Transform approach for operational risk modelling: VaR and TCE

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Abstract To quantify the aggregate losses from operational risk, we employ actuarial risk model, i.e. we consider compound Cox model of operational risk to deal with stochastic nature of its frequency rate in reality. A shot noise process is used for this purpose. A compound Poisson model is also considered as its counterpart for the case that operational loss frequency rate is deterministic. As the loss amounts arising due to mismanagement of operational risks are extremes in practice, we assume the loss sizes are Loggamma, Fréchet and truncated Gumbel. We also use an exponential distribution for the case of non-extreme losses. Employing loss distribution approach, we derive the analytical/explicit forms of the Laplace transform of the distribution of aggregate operational losses. The Value at Risk (VaR) and tail conditional expectation (TCE, also known as TailVaR) are used to evaluate the operational risk capital charge. Fast Fourier transform is used to approximate VaR and TCE numerically and the figures of the distributions of aggregate operational losses are provided. Numerical comparisons of VaRs and TCEs obtained using two compound processes are also made respectively.

Keywords: Operational risk; total loss; the compound Poisson/Cox process; shot noise process; loss distribution; VaR; tail conditional expectation (TCE); Fast Fourier transform.

1. Introduction

A capital charge for operational risk is required to the financial institutions. The Basel Committee for Banking Supervision (2006) defines operational risk as follows: "The risk of losses resulting from inadequate or failed internal processes, people and systems or from external events". A list of loss event types (level 1) of operational risks is shown in Table 1.1 that is adopted from Annex 9 of Basel Committee on Banking Supervision (2006).

Table 1.1

Event-Type Category (Level 1)	**Definition**
Internal fraud	Losses due to acts of a type intended to defraud, misappropriate property or circumvent regulations, the law or company policy, excluding diversity/ discrimination events, which involves at least one internal party
External fraud	Losses due to acts of a type intended to defraud, misappropriate property or circumvent law, by a third party
Employment Practices and Workplace Safety	Losses arising from acts inconsistent with employment, health or safety laws or agreement, from payment of personal injury claims, or from diversity/discrimination events
Clients, Products & Business Practices	Losses arising from an unintentional or negligent failure to meet a professional obligation to specific clients (including fiduciary and suitability requirements), or from the nature or design of a product
Damage to Physical Assets	Losses arising from loss or damage to physical assets from natural disaster or other events
Business disruption and system failures	Losses arising from distruption or sysytem failures
Execution, Delivery & Process Management	Losses from failed transaction processing or process management, from relations with trade counterparties and vendors

The collapse of Britain's Barings Bank in February 1995 is perhaps the quintessential tale of operational risk management gone wrong. A similar even more severe failure came to light in the last few weeks at the French bank Societe Generale. Both failures were completely unexpected. Over the course of days, Barings, Britain's oldest merchant bank, went from apparent strength to bankruptcy. In both cases the failure and upheaval was caused by the actions of a single trader. The estimated loss for Barings was £700million while first estimates of the Societe Generale loss are around $US 7bn.

To quantify the aggregate losses from operational risk, in this paper we use an actuarial risk model (Cramér 1930; Bühlmann 1970; Gerber 1979; Grandell 1976, 1991; Beard et al. 1984 and Asmussen 2000). Considering one line of business, let X_i, $i = 1, 2, \cdots$, be the loss amounts from type k operational risk, which are assumed to be independent and identically distributed with distribution function $H(x)$ $(x > 0)$, then the total loss arising from type k operational risk up to time t is defined by

$$L_t^{(k)} = \sum_{i=1}^{N_t} X_i, \tag{1.1}$$

where $k = 1, 2, \cdots , d$ and N_t is the total number of losses up to time t. We assume that the process N_t and the sequence $\{X_i\}_{i=1,2,\cdots}$ are independent each other. The grand total loss is hence given by

$$L_t = \sum_{k=1}^{d} L_t^{(k)}. \tag{1.2}$$

According to the Basel II Advanced Measurement Approach (AMA) guidelines, the financial institutions may use the Value at Risk (VaR or the q-quantile) as a risk measure to decide the capital amount required for next t years' operational risk, i.e.

$$\mathrm{VaR}_{99.9\%}(L_t).$$

However to obtain the $\mathrm{VaR}_{99.9\%}(L_t)$, it requires to derive the joint distribution of the total loss random vector $\left(L_t^{(1)},\ L_t^{(2)},\ \cdots,\ L_t^{(d)}\right)$, which is a challenging task. Accordingly, the Basel II AMA guidelines propose to use

$$\sum_{k=1}^{d} \mathrm{VaR}_{99.9\%}(L_t^{(k)}) \tag{1.3}$$

for a capital charge and consider a diversification effect under appropriate correlation assumptions, i.e.

$$\mathrm{VaR}_{99.9\%}(L_t) = \mathrm{VaR}_{99.9\%}\left(\sum_{k=1}^{d} L_t^{(k)}\right) \leq \sum_{k=1}^{d} \mathrm{VaR}_{99.9\%}(L_t^{(k)}). \tag{1.4}$$

This assumptions must be made persuadable to the local regulators. Numerous papers have looked at the modelling of operational losses arising from several sources and their dependence. The work by Nešlehová et al. (2006) and the paper by Chavez-Demoulin et al. (2006) contain numerous models to this effect. The three issues they address are:

Issue 1: The operational loss distribution is extremely heavy-tailed.

Issue 2: The operational loss arrival time is irregular and there exists a tendency to increase over time.

Issue 3: The problem of modelling the dependence between various operational risk sources that may lead to a reduction of the calculated risk capital.

For simplicity, in this paper, we ignore the correlation assumptions, i.e. we assume that $L_t^{(k)}$, $k = 1, 2, \cdots, d$ are independent each other but not identical. In order to calculate the each component of (1.3), i.e. $\mathrm{VaR}_{99.9\%}(L_t^{(k)})$, we need to calculate the distribution of the total loss, i.e.

$$\mathbb{P}\left(L_t^{(k)} \leq l\right).$$

However the calculation of $\mathbb{P}\left(L_t^{(k)} \leq l\right)$ in general is difficult and it cannot be derived explicitly. So in Section 2 we derive the explicit and analytical expressions of the Laplace transforms of the distributions of the total loss $L_t^{(k)}$ and invert their Fast Fourier transforms to calculate $\mathrm{VaR}_{99.9\%}(L_t^{(k)})$ numerically in Section 4. We also calculate the tail conditional expectation defined by

$$\mathbb{E}\left\{L_t^{(k)} \mid L_t^{(k)} \geq \mathrm{VaR}_{99.9\%}(L_t^{(k)})\right\} \tag{1.5}$$

as a coherent risk measure (Artzner et al. 1999) and calculate

$$\sum_{k=1}^{d} \mathbb{E}\left\{L_t^{(k)} \mid L_t^{(k)} \geq \mathrm{VaR}_{99.9\%}(L_t^{(k)})\right\} \tag{1.6}$$

as capital amount required for next t years' from all types of operational risk.

As examined in Moscadelli (2004) that losses arisen from the mismanagement of operational risk are heavy-tailed in practice, in Section 2 we employ Loggamma, Fréchet and truncated Gumbel as loss size distributions to deal with this issue. We also use an exponential distribution for the case of non-heavy-tail losses. A discussion on the techniques of extreme value theory; see for instance, Embrechts et al. (1997). To concern irregular arrival of operational losses and its tendency to increase over time, we use the Cox process with shot noise intensity λ_t for the loss arrival process N_t. A homogeneous Poisson process with loss frequency λ is also examined as its counterpart. In Section 3, we present the expressions for initial probabilities of the total loss and the expressions for initial value of its densities, which are required to improve the accuracy of the distributions of the total loss inverting the Fast Fourier transforms. We compare simulated numerical values of VaRs and TCEs obtained using compound Poisson and compound Cox model respectively in Section 4. Section 5 contains some concluding remarks.

2. The Laplace transform of the distribution of total loss

In order to evaluate the risk measures of VaR and TCE, it is necessary for us to calculate the distribution of total loss. However it is difficult to derive it explicitly. Hence for that purpose, we consider using the Laplace transform as it can be inverted to calculate relevant risk measures of (1.3) and (1.6) numerically.

2.1. Homogeneous Poisson process

As we can see in Table 1.1, fraud, business disruption, execution error and system failure etc. are primary events. In order to measure the occurrence of operational losses out of these primary events, we need a counting process to deal with deterministic or stochastic nature of their arrival rates in practice. Therefore it is natural to use point processes to consider series of operational losses. The simplest one is using a homogeneous Poisson process that has deterministic frequency.

Assuming that the loss arrival process N_t follows a homogeneous Poisson process with loss frequency λ and that $L_0^{(k)} = 0$, the Laplace transform of the the distribution of total loss $L_t^{(k)}$ is given by

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \exp\left[-\lambda t\left\{1 - \hat{m}(\nu)\right\}\right], \tag{2.1}$$

where $\nu \geq 0$ and

$$\hat{m}(\nu) = \int_0^\infty e^{-\nu x} dH(x) < \infty. \tag{2.2}$$

As it has been known that losses arisen from the operational risk are extremes in practice (Moscadelli, 2004), in this paper we consider three heavy-tailed distributions, i.e. a Loggamma,

$$h(x) = \frac{\beta^\alpha}{\sigma_2 \Gamma(\alpha)}\left\{\ln\left(\frac{x}{\sigma_2} + 1\right)\right\}^{\alpha-1}\left(\frac{x}{\sigma_2} + 1\right)^{-\beta-1}, \; x > 0, \; \sigma_2 > 0, \; \beta > 0 \text{ and } \alpha > 0, \tag{2.3}$$

a Fréchet,

$$h(x) = \frac{\varsigma}{\sigma_3}\left(\frac{x}{\sigma_3}\right)^{-\varsigma-1} \exp\left\{-\left(\frac{x}{\sigma_3}\right)^{-\varsigma}\right\}, \; x \geq 0, \; \sigma_3 > 0 \text{ and } \varsigma > 0, \tag{2.4}$$

and a truncated Gumbel,

$$h(x) = \frac{\exp\{\exp(\zeta/\eta)\}}{\exp\{\exp(\zeta/\eta)\} - 1}\frac{1}{\eta}\exp\left\{-\frac{x-\zeta}{\eta} - \exp\left(-\frac{x-\zeta}{\eta}\right)\right\},\ x \geq 0,\ \zeta > 0 \text{ and } \eta > 0. \quad (2.5)$$

If the loss amounts arising due to mismanagement of operational risk are not extremes, we may consider using an exponential for loss size distribution, i.e.

$$h(x) = \frac{1}{\sigma_1}\exp\left(-\frac{1}{\sigma_1}x\right),\ x \geq 0,\ \sigma_1 > 0. \quad (2.6)$$

Using (2.2)-(2.6), we can easily obtain the corresponding expressions for the Laplace transform of the distribution of total loss $L_t^{(k)}$, i.e.

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \exp\left[-\lambda t + \frac{\lambda t}{\alpha\Gamma(\alpha)}\int_0^\infty \exp\left\{-\nu\sigma_2\left(\exp(\beta^{-1}z^{1/\alpha}) - 1\right) - z^{1/\alpha}\right\}dz\right], \quad (2.7)$$

where $z = \left\{\beta\ln\left(\frac{x}{\sigma_2}+1\right)\right\}^\alpha$,

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \exp\left\{-\lambda t + \lambda t\int_0^\infty \exp\left(-\nu\sigma_3 z^{-1/\varsigma} - z\right)dz\right\}, \quad (2.8)$$

where $z = \left(\frac{x}{\sigma_3}\right)^{-\varsigma}$,

$$\begin{aligned}\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} &= \exp\left[-\lambda t + \lambda t\left\{\frac{\exp\{\exp(\zeta/\eta)\}}{\exp\{\exp(\zeta/\eta)\} - 1}\right\}\exp(-\nu\zeta)\,\Gamma(\nu\eta+1; e^{\zeta/\eta})\right] \\ &= \exp\left\{-\lambda t + \frac{\lambda t c^{-\nu\sigma_4}}{1-e^{-c}}\,\Gamma(\nu\sigma_4+1; c)\right\} \quad (2.9)\end{aligned}$$

where $\eta = \sigma_4$, $c = e^{\zeta/\eta}$, $\Gamma(\phi;\varphi) \equiv \int_0^\phi z^{\phi-1}e^{-z}\,dz$, $z = \exp\left(-\frac{x-\zeta}{\eta}\right)$ and

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \exp\left\{-\lambda t\left(\frac{\sigma_1\nu}{1+\sigma_1\nu}\right)\right\}. \quad (2.10)$$

2.2. Shot-noise Cox process

To deal with stochastic nature of operational loss arrival in practice, we consider a Cox process as an alternative point process. The Cox process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stocastic process. Therefore the Cox process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process.

Losses arising from the mismanagement of operational risks depend on the intensity of primary events. One of the processes that can be used to measure the impact of primary events is the shot noise process. Some works of insurance application using shot noise process and a Cox process with shot noise intensity can be found in Klüppelberg & Mikosch (1995), Dassios & Jang (2003) and Jang & Krvavych (2004). The shot noise process is particularly useful in loss arrival process as it measures the frequency, magnitude and time period needed to determine the effect of primary

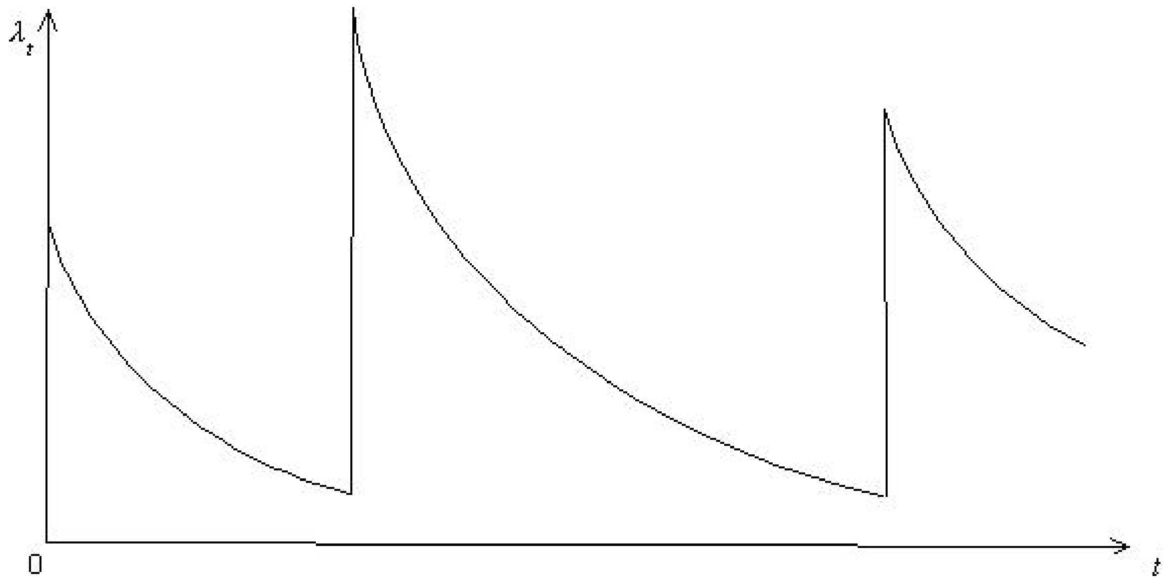

Figure 1: Graph illustrating shot noise process

events. As time passes, the shot noise process decreases as more and more losses are figured out. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of a Cox process to measure the number of operational losses, i.e. we will use it as an intensity function to generate a Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}$$

where:

- λ_0 is the initial value of λ_t that is carried on from primary events incurred previously;
- $\{Y_i\}_{i=1,2,\cdots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ $(y > 0)$ and $E(Y) < \infty$ (i.e. magnitude of contribution of primary event i to intensity);
- $\{S_i\}_{i=1,2,\cdots}$ is the sequence representing the event times of a Poisson process with constant intensity ρ;
- δ is the rate of exponential decay.

Some events such as internal fraud, may take much longer to materialise than others so the decay rate may not be exponential. It is assumed to be of this form for a matter of convenience, i.e. closed-form expressions of final results are easily derived. We also make the additional assumption that the Poisson process M_t and the sequences $\{Y_i\}_{i=1,2,\cdots}$ and $\{X_i\}_{i=1,2,\cdots}$ are independent of each other. Figure 1 illustrates shot noise process.

Now let us assume that the loss arrival process N_t follows a Cox process with its intensity λ_t. Figure 2 illustrates a Cox process with shot noise intensity.

Similar to a homogeneous Poisson process for N_t, the Laplace transform of the the distribution of total loss $L_t^{(k)}$ is given by

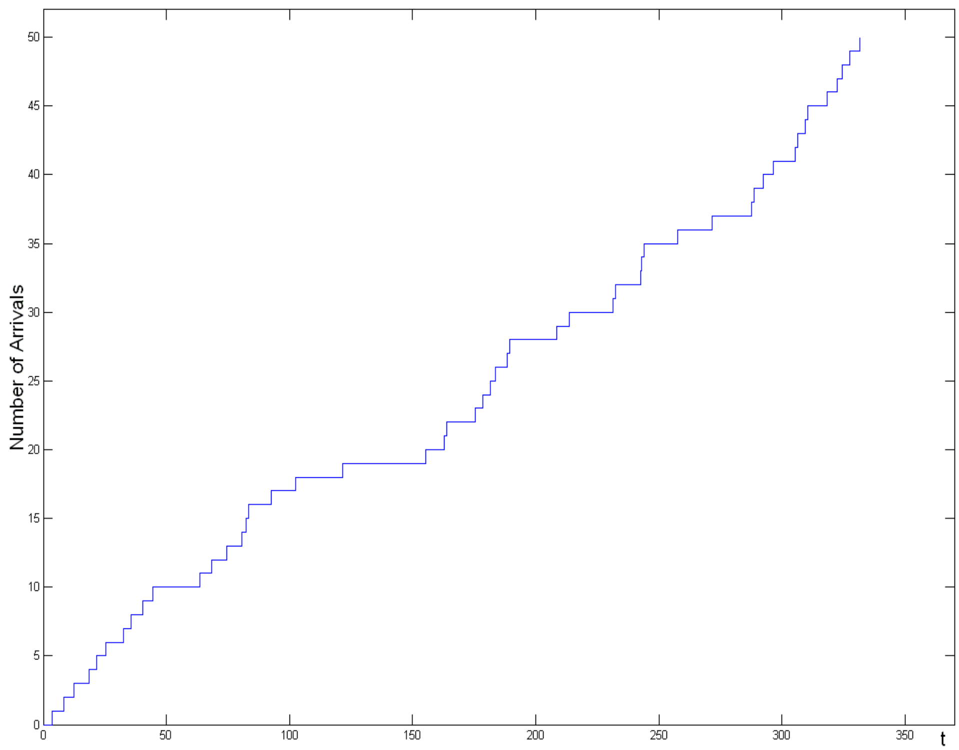

Figure 2: Graph illustrating the Cox process with shot noise intensity

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}} \mid \lambda_0\right\} = \mathbb{E}\left[\exp\left\{-\left\{1-\hat{m}(\nu)\right\}\Lambda_t\right\} \mid \lambda_0\right], \tag{2.11}$$

where λ_0 is assumed to be known. The equation (2.11) suggests that the problem of finding the Laplace transform of distribution of $L_t^{(k)}$, is equivalent to the problem of finding the Laplace transform of distribution of $\Lambda_t = \int_0^t \lambda_s ds$, the aggregated process.

Assuming that jump size of primary event follows an exponential distribution, i.e. $g(y) = b\exp(-by)$, $y > 0$, $b > 0$ and λ_t is stationary, the explicit expression of (2.11) is given by

$$\mathbb{E}\left\{e^{-\nu L_t^{(k)}}\right\} = \left(\frac{\delta b + \left\{1-\hat{m}(\nu)\right\}\left(1-e^{-\delta t}\right)}{\delta b e^{-\delta t}}\right)^{\frac{b\rho}{\delta b+\left\{1-\hat{m}(\nu)\right\}}-\frac{\rho}{\delta}}. \tag{2.12}$$

For details of the above expression, we refer the reader to Dassios and Jang (2003). We omit the corresponding expressions for the Laplace transform of the distribution of total loss $L_t^{(k)}$ using (2.3)-(2.6) as they can be easily obtained.

If $\{Y_i\}_{i=1,2,\ldots}$, which are the magnitude of contribution of primary event to intensity λ_t, are high, we need to consider heavy-tailed distributions for jump size of primary event $G(y)$. It causes higher number of operational loss consequently and eventually the financial institutions need to prepare higher operational risk capital charge as the risk measures of VaR and TCE become higher. This primary event jump size measure $G(y)$ also can be related with loss size measure $H(x)$ if there exists dependence between them, e.g. the higher the magnitude of contribution of primary event is, the higher losses from the operational risk arise. Compared to (2.1), the above Laplace transform provides the financial institutions with more flexibility in operational risk modelling as it contains stochastic intensity with three parameters of δ, ρ and $G(y)$.

3. Total loss distribution via the Fast Fourier transform

In order to calculate the risk measures of (1.3) and (1.6), we invert the Fast Fourier transforms from the Laplace transforms of $L_t^{(k)}$ obtained in Section 2. For details on how to invert the Fast Fourier transform, we refer you Heston (1993), Duffie et al. (2000), Castleman (1996), Gonzalez and Woods (2002) and Gonzalez et al. (2004). Before we show the calculations of risk measures in Section 4, we present the expressions for initial probabilities of total loss and the expressions for initial value of its densities. These are required to improve the accuracy of the distributions of the total loss inverting the Fast Fourier transforms.

If we let $\nu \to \infty$ in (2.1), we have the expression for initial probability of total loss, i.e.

$$\mathbb{P}\left(L_t^{(k)} = 0\right) = e^{-\lambda t}. \tag{3.1}$$

Regardless of loss size distributions, we have the same initial probability of total loss when the loss arrival process N_t follows a homogeneous Poisson process with loss frequency λ. If we set

$$\lim_{\nu\to\infty} \nu \exp\left[-\lambda t\left\{1-\hat{m}(\nu)\right\}\right],$$

we have the expression for initial value of the density of total loss, i.e.

$$f\left(L_t^{(k)} = 0\right) = \lambda t e^{-\lambda t} h(0), \tag{3.2}$$

where $f\left(L_t^{(k)}\right)$ is the density function of total loss.

Based on (3.2), we can easily obtain the expressions for initial probabilities of total loss, i.e. for an exponential loss size,

$$f\left(L_t^{(k)}=0\right)=\frac{\lambda t e^{-\lambda t}}{\sigma_1}, \tag{3.3}$$

for a Loggamma loss size

$$f\left(L_t^{(k)}=0\right)=\left\{\begin{array}{c} 0, \quad \alpha>1 \\ \frac{\lambda t\beta e^{-\lambda t}}{\sigma_2}, \ \alpha=1 \\ \infty, \quad \alpha<1 \end{array}\right\}, \tag{3.4}$$

for a Fréchet loss size

$$f\left(L_t^{(k)}=0\right)=0, \tag{3.5}$$

and for a truncated Gumbel loss size

$$f\left(L_t^{(k)}=0\right)=\frac{\lambda t c}{\left(e^c-1\right)e^{\lambda t}\sigma_4}. \tag{3.6}$$

Similarly, if we let $\nu\to\infty$ in (2.11), we have the expression for initial probability of total loss, i.e.

$$\mathbb{P}\left(L_t^{(k)}=0\right)=\left(\frac{\delta b e^{-\delta t}}{1-e^{-\delta t}+\delta b}\right)^{\frac{\rho}{\delta(1+\delta b)}}, \tag{3.7}$$

Regardless of loss size distributions, we also have the same initial probability of total loss when the loss arrival process N_t follows the Cox process with shot noise intensity λ_t. If we set

$$\lim_{\nu\to\infty}\nu\left(\frac{\delta b+\left\{1-\hat{m}(\nu)\right\}\left(1-e^{-\delta t}\right)}{\delta b e^{-\delta t}}\right)^{\frac{b\rho}{\delta b+\left\{1-\hat{m}(\nu)\right\}}-\frac{\rho}{\delta}},$$

we have the expression for initial value of the density of total loss, i.e.

$$\begin{aligned} f\left(L_t^{(k)}=0\right) &= \frac{h(0)\ \rho}{1+\delta b}\left(\frac{\delta b e^{-\delta t}}{1-e^{-\delta t}+\delta b}\right)^{\frac{\rho}{\delta(1+\delta b)}} \\ &\quad\times\left\{\left(\frac{b}{1+\delta b}\right)\ln\left(\frac{1-e^{-\delta t}+\delta b}{\delta b e^{-\delta t}}\right)+\frac{1-e^{-\delta t}}{\delta\left(1-e^{-\delta t}+\delta b\right)}\right\}. \end{aligned} \tag{3.8}$$

Based on (3.8), we can easily obtain the expressions for initial probabilities of total loss, i.e. for an exponential loss size,

$$\begin{aligned} f\left(L_t^{(k)}=0\right) &= \frac{\rho}{\sigma_1\left(1+\delta b\right)}\left(\frac{\delta b e^{-\delta t}}{1-e^{-\delta t}+\delta b}\right)^{\frac{\rho}{\delta(1+\delta b)}} \\ &\quad\times\left\{\left(\frac{b}{1+\delta b}\right)\ln\left(\frac{1-e^{-\delta t}+\delta b}{\delta b e^{-\delta t}}\right)+\frac{1-e^{-\delta t}}{\delta\left(1-e^{-\delta t}+\delta b\right)}\right\}, \end{aligned} \tag{3.9}$$

for a Loggamma loss size

$$f\left(L_t^{(k)}=0\right)=\left\{\begin{array}{cc} 0, & \alpha>1 \\ \frac{\beta\rho}{\sigma_2(1+\delta b)}\left(\frac{\delta b e^{-\delta t}}{1-e^{-\delta t}+\delta b}\right)^{\frac{\rho}{\delta(1+\delta b)}}\left\{\left(\frac{b}{1+\delta b}\right)\ln\left(\frac{1-e^{-\delta t}+\delta b}{\delta b e^{-\delta t}}\right)+\frac{1-e^{-\delta t}}{\delta\left(1-e^{-\delta t}+\delta b\right)}\right\}, & \alpha=1 \\ \infty, & \alpha<1 \end{array}\right\}, \tag{3.10}$$

for a Fréchet loss size

$$f\left(L_t^{(k)}=0\right)=0, \tag{3.11}$$

and for a truncated Gumbel loss size

$$\begin{aligned} f\left(L_t^{(k)}=0\right) &= \frac{c\rho}{\sigma_4\left(e^c-1\right)(1+\delta b)}\left(\frac{\delta b e^{-\delta t}}{1-e^{-\delta t}+\delta b}\right)^{\frac{\rho}{\delta(1+\delta b)}} \\ &\quad\times\left\{\left(\frac{b}{1+\delta b}\right)\ln\left(\frac{1-e^{-\delta t}+\delta b}{\delta b e^{-\delta t}}\right)+\frac{1-e^{-\delta t}}{\delta\left(1-e^{-\delta t}+\delta b\right)}\right\}. \end{aligned} \tag{3.12}$$

Figure 3-6 are the distributions of total loss with respect to a Poisson process and to a Cox process for N_t respectively, where loss size distributions are Exponential, Loggamma, Fréchet and truncated Gumbel. It shows that the distributions of total loss with respect to a Cox process have heavier tail than their counterparts with respect to a Poisson process. It will become apparent by numerical values of VaRs and TCEs in Example 4.1-4.4.

Since we derive the probability densities for total loss numerically via the Fast Fourier transform, all values of the probability densities in Figure 3-6 are approximated values except the first point, $f\left(L_t^{(k)}=0\right)$ and $\mathbb{P}\left(L_t^{(k)}=0\right)$. These two values are calculated using the explicit formulae above. The first point, $f\left(L_t^{(k)}=0\right)$ is usually distorted after the Fast Fourier transform so we replace these distorted values with the values obtained from the explicit formulae of $f\left(L_t^{(k)}=0\right)$.

4. **Calculating risk measures**

Now with two risk measures, i.e.

$$\mathrm{VaR}_q(L_t^{(k)})=\inf\left\{l\in\mathbb{R}:\ P(L_t^{(k)}>l)\leq 1-q\right\} \tag{4.1}$$

and

$$\mathrm{TCE}_q(L_t^{(k)})=\mathbb{E}\left\{L_t^{(k)}\mid L_t^{(k)}\geq \mathrm{VaR}_q(L_t^{(k)})\right\}=\frac{\mathbb{E}\left[L_t^{(k)}\ I\left\{L_t^{(k)}\geq \mathrm{VaR}_q(L_t^{(k)})\right\}\right]}{(1-q)} \tag{4.2}$$

where $I(\cdot)$ is the indicator function, let us illustrate their numerical values from the inversion of the Fast Fourier transforms.

The parameter values used to simulate N_t and calculate the above risk measures are

$$\lambda=10,\ \rho=4,\ b=1,\ \delta=0.4 \text{ and } t=1.$$

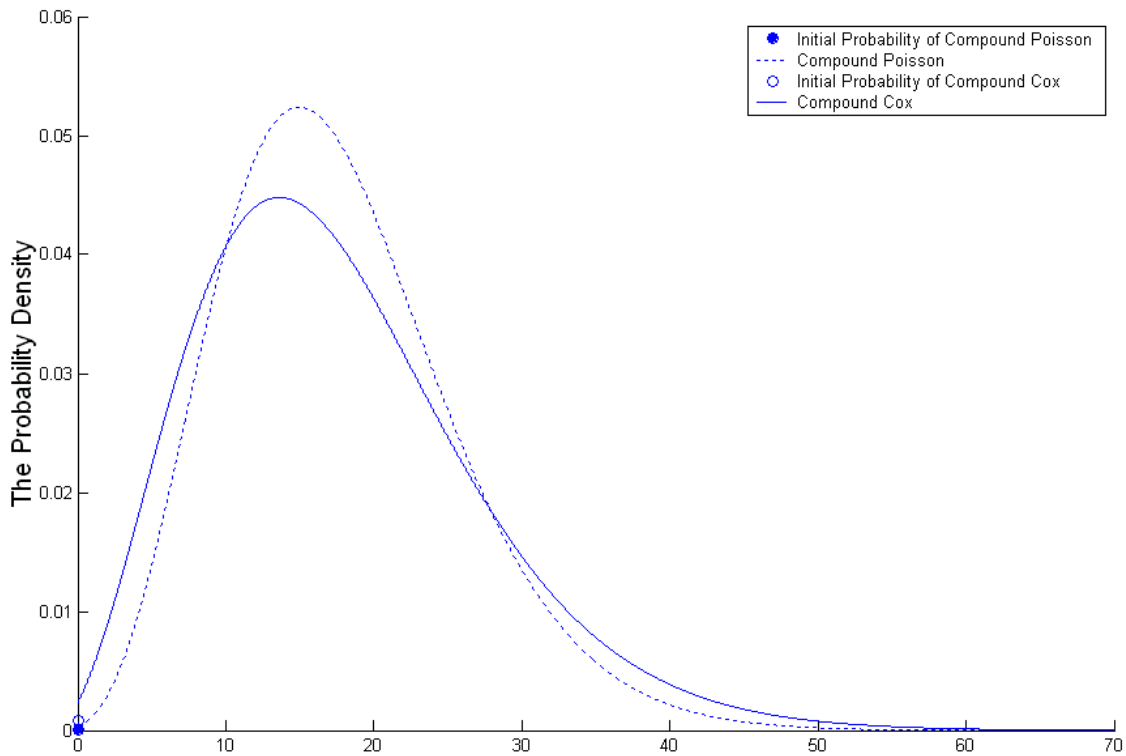

Figure 3: The distribution of total loss with respect to Poisson/Cox process with Exponential loss size distribution

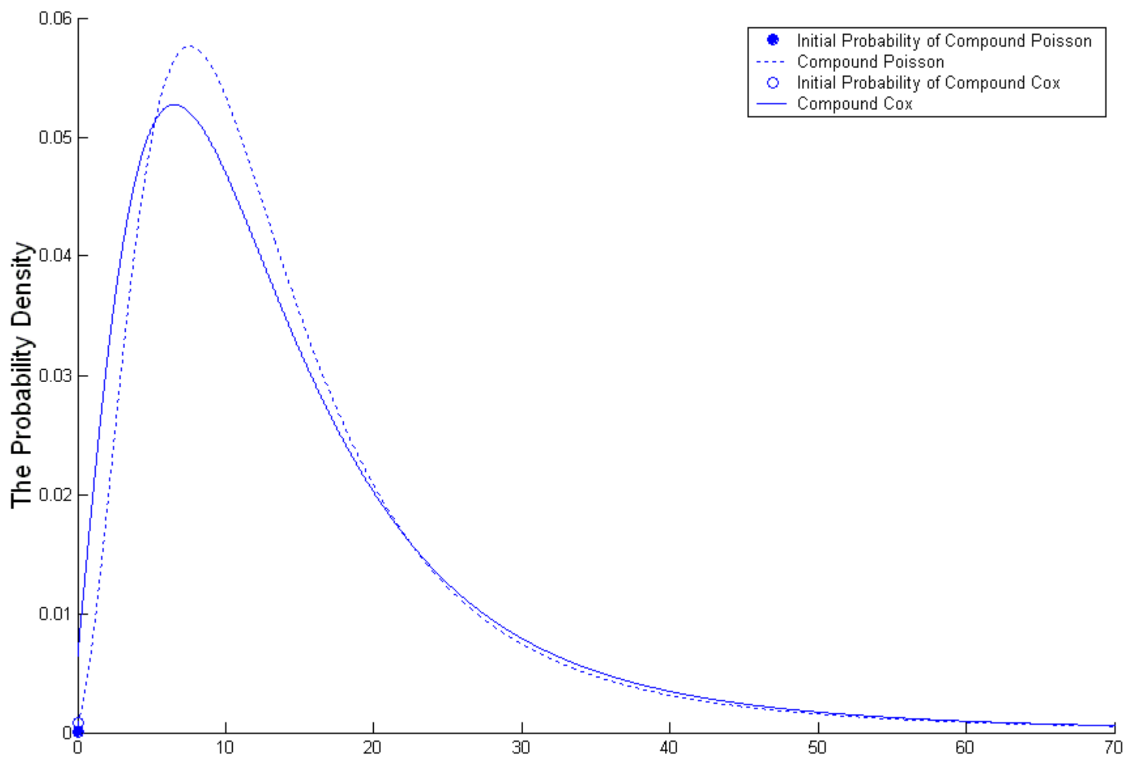

Figure 4: The distribution of total loss with respect to Poisson/Cox process with Loggamma loss size distribution

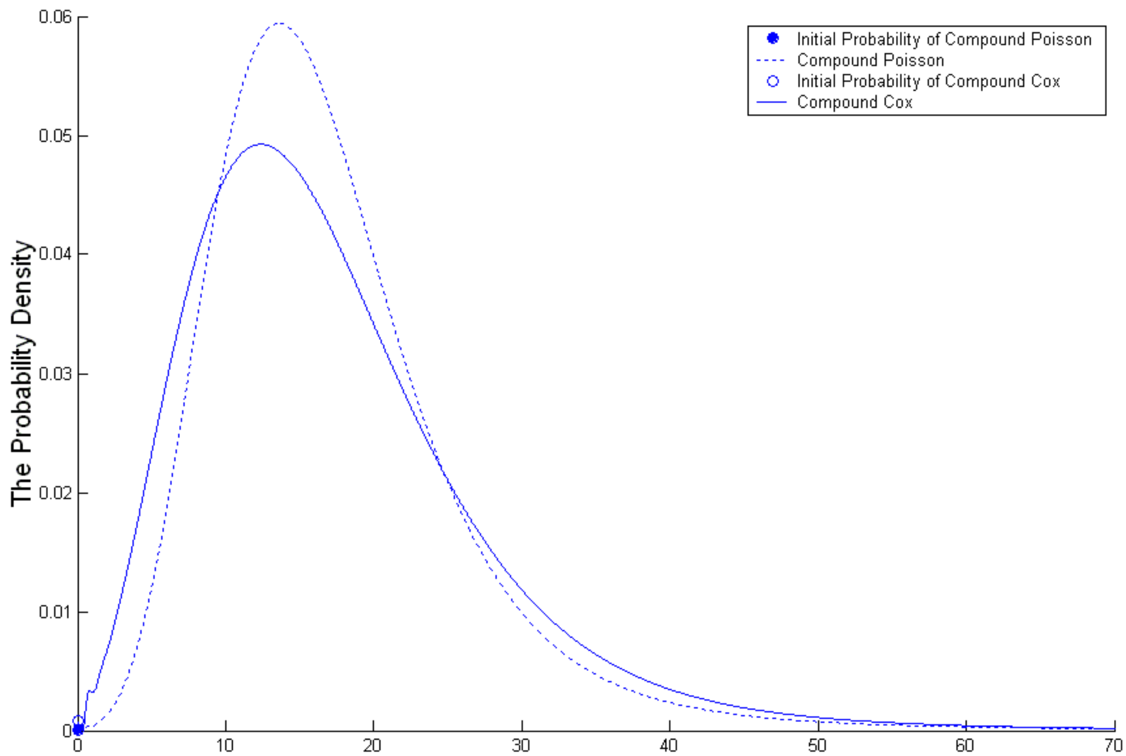

Figure 5: The distribution of total loss with respect to Poisson/Cox process with Fréchet loss size distribution

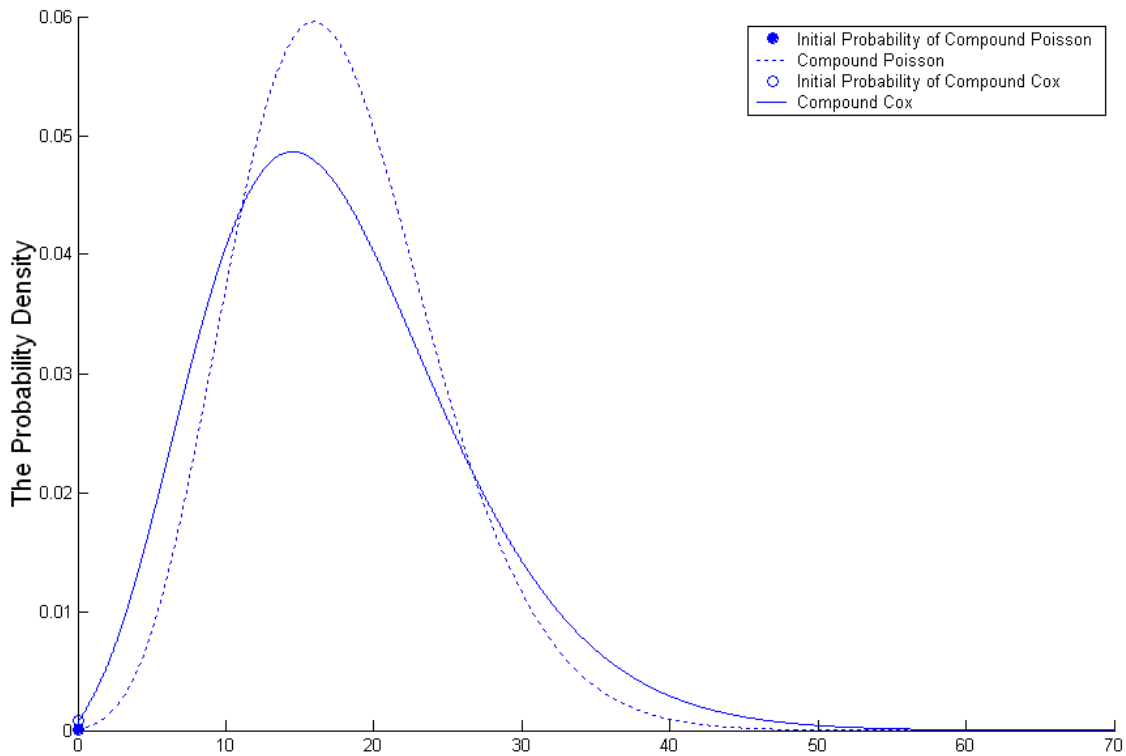

Figure 6: The distribution of total loss with respect to Poisson/Cox process with truncated Gumbel loss size distribution

We use the above parameter values that provide us with the same means of total loss regardless of the specification of the loss arrival process N_t to see the differences of the VaRs and TCEs due to the tails of the loss size distributions, i.e.

$$\mathbb{E}^{\text{Poisson}}\left\{L_t^{(k)}\right\} = \mathbb{E}^{\text{Cox}}\left\{L_t^{(k)}\right\}.$$

In order to make the computing easier, we also choose

$$\mathbb{E}^{\text{Exponential}}(X) = \mathbb{E}^{\text{Loggamma}}(X) = \mathbb{E}^{\text{Fréchet}}(X) = \mathbb{E}^{\text{truncated Gumbel}}(X) = \sqrt{\pi},$$

i.e.

$$\sigma_1 = \sigma_2\left\{\left(\frac{\beta}{\beta-1}\right)^{\alpha} - 1\right\} = \sigma_3\,\Gamma\left(1-\frac{1}{\varsigma}\right) = \sigma_4\left\{(\ln c) - \frac{1}{1-e^{-c}}\int_0^c (\ln y)\,e^{-y}dy\right\} = \sqrt{\pi} \quad (4.3)$$

and

$$\sigma_1 = \sqrt{\pi} \text{ and } \sigma_2 = \sigma_3 = \sigma_4 = 1.$$

From (4.3), we have the relationship for the parameters, i.e.

$$\begin{aligned} \left(\frac{\beta}{\beta-1}\right)^{\alpha} &= \sqrt{\pi}+1, \quad \beta > 1 \text{ and } \alpha \geq 1, \\ \Gamma\left(1-\frac{1}{\varsigma}\right) &= \sqrt{\pi}, \quad \varsigma > 1, \\ (\ln c) - \frac{1}{1-e^{-c}}\int_0^c (\ln y)\,e^{-y}dy &= \sqrt{\pi}. \end{aligned}$$

Using *Matlab*, the VaRs and TCEs for each loss size distribution with respect to a Poisson/a Cox process are shown in Table 4.1-4.8.

Example 4.1: Exponential

The calculations of the two risk measures as capital charges from type k operational risk up to time t when loss size follows an exponential are shown in Table 4.1 and 4.2, where $Var(X) = \pi$.

Table 4.1: Poisson process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	49.5368	53.3579
0.99	39.8691	44.1164
0.95	32.1210	36.8966
0.9	28.3286	33.4675
0.5	16.8305	23.9688
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = 20\pi$		

Table 4.2: Cox process

q	$\text{VaR}_q(L_t^{(k)})$	$\text{TCE}_q(L_t^{(k)})$
0.999	57.9834	63.0779
0.99	45.2875	50.8550
0.95	35.3129	41.4676
0.9	30.5114	37.0697
0.5	16.4034	25.1592
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = 90.45$		

Table 4.1 and 4.2 show that there is no significant increase in two risk measures respectively by changing N_t from a homogenous Poisson process to a shot-noise Cox process as loss size measure $H(x)$ is an exponential which is not a heavy-tailed distribution. It also shows that TCEs are slightly higher than VaRs regardless of the loss arrival process N_t.

Example 4.2: Loggamma

The calculations of the two risk measures as capital charges from type k operational risk up to time t when loss size follows a Loggamma are shown in Table 4.3 and 4.4, where $\alpha = 1$, $\beta = \frac{\sqrt{\pi}+1}{\sqrt{\pi}}$ and $Var\,(X) = \infty$.

Table 4.3: Poisson process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	337.2020	1017.4245
0.99	99.0571	246.1027
0.95	43.7821	97.4792
0.9	31.3680	67.0061
0.5	11.9483	28.2826
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = \infty$		

Table 4.4: Cox process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	378.9434	1018.9412
0.99	100.9566	247.8994
0.95	45.5266	99.3334
0.9	32.7244	68.7072
0.5	11.7146	28.9562
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = \infty$		

Table 4.3 and 4.4 show that there is significant increase in two risk measures respectively by changing N_t from a homogenous Poisson process to a shot-noise Cox process as loss size measure $H\,(x)$ is a Loggamma which is a heavy-tailed distribution. It also shows that TCEs are much higher than VaRs regardless of the loss arrival process N_t.

Example 4.3: Fréchet

The calculations of the two risk measures as capital charges from type k operational risk up to time t when loss size follows a Fréchet are shown in Table 4.5 and 4.6, where $\varsigma = 2$ and $Var\,(X) = \infty$.

Table 4.5: Poisson process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	118.9029	218.5851
0.99	51.6413	82.8136
0.95	33.4748	48.0499
0.9	28.0027	39.2087
0.5	15.8570	24.2760
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = \infty$		

Table 4.6: Cox process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	121.1252	220.5757
0.99	55.1842	85.8063
0.95	36.5327	51.3762
0.9	30.2885	42.1960
0.5	15.5337	25.4611
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = \infty$		

Similar to Loggamma case, we can see in Table 4.5 and 4.6 that two risk measures increase respectively by changing N_t from a homogenous Poisson process to a shot-noise Cox process. It also shows that TCEs are higher than VaRs regardless of the loss arrival process N_t.

Example 4.4: Truncated Gumbel

The calculations of the two risk measures as capital charges from type k operational risk up to time t when loss size follows a truncated Gumbel are shown in Table 4.7 and 4.8, where $c = 2.97957$ and $Var\,(X) = 1.51625$.

Table 4.7: Poisson process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	43.5250	46.4355
0.99	36.0213	39.3241
0.95	29.8658	33.6550
0.9	26.7942	30.9191
0.5	17.1443	23.1380
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = 46.58$		

Table 4.8: Cox process

q	$\mathrm{VaR}_q(L_t^{(k)})$	$\mathrm{TCE}_q(L_t^{(k)})$
0.999	53.0242	57.3668
0.99	42.1127	46.9002
0.95	33.4607	38.7967
0.9	29.2596	34.9745
0.5	16.6829	24.4921
where $\mathbb{E}\left\{L_t^{(k)}\right\} = 10\sqrt{\pi}$, $Var\left\{L_t^{(k)}\right\} = 74.19$		

Table 4.7 and 4.8 show that there is no significant increase in two risk measures respectively by changing N_t from a homogenous Poisson process to a shot-noise Cox process. It also shows that TCEs are slightly higher than VaRs regardless of the loss arrival process N_t. Interestingly, the values of two risk measures are lower than their counterparts calculated using exponential loss size distribution in Example 4.1 when $q \geq 0.9$. When $q = 0.5$, the VaRs/TCEs are slightly higher/lower than their counterparts calculated using exponential loss size distribution in Example 4.1.

5. Conclusion

We used a compound Cox process to model total losses arising from operational risk to accommodate stochastic nature of their frequency rates in practice. The shot noise process was used as an intensity of a Cox process as the number of losses arising from operational risk depends on the frequency and magnitude of primary events and time period needed to determine the effect of primary events. We also examined a compound Poisson process as it counterpart.

To deal with an issue raised by Moscadelli (2004) that the losses arisen from the mismanagement of operational risk are heavy-tailed in practice, we considered Loggamma, Fréchet and truncated Gumbel as loss size distributions. We also used an exponential distribution for the case of non-heavy-tail losses.

As it is difficult to calculate the distributions of total loss, we derived their Laplace transforms and inverted their Fast Fourier transforms numerically to calculate relevant risk measures, i.e. VaR and TCE. We presented the expressions for initial probabilities of the total loss and the expressions for initial value of its densities, which were used to improve the accuracy of the distributions of the total loss inverting the Fast Fourier transforms We also compared simulated numerical values of VaRs and TCEs obtained using compound Poisson and compound Cox model respectively.

We examined four different loss size distributions with two counting processes to treat the issues faced by the practitioners in bank and financial institutions. Risk measures considered to obtain the operational risk capital charge were VaRs and TCEs. We hope that what we presented in this paper provides the practitioners with feasible models to measure operational risk capital charge with flexibility using real data available. There are several approaches to model interdependence between operational loss processes, e.g. linear correlation or copula-based non-linear correlation. For simplicity, we assumed no dependence between operational risk types so we leave it as a further research.

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