Valuation of Power Options under Heston's Stochastic Volatility Model

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Abstract

Options are traded on all of the world's major exchanges. Among them, we study an important class of exotic options, power options, the payoff depending on the price of the underlying asset raised to some power. The feature of nonlinear payoffs of power options not only offer great flexibility to investors but also apply to many applications. However, a number of empirical studies have shown that the classical Black-Scholes model has systematic biases across moneyness and maturity. We derive semi-closed form solutions for the values of various power options under Heston's stochastic volatility model. The analytic solutions are derived using stochastic calculus and the Fourier transform.

Keywords. option pricing, power options, stochastic volatility, Heston model.

1 Introduction

Based on no arbitrage arguments, Black and Scholes (1973) and Merton (1973) derived a partial differential equation for the valuation of European stock options. The Black-Scholes (B-S) model assumed that the asset price follows a geometric Brownian motion with a constant volatility. Because of its simplicity and analytical tractability, B-S model has been widely used among practitioners for pricing options.

However, a number of empirical studies have shown that the B-S model has systematic biases across moneyness and maturity, see Fouque, Papanicolaou and Sircar (2000), Lewis (2000) and references therein. One of major direction,

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to improve these deficiencies of the B-S framework, is to employ stochastic volatility (SV) models due to random characteristics of the volatility. Various SV models have been proposed and successful to reduce these biases, such as Hull and White (1987), Scott (1987), Wiggins (1987), Stein and Stein (1991), Heston (1993), and Melino and Turnbull (1990, 1991). Among them, we shall focus on Heston's SV model since it is analytically tractable and does not allow negative volatility. Also it allows arbitrary correlation between underlying asset return and its volatility.

Even though the success of SV models has been recognized among specialists, the values of exotic options under SV models have few analytic solutions. We study an important class of exotic options, power options, i.e. the payoff depending on the price of the underlying asset raised to some power. The feature of nonlinear payoffs of power options not only offer great flexibility to investors but also apply to many applications. As Tompkins (1999/2000) pointed out, the leverage nature of power options are very useful for hedging of nonlinear price risk and of changes in implied volatility. Tompkins (1999/2000) also mentioned examples that the power option pays less than the package of several European options to get the same level of payoff at expiry. The closed-form solution of the power options under B-S model has been studied by Heynen and Kat (1996), Tompkins (1999/2000), Wilmott (1998), and Esser (2003). Esser (2003) also derived semi-closed form solutions of the power options in the case of an mean reverting Ornstein-Uhlenbeck process as a volatility process. Macovschi and Quittard-Pinon (2006) also mentioned the power options under Heston's SV models without detailed calculations of characteristic functions. Based on the formulas of the power options, Macovschi and Quittard-Pinon (2006) introduced polynomial options under some special assumptions.

We derive semi-closed form solutions for values of various types of power options under Heston's SV model. Inspired by Scott (1997), we use stochastic calculus and the Fourier inversion formula to price the power options.

The paper is organized as follows: In Section 2, we describe the general settings of Heston's SV model. In Section 3 we drive semi-closed form solutions for the values of power options and the valuation of various applications of power options, such as capped power, powered options, and parabolic options. Finally in Section 4 we summarize the results. All detailed proofs are presented in the Appendix.

2 The Heston Stochastic Volatility Model

Let $\{S_t\}_{t\geq 0}$ denote the price of the underlying asset on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us assume that the evolution of the underlying asset satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t, \tag{1}$$

where $\sqrt{v_t}$ is a volatility process and μ is a constant expected rate of return. We assume that the process v_t follows a mean-reverting square-root process, i.e.,

$$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma\sqrt{v_t}dZ_t, \tag{2}$$

where θ is a constant long-run average of v_t, κ is a constant rate of mean reversion, σ is a constant volatility of volatility and these parameters satisfy

$$2\kappa\theta/\sigma^2 > 1 \tag{3}$$

to ensure that v_t stays almost surely positive. Here W_t and Z_t are two Brownian motions with correlation coefficient $\rho \in [-1, 1]$, i.e. $dW_t dZ_t = \rho dt$.

Based on no arbitrage principle, the value of any option $U(s, v, t)$ satisfies the following partial differential equation:

$$\frac{\partial U}{\partial t} + \frac{1}{2}vs^2\frac{\partial^2 U}{\partial s^2} + \rho\sigma vs\frac{\partial^2 U}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rs\frac{\partial U}{\partial s} - rU \tag{4}$$

$$+\left\{\kappa\left(\theta - v\right) - \Lambda(s, v, t)\sigma\sqrt{v}\right\}\frac{\partial U}{\partial v} = 0 \quad ,$$

where $r > 0$ is a constant riskless interest rate. Here $\Lambda(s, v, t)$ is a market price of volatility risk. Heston (1993) chose the market price of volatility risk to be proportional to the volatility, i.e. $\Lambda(s, v, t) = k\sqrt{v}$ for a constant $k > 0$. So, the coefficient of $\frac{\partial U}{\partial v}$ in (4) becomes $\{\kappa(\theta - v) - \lambda v\}$ where $\lambda = k\sigma$. The final condition is

$$U(s, v, T) = g(s),$$

where $g(s)$ is a payoff function at expiry, T. For instance, European vanilla call option has a payoff function $g(s) = \max(s - K, 0)$ with a given strike price K.

Under a risk-neutral martingale measure, Q, the equation (1) and (2) can be written as

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t}S_t dW_t \\ dv_t &= \kappa^*\left(\theta^* - v_t\right)dt + \sigma\sqrt{v_t}dZ_t, \end{aligned} \tag{5}$$

where

$$\kappa^* = \kappa + \lambda, \qquad \theta^* = \frac{\kappa\theta}{\kappa + \lambda}, \qquad dW_t dZ_t = \rho dt.$$

We use the stochastic differential equations (5) as the evolutionary model of the underlying asset under Heston's SV model. It is convenient to write

$$W_t = \rho Z_t + \sqrt{1 - \rho^2}\hat{Z}_t, \tag{6}$$

where $\hat{Z}_t$ is a standard Brownian motion independent of Z_t.

3 Power Options

Let us assume that the process $\{S_t\}_{t\geq 0}$, the underlying asset price, follows the stochastic differential equation (5) under the risk-neutral martingale measure, Q. Consider a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t$ in the smallest σ-algebra generated by $\{W_s, Z_s : s \leq t\}$.

There are two main categories of power options: standard power option and powered option. The standard power options have the payoff depending on the price of the underlying asset raised to some power, while the powered options have the standard payoff raised to some power. Macovschi and Quittard-Pinon (2006) introduced some cases of polynomial payoffs including parabolic and best options. The pricing formula of the standard power options derived in Section 3.1 can be readily applied to those classes of polynomial payoffs. We first derive the formula for standard power options in Section 3.1 and discuss powered option in Section 3.2. The polynomial options are mentioned in Section 3.3.

3.1 Standard Power Options

The payoff of a standard power option is like a standard European option with the price of the underlying asset raised to the power $m > 0$. Then under the risk-neutral measure Q, the value of m-th power call option, C^m_{power}, can be expressed as

$$\begin{aligned} C^m_{power}(S_t, v_t, t) &= E^Q[e^{-r(T-t)} \max(S_T^m - K, 0)|\mathcal{F}_t] \\ &= E^Q[e^{-r(T-t)} S_T^m \mathbf{1}_{\{S_T^m > K\}}|\mathcal{F}_t] \\ &\quad - KE^Q[e^{-r(T-t)} \mathbf{1}_{\{S_T^m > K\}}|\mathcal{F}_t] \end{aligned} \tag{7}$$

for a constant $m > 0$. Let us first fix a parameter $\hat{K}$ by $\hat{K} = K^{1/m}$ depending on a given constant $m > 0$ and a strike price K. Let us denote the first expectation in (7) by

$$C_m(S_t, v_t, t) := E^Q[e^{-r(T-t)} S_T^m \mathbf{1}_{\{S_T > \hat{K}\}}|\mathcal{F}_t], \tag{8}$$

with a given constant $m > 0$. Using the above expressions, we can rewrite the value of power option for a constant $m > 0$ by

$$C^m_{power}(S_t, v_t, t) \ = C_m(S_t, v_t, t) - KC_0(S_t, v_t, t),$$

where

$$C_0(S_t, v_t, t) = E^Q[e^{-r(T-t)} \mathbf{1}_{\{S_T > \hat{K}\}}|\mathcal{F}_t].$$

In order to price various types of options incluing power options, we need to compute the values of the expectation C_m in (8).

Using the techniques of stochastic calculus and the Fourier inversion formula, we can compute the value of power options as follows.

Theorem 3.1 *Assume that the price of the underlying asset satisfies* (5)*. Then for a constant m satisfying the conditions in* (18)*, the value of standard power option can be written*

$$C^m_{power}(S_t, v_t, t) = C_m(S_t, v_t, t) - KC_0(S_t, v_t, t)$$

where

$$C_m(S_t, v_t, t) = e^{(m-1)r(T-t)-m\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))} \times S_t^m e^{A_m(t,T;\hat{s_1},\hat{s_2})v_t + B_m(t,T;\hat{s_1},\hat{s_2})} F^1_m, \qquad (9)$$

and $C_0(S_t, v_t, t) = e^{-r(T-t)}F^1_0$ *with the same* $\hat{K}$ *as* C_m*. Here*

$$F^1_m = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(f^1_m(\phi)\frac{\exp(-i\phi\ln\hat{K})}{i\phi}\right) d\phi,$$

$$F^1_0 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(f^1_0(\phi)\frac{\exp(-i\phi\ln\hat{K})}{i\phi}\right) d\phi$$

with $\hat{K} = K^{1/m}$ *for* $m > 0$*, and for* $m \geq 0$

$$f^1_m(\phi) = e^{i\phi(r(T-t)+x(t))-i\phi\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))} \times e^{(A_m(t,T;s_1,s_2,\gamma)-A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma}))v_t + B_m(t,T;s_1,s_2,\gamma)-B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})}, \qquad (10)$$

where

$$A_m(t,T;s_1,s_2,\gamma) = \frac{s_2\left(\gamma+\kappa^*\right) + s_2 e^{\gamma(T-t)}\left(\gamma-\kappa^*\right) + 2s_1(e^{\gamma(T-t)}-1)}{-s_2(e^{\gamma(T-t)}-1)\sigma^2 + \gamma - \kappa^* + e^{\gamma(T-t)}\left(\gamma+\kappa^*\right)}, \qquad (11)$$

and

$$B_m(t,T;s_1,s_2,\gamma) = \frac{2\theta^*\kappa^*}{\sigma^2}\ln\left[\frac{2\gamma e^{(T-t)(\gamma+\kappa^*)/2}}{-s_2(e^{\gamma(T-t)}-1)\sigma^2 + \gamma - \kappa^* + e^{\gamma(T-t)}\left(\gamma+\kappa^*\right)}\right] \qquad (12)$$

with

$$\begin{aligned} &s_1 = (m+i\phi)(\tfrac{k^*\rho}{\sigma} - \tfrac{1}{2}) + \tfrac{1}{2}(m+i\phi)^2(1-\rho^2), \qquad s_2 = (m+i\phi)\tfrac{\rho}{\sigma} \\ &\hat{s_1} = m(\tfrac{k^*\rho}{\sigma} - \tfrac{1}{2}) + \tfrac{1}{2}m^2(1-\rho^2), \quad \hat{s_2} = m\tfrac{\rho}{\sigma} \\ &\gamma = \sqrt{\kappa^{*2} - 2s_1\sigma^2}, \quad \hat{\gamma} = \sqrt{\kappa^{*2} - 2\hat{s_1}\sigma^2}. \end{aligned} \qquad (13)$$

Proof. We derive the valuation of the expectation C_m in (9). Let us first apply the change of numeraire with the Radon-Nikodym derivative

$$\frac{dQ^*}{dQ} = e^{m\int_0^T \sqrt{v_s}dW_s - \int_0^T \frac{1}{2}m^2 v_s ds} =: \xi_T. \qquad (14)$$

Since the asset price satisfies (5), we have

$$\begin{aligned} S^m_T &= S^m_t \exp\left(m\int_t^T (r - \frac{1}{2}v_s)ds + m\int_t^T \sqrt{v_s}dW_s\right) \\ &= S^m_t \exp(mr(T-t))\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds\right) \frac{\xi_T}{\xi_t}. \end{aligned}$$

Therefore we can write

$$\begin{aligned}
C_m(S_t, v_t, t) &= E^Q[e^{-r(T-t)} S_T^m \mathbf{1}_{\{S(T)>\hat{K}\}} | \mathcal{F}_t] \\
&= E^{Q^*}[e^{(m-1)r(T-t)} S_t^m e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} \mathbf{1}_{\{S(T)>\hat{K}\}} | \mathcal{F}_t] \\
&= e^{(m-1)r(T-t)} S_t^m E^{Q^*}[e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} \mathbf{1}_{\{S(T)>\hat{K}\}} | \mathcal{F}_t]
\end{aligned}$$

Now let us define a measure Q_m^1 on $\mathcal{F}_T$ such that for any $A \in \mathcal{F}_T$,

$$E^{Q_m^1}[\mathbf{1}_A] = \frac{E^{Q^*}[\mathbf{1}_A \exp\left(\int_t^T \frac{1}{2} m(m-1) v_s ds\right) | \mathcal{F}_t]}{E^{Q^*}[\exp\left(\int_t^T \frac{1}{2} m(m-1) v_s ds\right) | \mathcal{F}_t]}.$$

Therefore we have

$$\begin{aligned}
C_m(S, v, t) &= e^{(m-1)r(T-t)} S_t^m E^{Q^*}[e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} \mathbf{1}_{\{S(T)>\hat{K}\}} | \mathcal{F}_t] \\
&= e^{(m-1)r(T-t)} S_t^m E^{Q^*}[e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} | \mathcal{F}_t] Q_m^1(S_T > \hat{K})
\end{aligned}$$

Let us denote the log of the stock price $x(t) = \ln S(t)$. Then applying Ito's formula, the the stochastic differential equation (5) can be written by

$$dx(t) = (r - \frac{1}{2} v_t) dt + \sqrt{v_t} dW_t. \tag{15}$$

We now define the characteristic function $f_m^1(\phi)$ of x_T under Q_m^1 by the following conditional expectation

$$\begin{aligned}
f_m^1(\phi) &= E^{Q_m^1}[e^{i\phi x_T}] \\
&= \frac{E^{Q^*}[\exp\left(\int_t^T \frac{1}{2} m(m-1) v_s ds + i\phi x_T\right) | \mathcal{F}_t]}{E^{Q^*}[\exp\left(\int_t^T \frac{1}{2} m(m-1) v_s ds\right) | \mathcal{F}_t]}
\end{aligned} \tag{16}$$

and the probability

$$\begin{aligned}
F_m^1 =: F^{Q_m^1} &= Q_m^1(S_T > \hat{K}) \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(f_m^1(\phi) \frac{\exp(-i\phi \ln \hat{K})}{i\phi} \right) d\phi.
\end{aligned}$$

The value of $f_m^1(\phi)$ in (16) have the formula (10) following Lemma 5.1 - 5.3 in Appendix. Also from Lemma 5.3, we have for $\phi = 0$

$$E^{Q^*}[e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} | \mathcal{F}_t] = e^{-m\frac{\rho}{\sigma}(v_t + \kappa^* \theta^*(T-t))} e^{A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma}) v_t + B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})}.$$

Consequently we obtain

$$\begin{aligned}
C_m(S_t, v_t, t) &= e^{(m-1)r(T-t)} S_t^m E^{Q^*}[e^{\int_t^T \frac{1}{2} m(m-1) v_s ds} | \mathcal{F}_t] Q_m^1(S_T > \hat{K}) \\
&= e^{(m-1)r(T-t)} S_t^m e^{-m\frac{\rho}{\sigma}(v_t + \kappa^* \theta^*(T-t))} \\
&\quad \times e^{A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma}) v_t + B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})} Q_m^1(S_T > \hat{K}).
\end{aligned}$$

Therefore we conclude the theorem. □

Remark 3.2 *As Kraft (2005, Proposition 5.1) proved, the characteristic functions are well defined if*

$$\begin{array}{l} \hat{s_1} \le \frac{\kappa^{*2}}{2\sigma^2} \quad and \quad \hat{s_2} \le \frac{\kappa^* + \hat{\gamma}}{\sigma^2}, \\ Re(s_1) \le \frac{\kappa^{*2}}{2\sigma^2} \quad and \quad Re(s_2) \le \frac{\kappa^* + \gamma}{\sigma^2}, \end{array} \tag{17}$$

where $s_1, s_2, \gamma, \hat{s_1}, \hat{s_2}, \hat{\gamma}$ are defined in (13)*. In other words, the power constant $m > 0$ should satisfy the following two equations:*

$$\begin{array}{l} (1-\rho^2)m^2 + (\frac{2\kappa^*\rho}{\sigma} - 1)m - \frac{\kappa^{*2}}{\sigma^2} \le 0, \\ m\frac{\rho}{\sigma} \le \frac{\kappa^* + \hat{\gamma}}{\sigma^2}. \end{array} \tag{18}$$

Applying the above theorem, the value of other exotic options can be easily computed. For instance, the gap call option can be computed

$$\text{Gap call } = E^Q[e^{-r(T-t)}(S(T) - K_1)\mathbf{1}_{\{S_T > K_2\}}],$$

where K_1, K_2 are given constants, see Haug (2007). Note that the payoff from this option can be negative, depending on the values of K_1 and K_2. Let us denote $K = K_2$ then we have the value of gap options:

$$\text{Gap call } = C_1(S_t, v_t, t) - K_1 C_0(S_t, v_t, t), \tag{19}$$

where C_0 and C_1 are defined in Theorem 3.1.

Power options can lead to very high leverage and thus can cause very large losses for the writer of these options. It is therefore common to cap the payoff, $\min(\max(S_T^m - K, 0), \overline{C})$, where a predefined constant $\overline{C}$ is the upper bound of the payoff at expiry. For the closed-form solutions under B-S model see Esser (2003), Haug (2007). The value of this option under Heston's SV model has the form

$$\begin{aligned} \textit{Capped power call} &= E^Q[e^{-r(T-t)} \min(\max(S_T^m - K, 0), \overline{C})|\mathcal{F}_t] \\ &= E^Q[e^{-r(T-t)}(S_T^m - K)\mathbf{1}_{\{S_T^m > K\}}|\mathcal{F}_t] \\ &\quad - E^Q[e^{-r(T-t)}(S_T^m - (\overline{C} + K))\mathbf{1}_{\{S_T^m > \overline{C} + K\}}|\mathcal{F}_t]. \end{aligned}$$

Using the result in Theorem 3.1, the value of the capped power option can be computed as follows:

Corollary 3.3 *Assume that the price of the underlying asset satisfies* (5)*. Then for a constant m satisfying the conditions in* (18)*, the value of capped power option can be written*

$$\begin{aligned} \textit{Capped power call} &= C_m^1(S_t, v_t, t) - KC_0^1(S_t, v_t, t) \\ &\quad - C_m^2(S_t, v_t, t) + (\overline{C} + K)C_0^2(S_t, v_t, t), \end{aligned} \tag{20}$$

where C_m^1 and C_0^1 have the strike $\hat{K} = K^{1/m}$ like standard power option and C_m^2 and C_0^2 have the strike $\hat{K} = (\overline{C} + K)^{1/m}$ for a constant $m > 0$.

3.2 Powered Options

At maturity, a powered call option has the payoff $\max(S_T - K, 0)^m$ for a constant $m > 0$ satisfying the conditions in (18). Then the value of powered call option can be written

$$\begin{aligned} \textit{Powered call} &= E^Q[e^{-r(T-t)} \max(S_T - K, 0)^m | \mathcal{F}_t] \\ &= E^Q[e^{-r(T-t)} (S_T - K)^m \mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t] \\ &= \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) (-1)^j K^j E^Q[e^{-r(T-t)} S_T^{m-j} \mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t] \\ &= \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) (-1)^j K^j C_{m-j}(S_t, v_t, t), \end{aligned} \tag{21}$$

where the functions $C_{m-j}(S_t, v_t, t)$ are defined in (9) for $j = 0, ..., m$.

3.3 Parabolic Options

We may extend the pricing formula of power options to ploynomial payoff such as

$$\max(R(S_T) - K, 0),$$

where $R(x)$ is a real polynomial. As Macovschi and Quittard-Pinon (2006) showed closed-form formulas for some special cases when $R(x) - K$ has exactly n strictly positive roots, $\lambda_1, \lambda_2, ..., \lambda_n$ such as $\lambda_1 < \lambda_2 < ... < \lambda_n$ and $R(x) - K$ alternates its sign between two consecutive roots with $R(x) - K \leq 0$ for $0 \leq x \leq \lambda_1$ (See Theorem 3 in Macovschi and Quittard-Pinon (2006)).

From the formula of $C_m(S_t, v_t, t)$ in (9), we can readily extend the pricng formula to polynomial payoffs under Heston's SV model. For instance consider with degree 2, which introduced as parabolic option in Macovschi and Quittard-Pinon (2006):

$$\textit{Parabolic call} = E^Q[e^{-r(T-t)} \max(h(S_T - K_1)(K_2 - S_T), 0) | \mathcal{F}_t]$$

where $0 < K_1 < K_2$ and a constant $h > 0$. From the formula $C_m(S_t, v_t, t)$ in (9), we have the value of parabolic option by

$$\begin{aligned} \textit{Parabolic call} &= h\{C_2^2(S_t, v_t, t) - C_2^1(S_t, v_t, t) \\ &\quad + (K_1 + K_2)(C_1^1(S_t, v_t, t) - C_1^2(S_t, v_t, t))\}, \end{aligned} \tag{22}$$

where C_1^1 and C_2^1 have the strike $\hat{K} = K_1^{1/m}$ and C_1^2 and C_2^2 have the strike $\hat{K} = K_2^{1/m}$. One may set the constant $h = 1/(K_2 - K_1)$ to standardize an option. From the result in Theorem 3.1, other options introduced in Macovschi and Quittard-Pinon (2006), such as best option, balloon option, can also readily be extended to Heston's SV model.

4 Conclusions

We derive the fast closed form solutions for the values of power options under Heston's stochastic volatility model. The semi-closed solutions are derived using stochastic calculus and the Fourier transform.

5 Appendix

Lemma 5.1 *Assume that the value of the underlying asset price satisfies* (5). *Then for a constant* m *satisfying the conditions in* (18)*,*

$$\begin{aligned} & E^{Q^*}[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi x_T\right)|\mathcal{F}_t] \\ = \;& e^{i\phi(r(T-t)+x(t))-(m+i\phi)\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))}E^Q[e^{s_2 v(T)+s_1\int_t^T v_s ds}|\mathcal{F}_t], \end{aligned} \tag{23}$$

where

$$s_1 = (m+i\phi)(\frac{k^*\rho}{\sigma}-\frac{1}{2})+\frac{1}{2}(m+i\phi)^2(1-\rho^2), \qquad s_2 = (m+i\phi)\frac{\rho}{\sigma}.$$

Proof. From the definition of the Radon-Nikodym derivative, ξ_T in (14), we have

$$\begin{aligned} & E^{Q^*}[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi x_T\right)|\mathcal{F}_t] \\ &= E^Q[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi x_T\right)\frac{\xi_T}{\xi_t}|\mathcal{F}_t] \\ &= e^{i\phi x_t}E^Q[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi(x_T-x_t)\right. \\ &\qquad \left. +m\int_t^T \sqrt{v_s}dW_s - \int_t^T \frac{1}{2}m^2 v_s ds\right)|\mathcal{F}_t]. \end{aligned}$$

From the evolution of the asset price (15), we have

$$\begin{aligned} & E^Q[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi(x_T-x_t) + m\int_t^T \sqrt{v_s}dW_s - \int_t^T \frac{1}{2}m^2 v_s ds\right)|\mathcal{F}_t] \\ &= E^Q[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi(\int_t^T r - \frac{1}{2}v_s ds + \int_t^T \sqrt{v_s}dW_s)\right. \\ &\qquad \left. + m\int_t^T \sqrt{v_s}dW_s - \int_t^T \frac{1}{2}m^2 v_s ds\right)|\mathcal{F}_t] \\ &= e^{i\phi r(T-t)}E^Q[\exp\left(\int_t^T -\frac{1}{2}(m+i\phi)v_s ds + (m+i\phi)\int_t^T \sqrt{v_s}dW_s\right)|\mathcal{F}_t]. \end{aligned} \tag{24}$$

Using the relation (6), we can write

$$\int_t^T \sqrt{v_s} dW_s = \int_t^T \sqrt{v_s} \left(\rho dZ_s + \sqrt{1-\rho^2} d\hat{Z}_s \right),$$

where Z_s and $\hat{Z}_s$ are independent Brownian motions. Introduce another filtration $\tilde{\mathcal{F}}_t$, the smallest σ-algebra generated by $\{W_s, s \le t, Z_\tau, \tau \le T\}$ then the expectation in (24) can be written

$$\begin{aligned} & E^Q \left[E[\exp(\int_t^T -\frac{1}{2}(m+i\phi)v_s ds) \exp((m+i\phi)\int_t^T \sqrt{v_s} dW_s) | \tilde{\mathcal{F}}_t] | \mathcal{F}_t \right] \\ = & E^Q \left[\exp\left(\int_t^T (-\frac{1}{2}(m+i\phi)v_s ds \right) \exp((m+i\phi)\rho \int_t^T \sqrt{v_s} dZ_s) \right. \\ & \left. E^Q[\exp((m+i\phi)\sqrt{1-\rho^2}\int_t^T \sqrt{v_s} d\hat{Z}_s) | \tilde{\mathcal{F}}_t] | \mathcal{F}_t \right]. \end{aligned}$$

Since we have

$$\int_t^T \sqrt{1-\rho^2}\sqrt{v_s} d\hat{Z}_s \sim \mathcal{N}\left(0, \int_t^T (1-\rho^2) v_s ds\right),$$

and the independence of $\tilde{\mathcal{F}}_t$ and $\hat{Z}$, we obtain

$$E\left[\exp((m+i\phi)\sqrt{1-\rho^2}\int_t^T \sqrt{v_s} d\hat{Z}_s) | \tilde{\mathcal{F}}_t\right] = \exp\left(\frac{1}{2}(m+i\phi)^2(1-\rho^2)\int_t^T v_s ds\right).$$

Also from (5), we have

$$\begin{aligned} & (m+i\phi)\rho \int_t^T \sqrt{v_s} dZ_s = (m+i\phi)\frac{\rho}{\sigma}\left(\int_t^T dv_s - \int_t^T k^*(\theta^* - v_s) ds\right) \\ = & (m+i\phi)\frac{\rho}{\sigma}(v(T) - v(t) - k^*\theta^*(T-t)) + (m+i\phi)\frac{k^*\rho}{\sigma}\int_t^T v_s ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned} & E^{Q^*}[\exp\left(\int_t^T \frac{1}{2} m(m-1) v_s ds + i\phi x_T\right) | \mathcal{F}_t] \\ & = e^{i\phi(x_t + r(T-t)) - (m+i\phi)\frac{\rho}{\sigma}(v(t) + k^*\theta^*(T-t))} E^Q[e^{s_2 v(T)} e^{s_1 \int_t^T v_s ds} | \mathcal{F}_t], \end{aligned}$$

where

$$s_1 = (m+i\phi)(\frac{k^*\rho}{\sigma} - \frac{1}{2}) + \frac{1}{2}(m+i\phi)^2(1-\rho^2), \qquad s_2 = (m+i\phi)\frac{\rho}{\sigma}. \tag{25}$$

□

The next Lemma 5.2 gives the concrete value of the expectation in the right hand side of (23).

Lemma 5.2 *For complex values s_1 and s_2 satisfying* (17)*, we have*

$$E^Q[e^{s_2 v(T)+s_1\int_t^T v_s ds}|\mathcal{F}_t] = e^{A(t,T)v_t+B(t,T)}, \tag{26}$$

where the functions $A(t,T;s_1,s_2,\gamma)$ and $B(t,T;s_1,s_2,\gamma)$ satisfy

$$A(t,T;s_1,s_2,\gamma) = \frac{s_2\,(\gamma+\kappa^*) + s_2 e^{\gamma(T-t)}\,(\gamma-\kappa^*) + 2s_1(e^{(T-t)\gamma}-1)}{-s_2(e^{(T-t)\gamma}-1)\sigma^2 + \gamma - \kappa^* + e^{(T-t)\gamma}\,(\gamma+\kappa^*)}, \tag{27}$$

and

$$B(t,T;s_1,s_2,\gamma) = \frac{2\theta^*\kappa^*}{\sigma^2}\ln\left[\frac{2\gamma e^{(T-t)(\gamma+\kappa^*)/2}}{-s_2(e^{(T-t)\gamma}-1)\sigma^2 + \gamma - \kappa^* + e^{(T-t)\gamma}\,(\gamma+\kappa^*)}\right] \tag{28}$$

with

$$\gamma = \sqrt{k^{*2} - 2s_1\sigma^2}.$$

Proof. Let us define

$$y(v_t,t,T) = E^Q[e^{s_2 v(T)} e^{s_1\int_t^T v_s ds}|\mathcal{F}_t].$$

Let us first assume that s_1 and s_2 are real values satisfying (17) i.e., $s_1 \le \kappa^{*2}/(2\sigma^2)$, $s_2 \le (\kappa^* + \sqrt{\kappa^{*2} - 2s_1\sigma^2})/\sigma^2$. Then the process y is well-defined and the Feynman-Kac stochastic representation theorem provides us that y is the solution of the following partial differential equation

$$\frac{\partial y}{\partial t} + k^*(\theta^* - v)\frac{\partial y}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 y}{\partial v^2} + s_1 v y = 0 \tag{29}$$

with an initial condition

$$y(v,T,T) = e^{s_2 v}.$$

By analogy with Black-Scholes formula, we guess a solution of the form:

$$y(v,t,T) = e^{A(t,T)v+B(t,T)}.$$

By substituting the proposed value in (29), we obtain the following ordinary differntial equations(ODEs) for $A(t,T)$ and $B(t,T)$:

$$\begin{aligned} A' &= k^*A - \frac{1}{2}\sigma^2A^2 - s_1, \quad A(0) = s_2, \\ B' &= -k^*\theta^*A, \quad B(0) = 0. \end{aligned}$$

The solution of the above Riccati equation can be written (27) for $A(t,T)$, for example, see the analysis in Chesney, Elliott & Gibson (1993), Pitman and Yor (1982). From an integration, we have (28) for $B(t,T)$.

Now consider complex values s_1 and s_2 satisfying

$$Re(s_1) \le \frac{\kappa^{*2}}{2\sigma^2}, \quad Re(s_2) \le \frac{\kappa^* + \sqrt{\kappa^{*2} - 2Re(s_1)\sigma^2}}{\sigma^2}.$$

Then the left hand side of (26) is well-defined and similarly we can show that the equation (26) holds for complex values s_1, s_2 defined in (25). Therefore we conclude the lemma. $\square$

From Lemma 5.1 and Lemma 5.2, we consequently obtain the following formula:

Lemma 5.3 *Assume that the value of the underlying asset price satisfies* (5). *Then for a constant* m *satisfying the conditions in* (18),

$$Q^1_m(S_T > \hat{K}) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(f^1_m(\phi)\frac{\exp(-i\phi\ln\hat{K})}{i\phi}\right)d\phi. \tag{30}$$

where

$$\begin{aligned} f^1_m(\phi) &= e^{i\phi(r(T-t)+x(t))-i\phi\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))} \\ &\quad \times e^{(A_m(t,T;s_1,s_2,\gamma)-A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma}))v_t+B_m(t,T;s_1,s_2,\gamma)-B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})}. \end{aligned} \tag{31}$$

Here the functions $A_m(t,T;s_1,s_2,\gamma)$ *and* $B_m(t,T;s_1,s_2,\gamma)$ *satisfy*

$$A_m(t,T;s_1,s_2,\gamma) = \frac{s_2(\gamma+\kappa^*) + s_2 e^{\gamma(T-t)}(\gamma-\kappa^*) + 2s_1(e^{(T-t)\gamma}-1)}{-s_2(e^{(T-t)\gamma}-1)\sigma^2 + \gamma - \kappa^* + e^{(T-t)\gamma}(\gamma+\kappa^*)},$$

and

$$B_m(t,T;s_1,s_2,\gamma) = \frac{2\theta^*\kappa^*}{\sigma^2}\ln\left[\frac{2\gamma e^{(T-t)(\gamma+\kappa^*)/2}}{-s_2(e^{(T-t)\gamma}-1)\sigma^2 + \gamma - \kappa^* + e^{(T-t)\gamma}(\gamma+\kappa^*)}\right]$$

with

$$\begin{aligned} &s_1 = (m+i\phi)\left(\tfrac{k^*\rho}{\sigma} - \tfrac{1}{2}\right) + \tfrac{1}{2}(m+i\phi)^2(1-\rho^2), \qquad s_2 = (m+i\phi)\tfrac{\rho}{\sigma} \\ &\hat{s_1} = m\left(\tfrac{k^*\rho}{\sigma} - \tfrac{1}{2}\right) + \tfrac{1}{2}m^2(1-\rho^2), \quad \hat{s_2} = m\tfrac{\rho}{\sigma} \\ &\gamma = \sqrt{\kappa^{*2} - 2s_1\sigma^2}, \quad \hat{\gamma} = \sqrt{\kappa^{*2} - 2\hat{s_1}\sigma^2}. \end{aligned}$$

Proof. From the relation (16) in Theorem 3.1,

$$f^1_m(\phi) = \frac{E^{Q^*}[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds + i\phi x_T\right)|\mathcal{F}_t]}{E^{Q^*}[\exp\left(\int_t^T \frac{1}{2}m(m-1)v_s ds\right)|\mathcal{F}_t]}.$$

Note that the expectation in the denomicator is the special case of the expectation in numerator with $\phi = 0$. This gives the values $\hat{s_1}$ and $\hat{s_2}$ from s_1 and s_2 respectively. Therefore we get

$$\begin{aligned} f^1_m(\phi) &= \frac{e^{i\phi(r(T-t)+x(t))-(m+i\phi)\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))}e^{A_m(t,T;s_1,s_2,\gamma)v_t+B_m(t,T;s_1,s_2,\gamma)}}{e^{-m\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))}e^{A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})v_t+B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})}} \\ &= e^{i\phi(r(T-t)+x(t))-i\phi\frac{\rho}{\sigma}(v_t+\kappa^*\theta^*(T-t))} \\ &\quad \times e^{(A_m(t,T;s_1,s_2,\gamma)-A_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma}))v_t+B_m(t,T;s_1,s_2,\gamma)-B_m(t,T;\hat{s_1},\hat{s_2},\hat{\gamma})}. \end{aligned}$$

$\square$

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