Simple Approximations of the Asian Option: An Alternative Numerical Technique

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Abstract
In this paper, we make use of a certain alternative numerical technique to solve the problem of pricing an Asian option. Rogers and Shi (1995) have also looked at this problem and obtained bounds to the price. In fact, the lower bound that they obtain are so close that it can be regarded as the true price itself. However, they make use of a numerical integration technique to solve the problem. Now, this can be time consuming and also might require sophisticated hardware and software. In this paper, we make use of a simple expansion technique to solve he problem and avoid the numerical integration by replacing it with a set of exact integrations. For the expansions, we use the algebraic package MAPLE.

Keywords: Asian option, Brownian Motion, Strike price

AMS 1991 subject classification: 90A09, 60G15

1 Introduction

In this paper, we try to price the Asian option directly through a set of exact integrals. The problem is similar to the problem tackled by Rogers and Shi (1995). In fact, the lower bounds that they obtain are so close that it can be regarded as the true price itself. However, they make use of a numerical integration technique to solve the problem.

2 Problem Definition

Rogers and Shi assumes that at time t, the price of a risky asset S_t is given by

$$S_t = S_0 \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t + ct\right), \tag{1}$$

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where, B_t is a standard Brownian motion, σ^2 is the instantaneous variance. Also, c is a constant. They assume also that under an equivalent martingale measure $c = r$, the riskless interest rate (see Harrison and Kreps (1979) and Harrison and Pliska (1981)). The problem that Rogers and Shi looked at is that of computing the value of an Asian (call) option with maturity T and the strike price F written on the risky asset S_t. Mathematically, this is the same as calculating

$$E(Y - F)^+, \tag{2}$$

where, Y is defined by

$$Y = \int_0^T S_u du. \tag{3}$$

and S_t is defined by equation 1.

3 Calculations

Without loss of generality, we take t = 1. Like Rogers and Shi, we define the function of interest as

$$f(x) = (x - F)^+ = \max([x - F], 0).$$

Here f is convex in nature and defined exactly in the same manner as by Rogers and Shi. We are interested in finding

$$E(f(Y)),$$

where Y is defined by equation 3. We try to find a lower bound to the price of the option (similar to Rogers and Shi), rather than calculating the price of option directly. We thus have

$$E(f(Y)) = E[E(f(Y)|Z)] \geq E[f(E(Y|Z))], \tag{4}$$

where, Z is the conditioning factor used. Rogers and Shi discusses some choice of Z from an empirical stand point, but a more detailed discussion on the choice of Z is available in Basu (1999) - an outline is provided in the *Appendix* at the end of this paper. We shall further assume that Z, the conditioning factor, is suitably normalised. The conditioning factor used here is similar to the one used by Rogers and Shi with $t = 1$ and is given by

$$Z = \frac{\int_0^1 B_s ds}{\sqrt{\mathrm{Var}(\int_0^1 B_s ds)}},$$

where $\mathrm{Var}(\int_0^1 B_s ds) = \frac{1}{3}$.

The lower bound to the price of the option as calculated by Rogers and Shi is given by

$$E[E(f(Y|Z))] = E\left\{\left[E\left(\int_0^1 \exp\left(\sigma B_u - \frac{1}{2}\sigma^2 u + ru\right) du|Z\right) - F\right]^+\right\}. \tag{5}$$

Thus, like Rogers and Shi, we are interested in finding

$$E\left[\int_0^1 \exp\left(\sigma B_u - \frac{1}{2}\sigma^2 u + ru\right) du|Z\right]. \tag{6}$$

This is similar to the lower bound of the price as found by Rogers and Shi.

Now, to find the expectation as defined by equation (6), we first find the following.

$$E(B_u|Z) = k_u Z, \tag{7}$$

$$k_u = \mathrm{Cov}(B_u, Z) = \frac{\mathrm{Cov}(B_u, \int_0^1 B_s ds)}{\sqrt{\mathrm{Var}(\int_0^1 B_s ds}} = \sqrt{3}(u - \frac{u^2}{2}), \tag{8}$$

$$\mathrm{Var}(B_u|Z) = \sigma^2(u - k_u^2). \tag{9}$$

Once we have these values, we are then interested in finding, conditionally on Z, the expected value of

$$E[\int_0^1 e^{\sigma B_u - \frac{1}{2}\sigma^2 u + ru} du|Z] = \int_0^1 \exp\left(ru + \sigma k_u Z - \frac{3}{2}\sigma^2 (u - \frac{u^2}{2})^2\right) du$$

$$= \int_0^1 \left\{ exp\left(ru + \sigma\sqrt{3}\left(u - \frac{u^2}{2}\right) Z - \frac{3}{2}\sigma^2\left(u - \frac{u^2}{2}\right)^2\right)\right\} du.$$

Writing $k = \frac{r}{\sigma}$, we have the lower bound to the price of the asset, conditionally on Z, as

$$\int_0^1 \left\{ exp\left(k\sigma u + \sigma\sqrt{3}\left(u - \frac{u^2}{2}\right) Z - \frac{3}{2}\sigma^2\left(u - \frac{u^2}{2}\right)^2\right)\right\} du = \int_0^1 g(k, \sigma, u, z) du \quad \text{say.} \tag{10}$$

Rogers and Shi performed a numerical integration at this stage in order to obtain the price of the option conditionally on Z and then finally the expectation is taken over Z to obtain the final price of the option. However, at this stage that we make use of an expansion argument and differ from the approach of Rogers and Shi. This is done so as to allow us to avoid the numerical integrations involved.

We expand the exponential term, $g(k, \sigma, u, z)$, in equation (10) in terms of σ, and retain terms up to the fourth power of σ. Thus, we have, conditionally on Z = z,

$$g(k, \sigma, u, z) = g_1(k, \sigma, u, z) + O(\sigma^5),$$

where,

$$g_1(k, \sigma, u, z) = 1 + \left(ku + \sqrt{3}zu - \frac{1}{2}\sqrt{3}zu^2\right)\sigma$$

$$+ \left\{-\frac{3}{2}u^2 + \frac{3}{2}u^3 - \frac{3}{8}u^4 + \frac{1}{2}k^2u^2 + ku^2\sqrt{3}z - \frac{1}{2}ku^3\sqrt{3}z + \frac{3}{2}z^2u^2 - \frac{3}{2}z^2u^3 + \frac{3}{8}z^2u^4\right\}\sigma^2$$

$$+ \left\{-\frac{1}{16}z^3u^6\sqrt{3} + \frac{3}{8}z^2u^5k - \frac{3}{2}z^2u^4k + \frac{3}{8}z^3u^5\sqrt{3} + \frac{1}{2}k^2u^3\sqrt{3}z - \frac{1}{4}k^2u^4\sqrt{3}z\right.$$

$$+\frac{3}{2}z^2u^3k + \frac{1}{2}z^3u^3\sqrt{3} - \frac{3}{4}z^3u^4\sqrt{3} + \frac{1}{6}k^3u^3 - \frac{3}{8}ku^5 + \frac{3}{16}\sqrt{3}zu^6$$

$$\left.-\frac{3}{2}ku^3 + \frac{3}{2}ku^4 - \frac{3}{2}\sqrt{3}zu^3 + \frac{9}{4}\sqrt{3}zu^4 - \frac{9}{8}\sqrt{3}zu^5\right\}\sigma^3$$

$$+\Big\{-\frac{9}{4}z^2u^4+\frac{9}{8}u^4-\frac{9}{4}u^5+\frac{27}{16}u^6-\frac{3}{4}k^2u^4+\frac{3}{4}k^2u^5-\frac{3}{16}k^2u^6+\frac{9}{2}z^2u^5+\frac{3}{16}z^2u^6k^2-\frac{3}{4}z^2u^5k^2$$
$$+\frac{3}{16}ku^7\sqrt{3}z-\frac{3}{2}ku^4\sqrt{3}z+\frac{9}{4}ku^5\sqrt{3}z-\frac{3}{16}z^4u^7-\frac{1}{16}z^3u^7\sqrt{3}k-\frac{9}{8}ku^6\sqrt{3}z-\frac{3}{4}z^4u^5+$$
$$\frac{9}{16}z^4u^6+\frac{3}{8}z^4u^4-\frac{3}{4}z^3u^5k\sqrt{3}+\frac{3}{8}z^3u^6k\sqrt{3}-\frac{1}{12}k^3u^5\sqrt{3}z+\frac{1}{2}z^3u^4k\sqrt{3}+\frac{1}{6}k^3u^4\sqrt{3}z$$
$$+\frac{3}{128}z^4u^8+\frac{3}{4}z^2u^4k^2-\frac{27}{8}z^2u^6+\frac{9}{8}z^2u^7-\frac{9}{64}z^2u^8+\frac{1}{24}k^4u^4+\frac{9}{128}u^8-\frac{9}{16}u^7\Big\}\sigma^4. \tag{11}$$

Next we integrate out u from (11) and re-arrange the equation so that we have a polynomial in z. Thus, we have

$$\int_0^1 g_1(k,\sigma,u,z)du = \frac{1}{105}\sigma^4z^4+\left(\sqrt{3}[\frac{93}{4480}\sigma^4k+\frac{1}{35}\sigma^3]\right)z^3$$
$$+\left(-\frac{2}{35}\sigma^4+\frac{29}{560}\sigma^4k^2+\frac{11}{80}\sigma^3k+\frac{1}{5}\sigma^2\right)z^2$$
$$+\left(\sqrt{3}[\frac{5}{24}\sigma^2k+\frac{1}{3}\sigma+\frac{3}{40}\sigma^3k^2-\frac{279}{4480}\sigma^4k-\frac{3}{35}\sigma^3+\frac{7}{360}\sigma^4k^3]\right)z$$
$$+\left(1-\frac{1}{5}\sigma^2+\frac{1}{35}\sigma^4-\frac{29}{560}\sigma^4k^2+\frac{1}{24}\sigma^3k^3-\frac{11}{80}\sigma^3k+\frac{1}{6}\sigma^2k^2+\frac{1}{120}\sigma^4k^4+\frac{1}{2}k\right)$$
$$= g_2(k,\sigma,z) \quad \text{say.}$$

We are thus left with expressions in terms of k, σ and z. Treating k and σ as constants, or known values, we thus have a 4^{th} degree polynomial in z.

As stated earlier, we are interested in finding the lower bound to price of the option (as was Rogers and Shi) given by

$$E(E(Y-F)^+|Z) = E[g_2(k,\sigma,z)-F]^+,$$

where F is the strike price of the option. Now, the strike price value is grouped with the coefficient of z^0 in the polynomial $g_2(k,\sigma,z)$. The next thing that we need to do is to find the roots of this 4^{th} degree polynomial in z. Now, being a 4^{th} degree polynomial, it can have at most 4 real roots. Let these be ρ_1, ρ_2, ρ_3 and ρ_4. Without loss of generality, let us assume that

$$\rho_1 \le \rho_2 \le \rho_3 \le \rho_4.$$

Our objective is to calculate the price of the option in the region where the function $E(Y-F|Z)$ is positive. This is the area defined by the intervals $(-\infty,\rho_1)$, (ρ_2,ρ_3) and (ρ_4,∞). In case, the polynomial has some imaginary roots, we ignore them and concentrate on the real roots only.

Let us define the coefficient of z^j by a_j for j = 0, 1, 2, 3, 4. Thus, we have,

$a_0 = 1-\frac{1}{5}\sigma^2+\frac{1}{35}\sigma^4-\frac{29}{560}\sigma^4k^2+\frac{1}{24}\sigma^3k^3-\frac{11}{80}\sigma^3k+\frac{1}{6}\sigma^2k^2+\frac{1}{120}\sigma^4k^4+\frac{1}{2}k-F,$
$a_1 = \sqrt{3}[\frac{5}{24}\sigma^2k+\frac{1}{3}\sigma+\frac{3}{40}\sigma^3k^2-\frac{279}{4480}\sigma^4k-\frac{3}{35}\sigma^3+\frac{7}{360}\sigma^4k^3],$
$a_2 = -\frac{2}{35}\sigma^4+\frac{29}{560}\sigma^4k^2+\frac{11}{80}\sigma^3k+\frac{1}{5}\sigma^2,$

$a_3 = \sqrt{3}[\frac{93}{4480}\sigma^4 k + \frac{1}{35}\sigma^3]$,

$a_4 = \frac{1}{105}\sigma^4$.

Knowing the values of r, σ and F, we know $k = \frac{r}{\sigma}$. Once we know the values of k, σ and F, we can easily find the roots of the polynomial in z. Having obtained the value of ρ_1, ρ_2, ρ_3 and ρ_4 to calculate the value of the option, we then need to calculate

$$\sum_{j=0}^{4}\int_{-\infty}^{\rho_1} a_j z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=0}^{4}\int_{\rho_2}^{\rho_3} a_j z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=0}^{4}\int_{\rho_4}^{\infty} a_j z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$= \sum_{j=0}^{4} a_j \int_{-\infty}^{\rho_1} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=0}^{4} a_j \int_{\rho_2}^{\rho_3} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

$$\geq \sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{12}$$

Here, a_j is the co-efficient of z^j and being independent of z can be taken outside the integral.

Since we are interested in the lower bound to the price, we look at

$$\sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz, \tag{13}$$

where ρ_4 is the largest of the real roots. Furthermore, in practice the contribution from

$$\sum_{j=0}^{4} a_j \int_{-\infty}^{\rho_1} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=0}^{4} a_j \int_{\rho_2}^{\rho_3} z^j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz,$$

is negligible and hence can be ignored. This fact is also reflected in the results obtained, as shown in the tables (Tables 1 - 4).

Being interested in the lower bound of the price as given by equation (13), we are thus interested in the following integrals;

$$\int_{\rho}^{\infty} z^0 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = 1 - \Phi(\rho),$$

$$\int_{\rho}^{\infty} z \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}e^{-\frac{\rho^2}{2}},$$

$$\int_{\rho}^{\infty} z^2 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{\rho e^{-\frac{\rho^2}{2}} + \sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}},$$

$$\int_{\rho}^{\infty} z^3 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{2+\rho^2}{\sqrt{2\pi}}e^{-\frac{\rho^2}{2}},$$

$$\int_{\rho}^{\infty} z^4 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{\rho^3 e^{-\frac{\rho^2}{2}} + 3\rho e^{-\frac{\rho^2}{2}} + 3\sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}}.$$

Finally, the only thing that remains to be done to calculate the value of the lower bound of the option is to multiply the appropriate coefficients of z, a_j (j = 0, 1,2,3,4) with the corresponding values of the integrals and add them up. To obtain an approximation to the price of the asset, all that needs to be done is to put the strike price of the option at 0. Thus, in effect one needs to calculate

$$a_0[1-\Phi(\rho_4)] + a_1\frac{1}{\sqrt{2\pi}}e^{-\frac{\rho_4^2}{2}} + a_2[\frac{\rho e^{-\frac{\rho_4^2}{2}} + \sqrt{2\pi}(1-\Phi(\rho_4))}{\sqrt{2\pi}}] + a_3[\frac{2+\rho_4^2}{\sqrt{2\pi}}e^{-\frac{\rho_4^2}{2}}]$$

$$+a_4[\frac{\rho_4^3 e^{-\frac{\rho_4^2}{2}} + 3\rho_4^{-\frac{\rho_4^2}{2}} + 3\sqrt{2\pi}(1-\Phi(\rho_4))}{\sqrt{2\pi}}] = \Omega(r,\sigma,F) \quad \text{say.} \tag{14}$$

This is because $k = \frac{r}{\sigma}$ and ρ is a function of k, σ and F.

The values obtained using this method is given in Tables 1 to 4. The values of σ and r as well as the strike price b are exactly the same as the ones used by Rogers and Shi (1995). In fact, we also give the values of the Asian option obtained by Rogers and Shi. We give the values which they denote by LB_2 - according to them, it is the closest approximation to the true price.

4 Conclusion and Remarks

The prices calculated by using this approach for the Asian options are exactly similar to the ones calculated by using the conditioning factor approach - one can easily see that in the data shown in tables 1 - 4 where the price obtained by this method is compared with the Rogers and Shi (using conditioning factor) price.

This method has a few distinct advantages. First of all, it is very fast and can provide output in real time and does not need to perform any numerical integration. Secondly, and more importantly, all calculation in this approach can be carried out on such simple machines as a programmable calculator. The only care that needs to taken is to ensure that it has the facility to calculate the roots of a polynomial. Though the method involves the calculation of the roots of a 4^{th} degree polynomial, packages exist for it and can be done very easily. Further, the alternative would be to make use of two numerical integrations and thus obtaining the roots of the polynomial in z seems to be much better option.

This method can be easily extended to price zero coupon bonds as well as options on zero coupon bonds (see Basu (1999) for further details).

5 References

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6 Tables

The following tables show the comparison of the values obtained by the alternative method of valuing an Asian option as contrasted to the values obtained by Rogers and Shi (1995). The values obtained by the alternative method is given as in the calculated price.

Table 1 : σ = 0.05

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	95	7.178	7.178
	100	2.716	2.716
	105	0.337	0.337
0.09	95	8.809	8.809
	100	4.308	4.308
	105	0.958	0.958
0.15	95	11.094	11.094
	100	6.794	6.794
	105	2.744	2.744

Table 2 : σ = 0.1

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	11.951	11.951
	100	3.641	3.641
	110	0.331	0.331
0.09	90	13.385	13.385
	100	4.915	4.915
	110	0.630	0.630
0.15	90	15.399	15.399
	100	7.028	7.028
	110	1.413	1.413

Table 3 : σ = 0.2

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	12.596	12.595
	100	5.763	5.762
	110	1.989	1.989
0.09	90	13.831	13.831
	100	6.777	6.777
	110	2.545	2.545
0.15	90	15.642	15.641
	100	8.408	8.408
	110	3.555	3.554

Table 4 : σ = 0.3

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	13.952	13.952
	100	7.944	7.944
	110	4.070	4.070
0.09	90	14.983	14.983
	100	8.827	8.827
	110	4.695	4.695
0.15	90	16.512	16.512
	100	10.208	10.208
	110	5.728	5.728

7 Appendix: Choice of an appropriate conditioning factor

Earlier in the paper, we have used a conditioning factor argument to price the option - rather obtain the lower bound to the price of the option. The motivation of using the conditioning factor approach was derived from the use of a similar technique by Rogers and Shi to value an Asian option. Rogers and Shi have not given any mathematical justification for the choice of the conditioning factor - they just state that they tried a number of conditioning factors and the one used by them was found to perform the best. Here, we try to provide a mathematical justification to the conditioning factor used - incidentally the one used in this paper based on the justification that follows is similar to the one used by Rogers and Shi. We first try try to find a general form of the conditioning factor for a general Gaussian distribution and then show the exact form for the Brownian Motion case.

Let us define

$$f(U) = [U - K]^+ = \max([U - K], 0) \tag{15}$$

where U is a random variable and K is a constant (the strike price). Note that f is convex. In general, we are interested in finding

$$E(f(U)];$$

and in particular, in the case of pricing of Rogers and Shi problem of valuing of Asian options,

$$U = X.$$

Further, X is as

$$X = \int_0^1 e^{\sigma Y_s} ds$$

where $\{Y_s, 0 \leq s \leq 1\}$ is a stochastic process and σ is the instantaneous variance of the process Y_s.

Using the fact that the unconditional expectation is the expected value of the conditional expectation and also Jensen's Inequality, we have

$$E[f(U)] = E[E\{f(U)|Z\}] \geq E[f(E\{U|Z\})] \tag{16}$$

where, Z is another suitably normalised random variable used for conditioning purposes.

The lower bound in the equation (16) is not guaranteed to be good. However, an estimate the error made using the following argument. For any random variable U, we have,

$$\begin{aligned}
0 &\leq E(U^+) - E(U)^+ \\
&= \frac{1}{2}(E(|U|) - |E(U)|) \\
&\leq \frac{1}{2}E(|U - E(U)|) \\
&\leq \frac{1}{2}\sqrt{\mathrm{Var}(U)}.
\end{aligned}$$

This implies that for the Rogers and Shi case, we have

$$0 \leq E\left[E([X - K]^+|Z) - E([X - K]|Z)^+\right] \leq \frac{1}{2}E\left[\sqrt{\mathrm{Var}([X - K]|Z)}\right]. \tag{17}$$

Further, using Cauchy - Schwarz inequality, we have from equation (17)

$$\frac{1}{2}E\left[\sqrt{\mathrm{Var}([X - K]|Z)}\right] \leq \frac{1}{2}\sqrt{E\left[\mathrm{Var}([X - K]|Z)\right]} = \frac{1}{2}\sqrt{E\left[\mathrm{Var}(X|Z)\right]}. \tag{18}$$

Thus, in order to minimise the error made by using the lower bound as an approximation to the true value as given in equation (16), we try to choose the conditioning factor Z such that

$$E\left[\mathrm{Var}(X|Z)\right] \tag{19}$$

is minimised.

We now look at the exact form of the conditioning factor that minimises the expected value of the conditional variance. We look at a general Gaussian process and try to obtain the conditioning factor that minimises the expected conditional variance.

Let $\{Y_s, 0 \leq s \leq 1\}$ be a general Gaussian process, where $Y_s = \int_{-\infty}^{\infty} L(s,u)dB_u$ subject to the constraint $\sup_s(\int_{-\infty}^{\infty} L^2(s,u)du \leq \infty)$. Also, let the conditioning variable, in general, be Z, where

$$Z = \int_{-\infty}^{\infty} a(u)dB_u, \tag{20}$$

a($\bullet$) is so chosen that it satisfies the condition $\int_{-\infty}^{\infty} a^2(u)du = 1$. This condition ensures that the variance of the conditioning factor is 1. B_u is a standard Brownian motion. We are interested in finding

$$E(\int_0^t e^{\sigma Y_s} ds|Z) \quad \text{and} \quad \text{Var}(\int_0^t e^{\sigma Y_s} ds|Z) \tag{21}$$

where σ^2 is the instantaneous variance of the process. For this, we require the following terms : $E(Y_s|Z)$, $\text{Var}(Y_s|Z)$ and $\text{Cov}(Y_s, Y_v|Z)$.

Our objective is to find Z such that $\text{E}(\text{Var}(\int_0^1 e^{\sigma Y_s} ds|Z))$ minimum.

Now, for $0 \leq s \leq 1$, we have

$$E(Y_s|Z) = Z\int_{-\infty}^{\infty} L(s,u)a(u)du \tag{22}$$

$$\text{Var}(Y_s|Z) = \int_{-\infty}^{\infty} L^2(s,u)du - \left(\int_{-\infty}^{\infty} L(s,u)a(u)duds\right)^2 \tag{23}$$

$$\text{Cov}(Y_s, Y_v|Z) = \int_{-\infty}^{\infty} L(s,u)L(v,u)du - \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du \tag{24}$$

Thus, we can easily compute $E(\int_0^1 e^{\sigma Y_s} ds|Z)$ and $E(\int_0^1 e^{\sigma Y_s} ds \int_0^1 e^{\sigma Y_v} dv|Z)$. This will led us to the computation of $\text{Var}(\int_0^1 e^{\sigma Y_s} ds|Z)$ and then subsequently $E(Var(\int_0^1 e^{\sigma Y_s} ds|Z))$.

Simplifying, we thus get

$$E(Var(\int_0^1 e^{\sigma Y_s} ds|Z)) = A - B \tag{25}$$

$$A = \left[\left\{\int_0^1 \int_0^1 \left\{exp\left(\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right]\right) exp\left(\sigma^2 \int_{-\infty}^{\infty} L(s,u)L(v,u)du\right)\right\} dvds\right\}\right.$$

and

$$B = \int_0^1 \int_0^1 exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right]\right\} exp\left\{\sigma^2 \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du\right\} dvds$$

Now to minimise the expected value of the conditional variance, we need to maximise the second term of equation (25), given by B. This is because the other part of equation (25) does not involve any $a(u)$ and hence is fixed for fixed values of σ. Further σ is assumed to be small, thereby allowing the linearisation to be carried out. On linearisation of the integrand in B, we have

$$\int_0^1 \int_0^1 \left\{1 + \frac{\sigma^2}{2}\left(\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right) + \sigma^2 \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du + O(\sigma^4)\right\} dvds. \tag{26}$$

Now, equation (26) contains some terms independent of $a(s)$. These terms are fixed and hence B is maximised by maximising the terms involving $a(u)$ in equation (26). This is the same as maximising

$$\left(\int_0^1 \int_{-\infty}^{\infty} L(s,u)a(u)duds\right)^2,$$

which is the same as maximising

$$\int_0^1 \int_{-\infty}^{\infty} L(s,u)a(u)duds$$

subject to the constraint $\int_{-\infty}^{\infty} a^2(s)ds = 1$, i.e. the variance of the conditioning factor is 1. On changing the order of integration of the function to be maximised, we are required to maximise $\int_{-\infty}^{\infty} \int_0^1 L(s,u)dsa(u)du$ subject to the constraint $\int_{-\infty}^{\infty} a^2(s)ds = 1$. This implies that the optimal

$$a(u) \propto \int_0^1 L(s,u)ds \quad : \quad u \leq 1$$

$$\Rightarrow Z = \int_{-\infty}^{\infty} a(u)dB_u = \int_{-\infty}^{\infty} \int_0^1 L(s,u)dsdB_u \quad : \quad u \leq 1.$$

Finally, we use the theory described above to obtain the exact form of the conditioning factor Z when the Gaussian process follows a Brownian Motion. Here,

$$L(s,u) = \begin{cases} 1 & : \quad 0 \leq u \leq s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

Thus, we have, $Z = \int_0^1 \int_u^1 dsdB_u = \int_0^1 (1-u)dB_u = \int_0^1 Y_u du$, where $\{Y_u, 0 \leq u \leq 1\}$ is a Brownian Motion.