1)$. Since $(RR_t - 1) \leq 0$, the ERR_t is negatively related to PD_t

this section. They are the factor copula method and the Fourier transform method (the FTM).[12]

B1. Factor Copula method

Since the single firm cash flow model in (3) is set as a factor model indicating that a firm's free cash flow to firm is influenced by a set of systematic factors and a firm specific effect, conditioning on the realization of a set of state vector path, the diffusion terms of component firms' cash flow process are independent. It implies that the firm value distributions of the component firms are independent given a specific state vector path. We can then obtain, at time *t*, the portfolio's conditional joint value distribution given a realized state vector path can be expressed as $\prod_{j=1}^{n} f_t^j\left(V_t^j \middle| \tilde{F}=\bar{F}\right)$, where $f_t^j\left(V_t^j \middle| \tilde{F}=\bar{F}\right)$ indicates the probability density function of the component firm *j's* value at time *t* given the *F* state vector path. Employing the factor copula method, the conditional cumulative joint value distribution of the credit portfolio for each future time point *t* is as in (9), where $f_t(\widetilde{F})$ is the probability density function of the state vector $\widetilde{F}$ at time *t*. The conditional joint probability density function of the portfolio component firm's value for each future time point *t* is as in (10). In discrete cases, when simulating a large number of state paths (*S* paths) by the state process in (4), the (10) can be approximated by (11) according to the Law of Large Number and the Central Limit Theorem. In (11), $P_t\left(F_i\right)$ indicates the probability of state vector path F_i at time *t* and

[12] See for example Li (2000) and Laurent and Jon (2003) for factor copula method and see Debuysscher and Szegö (2003) for the Fourier transform method.

is equal to *1/S*. With the conditional joint probability density function of the credit portfolio component firm's value at time *t*, we can obtain the portfolio's *PD*, *RR* and *ERR* at time t endogenously.

$$\begin{aligned} F^P(V_t^i,\ldots,V_t^n|\tilde{F}) &= \int \prod_{i=1}^{n} \left(\int f_t\left(V_t^i \middle| \tilde{F}=\overline{F}\right) dV^i \right) f\left(\tilde{F}\right) d\tilde{F} \\ &= \int \prod_{i=1}^{n} F_i\left(V_t^i \middle| \tilde{F}=\overline{F}\right) f\left(\tilde{F}\right) d\tilde{F} \\ &= C\left(F_1\left(V_t^1 \middle| \tilde{F}\right),\ldots,F_n\left(V_t^n \middle| \tilde{F}\right)\right) \end{aligned} \tag{9}$$

$$f^P\left(V_t^1,\ldots,V_t^n \middle| \tilde{F}\right) = \frac{\partial^n F^p\left(V_t^1,\ldots,V_t^n \middle| \tilde{F}\right)}{\partial(V_t^1)\cdots\partial(V_t^n)} = \int f\left(\tilde{F}\right) d\tilde{F} \cdot \prod_i f_i\left(V_t^i \middle| \tilde{F}\right) \tag{10}$$

$$\begin{aligned} f_t^P\left(V_t^1,\ldots,V_t^n \middle| \tilde{F}\right) &\approx \sum_{i=1}^{S} \prod_{j=1}^{n} f_t^{ij}\left(V_t^{ij} \middle| \tilde{F}=F_i\right) P_t\left(F_i\right) \\ &\approx \frac{1}{S} \sum_{i=1}^{S} \prod_{j=1}^{n} f_t^{ij}\left(V_t^{ij} \middle| \tilde{F}=F_i\right) \end{aligned} \tag{11}$$

When assessing a firm's credit risk, we need to determine the value of the default boundary L_i for each firm. We assume that a firm defaults when its value falls below a certain default boundary.

In order to construct portfolio loss distribution, we simulate a free cash flow path conditionally on a state vector path realization, we use (5) to calculate firm value at any future time point t, and we can compute portfolio loss by comparing the firm value to the default boundary. And then, we weight them equally to get a conditional portfolio loss at any future point in time. By generating many state paths, we can obtain multi-period portfolio loss distribution.

B2. Fourier Transform method

Fourier transform method (FTM) is a powerful technique to handle complicated joint probability distribution. By changing the original domain of default probabilities into Fourier domain, the joint portfolio default probability can be as shown in simple product expression. Applying an inverse Fourier transform, we can obtain the portfolio default distribution in the original domain easily. The FTM can be adapted to heterogeneities assets with small to large number of assets, especially for credit analysis. In addition, the FTM conducts accurately in numerical computation with great computation speed over traditional Monte Carlo simulation.

As shown in (12) and (13)), the function $f(x)$ can be transformed by (12), which called Fourier transform, into τ domain (the Fourier domain) and then be inversely transformed back from *t* domain to X domain by (13).

$$\hat{f}(\tau) = \int_{-\infty}^{+\infty} f(x) \exp(-i\tau x) dx \tag{12}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \exp(i\tau x) dt \tag{13}$$

If we take $f(x)$ as probability density function of random variable X, $\hat{f}(\tau)$ would be X's characteristic function $\psi(\tau)$.

$$\psi(\tau) = E\left(e^{-i\tau X}\right) = \int_{-\infty}^{+\infty} e^{-i\tau x} dF(x) = \int_{-\infty}^{+\infty} e^{-i\tau x} f(x) dx = \hat{f}(\tau) \tag{14}$$

Combining with a factor model setting for the default indicator of a credit asset, Debuysscher and Szegö (2003) has shown that the default correlation between component assets of a credit portfolio can be analyzed by the FTM. They have shown that for a given state of scenario $\widetilde{F} = F$, the conditional Fourier transform of the portfolio default distribution is as (15), where w_k indicates the weight of the k^{th} asset in total portfolio, $p_k(F)$ indicates the k^{th} asset's conditional default probability, and *n* indicates the number of credit asset in the credit portfolio.

$$\hat{f}_{PDR|\widetilde{F}=F}(\tau) = \prod_{k=1}^{n}\left[1 + p_k(F)\left(e^{-i\tau w_k} - 1\right)\right] \tag{15}$$

They obtained the unconditional Fourier transform of portfolio's default distribution as in (16) by considering all scenarios of the state of the economy $\widetilde{F}$, of which the density function is $\phi(F)$.

$$\hat{f}_{PDR}(\tau) = \int_{-\infty}^{\infty}\prod_{k=1}^{n}\left[1 + p_k(F)\left(e^{-i\tau w_k} - 1\right)\right]\phi(F)dF \tag{16}$$

In discrete cases, when simulating *S* state paths by the state process in (4), the (16) can be approximated by (17) according to the Law of Large Numbers and the Central Limit Theorem. In (17), $P_t\left(F_i\right)$ indicates the probability of state vector F_i path at time *t* and is equal to *1/S*.

$$\begin{aligned} \hat{f}_{PDR}(\tau) &= \sum_{i=1}^{S}\prod_{k=1}^{n}\left[1 + p_k(F)\left(e^{-i\tau w_k} - 1\right)\right]P(F_i) \\ &\approx \frac{1}{S}\sum_{i=1}^{S}\prod_{k=1}^{n}\left[1 + p_k(F)\left(e^{-i\tau w_k} - 1\right)\right] \end{aligned} \tag{17}$$

Similary, Debuysscher and Szegö (2003) employed the FTM to determine the loss distribution and calculate the portfolio expected recovery rate. The conditional portfolio Fourier transform of the portfolio loss distribution under given state *F* is as (18):

$$\hat{f}_{PLR|\tilde{F}=F}(\tau)=\prod_{k=1}^{n}\left[1+p_k(F)\cdot\left(e^{-i\tau LGD_k w_k}-1\right)\right] \tag{18}$$

They obtained the unconditional Fourier transform of portfolio's loss distribution as in (19) by considering all scenarios of the state of economy $\tilde{F}$, of which the density function is $\phi(F)$.

$$\hat{f}_{PLR}(\tau)=\int_{-\infty}^{\infty}\prod_{k=1}^{n}\left[1+p_k(F)\left(e^{-i\tau LGD_k w_k}-1\right)\right]\phi(F)dF \tag{19}$$

In discrete cases, when simulating a large number of state paths (*S* paths) by the state process in (4), the (19) can be approximated by (20) according to the Law of Large Numbers and the Central Limit Theorem. In (20), $P_t(F_i)$ indicates the probability of state vector F_i path at time *t* and is equal to *1/S.*

$$\begin{aligned}\hat{f}_{PLR}(\tau)&=\sum_{i=1}^{S}\prod_{k=1}^{n}\left[1+p_k(F)\left(e^{-i\tau LGD_k w_k}-1\right)\right]P(F_i)\\&\approx\frac{1}{S}\sum_{i=1}^{S}\prod_{k=1}^{n}\left[1+p_k(F)\left(e^{-i\tau LGD_k w_k}-1\right)\right]\end{aligned} \tag{20}$$

Applying inverse Fourier transform, we can get the expected portfolio loss rate *PLR*. The portfolio expected recovery rate (*ERR*), by definition, is:

$$ERR=1-PLR \tag{21}$$

IV. Examples of the Model's Applications

In this section, we present an example of using the proposed approach to assess the credit risk of a bond portfolio comprising 15 U.S. straight corporate bonds. In the following, we introduce the data, the parameter estimation of the cash flow model and state model, and then the results of the credit assessments. For simplicity, the presentation of the example will major base on the factor copula method in the portfolio loss related portion. However, for comparison, we will also show the major results obtained from the Fourier transform method.

A. The Data

The example bond portfolio comprises 15 straight corporate bonds issued by different U.S. firms. Our criteria of selecting the component bonds are as follows. First, we select bonds with long-term corporate credit rating. Second, we select corporate bonds of non-financial firms with maturity less than ten years. Third, we exclude firms that have missing financial data, Screened by these we have 30 bonds left. For simplicity and the consideration of industry diversification, we select 15 of the 30 available bonds to construct the example portfolio. All firm-related financial information is from COMPUSTAT database and the credit rating information of the sample firms is from the Bloomberg. The information of the sample portfolio is illustrated in Table 1 and 2. We set our pricing time at December 31, 2004.

[Insert Table 1 approximately here]

[Insert Table 2 approximately here]

B. Factor Analysis and Parameter Estimation of the State Model

We employ factor analysis to extract state factors underlying the quarterly free cash flows of the 15 portfolio component firms. The estimation period for the parameters of the state factor model is from 1995 Q1 to 2004 Q4. We have found 4 state factors that can explain about 81.94% of these firms free cash flow variation. The estimated parameters of stochastic state model are illustrated in Table 3. .

[Insert Table 3 approximately here]

C. Estimation of a Firm's Weighted Average Cost of Capital γ_A [13]

For simplicity, we employ a one-factor CAPM to estimate a firm's equity required return r_e. The needed parameters are: the risk free rate, the market risk premium, and a company's market β. The market risk premium of the U.S. market is set as 7.5% according to *Ibbostson Associates, annual* [14]. The market β of each sample company is obtained from CMPUTSTAT and the market rate of five-year U.S. government notes as risk-free rate proxy obtained from Datastream. Regarding the cost of debt, r_d, for simplicity, we assume it is constant and use the market rate of the corporate bond that has the same credit rating as that of the sample company[15]. With all the information, we can estimate the γ_A for each portfolio component company.

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[13] As mentioned in footnote 10, if we are more confident in the estimation of the constant growth g than the weighted average cost of capital γ_A,we can calculated the implied γ_A that makes the following equation exist.

$$V_0 = \frac{1}{N}\sum_{i=1}^{N} V_{i,0}$$

Where there is N simulated cash flow paths and V_t represents the current market value of the firm's asset.

[14] Source: *Stocks, Bonds, Bills and Inflation* (Chicago, III.: *Ibbostson Associates, Annual*).

[15] It is not difficult to relax this assumption by using interest rates generated by a more complicated interest rate model.

D. Estimation of Constant Growth Rate of a Firm

In this study, we assume a firm grows at a constant rate after 10 years from our pricing time, that is, the beginning time of constant growth, T, in equation (5) is set as 10 years. We estimate the market implied growth rate as the proxy for the constant growth rate of a firm. We employ method of the Merton(1974) to transform a firm's equity market value to its market asset value.[16] The result of estimation of constant growth rate is showed in Table 4.

[Insert Table 4 approximately here]

E. Credit Rating Analysis of Portfolio Component Firms

The empirical results of credit rating analysis for the sample firms are illustrated in Table 4. The second column of Table 4 denotes the unconditional probability of default for each sample firm estimated by the cash flow based credit model (denoted as "model *PD*") for the next ten years. The third column represents the corresponding range of credit rating for each sample firm obtained by matching the model PD to the ten-year average cumulative default rates provided by the Standard and Poor's (denoted as "model rating"). The fourth column reports actual rating for each portfolio firm at the date December 31, 2004, which is from Standard and Poor's.

According to the empirical results illustrated in Table 4, eleven out of fifteen sample firms

[16]We obtained the market stock price of a firm from the CRSP database.

(about 73.3%) are rated correctly by our model; two firms (KO and EMR) are upgraded and other two firms (IP and BLS) are downgraded. The model's applicability seems acceptable in this empirical case.

[Insert Table 5 approximately here]

Due to contagion effect among sample firms, there are small gaps between model rating and accrual rating for four sample firms. Besides, if we observe the patterns of component firms' free cash flows, we can see company KO and company EMR have relatively stable free cash flows so that smaller probabilities of default are estimated by our model.

F. Estimation of the Portfolio Loss Distribution

To assess multi-period long-term credit risk of the example bond portfolio, we first simulate 1,000 cash flow paths under a state vector path realization to get the conditional loss level of each component firm at any future point (quarter) in time. And then we calculate the conditional portfolio loss level by summing all firms' loss up. By generating 1,000 state vector paths, we can construct the multi-period portfolio loss distribution. Since empirically loss distributions have the characteristics of scale and shape, of distributions such as Beta and Gamma, we also do distribution fittings for the example portfolio. The results of the parameter estimates and their standard deviations in Table 5 show that the fitting of Gamma, Birnbaum-Sauders, and Weibull distributions are efficient for loss distributions estimated by either the factor copula or the Fourier transform method. Figure 1 and Figure 2 illustrate single-period (a quarter ahead) portfolio loss distributions estimated both the factor copula the Fourier transform methods. Figure 3 shows the multi-period (one to forty quarters ahead) portfolio loss distribution of the example portfolio estimated by the factor copula method.

[Insert Table 6 approximately here]

[Insert Figure 1 Figure 2 and Figure3 approximately here]

G. Applications in Tranching and Pricing of a CBO

When doing CBO credit tranching, the required information includes the expected loss of the collateralized credit portfolio, the market demanded credit tranches and sizes (as the percentage of total portfolio; also called tranche weight). Using the bonds issued by the component firms of the example portfolio as the collateral assets of a CBO, we show the application of our approach in its tranching and pricing. The bonds data underlying forming the CBO is showed in Table 6. We assume the collateralized bond portfolio is formed equally weighted. Since the weighted average maturity of the bond portfolio is 5.945 years, we assume the maturity of CBO tranches is 6 years. Therefore, we consider the estimated multi-period portfolio loss distribution to the 6^{th} years (24^{th} quarters) to estimate the accumulated CBO's expected loss rate, which is 2.37% and is endogenously obtained. Based on the portfolio expected loss rate and the market demanded credit tranches and sizes, we can structure the CBO.

A tranching example is demonstrated as follows. Provided that the major market demanded for tranche ratings are Aaa and A2 with tranch weights 50% and 25% respectively and the issuer decides to issue two junior tranches, Baa2 and equity. Since we can obtain the corresponding six-year expected loss rate for the three rated tranches from Moody's idealized expected loss table[17], we have only two unknown variables to determine, the tranche weights of the Baa2 and the equity tranches. The tranche weights can be solved through the two constraints including that the total expected portfolio loss rate is equal to the endogenously obtained portfolio expected loss rate 2.37% and that the sum of all tranche weights must be equal to 1. The results of the tranching example are illustrated in Table 7. The sustainable loss rate is calculated by multiplying the expected loss rate by the tranche weight. In this numerical example, the tranche weights of the Baa2 and the equity tranches sizes are 22.97% and 2.03% respectively and the sustaintable loss rate of the equity tranche is 2.03%. For comparison, the tranching results by the Fourier transform method are illustrated in Table 8. We found in this example that the two CID methods have very similar results in corporate credit analysis.

[Insert Table 7 approximately here]

[17] See "Moody's approach to rating synthetic CDOs", 2003.

[Insert Table 8 approximately here]

Regarding credit portfolio pricing, there are two stages of pricing issues. They are the issuer pricing for the primary market and the investor pricing for the secondary market. Regarding the primary market pricing, the expected loss rate of a tranche represents its credit risk needed to be compensated, that is the required credit spread. Therefore, each tranche's expected loss rate indicates the credit spread required in the primary market pricing. Regarding the secondary market pricing, the credit risk may change over time. We have to employ the multi-period portfolio loss distribution to identify the expected loss rate for each point in time having cash inflow form the bond portfolio. The method employed here is suggested by Geske (1977) and Jarrow and Turnbull (1995). They price zero-coupon bonds involving credit risk by transforming them into defaut-free zero coupon bonds through the information of recovery rate. In the case of CBO tranche pricing, we can treat each CBO tranche as a N-year coupon bond, and split it into N zero coupon bonds with the maturity dates equal to the dates that the coupons are received. Thus, with the multi-period expected recovery rate estimated from our model, we can transform each zero coupon bond into default-free zero coupon bond. And the tranche price at time t P_t can be evaluated by the (22) below:

$$P_t = \left[\sum_{\tau=t+1}^{T-1} \frac{C \cdot ERR_\tau}{(1 + r_{t,\tau})^{\tau-t}} \right] + \frac{(C + F) \cdot ERR_{T-t}}{(1 + r_{t,\mathrm{T}})^{T-t}} \tag{22}$$

where C is the tranche's coupon rate, F is tranche's face value; and $r_{t,\tau}$ represents the term structure of risk-free interest rate; T is the tranche's maturity.

V. Conclusions

Most existing literature on portfolio credit analysis are reduced from models and focus on handling the default correlation issue between component assets comprising a credit portfolio. Few of them take into considerations of the dynamics of risk structure and none is able to endogenously estimate portfolio recovery (loss) rate. Within the framework of structural form credit models, this paper suggests combining a cash flow based firm valuation method and conditional independent default approaches, the factor copula and the Fourier transform methods, to estimate multi-period credit risk of a corporate credit portfolio. The proposed approach differs from most existing literatures in that it takes into considerations of the dynamics of risk structure and is able to endogenously estimate the portfolio recovery (loss) rate. We have exemplified how the proposed approach is applied to credit tranching and pricing of a cash funded CBO. We found in this example that the two CID methods have very similar results in corporate credit analysis. Besides, the example shows that the proposed approach is quite applicable.

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Table 1. The industry information of example sample

The example bond portfolio comprises 15 straight corporate bonds issued by different U.S. firms. Our criteria of selecting the component bonds are with long-term corporate credit rating, of non-financial firms with maturity less than ten years and exclude firms that have missing financial data. Screened by these we have 30 bonds left. For simplicity and the consideration of industry diversification, we select 15 of the 30 available bonds to construct the example portfolio. All firm-related financial information is from COMPUSTAT database and the credit rating information of the sample firms is from the Bloomberg.

Firm	Industry	Rating (2004/12/31)	Ticker
1. ALCOA INC	Basic Material - Aluminum	A-	AA
2. Black & Decker Corp.	Small Tools & Accessories	BBB	BDK
3. BellSouth Corp.	Telecom Service	A	BLS
4. Clear Channel Communications Inc.	Broadcasting - Radio	BBB-	CCU
5. Emerson Electric Co.	Industrial Electrical Equipment	A	EMR
6. International Business Machines Corp.	Diversified Computer Systems	A+	IBM
7. International Paper Co.	Paper & Paper Products	BBB	IP
8. Johnson & Johnson	Drug Manufacturers	AAA	JNJ
9. Kellogg Co.	Food - Major Diversified	BBB+	K
10. Coca-Cola Co.	Beverages	A+	KO
11. Masco Corp.	Industrial Equipment & Components	BBB+	MAS
12. McDonald's Corp.	Restaurants Service	A	MCD
13. Merck & Co. Inc.	Drug Manufacturers	AA-	MRK
14. Safeway Inc.	Grocery Stores	BBB	SWY
15. AT&T	Telecom Service	BB	T

Table 2. The Information of Asset Pool

The asset pool is formed by the criteria on table 1. All related corporate bonds' information of the sample firms is from the Bloomberg and YAHOO.Finance.

Firm	Bond	Rating	Coupon (%)	Maturity
AA	AA 7.375 08/01/10 Corp	A	7.375	2010/8/1
BDK	BDK 7.125 06/01/11 Corp	BBB	7.125	2011/6/1
BLS	BLS 6.000 10/15/11 Corp	A	6.000	2011/10/15
CCU	CCU 7.650 09/15/10 Corp	BBB	7.650	2010/9/15
EMR	EMR 4.625 10/15/12 Corp	A	4.625	2012/10/15
IBM	IBM 4.250 09/15/09 Corp	A	4.250	2009/9/15
IP	IP 4 04/01/10 Corp	BBB	4.000	2010/4/1
JNJ	JNJ 6.625 09/01/09 Corp	AAA	6.625	2009/9/1
K	K 2.875 06/01/08 Corp	A	2.875	2008/6/1
KO	KO 5.750 03/15/11 Corp	A	5.750	2011/3/15
MAS	MAS 5.875 07/15/12 Corp	BBB	5.875	2012/7/15
MCD	MCD 6.00 04/15/11 Corp	A	6.000	2011/4/15
MRK	MRK 4.375 02/15/13 Corp	BBB	4.375	2013/2/15
SWY	SWY 4.950 08/16/10 Corp	BBB	4.950	2010/8/16
T	T 6.000 03/15/09 Corp	BB	6.000	2009/3/15

Table 3. Parameters estimation of stochastic state model

We use the maximum likelihood estimation (MLE) method to estimate parameters for equation (4) and the state factor values are calculated from factor analysis method. The estimating results in table 2 are local optimizations based on Newton method.

	a_F	b_F	σ_F
Factor 1	0.1364	0.0856	0.4714
Factor 2	0.0968	0.2139	0.3940
Factor 3	0.1644	0.2695	0.5578
Factor 4	0.1889	0.0621	0.5570

Table 4. The Estimation Results of the Implied g for the Sample Firms (Quarterly)

The asset market value is estimated by Merton's(1974). Given the estimated weight average capital of cost (WACC) and future cash flows, the implied constant growth rate is estimated using (5) by an optimization technique. Equation (5) is as follows:

$$V_{it} = \left[\sum_{\tau=t+1}^{T} \frac{C_{i\tau}}{(1+\gamma_A)^{\tau-t}} \right] + \frac{C_{iT}(1+g)}{(1+\gamma_A)^{T-t}(\gamma_A - g)},$$

where, V_{it} is a firm's present value for the i^{th} free cash flow path at time t (the end of period t), $C_{i\tau}$ is the firm's $C_{i\tau}$ for the i^{th} cash flow path at time τ, T is the beginning time of constant growth, γ_A is the firm's weighted average cost of capital, and g is the firm's constant growth rate after time T. The mathematical expression is to estimate the g that makes the following equation exist:

$$V_t = \frac{1}{N} \sum_{i=1}^{N} V_{it}$$

Where there is N simulated cash flow paths and V_t represents the current market value of the firm's asset. Each firm's asset market value per book value is in 2004Q4.

Ticker	Equity value (million)	Asset value (million)	Market implied g
AA	$27,350.45	$40,038.28	1.39%
BDK	7,131.76	9,988.08	0.89%
BLS	50,904.64	74,093.25	0.56%
CCU	19,207.82	25,463.01	1.47%
EMR	29,422.86	36,093.05	1.16%
IBM	164,105.83	223,184.26	1.16%
IP	20,436.65	35,714.87	1.31%
JNJ	188,213.18	205,768.64	0.15%
K	18,440.56	24,078.76	0.55%
KO	100,745.31	113,807.73	0.97%
MAS	16,365.44	20,956.09	1.09%
MCD	40,305.99	48,806.75	1.15%
MRK	71,273.18	89,620.27	0.30%
SWY	8,833.65	16,197.82	1.11%
T	15,169.26	32,448.24	0.49%

Table 5. Rating results of multi-period credit model

We use KMV's definition for critical point of default, which is "current liability + 0.5(long-term liability)". Each firm's actual rating information is acquired from S&P. The results of model's PD are calculated from 1,000 times simulation of state vector model. We match model's PD to U.S. ten-year average cumulative default rates provided by S&P to obtain the range of credit rating for each firm. The assessment time point is 2004/12/3.

Ticker	Model PD(%)	Model Rating	Actual Rating(Issuer)
AA*	6.38	BBB~A	A-
BDK*	8.23	BB~BBB	BBB
BLS**	12.74	BB~BBB	A
CCU*	4.58	BBB~A	BBB-
EMR***	0.00	AAA	A
IBM*	1.71	A~AA	A+
IP*	20.27	BB~BBB	BBB
JNJ*	0.00	AAA	AAA
K*	5.93	BB~BBB	BBB+
KO***	0.01	AAA	A+
MAS*	8.19	BB~BBB	BBB+
MCD*	2.97	BBB~A	A
MRK*	0.74	AA~AAA	AA-
SWY*	3.66	BBB~A	BBB
T*	38.16	B~BB	BB

*: within the range of model's rating; **: above the range of model's rating; ***: below the range of model's rating

Table 6. Parameter Estimation of the fitting loss distributions

We fit the loss distribution to the specified distributions by Matlab. The value in parenthesis is the standard deviation of the parameter. The results of the parameter estimates and their standard deviations show that the fitting distributions are efficient.

Fitting Distribution	Gamma		Birnbaum-Saunders		Weibull	
	Loss distribution estimated by the factor copula method					
Parameters	a: shape	b: scale	beta: scale	gamma: shape	a: scale	b: shape
Estimates (Std. Err.)	0.7933 (0.0306)	13576.8 (711.598)	2690.3 (118.65)	2.1803 (0.0504)	10031.4 (385.402)	0.8658 (0.0216)
	Loss distribution estimated by the Fourier transform method					
Parameters	a: shape	b: scale	beta: scale	gamma: shape	a: scale	b: shape
Estimates (Std. Err.)	1.2352 (0.0495)	0.0197 (0.0010)	0.0147 (0.0004)	1.1282 (0.0252)	0.0256 (0.0007)	1.1565 (0.0296)

Table 7. The tranching results of the example CBO by the factor copula method

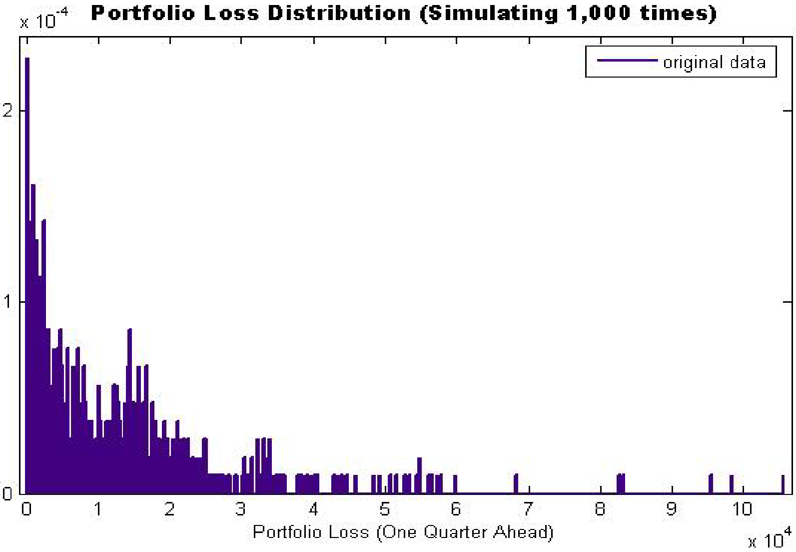
The sustainable loss rate of each tranche is calculated by multiplying the expected loss rate by its tranche weight. We can obtain the tranche weights of Baa2 and equity tranches by considering following two constraints. The first constraint is the sum of the sustainable loss rate of all tranches must be equal to the expected loss rate of the CBO, 2.37%. The second constraint is that tranche weights must sum to 1. In this example, the tranche weights of the Baa2 and the equity tranches are 22.97% and 2.03%. The expected loss rate of each credit rating is obtained from Moody's idealized expected loss table.

Tranches	A	B	C	Equity	Total
Rating	Aaa	A2	Baa2	N.A.	
Tranche weight	50%	25%	**22.97% ***	**2.03%***	100%
Expected Loss Rate	0.0022%	0.3207%	1.0835%	100%	
Sustainable Loss (%)	0.0011%	0.0802%	0.2489%	2.0300%	2.37%

Table 8. The tranching results of the example CBO by the Fourier transform method

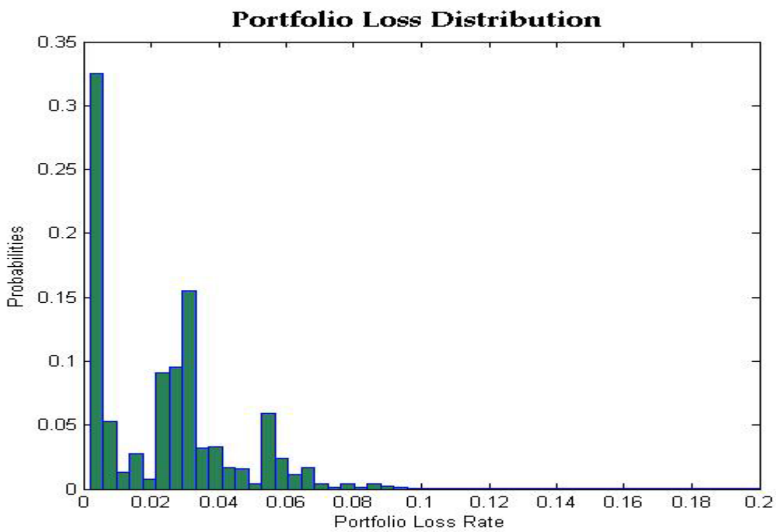
The sustainable loss rate of each tranche is calculated by multiplying the expected loss rate by its tranche weight. We can obtain the tranche weights of Baa2 and equity tranches by considering following two constraints. The first constraint is the sum of the sustainable loss rate of all tranches must be equal to the expected loss rate of the CBO, 1.95%, which is estimated by the Fourier transform method. The second constraint is that tranche weights must sum to 1. In this example, the tranche weights of the Baa2 and the equity tranches are 23.37% and 1.63%. The expected loss rate of each credit rating is obtained from Moody's idealized expected loss table.

Tranches	A	B	C	Equity	Total
Rating	Aaa	A2	Baa2	N.A.	
Tranche weight	50%	25%	**23.37%**	**1.63%**	100%
Expected Loss Rate	0.0022%	0.3207%	1.0835%	100%	
Sustainable Loss (%)	0.0011%	0.0802%	0.2532%	1.63%	1.95%



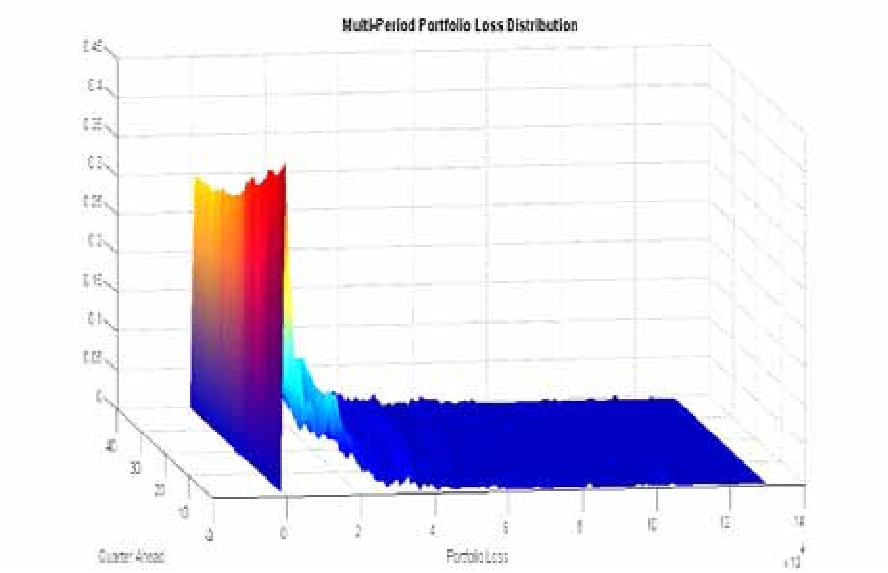
This vertical axis indicates the probability corresponding to the loss amount (in millions) shown in the horizontal axis. The integral of the distribution is the expected loss amount of the credit portfolio.

Figure 1. Single-Period Portfolio Loss Distribution Estimated by the Factor Copula Method



This vertical axis indicates the probability corresponding to the portfolio loss rate shown in the horizontal axis. The integral of the distribution is the expected loss rate of the credit portfolio.

Figure 2. Single-Period Portfolio Loss Distribution Estimated by the Fourier Transform Method



This z axis indicates the probability corresponding to the loss amount (in millions) shown in the x axis. The y axis indicates the future points in time (quarterly). The integral of the distributions are the multi-period expected loss of the credit portfolio.

Figure 3. Multi-Period Portfolio Loss Distributions