A CLOSED-FORM OPTION VALUATION FORMULA IN MARKOV JUMP DIFFUSION MODELS

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To improve the empirical performance of the Black-Scholes model, alternative models have been proposed to address the leptokurtic feature of the asset return distribution, volatility smile, and the effects of the volatility clustering phenomenon. However, the analytical tractability of the option valuation remains problematic for most of the alternative models. In this paper, we propose a Markov jump diffusion model that can not only incorporate both the leptokurtic feature and volatility smile but also present the economic features of volatility clustering. To evaluate the price of derivatives, we apply Lucas's general equilibrium framework to provide closed-form formulas for option and futures prices. When the jump size follows a specific distribution, such as a lognormal distribution or a default probability, we devise explicit analytic formulas for the equilibrium prices. Through these formulas, we illustrate the effect of jumps on implied volatility and volatility surface via stochastic intensity as well as sensitivity analysis in stock option prices.

KEY WORDS: contingent claims, equilibrium analysis, European call option, Markov jump diffusion model, Markov modulated Poisson process, rational expectations, volatility clustering

1. INTRODUCTION

The characterization of the arbitrage-free dynamics of stocks and interest rates in the presence of both jumps and diffusion has been developed by many authors in the financial literature. Some examples include option pricing with Poisson-type jumps (cf. Merton, 1976; Naik and Lee, 1990, and Kou, 2002), the pricing of interest rate derivatives (cf. Duffie, Pan, and Singleton, 2000, and Jarrow and Madan, 1995, 1999), and the marked point process framework (cf. Björk, Kabanov, and Runggaldier, 1997, and Glasserman and Kou, 2003). Empirical evidence and estimation methods for jump diffusion models can be found in Chernov and Ghysels (2000), Pan (2002), and Eraker (2004), among others. A good summary of jump processes and option prices can be found in Chapter 11 of Shreve (2004). The motivation for including jumps along with diffusion models is also explained in the above articles and the references therein.

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In the general framework of the marked point process developed by Björk, Kabanov, and Runggaldier (1997), the arbitrage and the completeness theory were investigated, and the existence and uniqueness of a martingale measure was proved. Further developments for interest rates were made by Glasserman and Kou (2003), allowing randomness in jump sizes and dependence between jump sizes, jump times, and interest rates. They also proved option pricing formulas via the arbitrage theory in the setting of Poisson-type jump diffusion models, to which some clear option price formulas for stock market have been established by Merton (1976), Naik and Lee (1990), and Kou (2002). Although the Poisson-type jump diffusion model can reveal the empirical phenomena of both the leptokurtic feature and the volatility smile, it is unable to explain the volatility clustering observed in empirical studies, due to the independent increment assumption for both the diffusion and the jump. The primary goal of this paper is to bridge this gap. We propose a Markov jump diffusion model, specified in (2.5), for a stock log return distribution. It can not only capture the empirical features, including leptokurtic phenomena, volatility smile/surface and volatility clustering but also provide fully expressed option pricing formulas in the framework of Lucas's general equilibrium setting. When the jump size follows a specific distribution, such as a lognormal distribution or a default probability, we devise explicit analytic formulas of the equilibrium prices for European call option and futures.

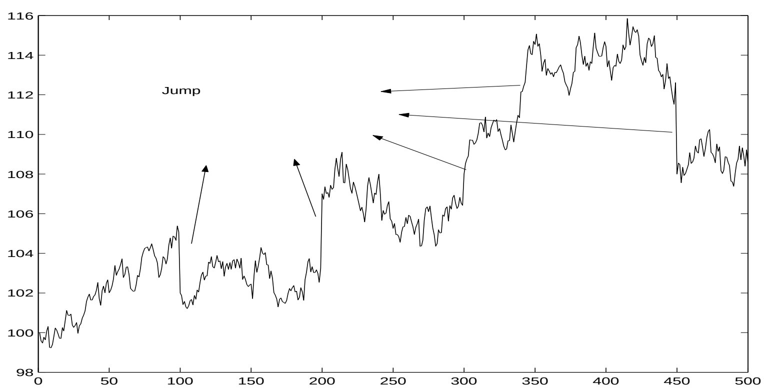

Figure 1: The dynamic process of the underlying asset price under a jump diffusion model

There are two other aspects to be studied in the Markov jump diffusion model beyond the motivation of capturing empirical phenomena and that of having closed-form option price formulas. First, the arrival rates of new information, good or bad news, are different from the "abnormal" vibrations of the asset price that are dependent on the current situation. In the jump diffusion model, as described in Merton (1976) and Kou (2002), the abnormal vibrations in price occur only due to the arrival of important information about the stock that has more than a marginal effect on price. The Markov jump diffusion model with two states, the so-called switched jump diffusion model, depends on the status of the economy, such as expansion or contraction. In Figures 1 and 2, we compare the dynamic processes of the asset price under a jump diffusion model and a switched jump diffusion model. In the jump diffusion model, the jump rate is averaged in the years as shown in Figure 1. On the other hand, in the switched jump diffusion model, the jump rates are different in different states as shown in Figure 2. The jump rates are large in one state and small in the other.

Second, the jump diffusion model as well as the Markov jump diffusion model can be used

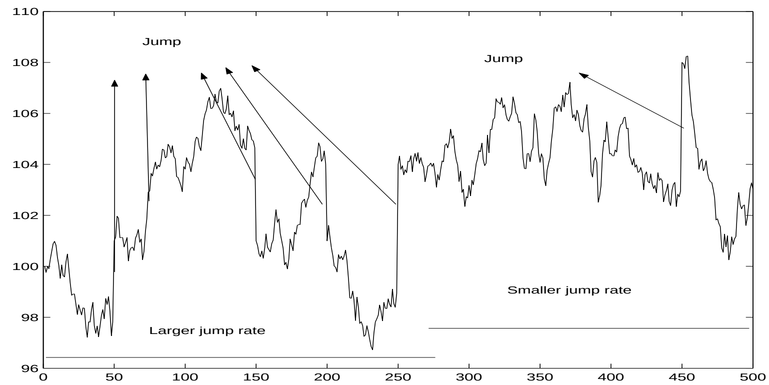

Figure 2: The dynamic process of the underlying asset price under a switched jump diffusion model

to describe defaultable risk in a financial market. This provides further motivation for including jumps. In the jump diffusion model, there is a positive probability (default probability) of immediate ruin, i.e., if the Poisson event occurs, the stock price falls to zero (cf. Samuelson, 1973; Merton, 1976, and Duffie and Singleton, 1999). In the switched jump diffusion model, we describe the risk with two different probabilities. There are high and low default probabilities of immediate ruin, and the jump rates are modeled by a two-state Markov modulated Poisson process.

The remainder of this paper is organized as follows. In Section 2, we introduce the structure of the model, and make the necessary assumptions in a general equilibrium framework. In Section 3, we explore the empirical phenomena of the switched jump diffusion model, which includes the leptokurtic and volatility clustering features. In Section 4, we first present a general equilibrium framework of Lucas (1978) in the Markov jump diffusion model; we then provide a formula for European call option price under the Markov jump diffusion model with a general jump size distribution. In particular, a closed-form solution is given in the case of a default risk and a lognormal jump size distribution. Further, using the closed-form option price formula, we study numerical analysis for implied volatility and volatility surface. A sensitivity analysis of the parameters to the option price is conducted as well. The conclusions are provided in Section 5. An essential point in this study is that we obtain the new transition probability for the new Markov jump diffusion model. All proofs are provided in the Appendices.

2. GENERAL FRAMEWORK OF THE MODEL

We consider the general equilibrium framework of Lucas (1978) in a frictionless market, where there is a representative consumer in a rational expectations economy that maximizes an objective function of the form

$$\max_c E\left[\int_0^\infty U(c(t),t)dt\right], \tag{2.1}$$

where E is the unconditional expectations operator and $U(c(t),t)$ is the utility function, which is continuously differentiable, strictly concave, and strictly increasing in consumption process $c(t)$.

Throughout this paper, for simplicity, we consider the power utility function. The assumptions are listed as follows.

Assumption 1. The power utility function. Let the utility function be

$$U(c,t) = \begin{cases} e^{-\theta t}\dfrac{c^a}{a} & \text{if } 0 < a < 1, \\ e^{-\theta t}\log c & \text{if } a = 0, \end{cases} \tag{2.2}$$

where θ is the positive discount rate and a is the risk aversion parameter.

We assume that there are predictable, locally bounded, and self-financing feasible trading strategies under the nonnegative wealth constraint at all times. For the agent, predictable trading strategies is an informational constraint to choose portfolios at any time t based only on the available information before t. Under locally bounded trading strategies, the cumulative mean and variance of the investor's portfolio remains finite in finite time so that the stochastic integral for wealth is well defined. In the case of self-financing trading strategies, the portfolio wealth at time t could be equal to the initial value of the portfolio plus trading gains, net of the value of consumption between 0 and t. For the nonnegative wealth constraint, we rule out the liability of the portfolio (borrowing without repayment) so that the agent's wealth is sufficiently high to cover it, as shown in Dybvig and Huang (1988), Naik and Lee (1990), and Kou (2002); this eliminates all arbitrage opportunities in an equilibrium price system.

There exists an exogenous endowment process denoted by $\delta(t)$ that is available to the investor. If $\delta(t)$ is Markovian, it can be shown (cf. Stokey and Lucas, 1989) that the rational expectations equilibrium price of the security $p(t)$ must satisfy the Euler equation

$$p(t) = \frac{E(U_c(\delta(T),T)p(T)|\mathcal{F}_t)}{U_c(\delta(t),t)}, \qquad \text{for all } T \in [t, T_0], \tag{2.3}$$

where U_c is the partial derivative of U with respect to c, and T_0 denotes a finite liquidation date of the security. Instead, in equilibrium, the investor finds it optimal to simply consume the exogenous endowment, $\delta(t)$, i.e., $c(t) = \delta(t)$ for all $t \geq 0$. Under Equation (2.2), or for more general utility functions, the rational expectations equilibrium price in Equation (2.3) becomes

$$p(t) = \frac{E(e^{-\theta T}(\delta(T))^{a-1}p(T)|\mathcal{F}_t)}{e^{-\theta t}(\delta(t))^{a-1}}. \tag{2.4}$$

Assumption 2. The stochastic differential equation of the endowment. Under the physical measure $\mathbb{P}$, the endowment follows a Markov jump diffusion model,

$$\frac{d\delta(t)}{\delta(t-)} = \mu_1(t)dt + \sigma_1 dW_1(t) + d\left(\sum_{n=1}^{\Phi(t)}(\widetilde{Y}_n - 1)\right), \tag{2.5}$$

where $\delta(t-)$ denotes the endowment at time $t-$, $\delta(t)$ denotes the endowment at time t, the drift $\mu_1(t)$ is the instantaneous return of $\delta(t)$ at time t, the volatility σ_1 of the stock price is assumed to be constant, $W_1(t)$ is assumed to be a one-dimensional standard Wiener process under the *physical measure $\mathbb{P}$, $\Phi(t)$ is a Markov modulated Poisson process with a finite state $\mathcal{X}$; further, $\widetilde{Y}_n$ is a sequence of jump sizes when the jump event occurs and is assumed to be independent for the sequence, where the endowment is from $\delta(t-)$ to $\widetilde{Y}_n\delta(t-)$.*

The resulting sample path for the endowment process will be continuous except at finite points in time, where jumps occur with the new information and the jump rate depends on the status of the economy. The drift term is a deterministic function of time in (2.5), and the volatility is assumed to be a constant. This setting extends the previous work by Naik and Lee (1990) and Kou (2002), in which case $\Phi(t) = N(t)$ is a Poisson process and $\mu_1(t) = \mu_1$ is a constant. We refer to a Markov jump diffusion model as a switched diffusion model when there are only two states in the underlying Markov chain. Under the general equilibrium framework, we consider the possibility that asset prices are influenced by an exogenous endowment through a series of correlated noise, which is characterized by the Brownian motion and jump size.

Assumption 3. The stochastic differential equation of the underlying asset price. The price of an underlying asset $S(t)$ also follows a Markov jump diffusion model, defined as

$$\frac{dS(t)}{S(t-)} = \mu(t)dt + \sigma dW(t) + d\left(\sum_{n=1}^{\Phi(t)}(Y_n - 1)\right) \tag{2.6}$$

$$= \mu(t)dt + \sigma\Big(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\Big) + d\left(\sum_{n=1}^{\Phi(t)}(\widetilde{Y}_n^b - 1)\right), \tag{2.7}$$

where $W_2(t)$ is a Brownian motion independent of $W_1(t)$, ρ is the constant correlation coefficient of the underlying asset and the endowment, the drift $\mu(t)$ is the instantaneous return of $S(t)$ at time t, σ denotes the constant volatility of the asset; further, $\{Y_n\}$ is a sequence of jump sizes when the jump event happens, which is connected through a power function $b \in (-\infty, \infty)$, where $Y_n = \widetilde{Y}_n^b$, and $\Phi(t)$ is the same Markov modulated Poisson process as the endowment process.

We observe that the same Markov modulated Poisson process $\Phi(t)$ affects both the endowment process and the asset price process, and the jump sizes are related through a power function, where power $b \in (-\infty, \infty)$ is an arbitrary constant, $Y_n = \widetilde{Y}_n^b$. The Markov jump diffusion model can be embedded in the rational expectations equilibrium requirement under a Markovian assumption of the Brownian motion and the jump size in Section 4.

The Markov modulated Poisson process $\Phi(t)$ forms a particular class of doubly stochastic Poisson processes where the underlying state is governed by a homogeneous Markov chain (cf. Last and Brandt, 1995). We particularly consider a series of nonnegative numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_I\}$, where λ_i denotes the intensity of the doubly stochastic Poisson process Φ if the underlying Markov chain $X(t)$ is at state i for time t. In this case, $\{X(t), \{P_i : i \in \mathcal{X}\}\}$ is a Markov jump process on the state space $\mathcal{X} = \{1, \cdots, I\}$, with transition rate $\Psi(i,j)$ defined as

$$\Psi(i,j) = \begin{cases} \alpha(i,j), & i \neq j, \\ -\sum_{j,j\neq i}\alpha(i,j), & \text{otherwise}, \end{cases} \tag{2.8}$$

for $i, j \in \mathcal{X}$. This Φ is called the Markov modulated Poisson process. In other words, the conditional distribution of a point process Φ is P-almost surely equal to the distribution of a

Poisson distribution with the intensity function $t \to \lambda_{X(t)}$. In other words,

$$P(\Phi(t) = n|X) = \frac{(\int_0^t \lambda_{X(s)} ds)^n}{n!} \exp[-\int_0^t \lambda_{X(s)} ds] \qquad P - a.s. \tag{2.9}$$

From the Markovian structure of $X(t)$, we can obtain the joint probability of $X(t)$ and $\Phi(t)$ via the Laplace inverse transform, which is given by $P_{ij}(n,t) := P_i(X(t) = j, \Phi(t) = n) := P(X(0) = i, X(t) = j, \Phi(t) = n)$ at time t with initial $X(0) = i$ for $n \in Z^+$. Define $\Psi := (\Psi(i,j))$ and $P(n,t) := (P_{ij}(n,t))$, and denote Λ as an $I \times I$ diagonal matrix with diagonal elements λ_i. For $0 \leq z \leq 1$, define

$$P^*(z,t) = \sum_{n=0}^{\infty} P(n,t) z^n \tag{2.10}$$

with $P(n,0) = (\mathbf{1}_{\{n=0\}} D_{ij})$, where $D_{ij} = 1$, if $i = j$ and 0, otherwise. Hence, $P^*(z,0) = (D_{ij})$. By using Kolmogorov's forward equation, the derivative of $P(n,t)$ becomes

$$\frac{d}{dt} P(n,t) = P(n,t)(\Psi - \Lambda) + \mathbf{1}_{\{n \geq 1\}} P(n-1,t)\Lambda.$$

Further, its unique solution is

$$P^*(z,t) = e^{[\Psi - (1-z)\Lambda]t}, \tag{2.11}$$

where

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

for any $(I \times I)$-matrix A and $A^0 := (D_{ij})$. Using the Laplace inverse transform (2.10) and the solution (2.11), we obtain the joint distribution of X and Φ at the time t as

$$P(n,t) = \frac{\partial^n}{n! \partial z^n} P^*(z,t)|_{z=0}. \tag{2.12}$$

To compute (2.11), we use the numerical inversion method proposed by Abate and Whitt (1992), which presents a version of the Fourier-series method for numerically inverting the probability generating function, and obtain a simple algorithm with a convenient error bound from the discrete Poisson summation formula.

Assumption 4. Jump size distribution. We define $\zeta_1^{(l)} := E(\widetilde{Y}^l - 1)$ *and assume that*

$$\zeta_1^{(a-1)} < \infty, \tag{2.13}$$

$$\zeta_1^{(a+b-1)} < \infty, \tag{2.14}$$

$$E(\prod_{n=0}^{\Phi(t)} \widetilde{Y}_n^{a-1}) = \sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{I} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t) < \infty, \quad \text{for all } t > 0, \tag{2.15}$$

and

$$E(\prod_{n=0}^{\Phi(t)} \widetilde{Y}_n^{a+b-1}) = \sum_{n=0}^{\infty} \sum_{i=1}^{I} \sum_{j=1}^{I} (\zeta_1^{(a+b-1)} + 1)^n \pi_i P_{ij}(n,t) < \infty, \quad \text{for all } t > 0, \tag{2.16}$$

where π_i denotes the stationary distribution at state i, and $a \in [0,1)$ and $b \in (-\infty,\infty)$ are defined in Equations (2.2) and (2.7), respectively.

The assumption of (2.13) and (2.14) implies that the means of the jump sizes are finite for the endowment and the asset prices under the first derivative of the power utility function. Equations (2.15) and (2.16) guarantee that the means of the jump sizes in the Markov modulated Poisson process are finite under the first derivative of the power utility function.

Two specific jump size distributions are considered in the following sections. First, we take the immediate ruin as the jump event occurs (cf. Samuelson, 1973, and Merton, 1976). In other words, if the Markov modulated Poisson event occurs, the stock price falls to zero. Thus,

$$\widetilde{Y}_n^b = \begin{cases} 0, & \text{if event occurs,} \\ 1, & \text{if event does not occur.} \end{cases} \tag{2.17}$$

Next, the random variable $\widetilde{Y}_n^b$ is assumed to have a lognormal distribution as the jump event occurs. Let σ_y^2 denote the variance of the logarithm of $\widetilde{Y}_1$, and μ_y the mean of the logarithm of $\widetilde{Y}_1$. Note that *Assumption 4* is satisfied in both cases.

Assumption 5. The discount rate θ of the utility function. The discount rate θ should be sufficiently large such that

$$\theta > -(1-a)\mu_1(t) + \frac{1}{2}\sigma_1^2(1-a)(2-a) + \tilde{\eta}(t), \qquad \text{for } t \in [0,T], \tag{2.18}$$

where $\tilde{\eta}(t) := d\log\{E[\prod_{n=1}^{\Phi(t)} \widetilde{Y}_n^{a-1}]\}/dt$.

This assumption guarantees that the term structure of the deterministic interest rate is positive; this will be discussed in detail in Section 4. Note that when $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$ and as $\tilde{\eta}(t) = d\log\{E[\prod_{n=1}^{\Phi(t)} \widetilde{Y}_n^{a-1}]\}/dt = \lambda\zeta_1^{(a-1)}$, *Assumption 5* reduces to $\theta > -(1-a)\mu_1(t) + \frac{1}{2}\sigma_1^2(1-a)(2-a) + \lambda\zeta_1^{(a-1)}$, and the parallel assumption appears in Kou (2002).

Assumption 6. The deterministic interest rate. Let $B(t,T)$ be the price of a zero-coupon bond with maturity date T. We assume that the interest rate

$$r(t) = \lim_{T \to t} \frac{-d\log(B(t,T))}{dT} \tag{2.19}$$

is a deterministic function of t. Therefore,

$$B(t,T) = e^{-\int_t^T r(s)ds}. \tag{2.20}$$

Note that in *Assumption 3*, the asset return contains the risk premium from the Brownian motion and the Markov jump risk, which is the function of t. In an equilibrium setting, we need to establish the relationship between the asset return and the interest rate; therefore, the interest rate is assumed to be a deterministic function of time in *Assumption 6*. The details are presented in Section 4.

3. EMPIRICAL PERFORMANCE

3.1 Leptokurtic features

Recall that $S(t)$ is defined in Equation (2.6). Solving this stochastic differential equation of the asset price yields the dynamics of the asset price as follows:

$$S(t) = S(0)\exp\left\{\int_0^t (\mu(s) - \frac{1}{2}\sigma^2)ds + \sigma W(t)\right\}\prod_{n=1}^{\Phi(t)} Y_n. \tag{3.1}$$

By using Equation (3.1), if the time interval Δt is small as in the case of daily observations, the return can be approximated by ignoring the terms with an order higher than Δt and by using the expansion $e^x \approx 1 + x + x^2/2$. The dynamic return of the asset for a small Δt is given by

$$\begin{aligned}\frac{\Delta S(t)}{S(t)} &\approx (\mu(t) - \tfrac{1}{2}\sigma^2)\Delta t + \sigma(W(t+\Delta t) - W(t)) + \sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n + \frac{1}{2}\sigma^2(W(t+\Delta t) - W(t))^2 \\ &\approx \mu(t)\Delta t + \sigma Z\sqrt{\Delta t} + \sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n,\end{aligned}$$

where Z is the standard normal random variable. The probability of the Markov modulated Poisson process $\Phi(t)$ with one jump from time t to $t + \Delta t$ is

$$\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P\left(X(t) = i, X(t+\Delta t) = j, \Phi(t+\Delta t) = 1\right) = \sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t).$$

In addition, the probability of having more than one jump is $o(\Delta t)$.

If the probability of one jump is smaller than $o(\Delta t)$, then we can ignore the multiple jumps and obtain

$$\sum_{n=\Phi(t)}^{\Phi(t+\Delta t)} \log Y_n \approx \begin{cases} \log Y_n, & \text{with probability } \displaystyle\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t), \\ 0, & \text{with probability } 1 - \displaystyle\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t). \end{cases}$$

The return, in Equation (3.1), can be approximately rewritten as

$$\frac{\Delta S(t)}{S(t)} \approx \mu(t)\Delta t + \sigma Z\sqrt{\Delta t} + HV, \tag{3.2}$$

where H is a Bernoulli random variable with $P(H = 1) = \sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)$, $P(H = 0) = 1 - \sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)$, and $V = \log Y$ is a normal random variable with mean $\mu_y = E(\log Y)$

and variance $\sigma_y^2 = \mathrm{var}(\log \mathrm{Y})$. Note that excluding the last term in Equation (2.6) reduces it to the classical model of the geometric Brownian motion, with the return $\Delta S(t)/S(t)$ being characterized approximately by a normal density.

The probability density of $\dfrac{\Delta S(t)}{S(t)}$ is given by

$$\begin{aligned} f(x) &= (1 - \sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t))\frac{1}{\sigma\sqrt{\Delta t}}\phi(\frac{x - \mu(t)\Delta t}{\sigma\sqrt{\Delta t}}) \\ &\quad +(\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t))\frac{1}{\sqrt{\sigma^2\Delta t + \sigma_y^2}}\phi(\frac{x - \mu(t)\Delta t - \mu_y}{\sqrt{\sigma^2\Delta t + \sigma_y^2}}), \end{aligned} \tag{3.3}$$

where $\phi(\cdot)$ is the standard normal density function. The mean of $\dfrac{\Delta S(t)}{S(t)}$ is

$$E(\frac{\Delta S(t)}{S(t)}) = \mu(t)\Delta t + \mu_y(\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)), \tag{3.4}$$

and the variance

$$\begin{aligned} \mathrm{var}(\frac{\Delta S(t)}{S(t)}) &= \sigma^2\Delta t + \sigma_y^2(\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)) \\ &\quad +\mu_y^2(\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t))(1 - (\sum_{i=1}^{I}\sum_{j=1}^{I} \pi_i P_{ij}(1, \Delta t)). \end{aligned} \tag{3.5}$$

An important feature of this asset pricing density is that as compared to the normal density with identical mean and variance, it has a higher peak around the mean and two heavier tails; in short, this asset pricing density has the leptokurtic feature. Moreover, the density is not symmetric if the mean jump size μ_y is not zero; in fact, it is skewed to the left if $\mu_y > 0$. These features have been favored by many empirical investigations.

We now consider the density of the discrete return in a switched jump diffusion model and compare the difference of this density with the normal density. In Equation (3.3), we consider the normal density with mean (3.4) and variance (3.5). Figure 3.1 compares the overall shapes of the two densities, Figure 3.2 details the shapes around the peak areas, and Figures 3.3 and 3.4 show the left and right tails. The dotted line is used for the normal density, and the solid line is used for the switched jump diffusion model with probability density function $f(x)$. The parameters used in this case are as follows; the state $I = 2$, $\Delta t = 1$ day $= 1/250$ year, $\sigma = 20\%$ per year, $\mu(t)\Delta t = 0.06\%$ per year, $\lambda_1 = 10$ per year, $\lambda_2 = 1$ per year, the transition rate $\alpha_1 = 0.9$, $\alpha_2 = 0.1$, the jump size $\mu_y = -2\%$, and the jump volatility $\sigma_y = 2\%$.

It appears that the jump parameters used are quite reasonable for a U.S. stock market. If the Markov chain remains at state 1, there are approximately 10 jumps per year with an average jump size of -2% and a jump volatility of 2%; if the Markov chain remains at state 2, there is approximately one jump per year with an average jump size of -2% and a jump volatility of

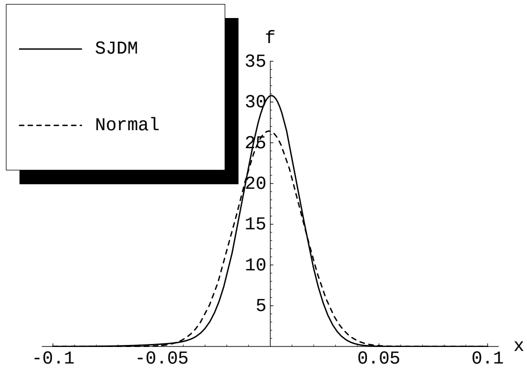

Figure 3.1 Overall comparison

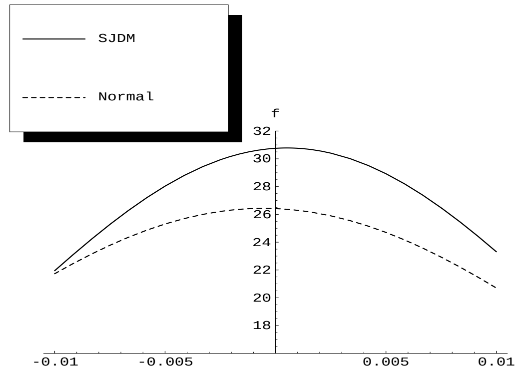

Figure 3.2 Peak comparison

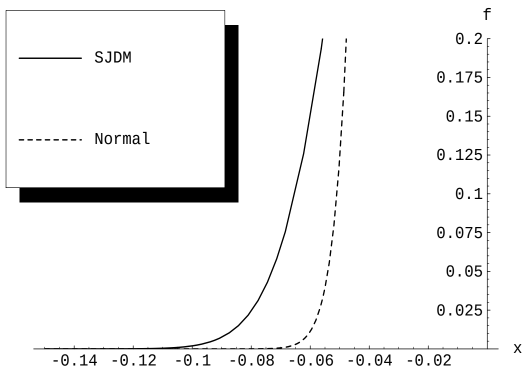

Figure 3.3 Left tail comparison

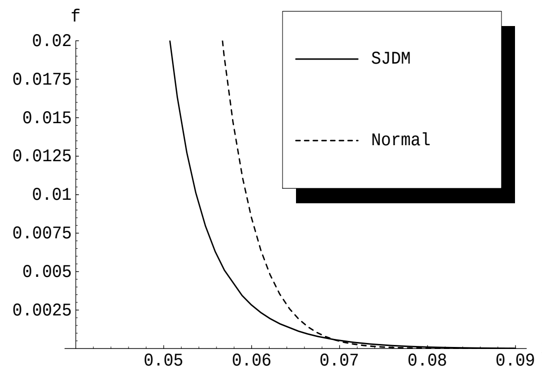

Figure 3.4 Right tail comparison

Figure 3: The dynamic process of the underlying asset price under a switched jump diffusion model

2%. The transition rate is $\alpha_1 = 0.9$ of leaving state i for $i = 1, 2$, and $\alpha_2 = 0.1$. The leptokurtic feature is highly evident under the switched jump diffusion model. The peak of the density $f(x)$ is approximately 30.9, whereas that of the normal density is approximately 29. The density $f(x)$ also has heavier tails than the normal density, particularly for the left tail, which could reach -10%, while the normal density is basically confined within -6%, as shown in Figures 3.1 to 3.4. Additional numerical plots suggest that the feature of the higher peak and heavier tails becomes more significant if either $|\mu_y|$ (the jump size), $\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}(1, \Delta t)$ (the transition probability), or σ_y (the jump volatility) increases.

3.2 Volatility clustering

A volatility clustering phenomena explored by Mandelbrot (1963) essentially implies that large values of volatility are usually followed by large values and that small values are followed by small ones. The GARCH models (Bollerslev, Chou, and Kroner, 1992; Engle, 1995) were

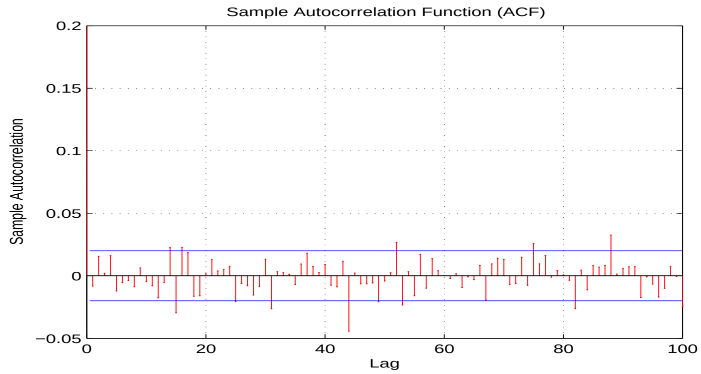

Figure 4.1 Autocorrelation of daily returns in the switched jump diffusion model

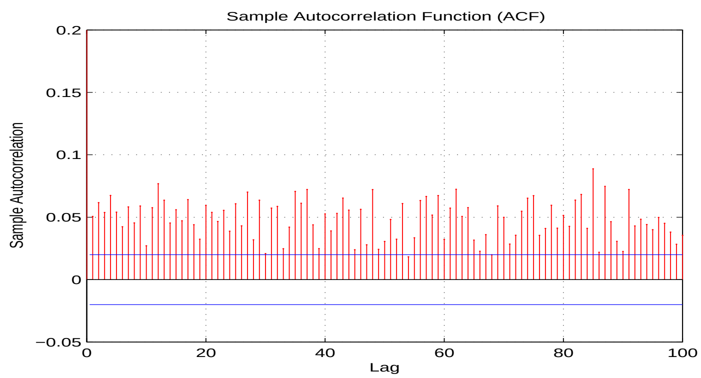

Figure 4.2 Autocorrelation of daily squared returns in the switched jump diffusion model

Figure 4: Clustering phenomenon under the switched jump diffusion model

among the first to produce the volatility clustering phenomenon. For a GARCH model,

$$\begin{cases} r_t = \sigma_t \varepsilon_t \\ \sigma_t^2 = a_0 + a_1 \sigma_{t-1}^2 + a_2 \varepsilon_t^2 \end{cases} \tag{3.6}$$

the squared volatility depends on the squared volatility in the last period and the squared residual in the current. The model is covariance stationary if $a1 + a_2\gamma_2 < 1$, given the finite second moment $E(\varepsilon_t^2) = \gamma_2$; then, the stationary volatility is

$$\sigma_t^2 = \frac{a_0}{1 - a_1 - a_2\gamma_2}. \tag{3.7}$$

Assuming $E(\varepsilon_t^4) = \gamma_4 < \infty$, the unconditional fourth moment of the return for the model exists if and only if $a_2^2\gamma_4 + 2a_1a_2\gamma_2 + a_1^2 < 1$. According to this assumption, the k-th autocorrelation of the squared return is

$$(a_2\gamma_2 + a_1)^k \frac{a_2\gamma_2(1 - a_1^2 - a_1a_2\gamma_2)}{(1 - a_1^2 - 2a_1a_2\gamma_2)},$$

leading to positive autocorrelation in the volatility process with a rate of decay governed by $a_2\gamma_2 + a_1$ given the assumptions $a_2^2\gamma_4 + 2a_1a_2\gamma_2 + a_1^2 < 1$ and $a_1 + a_2\gamma_2 < 1$. The closer $a_2\gamma_2 + a_1$ is to 1, the slower will be the decay of the autocorrelation of σ_t.

Cont (2005) investigates several economic mechanisms that have been proposed to explain the origin of this volaitlity clustering in terms of the behavior of market participants and the news arrival process. To apply the view of Cont (2005), a mechanism of heterogeneous arrival rates of information is proposed for the origin of volatility clustering in Markov jump diffusion models. Consider the case of a two-state Markov modulated Poisson process, a switched jump diffusion model, and rewrite Equation (2.6) in discrete time as follows:

$$R(t) = \begin{cases} \mu(t)\Delta t + \sigma \Delta t Z(t) + \sum_{n=1}^{N_1(\Delta t)} \log Y_n, & \text{if} \quad X(t) = 1, \\ \mu(t)\Delta t + \sigma \Delta t Z(t) + \sum_{n=1}^{N_2(\Delta t)} \log Y_n, & \text{if} \quad X(t) = 2, \end{cases} \tag{3.8}$$

where $Z(t) \sim N(0,1)$, $\log Y_n \sim N(\mu_y, \sigma_y^2)$, $N_1(\Delta t)$ is the Poisson process with jump rate $\lambda_1 \Delta t$ in the interval time Δt when the Markov chain $X(t)$ remains at state 1, and $N_2(\Delta t)$ is the Poisson process with jump rate $\lambda_2 \Delta t$ in the interval time Δt when the Markov chain $X(t)$ remains at state 2. Note that p_{11} (p_{22}) is the transition probability from state 1 (2) to state 1 (2) in the interval of time Δt. Δt may vary between a minute or seconds for tick data to several days. For example if $\Delta t = 1$ day, $R(t)$ denotes the return of one day. To simplify the notations, denote V_n, μ, σ, N_1, and N_2 as $\log Y_n$, $\mu(t)\Delta t$, $\sigma \Delta t$, $N_1(\Delta t)$ and $N_2(\Delta t)$, respectively. Let $\hat{\xi}(t) = (\mathbf{1}_{\{X(t)=1\}}, \mathbf{1}_{\{X(t)=2\}})'$, $\widehat{V}(t) = (\sum_{n=1}^{N_1} V_n, \sum_{n=1}^{N_2} V_n)'$, where $\mathbf{1}$ denotes the indicator function and $'$ denotes the transpose. In a matrix form, $R(t) = \mu + \sigma Z(t) + \widehat{\xi}'(t)\widehat{V}(t)$.

According to the matrix form, the autocorrelation function of the return is computed as

$$\rho_k = \frac{(p_{11} + p_{22} - 1)^k (\lambda_1 - \lambda_2)^2 \mu_y^2 \frac{(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})^2}}{\frac{(1-p_{11})}{(2-p_{11}-p_{22})}\lambda_1 \sigma_y^2 + \frac{(1-p_{22})}{(2-p_{11}-p_{22})}\lambda_2 \sigma_y^2 + \sigma^2}. \tag{3.9}$$

See Appendix A in details. When the jump rates are equal ($\lambda_1 = \lambda_2$), or $p_{11} + p_{22} = 1$, or the mean of the jump size is zero ($\mu_y = 0$), the autocorrelation function of the return is uncorrelated. Further, we let $\mu_y = 0$ and $\mu = 0$; then, the autocorreltation function of the squared return will be

$$\rho_k^* = \frac{(p_{11} + p_{22} - 1)^k (1 - p_{11})(1 - p_{22})(\lambda_1 - \lambda_2)^2 \sigma_y^4}{(2 - p_{11} - p_{22})^2 E}, \tag{3.10}$$

where

$$\begin{aligned} E &= 2\sigma^4 + 4\sigma^2\sigma_y^2\left(\frac{\lambda_1(1-p_{22})}{(2-p_{11}-p_{22})} + \frac{\lambda_2(1-p_{11})}{(2-p_{11}-p_{22})}\right) \\ &\quad + \sigma_y^4\left(\frac{\lambda_1(1-p_{22})}{(2-p_{11}-p_{22})} + \frac{\lambda_2(1-p_{11})}{(2-p_{11}-p_{22})} - \frac{2\lambda_1\lambda_2(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})^2}\right). \end{aligned}$$

The autocorrelation of the squared return is positive with a rate of decay governed by $p_{11}+p_{22}-1$. The closer $p_{11} + p_{22} - 1$ is to 1, the slower will be the decay of the autocorrelation of squared return. In other words, the feature of volatility clustering is a significant when $p_{11} + p_{22}$ is close to 2. Consider the equation of the dynamic return (3.8) with parameters $\lambda_1 = 0.02$, $\lambda_2 = 0.2$,

$\sigma = \frac{0.2}{\sqrt{250}}$, $p_{11} = 0.999$, $p_{22} = 0.999$, $\mu = 0$, $\mu_y = 0$, and $\sigma_y = 0.05$. The autocorrelation function of the squared return decays very slowly in Figure 4.2, while the autocorrelation function of the returns are almost zero in Figure 4.1.

4. OPTION PRICING: THEORY AND NUMERICAL ANALYSIS

4.1 General equilibrium for a Markov jump diffusion model

Suppose *Assumptions 1-6* hold. In this subsection, we study the relationship of the deterministic interest rate, the discount rate, and the return of endowment in Lucas's equilibrium setting. In addition, we obtain a risk neutral probability measure that depends on the utility function (2.2).

Proposition 1 (1) *The relationship of the deterministic interest rate and the endowment return in equilibrium is given by*

$$r(t) = \theta + (1-a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1-a)(2-a) - \tilde{\eta}(t) > 0 \quad \text{for all } t \in [0,T]. \tag{4.1}$$

(2) *Recall that* $\delta(t)$ *follows Equation* (2.5) *at time* t. *Let* $Z(t) := e^{\int_0^t (r(s)-\theta)ds} U_c(\delta(t),t) = e^{\int_0^t (r(s)-\theta)ds} (\delta(t))^{a-1}$. *Then,* $Z(t)$ *is a martingale under* $\mathbb{P}$ *at time* t, *and*

$$\frac{dZ(t)}{Z(t-)} = -\tilde{\eta}(t)dt + \sigma_1(a-1)dW_1(t) + d\left(\sum_{n=1}^{\Phi(t)} (\widetilde{Y}_n^{a-1} - 1)\right). \tag{4.2}$$

Using $Z(t)$, *one can define a new probability measure* $d\mathbb{P}^*/d\mathbb{P} := Z(t)/Z(0)$. *Under* $\mathbb{P}^*$, *the Euler equation defined in* (2.4) *holds if and only if the asset price satisfies*

$$S(t) = E^*(B(t,T)S(T)|\mathcal{F}_t), \qquad \text{for all } T \in [t, T_0]. \tag{4.3}$$

Furthermore, the rational expectations equilibrium price of an European option with payoff $\psi_s(T)$ *at maturity* T *is given by*

$$\psi_s(t) = E^*(B(t,T)\psi_s(T)|\mathcal{F}_t)), \qquad \text{for all } t \in [0,T]. \tag{4.4}$$

The proof of Proposition 1 is provided in Appendix B.

Remarks: 1. If $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, then $\tilde{\eta}(t) = \lambda\zeta_1^{(a-1)}$ is a constant. Let $Z(t) := e^{rt}U_c(\delta(t),t)$. Then, Equation (4.1) reduces to

$$r = \theta + (1-a)\mu_1 - \frac{1}{2}\sigma_1^2(1-a)(2-a) - \lambda\zeta_1^{(a-1)} > 0,$$

and Equation (4.2) reduces to

$$\frac{dZ(t)}{Z(t-)} = -\lambda\zeta_1^{(a-1)}dt + \sigma_1(a-1)dW_1(t) + d\left(\sum_{n=1}^{N(t)} (\widetilde{Y}_n^{a-1} - 1)\right).$$

This is the general equilibrium setting for jump diffusion models. cf. Kou (2002).

2. If $\alpha_k \to 0$ for some given $k \in \mathcal{X}$ and $\alpha_i \to \infty$, where $\{i : i \neq k, i \in \mathcal{X}\}$, then $\tilde{\eta}(t) = \lambda_k \zeta_1^{(a-1)}$, and the results in Remark 1 still hold.

Theorem 1 *Model* (2.6) *or* (2.7) *satisfies the equilibrium requirement* (2.4) *for the zero-coupon bond and the asset price if and only if*

$$\begin{aligned} \mu(t) &= r(t) + \sigma_1\sigma\rho(1-a) - \eta^*(t) \\ &= \theta + (1-a)(\mu_1(t) - \tfrac{1}{2}\sigma_1^2(2-a) + \sigma_1\sigma\rho) - \eta^*(t) - \tilde{\eta}(t), \end{aligned} \tag{4.5}$$

where $\eta^*(t) = d\log\left(E^*(\prod_{n=1}^{\Phi^*(t)} \widetilde{Y}_n^{*b})\right)/dt$*. If* (4.5) *is satisfied, then under* $\mathbb{P}^*$*, the equilibrium model of the asset price is*

$$\frac{dS(t)}{S(t-)} = r(t)dt - \eta^*(t)dt + \sigma dW^*(t) + d\left(\sum_{n=1}^{\Phi^*(t)} (\widetilde{Y}_n^{*b} - 1)\right). \tag{4.6}$$

Here, under $\mathbb{P}^*$*,* $W^*(t)$ *is a new Brownian motion,* $\Phi^*(t)$ *is a new Markov modulated Poisson process with transition probability given by*

$$Q_{ij}(m,t) = \frac{(\zeta_1^{(a-1)}+1)^m P_{ij}(m,t)}{\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}(\zeta_1^{(a-1)}+1)^n \pi_i P_{ij}(n,t)}, \tag{4.7}$$

and $\{\widetilde{Y}_n^*, n \geq 0\}$ *are i.i.d. random variables with* $\mathbb{P}^*$*-probability density function*

$$f_{\widetilde{Y}^*}(y) = \frac{1}{(\zeta_1^{(a-1)}+1)} y^{a-1} f_{\widetilde{Y}}(y).$$

The proof of Theorem 1 is provided in Appendix B.

Remarks: 3. Owing to the Markovian structure of the diffusion and $\Phi(t)$, which is also an essential aspect of Lucas's equilibrium setting, $\Phi^*(t)$ is still a Markov modulated Poisson process with transition probability $Q_{ij}(m,t)$. Therefore, it is an equilibrium model. Note that $P_{ij}(m,t)$ remains the same in the new transition probability $Q_{ij}(m,t)$ of (4.7), and the change is only affected by the moments of the jump size. This is also coherent with a generalization of the Girsanov theorem, as in Björk, Kabanov, and Runggaldier (1997), that a changing measure corresponds to a change of drift for the underlying Brownian motion and a change of the stochastic intensity for the Markov modulated Poisson process.

4. If $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, then $\eta^*(t) = \lambda(\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)})$ is a constant. Hence, under $\mathbb{P}^*$, Equation (4.6) reduces to

$$\frac{dS(t)}{S(t-)} = r(t)dt - \lambda(\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)})dt + \sigma dW^*(t) + d\left(\sum_{n=1}^{N^*(t)} (\widetilde{Y}_n^{*b} - 1)\right), \tag{4.8}$$

where $N^*(t)$ is the new Poisson process. This is the general equilibrium setting for jump diffusion models. cf. Kou (2002).

5. If $\alpha_k \to 0$ for some $k \in \mathcal{X}$ and $\alpha_i \to \infty$, where $\{i : i \neq k, i \in \mathcal{X}\}$, then the new Markov modulated Poisson process reduces to the new Poisson process with jump rate $\lambda_k(\zeta_1^{(a-1)} + 1)$. Hence, $\eta^*(t) = \lambda_k(\zeta_1^{(a+b-1)} - \zeta_1^{(a-1)})$, and the results in Remark 4 still hold.

Corollary 1 *Suppose family $\mathcal{Y}$ of the distributions of the jump size $\widetilde{Y}$ for the endowment process $\delta(t)$ satisfies*

$$\widetilde{Y}^b \in \mathcal{Y} \quad \text{and} \quad \frac{1}{\zeta_1^{(a-1)} + 1} \cdot y^{a-1} f_{\widetilde{Y}}(y) \in \mathcal{Y}, \tag{4.9}$$

then the jump sizes for the asset price $S(t)$ under $\mathbb{P}$, and the jump sizes for $S(t)$ under the rational expectations risk neutral probability $\mathbb{P}^*$ all belong to the same family $\mathcal{Y}$.

4.2 Option pricing formulas

In this subsection, we will derive the European call option price formula as well as a formula for a European call option on a futures contract using the results in Section 4.1. The European put option price formula can be obtained through put-call parity.

Denote $C(\cdot,\cdot,\cdot,\cdot,\cdot)$ as the Black-Scholes option price formula that includes five parameters, namely, the asset price $S(0)$ at time 0, strike price K, maturity T, interest rate $\frac{1}{T}\int_0^T r(t)dt$, and volatility σ. Further, denote $B(0,T)$ as the bond price in *Assumption* 6, $S(T)$ is the asset price at time T. Then

$$C(S(0), K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma) = S(0)\mathbb{N}(d(+)) - Ke^{-\int_0^T r(t)dt}\mathbb{N}(d(-)),$$

where $d(\pm) = \dfrac{\ln(\frac{S(0)}{KB(0,T)}) \pm 1/2\sigma^2 T}{\sigma\sqrt{T}}$. Let T^* be the delivery date, and denote the futures price $F(t, T^*)$ as

$$F(t,T^*) \;=\; e^{\int_t^{T^*} r(s)ds} S(t) = \frac{S(t)}{B(t,T^*)}, \tag{4.10}$$

and define $L(T) = e^{-\int_0^T \eta^*(t)dt}$.

Theorem 2

(1) *From Equation* (4.4)*, the European call option is given by*

$$MJ^c(0) \;=\; \sum_{m=0}^{\infty}\Bigg(E^*(C(S(0)L(T)\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|\Phi^*(T) = m) \sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg), \tag{4.11}$$

where $\widetilde{V}_m^b = \prod_{n=1}^{m} \widetilde{Y}_n^{*b}$.

(2) *The European call option on a futures contract is given by*

$$MJ_F^c(0) \;=\; \sum_{m=0}^{\infty}\Big(E^*(C(F(0,T^*)L(T)\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|\Phi^*(T)=m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Big). \tag{4.12}$$

The proof of Theorem 2 is provided in Appendix C.

We present three degenerated cases of Equations (4.11) and (4.12) as follows.

Corollary 2

(1) *If* $b \to 0$ *or* $\widetilde{Y}^b \to 1$ *with probability* 1, *then the pricing formulas* (4.11) *and* (4.12) *reduce to the corresponding Black-Scholes formulas*

$$MJ^c(0) \;\to\; C(S(0), K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma), \tag{4.13}$$

$$MJ_F^c(0) \;\to\; C(F(0,T^*), K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma). \tag{4.14}$$

(2) *When* $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, *then the pricing formulas* (4.11) *and* (4.12) *reduce to the Merton's formulas* (cf. Merton, 1976) *with jump rate* $\lambda(\zeta_1^{(a-1)}+1)$, *as given below:*

$$J^c(0) \;=\; \sum_{m=0}^{\infty}\Big(E^*(C(S(0)e^{-\lambda(\zeta_1^{(a+b-1)}-\zeta_1^{(a-1)})T}\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|N^*(T)=m) \frac{e^{-\lambda(\zeta_1^{(a-1)}+1)T}(\lambda(\zeta_1^{(a-1)}+1)T)^m}{m!}\Big), \tag{4.15}$$

$$J_F^c(0) \;=\; \sum_{m=0}^{\infty}\Big(E^*(C(F(0,T^*)e^{-\lambda(\zeta_1^{(a+b-1)}-\zeta_1^{(a-1)})T}\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|N^*(T)=m) \frac{e^{-\lambda(\zeta_1^{(a-1)}+1)T}(\lambda(\zeta_1^{(a-1)}+1)T)^m}{m!}\Big), \tag{4.16}$$

where $N^*(T)$ *is the new Poisson process with jump rate* $\lambda(\zeta_1^{(a-1)}+1)$.

If $\lambda = 0$, *then* (4.15) *and* (4.16) *reduce to the Black-Scholes formulas* (4.13) *and* (4.14).

(3) *If* $\alpha_k \to 0$ *for some* $k \in \mathcal{X}$, *and* $\alpha_i \to \infty$ *for* $i \in \mathcal{X}$ *and* $i \neq k$, *then the pricing formulas* (4.11) *and* (4.12) *reduce to* (4.15) *and* (4.16) *with intensity* λ_k.

When the jump size follows a default or a lognormal distribution, we provide explicit formulas for (4.11) and (4.12) in Corollaries 3 and 4, respectively.

Corollary 3 *Suppose that the jump size follows a positive default probability as Equation* (2.17)*.*

(1) *The price of the European call option is*

$$MJ_1^c(0) \;=\; C(S(0), K, T, \frac{1}{T}\int_0^T (r(t) - T\eta_1(t))dt, \sigma), \tag{4.17}$$

where $\eta_1(t) = d\log\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(0,t)\right)/dt.$

(2) *The price of the European call option on a futures contract is given by*

$$MJ_{F,1}^c(0) \;=\; C(F(0,T^*), K, T, \frac{1}{T}\int_0^T (r(t) - T\eta_1(t))dt, \sigma). \tag{4.18}$$

Note that when $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, the pricing formulas (4.17) and (4.18) reduce to the Merton's formulas with jump rate $\lambda(\zeta_1^{(a-1)} + 1)$, as given below:

$$J_1^c(0) \;=\; C(S(0), K, T, \frac{1}{T}\int_0^T r(t)dt + \lambda(\zeta_1^{(a-1)} + 1), \sigma), \tag{4.19}$$

$$J_{F,1}^c(0) \;=\; C(F(0,T^*), K, T, \frac{1}{T}\int_0^T r(t)dt + \lambda(\zeta_1^{(a-1)} + 1), \sigma). \tag{4.20}$$

In particular, if $\lambda = 0$, Equations (4.11) and (4.12) reduce to the Black-Scholes formulas (4.13) and (4.14), respectively. If $\alpha_k \to 0$ for $k \in \mathcal{X}$ and $\alpha_i \to \infty$ for $i \in \mathcal{X}$ and $i \neq k$, then the pricing formulas (4.17) and (4.18) reduce to (4.19) and (4.20), respectively, with jump rate $\lambda_k(\zeta_1^{(a-1)}+1)$.

We now consider two assets. One is the underlying asset that follows a diffusion model

$$\frac{dS_2(t)}{S_2(t)} \;=\; \mu_2(t)dt + \sigma dW(t), \tag{4.21}$$

and the other is that which follows a Markov jump diffusion model with the default probability as

$$\frac{dS(t)}{S(t-)} = \mu(t)dt + \sigma dW(t) + d\left(\sum_{n=1}^{\Phi(t)}(Y_n - 1)\right), \tag{4.22}$$

where $Y = \widetilde{Y}^b$ satisfies Equation (2.17). In equilibrium, the deterministic return is $\mu_2(t) = r(t) + \sigma_1\sigma\rho(1-a)$, and the European call option price formula is $C(S(0), T, K, \frac{1}{T}\int_0^T r(t)dt, \sigma)$ in the asset given by (4.21). In the asset expressed in (4.22), the deterministic return is $\mu(t) = r(t) + \sigma_1\sigma\rho(1-a) - \eta_1(t)$, where $\eta_1(t) < 0$. Therefore, we have $\mu(t) > \mu_2(t)$ for all

$t > 0$ in equilibrium, i.e., the second asset has the higher risk premium from the jump risk. Hence, in the Markov jump diffusion model, the European call option price formula Equation $C(S(0), T, K, \frac{1}{T}\int_0^T (r(t) - T\eta_1(t))dt, \sigma)$, with $\eta_1(t) < 0$, will be higher than that of the asset (4.21) with no jump. In particular, if $\lambda_1 = \lambda_2 = \cdots = \lambda$, the return of the asset (4.22) is $\mu(t) = r(t) + \sigma_1\sigma\rho(1-a) + \lambda(\zeta_1^{(a-1)} + 1) > \mu_2(t)$, and $C(S(0), T, K, \frac{1}{T}\int_0^T r(t)dt + \lambda(\zeta_1^{(a-1)} + 1), \sigma)$ $> C(S(0), T, K, \frac{1}{T}\int_0^T r(t)dt, \sigma)$. As shown in Merton (1973, 1976), the option price is an increasing function of the interest rate; therefore, the option on a stock with a positive default probability is more valuable than the one that has no default probability.

Corollary 4 *If the jump size follows a lognormal distribution with location parameter μ_y and scale parameter σ_y^2, then the following holds:*

(1) *The European call option price is given by*

$$MJ_2^c(0) = \sum_{m=0}^{\infty}\left(C(S(0), K, T, \frac{1}{T}\int_0^T r(m,t,T)dt, \sigma(m))\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}^*(m,T)\right), \qquad (4.23)$$

where the deterministic interest rate $r(m,t,T) = r(t) - T\eta_2(t) + m\gamma$ of the jump m times with the parameter $\gamma = \mu_y + \frac{1}{2}\sigma_y^2$, and the variance of the asset price $\sigma^2(m) = \sigma^2 + m\sigma_y^2/T$ with jump m times, $Q_{ij}^*(m,T)$ is the new transition probability of the jump m times from the state i at time 0 to the state j at time T, denoted as $Q_{ij}^*(m,T) = (\zeta+1)^m P_{ij}(m,T)/\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}(\zeta+1)^n\pi_i P_{ij}(n,T)$, and the predictable process is $\eta_2(t) = d\log\left(\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}(\zeta+1)^n\pi_i P_{ij}(n,t)\right)/dt$.

(2) *The European call option price on a futures contract is given by*

$$MJ_{F,2}^c(0) = \sum_{m=0}^{\infty}\left(C(F(0,T^*), K, T, \frac{1}{T}\int_0^T r(t,m,T)dt, \sigma(m))\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}^*(m,T)\right). \qquad (4.24)$$

Note that if $\zeta \to 0$ or $\mu_y \to 0$ and $\sigma_y^2 \to 0$, the pricing formulas (4.23) and (4.24) reduce to the Black-Scholes model expressed in (4.13) and (4.14), respectively.

4.3 Volatility smile and surface

The implied volatility should be constant, provided the Black-Scholes model is correct. However, volatility has a "smile" feature in many empirical phenomenon. In this subsection, we illustrate that the Markov jump diffusion model can produce a "volatility smile" in a real data set in the options market. The underlying asset is the IBM stock, priced at 62.66 on February 1; the maturity date is July 19, and the bond price is 0.9851 on February 1 according to the U.S. Treasury Bill. The exercise price and call value is obtained from the historical data. If we obtain the applicable parameter $\lambda_1 = 10$, $\lambda_2 = 5$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$, $\mu_y = -0.02$, and $\sigma_y = 0.02$ from the stock data, then we can show the "volatility smile" in Figure 5.1 using the option data

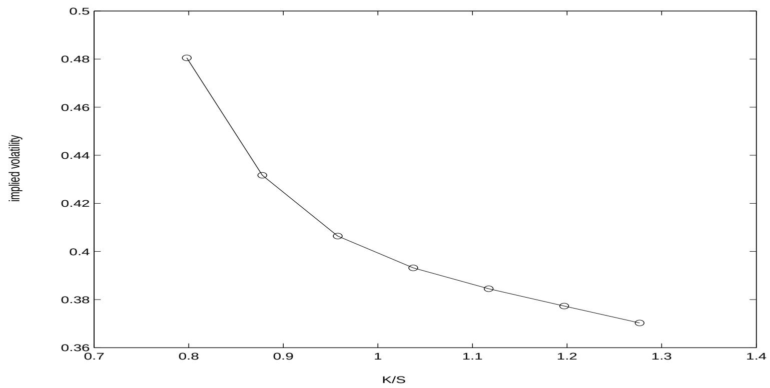

Figure 5.1 Implied volatility under the switched jump diffusion model

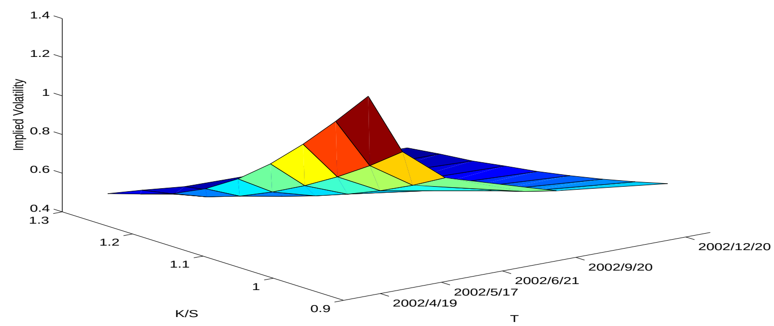

Figure 5.2 Volatility surface under the switched jump diffusion model

Figure 5: "Volatility smile" and "volatility surface" under the switched jump diffusion model

and the pricing formula in Corollary 4, under the switched jump diffusion model with lognormal distribution for the jump size.

Similarly, we can show the volatility surface against both maturity and strike in a three-dimensional plot. In other words, we can consider $\sigma(S,t)$ as a function of S and t. Figure 5.2 displays the price data of the IBM stock with five different maturity days. The call options were traded on March 15, 2002 with nine strikes of 100, 105, 110, 115, 120, 125, 130, 135, and 140. This implied surface represents the constant value of volatility that assigns each traded option a theoretical value equal to the market value. The time dependence in the implied volatility can be viewed as the time dependence of the volatility of the underlying asset. $\sigma(S,t)$ deduced from volatility surface at a specific time t^* can be considered as it the local volatility surface. This local volatility surface can be regarded as the market's view of the future value of volatility when the asset price is S at time t.

We should emphasize that the examples presented in Figures 5.1 and 5.2 are not empirical tests of the switched jump diffusion model; they are only illustrations to show that the model can produce a close fit to the empirical phenomenon.

4.4 Sensitivity analysis

The option price depends on several parameters in a Markov jump diffusion model shown in Equation (4.24) when the jump size follows a lognormal distribution. It would be helpful to understand the impact of each changing parameter on the option price. Table 1 reports the sensitivity of the parameters for the Markov jump diffusion model to the option prices. The parametric values listed in the table are the stock prices with small perturbation, a 10% increase, for the indicated parameter, with all other parameters fixed.

Table 1: Sensitivity analysis of the parameters to option prices

Base valuation 12.6040	Perturbed Parameter	NIM SJ^c	Difference
α_1	1.1	12.5998	-0.0042
α_2	1.1	12.6072	0.0032
λ_1	5.5	12.6220	0.0180
λ_2	1.1	12.6044	0.0004
$-\ln B(t,T)$	0.022	12.6721	0.0681
σ	0.22	13.0057	0.4017
μ_y	-0.022	12.6230	0.0190
σ_y	0.022	12.6168	0.0128
T	0.55	12.8726	0.2686

The parameters of the base valuation are $S(0) = 100$, $K = 90$, $T = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 5$, $\lambda_2 = 1$, $-\ln(B(0, 0.5)) = 0.02$, $\sigma = 0.2$, $\mu_y = -0.02$, and $\sigma_y = 0.02$ and are truncated by 15. SJ^c is the option price when the parameters increase by 10%, and "Difference" denotes the difference in the option price between the base valuation and SJ^c when the parameters increase by 10%.

In Table 1, the volatility σ has the most significant effect, although it is fixed and known in most time periods. If parameter α_1 increases by 10%, the Markov chain will leave state 1 rapidly such that the decrease in the jump rate reduces the option price, since the risk premium decreases in the jump risk. Similarly, if α_1 increases by 10%, the Markov chain will leave state 2 rapidly so that the increase in jump rate increases the option price. Regarding parameters λ_1 and λ_2, if one increases by 10% while the other one remains fixed, the increase in the jump rate increases the option price. If the yield $-\ln B(t,T) = 0.02$ increases to $-\ln B(t,T) = 0.022$, then the European option price increases. For the whole-time varying parameters such as λ_1, λ_2, α_1, α_2, $B(t,T)$, μ_y, and σ_y, we observe that parameter σ has the most effect among the others.

5. CONCLUSIONS

In this paper, we propose a Markov jump diffusion model, which can capture the leptokurtic and asymmetric features, volatility clustering, and the volatility smile. Under the general equilibrium setting of Lucas for the Markov jump diffusion model, closed-form formulas of the prices for a

European option and a futures contract have been developed. When the jump size of a switched jump diffusion model follows a lognormal distribution, we report numerical analysis in detail, and provide a computation method via the numerical inversion method to calculate the option prices.

For simplicity, in order to issue the formulas of (4.23) and (4.24) for the European call and futures with a lognormal jump size in a Markov jump diffusion model, we must estimate the parameters in the discrete time model, as indicated by Equation (3.8). We can apply the EM (expectation and maximization) algorithm to maximum likelihood methods to estimate these parameters. In the future, we will develop the EM algorithm in order to estimate and test the empirical performance for the Markov jump diffusion model in order that it will be convenient to value European options.

An exact closed-form formula provides useful insight into European option pricing in the Markov jump diffusion model. It not only explains the impact of regime switching in the jump rate on option pricing but also sheds light on analytical approximation, where it accelerates the computation of European option pricing in the Markov jump diffusion model. It has a number of further applications. For instance, it can be used to compute hedge ratios and implied Markov jump diffusion model parameters, i.e., to calibrate the parameters using the implied volatility surface. The approximation approach can facilitate empirical studies on index options, which are, in many cases, European in style. This feature is worthy of further exploration and may have many other applications.

APPENDIX A: Volatility Clustering in the Switched Jump Diffusion Model

Let $\lambda_1 \neq \lambda$, $\mu_y \neq 0$, and $p_{11} + p_{22} \neq 0$. The autocorrelation function of the return is

$$\begin{aligned}
\rho_k &= \frac{\operatorname{cov}(R(t), R(t-k))}{\sqrt{\operatorname{var}(R(t))}\sqrt{\operatorname{var}(R(t-k))}} \\
&= \frac{\operatorname{cov}(\mu + \sigma Z(t) + \hat{\xi}'(t)\widehat{V}(t), \mu + \sigma Z(t-k) + \hat{\xi}'(t-k)\widehat{V}(t-k))}{\sqrt{\operatorname{var}(R(t))}\sqrt{\operatorname{var}(R(t-k))}} \\
&= \frac{\operatorname{cov}(\hat{\xi}'(t)\widehat{V}(t), \hat{\xi}'(t-k)\widehat{V}(t-k))}{\sqrt{\operatorname{var}(R(t))}\sqrt{\operatorname{var}(R(t-k))}}. \qquad \text{(A.1)}
\end{aligned}$$

To evaluate (A.1), we first compute $\operatorname{var}(R(t)) = \operatorname{var}(\mu + \sigma Z(t) + \hat{\xi}'(t)\widehat{V}(t))$. Note that

$$\operatorname{var}(\mu + \sigma Z(t) + \hat{\xi}'(t)\widehat{V}(t)) = \sigma^2 + \operatorname{var}\left(\mathbf{1}_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + \mathbf{1}_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n\right)$$

$$= \sigma^2 + \frac{(1-p_{22})}{(2-p_{11}-p_{22})}\lambda_1\sigma_y^2 + \frac{(1-p_{11})}{(2-p_{11}-p_{22})}\lambda_2\sigma_y^2 + \frac{(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})^2}(\lambda_1-\lambda_2)^2\mu_y^2 \qquad \text{(A.2)}$$

Next, we compute $\text{cov}(\hat{\xi}'(t)\widehat{V}(t), \hat{\xi}'(t-k)\widehat{V}(t-k))$, which equals

$$
\begin{aligned}
&= E\Bigg((\mathbf{1}_{\{X(t-k)=1\}} \sum_{n=1}^{N_1} V_n + \mathbf{1}_{\{X(t-k)=2\}} \sum_{n=1}^{N_2} V_n))(\mathbf{1}_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + \mathbf{1}_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n \Bigg) \\
&\quad - E(\mathbf{1}_{\{X(t-j)=1\}} \sum_{n=1}^{N_1} V_n + \mathbf{1}_{\{X(t-k)=2\}} \sum_{n=1}^{N_2} V_n) E(\mathbf{1}_{\{X(t)=1\}} \sum_{n=1}^{N_1} V_n + \mathbf{1}_{\{X(t)=2\}} \sum_{n=1}^{N_2} V_n) \qquad (A.3)
\end{aligned}
$$

To compute (A.3), denote M_{ij}^k as the k-period-ahead transition probabilities from i to j, and let $\nu_i, \ i = 1, 2$, be the eigenvalues of the transition matrix

$$
\boldsymbol{P} = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}.
$$

A simple calculation yields that $\nu_1 = 1$, $\nu_2 = -1 + p_{11} + p_{22}$, and

$$
M_{ii}^k = \frac{(1 - p_{jj}) + \nu_2^k (1 - p_{ii})}{(2 - p_{11} - p_{22})} \text{ and } M_{ij}^k = \frac{(1 - p_{ii}) - \nu_2^k (1 - p_{ii})}{(2 - p_{11} - p_{22})}, \text{ for } i \neq j.
$$

By making use of a similar calculation as that in Hamilton (1994, p.683), we have

$$
(A.3) = \nu_2^k (\lambda_1 - \lambda_2)^2 \mu_y^2 \frac{(1 - p_{11})(1 - p_{22})}{(2 - p_{11} - p_{22})^2}. \tag{A.4}
$$

Putting equations (A.2) and (A.4) into equation (A.1), we obtain

$$
\begin{aligned}
\rho_k &= \frac{\text{cov}(\hat{\xi}'(t)\widehat{V}(t), \hat{\xi}'(t-k)\widehat{V}(t-k))}{\sqrt{\text{var}(R(t))}\sqrt{\text{var}(R(t-k))}} \\
&= \frac{(p_{11} + p_{22} - 1)^k (\lambda_1 - \lambda_2)^2 \mu_y^2 \dfrac{(1 - p_{11})(1 - p_{22})}{(2 - p_{11} - p_{22})^2}}{\sigma^2 + \dfrac{(1 - p_{22})}{(2 - p_{11} - p_{22})}\lambda_1 \sigma_y^2 + \dfrac{(1 - p_{11})}{(2 - p_{11} - p_{22})}\lambda_2 \sigma_y^2 + \dfrac{(1 - p_{11})(1 - p_{22})}{(2 - p_{11} - p_{22})^2}(\lambda_1 - \lambda_2)^2 \mu_y^2}.
\end{aligned}
$$

If the Δt is very small, the autocorrelation function of the return is almost uncorrelated. In particular, if we let $\mu_y = 0$; then the autocorrelation function of the return will be uncorrelated. Further, we let $\mu = 0$, then the autocorrelation of function of the squared return will be

$$
\begin{aligned}
\rho_k^* &= \frac{\text{cov}(R^2(t), R^2(t-k))}{\sqrt{\text{var}(R^2(t))}\sqrt{\text{var}(R^2(t-k))}} \\
&= \frac{\text{cov}((\mu + \sigma Z(t) + \hat{\xi}'(t)\widehat{V}(t)^2, (\mu + \sigma Z(t-k) + \hat{\xi}'(t-k)\widehat{V}(t-k))^2)}{\sqrt{\text{var}(R^2(t))}\sqrt{\text{var}(R^2(t-k))}} \\
&= \frac{\text{cov}((\hat{\xi}'(t)\widehat{V}(t))^2, (\hat{\xi}'(t-k)\widehat{V}(t-k))^2)}{\sqrt{\text{var}(R^2(t))}\sqrt{\text{var}(R^2(t-k))}} \\
&= \frac{(p_{11} + p_{22} - 1)^k (1 - p_{11})(1 - p_{22})(\lambda_1 - \lambda_2)^2 \sigma_y^4}{(2 - p_{11} - p_{22})^2 A}, \qquad (A.5)
\end{aligned}
$$

where

$$\begin{aligned}\mathrm{var}((R(t))^2) &= \mathrm{var}((\sigma Z + \hat{\xi}'(t)\widehat{V}(t))^2) \\ &= 2\sigma^4 + 4\sigma^2\sigma_y^2(\frac{\lambda_1(1-p_{22})}{(2-p_{11}-p_{22})} + \frac{\lambda_2(1-p_{11})}{(2-p_{11}-p_{22})}) \\ &\quad +\sigma_y^4(\frac{\lambda_1(1-p_{22})}{(2-p_{11}-p_{22})^2} + \frac{\lambda_2(1-p_{11})}{(2-p_{11}-p_{22})} - \frac{2\lambda_1\lambda_2(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})}) = A\end{aligned}$$

and

$$\begin{aligned}\mathrm{cov}(R^2(t), R^2(t-k)) &= \mathrm{cov}(\sigma Z(t) + \hat{\xi}'(t)\widehat{V}(t))^2, (\sigma Z(t-k) + \hat{\xi}'(t-k)\widehat{V}(t-k))^2) \\ &= \mathrm{cov}(\hat{\xi}'(t)\widehat{V}(t))^2, \hat{\xi}'(t-k)\widehat{V}(t-k))^2) \\ &= \frac{(p_{11}+p_{22}-1)^k(1-p_{11})(1-p_{22})}{(2-p_{11}-p_{22})^2}(\lambda_1-\lambda_2)^2\sigma_y^4\end{aligned}$$

APPENDIX B: General Equilibrium for Markov Jump Diffusion Models

Proof of Proposition 1.

(1) Since $B(T,T)=1$, Equation (2.4) yields

$$B(t,T) = e^{-\theta(T-t)}\frac{E((\delta(T))^{a-1})|\mathcal{F}_t)}{(\delta(t))^{a-1}}. \tag{B.1}$$

By making use of

$$(\frac{\delta(T)}{\delta(t)})^{a-1} = \exp\{(a-1)\int_t^T(\mu_1(s)-\frac{1}{2}\sigma_1^2)ds + \sigma_1(a-1)(W_1(T-t))\}\left(\prod_{n=1}^{\Phi(T-t)}\widetilde{Y}_n^{a-1}\right),$$

$$\begin{aligned}E(\left(\prod_{n=1}^{\Phi(T-t)}\widetilde{Y}_n^{a-1}\right)|\mathcal{F}_t) &= E(E(E(E(\left(\prod_{n=1}^{m}\widetilde{Y}_n^{a-1}\right)|X(0)=i, X(t)=j, \Phi(T-t)=m)))) \\ &= \sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n, T-t),\end{aligned}$$

we have

$$\begin{aligned}B(t,T) &= \exp\left\{-(T-t)\theta + \textstyle\int_t^T(a-1)(\mu_1(s)-\frac{1}{2}\sigma_1^2)ds + \frac{1}{2}\sigma_1^2(a-1)^2(T-t)\right\} \\ &\quad \times\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n, T-t)\right) \\ &= \exp\{-\textstyle\int_t^T r(s)ds\}.\end{aligned}$$

Taking the logarithm, differentiating T, and limiting $T \to t$, we get

$$\begin{aligned}
r(t) &= \theta + (1-a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1-a)(2-a) \\
&\quad - \lim_{T\to t} \frac{d\log\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n \pi_i P_{ij}(n,T-t)\right)}{dT} \\
&= \theta + (1-a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1-a)(2-a) - \lim_{T\to t}\frac{d\log\left(E(\prod_{n=1}^{\Phi(T-t)}\widetilde{Y}_n^{a-1})\right)}{dT} \\
&= \theta + (1-a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1-a)(2-a) - \lim_{T\to t}\frac{d\log\left(E(\prod_{n=1}^{\Phi(T-t)}\widetilde{Y}_n^{a-1})\right)}{dT-t}\frac{dT-t}{dT} \\
&= \theta + (1-a)\mu_1(t) - \frac{1}{2}\sigma_1^2(1-a)(2-a) - \tilde{\eta}(t) > 0,
\end{aligned}$$

where $\tilde{\eta}(t) = d\log\left(E(\prod_{n=1}^{\Phi(t)}\widetilde{Y}_n^{a-1})\right)/dt = d\log\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n,t)\right)/dt$ or $e^{-\int_0^t \tilde{\eta}(s)ds} = 1/\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n,t)\right)$.

(2) Note that Equation (B.1) implies that

$$e^{-\int_t^T r(s)ds} = E(U_c(\delta(T),T)/U_c(\delta(t),t)|\mathcal{F}_t). \tag{B.2}$$

By making use of *Assumption 2* and Equation (4.1), we have

$$\begin{aligned}
Z(t) &= (\delta(0))^{a-1}\exp\{\int_0^t (r(s)-\theta)ds + (a-1)\int_0^t(\mu_1(s)-1/2\sigma_1^2)ds + \sigma_1(a-1)W_1(t)\}\left(\prod_{n=1}^{\Phi(t)}\widetilde{Y}_n^{a-1}\right) \\
&= (\delta(0))^{a-1}\exp\{-\int_0^t\tilde{\eta}(s)ds - \frac{1}{2}\sigma_1^2(a-1)^2t + \sigma_1(a-1)W_1(t)\}\left(\prod_{n=1}^{\Phi(t)}\widetilde{Y}_n^{a-1}\right),
\end{aligned}$$

from which Equation (4.2) follows. Next, we need to prove that $Z(t)$ is a martingale. Given

$0 < u < t$,

$$
\begin{aligned}
& E(Z(t)|\mathcal{F}(u)) \\
= \; & E((\delta(0))^{a-1} \exp\{-\int_0^t \tilde{\eta}(s)ds - \frac{1}{2}\sigma_1^2(a-1)^2 t + \sigma_1(a-1)W_1(t)\} \left(\prod_{n=1}^{\Phi(t)} \widetilde{Y}_n^{a-1}\right) |\mathcal{F}(u)) \\
= \; & (\delta(0))^{a-1} \exp\{(-\int_0^u \tilde{\eta}(s)ds)\} \left(\prod_{n=1}^{\Phi(u)} \widetilde{Y}_n^{a-1}\right) \\
& \sum_{k=1}^{I} E(\exp\{(-\int_u^t \tilde{\eta}(s)ds)\} \left(\prod_{n=\Phi(u)+1}^{\Phi(t)} \widetilde{Y}_n^{a-1}\right), X(u) = k|\mathcal{F}(u)) \\
= \; & Z(u) \sum_{k=1}^{I} E(\exp\{(-\int_u^t \tilde{\eta}(s)ds)\} \left(\prod_{n=\Phi(u)+1}^{\Phi(t)} \widetilde{Y}_n^{a-1}\right), X(u) = k|\mathcal{F}(u)) \\
= \; & Z(u) \frac{\left(\sum_{i=1}^{I}\sum_{k=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n \pi_i P_{ik}(n,u)\right)}{\left(\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n \pi_i P_{ij}(n,t)\right)} \left(\sum_{j=1}^{I}\sum_{m=0}^{\infty}(\zeta_1^{(a-1)}+1)^m P_{kj}(n,t-u)\right). \\
= \; & Z(u).
\end{aligned}
$$

Hence, $Z(t)$ is martingale. Now, by Equations (2.4) and (B.2), we have

$$
\varphi_s(t) = \frac{E(U_c(\delta(T),T))}{U_c(\delta(t),t)} = e^{-\int_t^T r(s)ds} E(\frac{Z(T)}{Z(t)}\varphi_s(T)|\mathcal{F}_t) = B(t,T)E^*(\varphi_s(T)|\mathcal{F}_t).
$$

Proof of Theorem 1.

By using the Girsanov theorem for the Markov jump diffusion model (see Björk, Kabanov, and Runggaldier, 1997) we obtain, under $\mathbb{P}^*$, $W_1^*(t) := W_1(t) - \sigma_1(a-1)t$ as a Brownian motion. Further, under $\mathbb{P}^*$ the transition probability of $\Phi^*(t) = m$ given $X(0) = i$ and $X(t) = j$ is

$$Q_{ij}(m,t) = \frac{(\zeta_1^{(a-1)} + 1)^m P_{ij}(m,t)}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t)},$$

and $\widetilde{Y}_n^*$ has the probability density $f_{\widetilde{Y}}^*(y) = \frac{1}{(\zeta_1^{a-1}+1)} y^{a-1} f_{\widetilde{Y}}(y)$.

We compute the pricing measure via exponential embedding for given $\Phi(t) = m$, $X(0) = i$ and $X(t) = j$ as follows:

$$\begin{aligned}
& d\mathbb{P}^*(W_1^*(t), \Phi^*(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1^*, \cdots, \widetilde{Y}_m^*) \\
= \; & \exp\{-\tfrac{1}{2}\sigma_1^2(a-1)^2 t - \textstyle\int_0^t \tilde{\eta}(s)ds + \sigma_1(a-1)W_1(t)\} \left(\prod_{n=1}^{m} y_n^{a-1}\right) \\
& \cdot d\mathbb{P}(W_1(t), \Phi(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1, \cdots, \widetilde{Y}_m) \\
= \; & \exp\{-\tfrac{1}{2}\sigma_1^2(a-1)^2(t) + \sigma_1(a-1)W_1(t)\} d\mathbb{P}(W_1(t)) \\
& \cdot \frac{\left(\prod_{n=1}^{m} y_n^{a-1}\right)}{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t)} d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1, \cdots, \widetilde{Y}_m) \\
= \; & \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(W_1(t) - \sigma_1(a-1)t)^2}{2t}\} \\
& \cdot \frac{\left(\prod_{n=1}^{m} y_n^{a-1}\right)}{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t)} d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1, \cdots, \widetilde{Y}_m).
\end{aligned}$$

Then, the new Brownian motion will be $W_1^*(t) := W_1(t) - \sigma_1(a-1)t$, and we integrate the Brownian motion to obtain

$$\begin{aligned}
& d\mathbb{P}^*(\Phi^*(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1^*, \cdots, \widetilde{Y}_m^*) \\
= \; & \frac{\left(\prod_{n=1}^{m} y_n^{a-1}\right)}{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n,t)} d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1, \cdots, \widetilde{Y}_m).
\end{aligned}$$

For the jump sizes, note that $\{\widetilde{Y}_1, \cdots, \widetilde{Y}_m\}$ are i.i.d. random variables; therefore,

$$\begin{aligned} & d\mathbb{P}^*(\Phi^*(t) = m, X(0) = i, X(t) = j, \widetilde{Y}_1^*, \cdots, \widetilde{Y}_m^*) \\ = & \frac{1}{(\zeta_1^{(a-1)} + 1)} y_1^{(a-1)} f_{\widetilde{Y}}(y_1) \cdots \frac{1}{(\zeta_1^{(a-1)} + 1)} y_m^{(a-1)} f_{\widetilde{Y}}(y_m) \\ & \cdot \frac{(\zeta_1^{(a-1)} + 1)^m}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)} d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j). \end{aligned}$$

We integrate the behavior of $\widetilde{Y}_n$ to obtain

$$\begin{aligned} & d\mathbb{P}^*(\Phi^*(t) = m, X(0) = i, X(t) = j) \\ = & \frac{(\zeta_1^{(a-1)} + 1)^m}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)} d\mathbb{P}(\Phi(t) = m, X(0) = i, X(t) = j) \\ = & \frac{(\zeta_1^{(a-1)} + 1)^m P_{ij}(m, t)}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)}. \end{aligned}$$

Therefore, the new transition probability will be

$$Q_{ij}(m, t) = \frac{(\zeta_1^{(a-1)} + 1)^m P_{ij}(m, t)}{\sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\zeta_1^{(a-1)} + 1)^n \pi_i P_{ij}(n, t)}.$$

Further, the dynamics of $S(t)$ are given by

$$\begin{aligned} \frac{dS(t)}{S(t-)} & = \mu(t)dt + \sigma\{\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\} + d\left(\sum_{n=1}^{\Phi(t)} (\widetilde{Y}_n^b - 1) \right) \\ & = \{\mu(t) + \sigma_1 \sigma \rho (a-1)\} dt + \sigma\{\rho dW_1^*(t) + \sqrt{1-\rho^2} dW_2(t)\} + d\left(\sum_{n=1}^{\Phi(t)} (\widetilde{Y}_n^b - 1) \right). \end{aligned}$$

Since

$$\begin{aligned} E^*(\left(\prod_{n=1}^{\Phi^*(t)} \widetilde{Y}_n^{*b} \right) | \mathcal{F}_t) & = E^*(E^*(E^*(E^*(\left(\prod_{n=1}^{m} \widetilde{Y}_n^{*b} \right) | X(0) = i, X(t) = j, \Phi(t) = m)))) \\ & = \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{n=0}^{\infty} (\frac{(\zeta_1^{(a+b-1)} + 1)}{(\zeta_1^{(a-1)} + 1)})^n \pi_i Q_{ij}(n, t), \end{aligned}$$

we have

$$\eta^*(t) = \frac{d\log\{E^*\left(\left(\prod_{n=1}^{\Phi^*(t)} \widetilde{Y}_n^{*b}\right)\right)\}}{dt} = \frac{d\log\left(\dfrac{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a+b-1)}+1)^n\pi_i P_{ij}(n,t)}{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n,t)}\right)}{dt} \quad \text{(B.3)}$$

or

$$e^{-\int_0^t \eta^*(s)ds} = \frac{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a-1)}+1)^n\pi_i P_{ij}(n,t)}{\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{n=0}^{\infty}(\zeta_1^{(a+b-1)}+1)^n\pi_i P_{ij}(n,t)}.$$

Hence, under the new Markov jump diffusion model, the dynamic process of $S(t)$ is

$$\begin{aligned}\frac{dS(t)}{S(t-)} &= \{\mu(t)+\sigma_1\sigma\rho(a-1)+\eta^*(t)\}dt-\eta^*(t)dt+\sigma\{\rho dW_1^*(t)+\sqrt{1-\rho^2}dW_2(t)\} \\ &\quad + d\left(\sum_{n=1}^{\Phi^*(t)}(\widetilde{Y}_n^{*b}-1)\right). \end{aligned} \quad \text{(B.4)}$$

If $S(t)$ satisfies (B.4) in the equilibrium setting (4.3), we must have $\mu(t)+\sigma_1\sigma\rho(a-1)+\eta^*(t) = r(t)$, from which (4.5) follows. On the other hand, if (4.5) are satisfied under the measure $\mathbb{P}^*$, then the dynamics of $S(t)$ are given by

$$\frac{dS(t)}{S(t-)} = r(t)dt-\eta^*(t)dt+\sigma\{\rho dW_1^*(t)+\sqrt{1-\rho^2}dW_2(t)\}+d\left(\sum_{n=1}^{\Phi^*(t)}(\widetilde{Y}_n^{*b}-1)\right),$$

from which Equation (4.6) follows.

APPENDIX C: Option Pricing Formulas

Proof of Theorem 2.

(1) Under the measure $\mathbb{P}^*$, the dynamic process of the asset price $S(t)$ in Equation (4.6) becomes

$$S(T) = S(0)\exp\{\int_0^T (r(t)-\eta^*(t)-1/2\sigma^2)dt+\sigma W^*(T)\}\left(\prod_{n=1}^{\Phi^*(T)}\widetilde{Y}_n^{*b}\right), \quad \text{(C.1)}$$

where $W^*(T)$ is a normal random variable with mean 0 and variance T, and the jump sizes $\widetilde{Y}_n^*$ are i.i.d. random variables from distribution $f_{\widetilde{Y}^*}(y)$. Under the conditions of $X(0)=i$ and

$X(T) = j$ and given the conditional jump times $\Phi^*(T) = m$, $\widetilde{V}_m^b = \prod_{n=1}^m \widetilde{Y}_n^{*b}$, and Z is the standard normal distribution, we can rewrite (C.1) as

$$S(T) = S(0)\exp\{\textstyle\int_0^T (r(t) - \eta^*(t) - 1/2\sigma^2)dt + \sigma\sqrt{T}Z\}\widetilde{V}_m^b$$

with transition probability $Q_{ij}(m,T)$.

Hence, under the rational expectations setting, the equilibrium price (4.4) of the call option in the Markov jump diffusion model is

$$
\begin{aligned}
& MJ^c(0) = E^*(B(0,T)(S(T)-K)^+|\mathcal{F}_0) \\
= \;& B(0,T)E^*((S(T)\mathbf{1}_{\{S(T)>K\}}|\mathcal{F}_0) - B(0,T)KE^*\mathbf{1}_{\{S(T)>K\}}|\mathcal{F}_0) \\
= \;& \sum_{m=0}^{\infty}\big(B(0,T)E^*(S(T)\mathbf{1}_{\{S(T)>K\}}|X(0)=i, X(T)=j, \Phi(T)=m) \\
& -B(0,T)KE^*(\mathbf{1}_{\{S(T)>K\}}|X(0)=i, X(T)=j, \Phi(T)=m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Big) \\
= \;& \sum_{m=0}^{\infty}\Bigg(E^*E^*((S(0)\exp\{\int_0^T (\eta^*(t) - 1/2\sigma^2)dt + \sigma\sqrt{T}Z\}\widetilde{V}_m^b \\
& \quad \mathbf{1}_{\{Z<\frac{\ln(S(0)e^{-\int_0^T \eta^*(t)dt}\widetilde{V}_m^b/K)+\int_0^T r(t)dt - 1/2\sigma^2 T}{\sigma\sqrt{T}}\}}|\Phi(T)=m)) - Ke^{-\int_0^T r(t)dt} \\
& \quad E^*E^*((\mathbf{1}_{\{Z<\frac{\ln(S(0)e^{-\int_0^T \eta^*(t)dt}\widetilde{V}_m^b/K)+\int_0^T r(t)dt - 1/2\sigma^2 T}{\sigma\sqrt{T}}\}}|\Phi(T)=m))\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg) \\
= \;& \sum_{m=0}^{\infty}\Bigg(E^*(S(0)L(T)\widetilde{V}_m^b\mathbf{1}_{\{Z<\frac{\ln(S(0)L(T)\widetilde{V}_m^b/K+\int_0^T r(t)dt+1/2\sigma^2 T}{\sigma\sqrt{T}}\}}|\Phi^*(T)=m) \\
& -B(0,T)KE^*(\mathbf{1}_{\{Z<\frac{\ln(S(0)L(T)\widetilde{V}_m^b/K)+\int_0^T r(t)dt-1/2\sigma^2 T}{\sigma\sqrt{T}}\}}|\Phi^*(T)=m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg) \\
= \;& \sum_{m=0}^{\infty}\Bigg(E^*(S(0)\exp\{-\int_0^T \eta^*(t)dt\}\widetilde{V}_m^b\mathbb{N}(d^*(+)) - B(0,T)K\mathbb{N}(d^*(-))|\Phi^*(T)=m) \\
& \quad \sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg) \\
= \;& \sum_{m=0}^{\infty}\Bigg(E^*(C(S(0)L(T)\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|\Phi^*(T)=m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg).
\end{aligned}
$$

Here, $C(S(0)L(T)\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)$ is the option price in the Black-Scholes formula with the stock price $S(0)L(T)\widetilde{V}_m^b$, the strike price K, the maturity day T, the deterministic interest rate $r(t)$, and the volatility of the stock price σ. Define E^* as the expectation operator under the distribution $\widetilde{V}_m^b$, and

$$d^*(\pm) = \frac{\log\left(S(0)L(T)\widetilde{V}_m^b/K\right) + \int_0^T r(t)dt \pm 1/2\sigma^2 T}{\sigma\sqrt{T}}.$$

(2) Using Equations (4.10) and (4.4), we obtain

$$E^*(B(0,T)(F(T,T^*))) = E^*(B(0,T)(\frac{S(T)}{B(T,T^*)} - K)^+)$$

$$= \frac{1}{B(T,T^*)}E^*(B(0,T)(S(T) - B(T,T^*)K)^+)$$

$$= \frac{1}{B(T,T^*)}\sum_{m=0}^{\infty}\Bigg(E^*(S(0)\exp\{-\int_0^T \eta^*(t)dt\}\widetilde{V}_m^b\mathbb{N}(d_F^*(+)) - B(0,T)B(T,T^*)K\mathbb{N}(d_F^*(-))$$

$$|\Phi^*(T) = m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\Bigg)$$

$$= \sum_{m=0}^{\infty}\left(E^*(C(F(0,T^*)L(T)\widetilde{V}_m^b, K, T, \frac{1}{T}\int_0^T r(t)dt, \sigma)|\Phi^*(T) = m)\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i Q_{ij}(m,T)\right),$$

where

$$d_F^*(\pm) = \frac{\log\left(S(0)L(T)\widetilde{V}_m^b/(KB(0,T^*))\right) + \int_0^T r(t)dt \pm 1/2\sigma^2 T}{\sigma\sqrt{T}}$$

$$= \frac{\log\left(F(0,T^*)L(T)\widetilde{V}_m^b/K\right) + \int_0^T r(t)dt \pm 1/2\sigma^2 T}{\sigma\sqrt{T}}.$$

This completes the proof.

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