

Using a LIBOR Market Model with Jump Risk to Price Quanto Range Accrual Interest Rate Swap

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Abstract

As an important economic index, interest rates are assumed to be constant in the Black and Scholes model (1973); however, they actually fluctuate due to economic factors. Using a constant interest rate to evaluate derivatives in a stochastic model will produce biased results. This research derives the LIBOR market model with jump risk, assuming that interest rates follow a continuous time path and tend to jump in response to sudden economic shocks. We then use the LIBOR model with jump risk to price a Quanto Range Accrual Interest Rate Swap (QRAIRS). Given that the multiple jump processes are independent, we employ numerical analysis to further demonstrate the influence of jump size, jump volatility, and jump frequency on the pricing of QRAIRS. Our results show a negative relation between jump size, jump frequency, and the swap rate of QRAIRS, but a positive relation between jump volatility and the swap rate of QRAIRS. When new information emerges, the resulting increase in jump size reduces the value of LIBOR, which in turn lowers the value of QRAIRS. Similarly, the value of QRAIRS declines when the jump frequency of LIBOR increases. This is because jump frequency is associated with higher uncertainty risk, and the market pays out a premium for bearing such risk. On the other hand, when jump volatility increases, both parties must agree to a higher swap rate because the floating rate payer is subsidized by the fixed rate payer for bearing risk.

Keywords: Stochastic model in continuous time interest rate, LIBOR market model, jump risks, Quanto Range Accrual Interest Rate Swap (QRAIRS).

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1. Introduction

Interest rates constitute an important economic index, and unexpected movement in interest rates may cause changes in the pricing of financial assets. Therefore, interest rate swaps are often used by investors to alter their exposure to interest rate risk. Since interest rate swaps are a major instrument in global financial markets, the ability to price interest rate derivatives accurately is of great importance.

First introduced by Vasicek (1977), the short rate model following a mean reverting process may produce negative interest rates. To overcome this disadvantage, Cox, Ingersoll and Ross (1985) (CIR) introduce a model in which expectations, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining bond prices. By further assuming that all interest rate claims are priced contingent on only the short rate, CIR derive an equilibrium pricing model that relies on a continuous arbitrage argument.

Ho and Lee (1986) are the first to incorporate a discount function into the pricing of contingent claims. Given a term structure, their model derives the subsequent stochastic movement of the term structure such that the movement is arbitrage-free. However, the short rate movement is assumed to be constant, which results in a failure to derive an interest rate that is always consistent with the market interest rate. Therefore, Hull and White (1990) incorporate the discount function into the Vasicek model and relax the constant short rate assumptions of Ho and Lee (1986) to derive the one-factor equilibrium term structure model, which is capable of determining a short rate process that is consistent with the current term

structure of interest rates. Furthermore, Ho and Lee (1986), and Hull and White (1990) apply term structure equilibrium models to the pricing of discount bonds. In these models, all discount bonds are priced relative to the stochastic short rate such that there are no arbitrage opportunities in their trading. Researchers have since developed such long-term interest rate models as the instantaneous forward rate model (Heath, Jarrow and Morton, 1992) and the LIBOR market model (Brace, Gatarek and Musiela, 1997), all based on the no-arbitrage condition.

Merton (1976) points out that since stock price dynamics do not follow a continuous sample path, they should be modeled as a “jump” process with a non-continuous sample path that reflects the impact of the emergence of important new information. Later, research increasingly focused on the interest rate jump-diffusion model. Shikarawa (1991) employs a pure jump model to price derivatives; Björk, Kabanov and Runggaldier (1997) extend the Heath, Jarrow and Morton (1992) “drift condition” to incorporate marked point process to derive the instantaneous forward rate jump-diffusion model. The jump size in the marked point process is assumed to be drawn from a continuous distribution with multiple jump processes, each of which is associated with a constant jump value scaled by time-deterministic jump volatility. Such a model allows for solving for a no-arbitrage condition and a risk-neutral probability measurement. Glasserman and Kou (2003) studies the arbitrage-free dynamics of interest rates in the presence of jumps and diffusion. Using their formulas, they demonstrate the impact of jumps on the implied volatility of interest rate swaps. Finally, following the model of Björk, Kabanov and Runggaldier (1997), Chiarella and To (2003) propose an instantaneous forward rate jump-diffusion model. Assuming that multiple marked point processes are independent and follow a Poisson process, Chiarella and To (2003) derive no-arbitrage pricing of interest rate futures in a risk-neutral condition. Furthermore, they use the full-information-likelihood function to prove the pricing accuracy of interest rate futures

after incorporating the jump process.

The use of interest rate with jump risk is more consistent with real world observations (e.g., the emergence of important economic information often results in discontinuous interest rate movement). For example, China recently increased interest rates to cool down an overheated economy that resulted from the maintenance of low interest rates to lure foreign investment. Figure 1 shows that movements in LIBOR from 1998 to 2005 did not follow a smooth path and were discontinuous. Prior to December 23, 1998, fluctuations in interest rates tended to be small, ranging between 4.9% and 5.0%. However, interest rates suddenly jumped to 7.25%~7.27% in late-December 1998, returning to their previous level shortly thereafter. Hence, movements in interest rates are often not smooth but rather, discontinuous—they tend to jump following unexpected world events.

Therefore, using a constant interest rate over a continuous time period to evaluate derivatives in a stochastic model will produce biased results. The present research has two objectives. We first derive the LIBOR market model with jump risk assuming that interest rates tend to jump in response to sudden economic shocks. We then use this LIBOR model with jump risk to price a Quanto Range Accrual Interest Rate Swap (QRAIRS). Given that the multiple jump processes are independent of each other, we then employ numerical analysis to demonstrate the impact of jump size, jump volatility and jump frequency on the pricing of the QRAIRS. Our results show a negative relation between jump size, jump frequency, and the swap rate of QRAIRS, but a positive relation between jump volatility and the swap rate of QRAIRS.

This paper proceeds as follows. QRAIRS and the model assumptions are introduced in Section 2. Section 3 incorporates the interest rate jump process to derive the LIBOR market model with jump risk. The QRAIRS pricing model is then derived to incorporate the

influence of jump size, jump volatility, and jump frequency. Numerical analysis demonstrating the relationship between swap rate and jump size, jump frequency, and jump volatility are presented in Section 4. Section 5 details our conclusion.

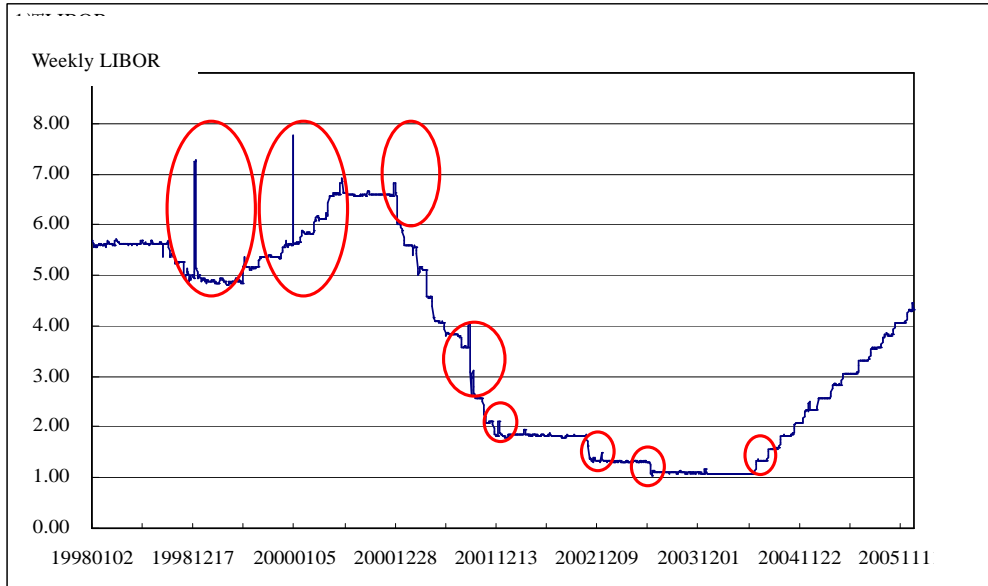


Figure1. Weekly LIBOR: US Dollars

Data source: Economagic.com: Economic Time Series, <http://www.economagic.com>.

2. The Quanto Range Accrual Interest Rate Swap (QRAIRS) Contract

Consider a hypothetical 1-year QRAIRS initiated on T_0 , with a termination date T_4 , between counterparties A and B. We assume A agrees to pay B at a fixed interest rate per annum on a notional principal amount of $\$F$, and in return B agrees to pay A the 3-Month USD LIBOR rate on the same notional principal amount. Hence A is the fixed-rate payer and B is the floating-rate payer. We further assume that this agreement specifies that payments are to be swapped every 3 months and that the interest rate, K , is compounded annually. There are four swaps each year. A schematic of this swap agreement is shown in Figure 2. If the LIBOR falls within K_1 and K_2 , A would pay B an amount equal to $\$F \times K/4$. This is the interest on

the $\$F$ principal for 3 months at interest rate K . On the other hand, B would pay A interest on the $\$F$ principal at the 3-month LIBOR rate. That is, B would pay A an amount equal to $\$F \times a \times N/D$. K_1 represents interest rate floor; K_2 is the interest rate cap; K denotes the fixed rate; F is the notional principal; N shows the total number of calendar days in the period between swaps; and D is the total number of calendar days in the year. (See Table 1 and Figure 2)

Table 1. The content of a Quanto Range Accrual Interest Rate Swap

| | |
|---|--|
| Notional principal | F |
| Currency | USD (\$) |
| Trade type | Quanto Range Accrual Interest Rate Swap |
| Trade date | T |
| Effective date | T_0 |
| Termination date | T_4 |
| Duration | 1 year |
| Fixed rate payer | A |
| Fixed rate | K |
| Fixed rate payment dates | Payments are to be swapped every 3 months; there are four exchanges of payment. |
| Fixed rate count convention | Actual/360 |
| Reset date | The first day of each trading date |
| Floating rate payer | B |
| Floating rate payment | $a \times \frac{N}{D}$, N is the total number of calendar days in the period between swaps, D is the total number of calendar days in a year. |
| Quanto range accrual interest rate interval | $[K_1, K_2]$ |
| Floating rate | 3-Month USD LIBOR |

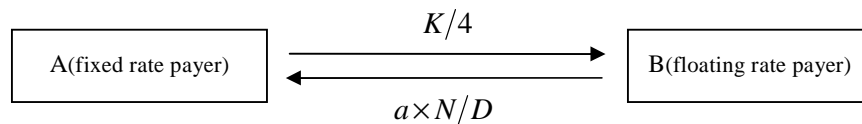


Figure 2. The contract form of QRAIRS

3. LIBOR Market Model with Jump Risk and the Pricing of QRAIRS

Merton (1976) argues that investors react swiftly to the arrival of important information, hence the sample path of stock prices does not satisfy the continuity property. To be sure, stock price dynamics can be decomposed into two components: (1) The “normal” variations in stock prices due to factors such as interest rate changes, temporary imbalances between supply and demand, and mild changes in the economic outlook. These factors often cause small changes in stock prices, which can be modeled by a standard geometric Brownian motion characterized by a constant variance per unit of time and a continuous sample path. This type of stock price change is called the “diffusion” process. (2) “Abnormal” variations in stock prices that result from the sudden emergence of important information, often resulting in dramatic stock price changes. Typically, such information is specific to the firm or industry; this type of price change is called a “jump” process.

The jump process is also applicable to interest rate changes. In the next section, we construct an instantaneous forward rate jump-diffusion model based on the framework of Björk, Kabanov, and Runggaldier (1997). Assuming that multiple marked point processes are independent and follow the Poisson process, we incorporate the jump dynamics in the LIBOR stochastic process such that the instantaneous forward rate process of $df(t, T)$ can be described as

$$df(t, T) = \underbrace{\alpha(t, T)dt + \sigma(t, T)dW(t)}_{1st} + \underbrace{\sum_{i=1}^r h_i(t, x_i, T)dN_i(t)}_{2nd}, \quad (1)$$

where α represents the changes in the instantaneous forward rate; σ is the standard

deviation of instantaneous forward rate; $dW(t)$ stands for the Wiener process of forward rate in t ; $dN_i(t)$ is the i^{th} Poisson process with expected value $\lambda_i dt$, where $i=1,2,\dots,r$; $h_i(t, x_i, T)$ is the i^{th} function of the forward rate variations, where $i=1,2,\dots,r$; and random variable x stands for the jump size of forward rate. The first component of Eq. (1) is the “normal” variation in interest rate caused by factors such as a temporary imbalance between the supply of and demand for deposits/loans, causing small changes in the instantaneous forward rate. This component can be modeled by a standard geometric Brownian motion. The second component of Eq. (1) is the “abnormal” variation in interest rate due to the emergence of important new information about interest rates, resulting in dramatic impact on the instantaneous forward rate. The second component is the jump term of the interest rate stochastic, which follows r independent Poisson processes.

Based upon Eq. (1), in the following subsections we: (1) derive the no-arbitrage condition of the LIBOR market model with r independent jump risks, assuming that the model follows Brownian motion and r jump processes; (2) convert the LIBOR market model from a risk-neutral measure to a forward measure; and (3) derive the pricing formula of QRAIRS based upon the forward measure in the LIBOR market model.

3.1. No-arbitrage condition and the stochastic process of LIBOR in risk-neutral probability

In this subsection, the no-arbitrage condition of bonds is derived from the instantaneous forward rate process. Based on this no-arbitrage condition and the relationship between the instantaneous forward rate and LIBOR, we then derive the LIBOR market process under the risk-neutral measure.

The relationship between the value of a zero coupon bond and the forward rate is given by

$$B(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\},$$

where $B(t,T)$ is the price of a default-free pure discount bond; therefore, it can be considered equivalent to the value of \$1 to be received at time T . Let

$$B(t,T) = \exp\left[-\int_t^T f(t,u)du\right] = \exp[V(t,T)],$$

then

$$\begin{aligned} dV &= d\left[-\int_t^T f(t,u)du\right] = -\left[-f(t,t)dt + \int_t^T df(t,u)du\right] \\ &= f(t,t)dt - \int_t^T (\alpha(t,u)dt)du - \int_t^T (\sigma(t,u) \cdot dW)du - \sum_{i=1}^r \int_t^T h_i(t, x_i, u)du \cdot dN_i(t) \\ &= f(t,t)dt - \alpha^*(t,T)dt - \sigma^*(t,T)dW - \sum_{i=1}^r \int_t^T h_i(t, x_i, u)du \cdot dN_i(t), \end{aligned} \quad (2)$$

where $\alpha^*(t,T) = \int_t^T \alpha(t,u)du$, $\sigma^*(t,T) = \int_t^T \sigma(t,u)du$. Based upon Björk, Kabanov and Runggaldier (1997), modified with i independent jump processes, we employ the Itô formula to derive the stochastic process value of the zero coupon bond as follows

$$\begin{aligned} dB(t,T) &= B(t,T)dx + \frac{1}{2}B(t,T)(dx)^2 \\ &\quad + \sum_{i=1}^r \left[\exp\left(-\int_t^T f(t,u)du - \int_t^T h_i(t, x_i, u)du\right) - \exp\left(-\int_t^T f(t,u)du\right) \right] dN_i(t). \end{aligned} \quad (3)$$

Substituting Eq. (1) into Eq. (3), we obtain

$$\begin{aligned} dB(t,T) &= B(t,T) \left\{ R(t) - \alpha^*(t,T) + \frac{1}{2} [\sigma^*(t,T)]^2 \right\} dt - B(t,T) \sigma^*(t,T) dW(t) \\ &\quad + \sum_{i=1}^r \exp\left[-\int_t^T f(t,u)du\right] \left\{ \exp\left[-\int_t^T h_i(t, x_i, u)du\right] - 1 \right\} dN_i(t). \end{aligned} \quad (4)$$

where $R(t)$ is the spot rate. Discounting the zero coupon bond to derive a no-arbitrage condition in risk-neutral probability, and substituting the no-arbitrage condition into Eq. (1), we obtain the forward stochastic process in the no-arbitrage condition. First, we define the discount factor $D(t)$ as

$$D(t) = \exp\left(-\int_0^t R(u)du\right).$$

Furthermore, $D(t)B(t,T)$ follows the Martingale process. That is, if we discount the zero coupon bond to time t in the risk-neutral probability, the discounted price will be identical to the current price at time t , implying the absence of arbitrage, hence the intercept must be equal to zero. Therefore,

$$\begin{aligned} d[D(t)B(t,T)] &= -D(t)B(t,T)\sigma^*(t,T)dW^\mathcal{Q}(t) \\ &\quad + D(t)B(t-,T)\sum_{i=1}^r\left\{\exp\left[-\int_t^T h_i(t,x_i,u)du\right]-1\right\}(dN_i(t)-\lambda_i^\mathcal{Q}dt). \end{aligned} \quad (5)$$

Incorporating the discount factor from Eq. (4),

$$\begin{aligned} d(D(t)B(t,T)) &= D(t)B(t,T)\left\{(-\alpha^*(t,T)+\frac{1}{2}[\sigma^*(t,T)]^2)dt-\sigma^*(t,T)dW(t)\right\} \\ &\quad + D(t)B(t-,T)\sum_{i=1}^r\left\{\exp\left[-\int_t^T h_i(t,x_i,u)du\right]-1\right\}dN_i(t). \end{aligned} \quad (6)$$

Since under the no arbitrage condition the intercept is equal to zero, comparing Eq. (5) and (6),

we obtain

$$-\alpha^*(t,T)+\frac{1}{2}[\sigma^*(t,T)]^2+\sum_{i=1}^r\left\{\exp\left[-\int_t^T h_i(t,x_i,u)du\right]-1\right\}\lambda_i^\mathcal{Q}f(x_i)dx_i=-\sigma^*(t,T)\theta(t), \text{ for all } t, \quad (7)$$

where $\theta(t)$ is the risk premium of the diffusion rate. Since each zero coupon bond in a term structure must correspond to the no-arbitrage condition in the risk-neutral probability, differentiating Eq. (7) with respect to T , the no-arbitrage condition is obtained as

$$-\alpha(t, T) + \sigma(t, T)\sigma^*(t, T) - \int_0^\infty h_i(t, x_i, T) \exp\left[-\int_t^T h_i(t, x_i, u) du\right] \lambda_i^Q f(x_i) dx_i = -\sigma(t, T)\theta(t), \quad \text{for all } t, \quad (8)$$

where $f(x_i)$ is the probability distribution of stochastic variable x_i . By substituting the no arbitrage condition into Eq. (1), the instantaneous forward rate with r independent jump-diffusion processes in risk-neutral probability becomes

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T) \cdot dW^Q(t) + \sum_{i=1}^r h_i(t, x_i, T) dN_i(t), \quad (9)$$

where $\alpha(t, T) = \sigma(t, T)\sigma^*(t, T) - \int_0^\infty \sum_{i=1}^r h_i(t, x_i, T) \exp\left[-\int_t^T h_i(t, x_i, u) du\right] \lambda_i^Q f(x_i) dx_i$.

Similar to Brace, Gatarak and Musiela (1997), the stochastic process of LIBOR in risk-neutral probability can be derived using the relationship between instantaneous forward rate and LIBOR. Therefore, the relationship between forward rate and LIBOR is given by

$$1 + \delta L_n(t, T_n) = \exp \int_{T_n}^{T_n + \delta} f(t, u) du, \quad (10)$$

where $dL_n(t, T_n)$ stands for the compounded forward LIBOR (from T_n to $T_n + \delta$) at time t , and

δ is the constant interval measured in year. Letting $Z(T_n, T_n + \delta) = \int_{T_n}^{T_n + \delta} f(t, u) du$,

$$\begin{aligned}
dZ &= \int_{T_n}^{T_{n+\delta}} df(t,u)du \\
&= \int_{T_n}^{T_{n+\delta}} \sigma(t,u)du \} dW^Q(t) + \int_0^\infty \sum_{i=1}^r \left[\int_{T_n}^{T_{n+\delta}} h_i(t, x_i, u)du \right] dN_i(t) f(x)dx \\
&\quad + \int_{T_n}^{T_{n+\delta}} \left\langle \frac{\partial}{\partial u} \left\{ \frac{1}{2} [\sigma^*(t,u)]^2 \right\} du \right\rangle dt + \int_0^\infty \sum_{i=1}^r \frac{\partial}{\partial u} \left\{ \exp \left[- \int_{T_n}^{T_{n+\delta}} h_i(t, x_i, u)du \right] \right\} \lambda_i^Q dt f(x)dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
dZ &= \int_{T_n}^{T_{n+\delta}} df(t,u)du + \sum_{i=1}^r \int_0^\infty \left\{ \exp \left[- \int_t^{T_{n+\delta}} h_i(t, x_i, u)du \right] - \exp \left[- \int_t^{T_n} h_i(t, x_i, u)du \right] \right\} \lambda_i^Q dt f(x)dx \\
&\quad + \int_0^\infty \sum_{i=1}^r \left[\int_{T_n}^{T_{n+\delta}} h_i(t, x_i, u)du \right] dN_i(t) f(x)dx.
\end{aligned}$$

According to the Itô formula and the jump stochastic process of Elliott (1982), the LIBOR jump-diffusion process follows

$$\begin{aligned}
dL_n(t,T) &= \frac{1}{\delta} \exp[Z(T_n, T_{n+\delta})] d[Z(T_n, T_{n+\delta})] + \frac{1}{2\delta} \exp[Z(T_n, T_{n+\delta})] d[Z(T_n, T_{n+\delta})]^2 \\
&\quad + \sum_{i=1}^r \frac{1}{\delta} \int_0^\infty \left\{ \exp[Z(T^-)] \cdot \exp \left\{ \int_{T_n}^{T_{n+\delta}} h_i(t, x_i, u)du \right\} - \exp[Z(T^-)] \right\} dN_i(t) f(x)dx.
\end{aligned} \tag{11}$$

Substituting $dZ(T_n, T_{n+\delta})$ into Eq. (11) yields the jump-diffusion stochastic process of LIBOR

as

$$\begin{aligned}
dL_n(t, T_n) &= \frac{1}{\delta} [1 + \delta L_n(t, T_n)] \left\{ \left[\sigma^*(t, T_{n+\delta}) \right]^2 - \sigma^*(t, T_{n+\delta}) \sigma^*(t, T_n) \right\} dt \\
&\quad + \frac{1}{\delta} [1 + \delta L_n(t, T_n)] \left\{ \sigma^*(t, T_{n+\delta}) - \sigma^*(t, T_n) \right\} dW^Q(t) \\
&\quad + \sum_{i=1}^r \frac{1}{\delta} \int_0^\infty [1 + \delta L_n(t, T_n)] \\
&\quad \left\{ \exp \left[- \int_t^{T_{n+\delta}} h_i(t, x_i, u)du \right] - \exp \left[- \int_t^{T_n} h_i(t, x_i, u)du \right] \right\} \lambda_i^Q dt f(x)dx \\
&\quad + \sum_{i=1}^r \frac{1}{\delta} \int_0^\infty [1 + \delta L_n(t^-, T_n)] \left\{ \exp \left[\int_{T_n}^{T_{n+\delta}} h_i(t, x_i, u)du \right] - 1 \right\} dN_i(t) f(x)dx.
\end{aligned} \tag{12}$$

From Brace, Gatarek and Musiela (1997), assuming that in each probability measure the stochastic process of LIBOR follows a lognormal distribution, the stochastic process of LIBOR becomes

$$dL_n(t, T_n) = m_n L_n(t, T_n) dt + \gamma_n L_n(t, T_n) dW^Q(t) + L_n(t, T_n) dJ_n(t), \quad (13)$$

where m_n is the drift rate, γ_n is the diffusion rate, and $J_n(t)$ is the jump rate of LIBOR.

Rearranging the diffusion rate component and comparing Eq. (12) with Eq. (13) we obtain

$$\sigma^*(t, T_{n+1}) - \sigma^*(t, T_n) = \frac{\gamma_n \delta L_n(t, T_n)}{1 + \delta L_n(t, T_n)}, \quad (14)$$

and

$$\sigma^*(t, T_{n+1}) \left[\sigma^*(t, T_{n+1}) - \sigma^*(t, T_n) \right] = \frac{\gamma_n \delta L_n(t, T_n)}{1 + \delta L_n(t, T_n)} \sigma^*(t, T_n) + \left[\frac{\gamma_n \delta L_n(t, T_n)}{1 + \delta L_n(t, T_n)} \right]^2.$$

If the interval from t to u is smaller than δ , i.e., $t < u < t + \delta$, $\sigma^*(t, T_n) = 0$, then

$$\gamma_n \delta L_n(t, T_n) \sigma^*(t, T_n) + \frac{\gamma_n^2 \delta L_n^2(t, T_n)}{1 + \delta L_n(t, T_n)} = \sum_{k=\eta(t)}^n \frac{\gamma_n \gamma_k \delta L_n(t, T_n) L_k(t, T_k)}{1 + \delta L_n(t, T_n)}. \quad (15)$$

Substituting Eq. (15) into Eq. (13), we obtain the LIBOR jump-diffusion process as

$$\begin{aligned}
dL_n(t, T_n) = & \left[\sum_{k=\eta(t)}^n \frac{\gamma_n \gamma_k \delta L_n(t, T_n) L_k(t, T_k)}{1 + \delta L_n(t, T_n)} \right] dt + \gamma_n L_n(t, T) dW^Q(t) \\
& + \sum_{i=1}^r \frac{1}{\delta} \int_0^\infty [1 + \delta L_n(t, T_n)] \\
& \left\{ \exp \left[- \int_t^{T_n+\delta} h_i(t, x_i, u) du \right] - \exp \left[- \int_t^{T_n} h_i(t, x_i, u) du \right] \right\} \lambda_i^Q dt f(x) dx \\
& + \sum_{i=1}^r \frac{1}{\delta} \int_0^\infty [1 + \delta L_n(t^-, T_n)] \left\{ \exp \left[\int_{T_n}^{T_n+\delta} h_i(t, x_i, u) du \right] - 1 \right\} dN_i(t) f(x) dx.
\end{aligned} \tag{16}$$

Rearranging the jump rate component, and let $H(Y)$ be the jump size function of LIBOR, then

$$\begin{aligned}
\frac{1}{\delta} [1 + \delta L_n(t^-, T_n)] \left\{ \exp \left[\int_{T_n}^{T_n+\delta} h_i(t, x_i, u) du \right] - 1 \right\} &= L(t^-, T_n) H_i(x_i) \\
&= \frac{1}{\delta} \left\{ \exp \left[\int_T^{T+\delta} f(t, u) du \right] - 1 \right\} H_i(x_i).
\end{aligned}$$

After transposing we obtain

$$H_i(x_i) = \frac{\left\{ \exp \left[\int_T^{T+\delta} f(t, u) du + \int_T^{T+\delta} h_i(t, x_i, u) du \right] \right\} - 1}{\left\{ \exp \left[\int_T^{T+\delta} f(t, u) du \right] - 1 \right\}} - 1 = Y_i - 1, \tag{17}$$

where Y_i stands for the jump ratio of LIBOR at the i^{th} jump, and $Y_i \geq 1$,

$$Y_i = \frac{\exp \left[\int_T^{T+\delta} f(t, u) du + \int_T^{T+\delta} h(t, x_i, u) du \right] - 1}{\exp \left[\int_T^{T+\delta} f(t, u) du \right] - 1}.$$

Influenced by the movement of the forward rate function $h(t, x_i, T)$, if jumps occur,

$H(Y_i) = Y_i - 1$; if not, $H(Y_i) = 0$, and the jump rate compound Poisson process of LIBOR is

$$\sum_{i=1}^r \int_0^\infty L_n(t^-, T_n) H_{ni}(x_i) dN_i(t) dx f(x) = L(t^-, T_n) d \sum_{i=1}^r \sum_{n=1}^{N_i(t)} H_{ni}(x_i), \quad (18)$$

and

$$\exp \left[\int_{T_n}^{T_n+\delta} h(t, x, u) du \right] = \frac{\delta L_n(t^-, T_n) H_n(x_i)}{1 + \delta L_n(t^-, T_n)} + 1 = \frac{1 + \delta L_n(t^-, T_n) [1 + H_n(x_i)]}{1 + \delta L_n(t^-, T_n)}.$$

Similar to the diffusion rate component, if $t < u < t + \delta$, $\exp \left[\int_{T_n}^u h(t, x_i, u) du \right] = 1$. Using this

result to simplify the second term of Eq. (12), we obtain

$$-\sum_{i=1}^r \int_0^\infty H_{ni}(x_i) L_n(t^-, T_n) \prod_{k=\eta(t)}^n \frac{1 + \delta L_k(t^-, T_k)}{1 + \delta L_k(t^-, T_k) [1 + H_{ki}(x_i)]} \lambda_i^Q dt f(x) dx. \quad (19)$$

Substituting Eq. (18) and Eq. (19) into Eq. (12) yields the stochastic process of LIBOR with risk-neutral probability as

$$\begin{aligned} dL_n(t, T_n) = & \left\{ \sum_{k=\eta(t)}^n \frac{\gamma_n \gamma_k \delta L_n(t, T_n) L_k(t, T_k)}{1 + \delta L_n(t, T_n)} \right. \\ & \left. - \sum_{i=1}^r \int_0^\infty H_{ni}(x_i) L_n(t^-, T_n) \prod_{k=\eta(t)}^n \frac{1 + \delta L_k(t^-, T_k)}{1 + \delta L_k(t^-, T_k) (1 + H_{ki}(x_i))} \lambda_i^Q df(x) dx \right\} dt \\ & + \gamma_n L_n(t, T) dW^Q(t) + L(t^-, T_n) d \sum_{i=1}^r \sum_{n=1}^{N_i(t)} H_{ni}(x_i). \end{aligned} \quad (20)$$

3.2. Forward measure of LIBOR market model

Because the stochastic process of LIBOR involves many parameters under risk-neutral probability, it is difficult to price interest rate derivatives incorporating multi-period LIBOR. To simplify the model, we must transform the diffusion rate and jump rate parameters to the last pricing period. To do so, we rearrange the diffusion rate of Eq.(12) as

$$dL_n(t, T_n) = \frac{1}{\delta} [1 + \delta L_n(t, T_n)] [\sigma^*(t, T_{n+\delta}) - \sigma^*(t, T_n)] [\sigma^*(t, T_{n+\delta}) dt + dW^Q(t)], \quad (21)$$

according to the theory of the transformation of probability measure

$$\tilde{W}^{T+\delta}(t) = \int_0^t \sigma^*(u, T+u) du + \tilde{W}(t),$$

then Eq. (21) becomes

$$dL_n(t, T_n) = \frac{1}{\delta} [1 + \delta L_n(t, T_n)] [\sigma^*(t, T_{n+\delta}) - \sigma^*(t, T_n)] \tilde{W}^{T_{n+\delta}}(t). \quad (22)$$

Similar to Brace, Gatarak and Musiela (1997), let

$$dW^{T_j}(t) = \sigma^*(t, T_j) dt + dW^Q(t),$$

$$dW^{T_{j+1}}(t) = \sigma^*(t, T_{j+1}) dt + dW^Q(t).$$

Comparing these two functions, we obtain

$$dW^{T_j} = [\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})] dt + dW^{T_{j+1}}(t),$$

If we substitute the term $[\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})]$ from Eq. (14), then

$$dW^{T_j} = -\frac{\gamma(t, T_j) \delta L(t, T_j)}{1 + \delta L(t, T_j)} dt + dW^{T_{j+1}}(t).$$

Letting $j = n$, we obtain

$$dW^{T_n} = -\frac{\gamma(t, T_n) \delta L(t, T_n)}{1 + \delta L(t, T_n)} dt + dW^{T_{n+1}}(t).$$

Letting $j = n - 1$, then

$$\begin{aligned} dW^{T_{n-1}} &= -\frac{\gamma(t, T_{n-1})\delta L(t, T_{n-1})}{1 + \delta L(t, T_{n-1})} dt + dW^{T_n}(t) \\ &= -\frac{\gamma(t, T_{n-1})\delta L(t, T_{n-1})}{1 + \delta L(t, T_{n-1})} dt - \frac{\gamma(t, T_n)\delta L(t, T_n)}{1 + \delta L(t, T_n)} + dW^{T_{n+1}}(t). \end{aligned}$$

Repeating this process, we have

$$dW^{T_{j+1}} = -\sum_{i=j+1}^n \frac{\gamma(t, T_i)\delta L(t, T_i)}{1 + \delta L(t, T_i)} dt + dW^{T_{n+1}}(t). \quad (23)$$

Substituting Eq. (23) into Eq. (12), and unifying the multiple-period Brownian motion measures, we obtain

$$\begin{aligned} dL_n(t, T_n) &= \frac{1}{\delta} [1 + \delta L_n(t, T_n)] [\sigma^*(t, T_{n+\delta}) - \sigma^*(t, T_n)] \left[-\sum_{i=n+1}^M \frac{\gamma(t, T_i)\delta L(t, T_i)}{1 + \delta L(t, T_i)} dt \right] \\ &\quad + \frac{1}{\delta} [1 + \delta L_n(t, T)] [\sigma^*(t, T_{n+1}) - \sigma^*(t, T_{M+1})] dW_{M+1}(t). \end{aligned} \quad (24)$$

According to Björk, Kabanov and Runggaldier (1997) the condition for transforming λ^ϱ to the last pricing period is

$$\lambda_{T+\delta} = e^{-\int_t^{T+\delta} h(t,x,u) du} \lambda^\varrho,$$

In Eq. (16) the transformation process of λ^ϱ measure is

$$\begin{aligned}
& \frac{1}{\delta} [1 + \delta L_n(t, T_n)] e^{\int_{T_n}^{T_{M+1}} h(t, x_i, u) du} \left[e^{-\int_{T_n}^{T_{n+1}} h(t, x_i, u) du} - 1 \right] \lambda_{M+1} dt \\
&= \frac{1}{\delta} [1 + \delta L_n(t, T_n)] e^{\int_{T_{n+1}}^{T_{M+1}} h(t, x_i, u) du} \left[1 - e^{\int_{T_n}^{T_{n+1}} h(t, x_i, u) du} \right] \lambda_{M+1} dt \\
&= -H(Y_i) L_n(t^-, T_n) e^{\int_{T_{n+1}}^{T_{M+1}} h(t, x_i, u) du} \lambda_{M+1} dt \\
&= -H(Y_i) L_n(t^-, T_n) \prod_{k=n+1}^M \frac{1 + \delta L_k(t^-, T_k)(1 + H(t_k))}{1 + \delta L_k(t^-, T_k)} \lambda_{M+1} dt.
\end{aligned} \tag{25}$$

Letting $n = M$ and substituting Eq. (25) into Eq. (13), we obtain the LIBOR market model of forward rate measurement with jump risk as

$$dL_n(t, T) = \alpha_n L_n(t, T) dt + \gamma_n L_n(t, T) dW_{M+1}(t) + L_n(t, T) dJ_n(t), \tag{26}$$

where

$$\begin{aligned}
\alpha_n &= - \sum_{k=n+1}^n \frac{\gamma_i(t, T_i) \gamma_k(t, T_k) \delta L(t, T_i)}{1 + \delta L(t, T_i)} \\
&\quad - \int_0^\infty \sum_{i=1}^r H_n(t) L_n(t^-, T_n) \prod_{k=n+1}^M \frac{1 + \delta L_k(t^-, T_k)[1 + H_k(t)]}{1 + \delta L_k(t^-, T_k)} \lambda_{M+1} f(x) dx,
\end{aligned}$$

$$dJ_n(t) = d \sum_{i=1}^r \sum_{n=1}^{N_i(t)} H_{ni}(x_i) f(x).$$

If $n = M$, then

$$\left[\sigma^*(t, T_{n+1}) - \sigma^*(t, T_{M+1}) \right] = 0,$$

$$-H_{ni}(x_i) L_n(t^-, T_n) e^{\int_{T_{n+1}}^{T_{M+1}} h(t, x_i, u) du} \lambda_{M+1} dt = -H_{ni}(x_i) L_n.$$

Eq. (26) can be simplified as

$$dL_M(t, T) / L_M(t, T) = \gamma_M(t) dW^{M+1}(t) + dJ_M(t) - \int_0^\infty \sum_{i=1}^r H_{M_i}(x_i) \lambda_{M+1}^i dt f(x) dx. \quad (27)$$

3.3. The pricing model of Quanto Range Accrual Interest Rate Swap

The value of cash inflows for floating-rate payer at the maturity of QRAIRS is

$$QRAIRS(t, T_i, T_{i+1}) = \left(\sum_{q=1}^4 \frac{a^* N_q}{D} - \sum_{q=1}^4 \frac{K}{4} \right) \times F,$$

where N_q is the number of calendar days between swaps, and K is the fixed rate paid by the fixed-rate payer. Assuming 30 trading days per month and 3 months per period, the number of trading days will be 90 per period. The pricing model of floating-rate payer in risk neutral probability is thus

$$QRAIRS(t, T_q, T_{q+1}) = \sum_{q=1}^4 \sum_{l=1}^{90} B(t, T_q + l) X \left\{ E^Q \left[\mathbf{1}_{\{K_2 > L(T_q + l, T_{q+1} + l) > K_1\}} \right] \right\} \times \frac{B(t, T_q)}{B(t, T_q + l)} \times F - \sum_{q=1}^4 B(t, T_q) \frac{K}{4} \times F, \quad (28)$$

where q is the number of swaps in the trading period; l is the number of days between swaps; $B(t, T_q) / B(t, T_q + l)$ stands for the discount factor of Quanto Range Accrual Bond in a Martingale process; $B(t, T_q + l)$ is the value of the zero-coupon bond at termination; X is the agreed swap rate, $X = a / D$ measured in years from today. Eq. (28) can be decomposed as

$$\begin{aligned}
QRAIRS(t, T_q, T_{q+1}) &= \sum_{q=1}^4 \sum_{l=1}^{90} B(t, T_q + l) \left\{ E^Q X \left[1_{\{K_2 > L(T_q, T_{q+1} + l) > K_1\}} \right] \right\} F \cdot \frac{B(t, T_q)}{B(t, T_q + l)} \\
&\quad - \sum_{q=1}^4 B(t, T_q) \frac{K}{4} F \\
&= F \left\{ \sum_{q=1}^4 \sum_{l=1}^{90} B(t, T_q + l) X \sum_{j=0}^{\infty} \frac{e^{-\lambda_i(T+l-t)} \lambda_i(T+l-t)^j}{j!} [N(d_{1,ij}) - N(d_{2,ij})] \right. \\
&\quad \left. \frac{B(t, T_q)}{B(t, T_q + l)} - \sum_{q=1}^4 B(t, T_q) \frac{K}{4} \right\} \\
&= F \times [\tilde{A} - \tilde{B}],
\end{aligned} \tag{29}$$

where $F \times \tilde{A}$ is the floating rate interest received by the fixed rate payer. On the other hand, $F \times \tilde{B}$ is the fixed rate interest received by the floating rate payer.

$$d_{1,ij} = \frac{\ln \frac{L(t, T_q + l)^{(ij)}}{K_1} - \frac{1}{2} \sigma_{ij}^2 (T_q + l - t)}{\sqrt{\sigma_{ij}^2 (T_q + l - t)}}, \quad d_{2,ij} = \frac{\ln \frac{L(t, T_q + l)^{(ij)}}{K_2} - \frac{1}{2} \sigma_{ij}^2 (T_q + l - t)}{\sqrt{\sigma_{ij}^2 (T_q + l - t)}},$$

$$L(t, T_q + l)^{(ij)} = L(t, T_q + l) \exp \left[\sum_{i=1}^r -\lambda_i m_{li} (T_q + l - t) \right] \prod_{i=1}^r (1 + m_{li})^j.$$

Let the price of QRAIRS be equal to zero, then the fixed interest rate K (swap rate) is

$$K = \frac{4}{\sum_{q=1}^4 B(t, T_q)} \sum_{q=1}^4 \sum_{l=1}^{90} B(t, T_q + l) X \sum_{j=0}^{\infty} \frac{e^{-\lambda_i(T+l-t)} \lambda_i(T+l-t)^j}{j!} [N(d_{1,j}) - N(d_{2,j})] \frac{B(t, T_q)}{B(t, T_q + l)}. \tag{30}$$

4. Numerical Analysis

In this section, we use a numerical analysis to demonstrate the relationship between swap rate and jump size, jump frequency, and jump volatility of a QRAIRS. Assume a value for QRAIRS equal to zero, a duration of contract of one year, 360 days in a year, and payments swapped every 3 months—four swaps per year. The interest rate floor is 0% and the cap is

2.5%, 3-month USD LIBOR is 1.8% at time $t=0$, the volatility of 3-month USD LIBOR is 0.05. Two independent jump processes are assumed in the pricing period. The first one is induced by stronger information, which has a dramatic influence on the jump process of LIBOR. On the contrary, the second is associated with weaker information. The corresponding jump sizes are $0.01(m_1 = 0.01)$ and $0.5 (m_2 = 0.5)$; the corresponding jump volatilities are $0.15 (s_1 = 0.15)$ and $0.45 (s_2 = 0.45)$; and the corresponding jump frequencies are $0.5 (\lambda_1 = 0.5)$ and $1.5 (\lambda_2 = 1.5)$, respectively. The value of the zero coupon bond declines \$0.0001 mark-to-market starting at \$0.90.

4.1. The impact of jump size on the swap rate of QRAIRS

From Table 2, when jump sizes m_1 and m_2 change (within an interval from 0 to 0.6), the swap rate of QRAIRS will decrease with the increase of the expected value of jump size, and the smaller the initial swap rate, the less valuable the QRAIRS will be. When evaluating the LIBOR in risk-neutral probability, potential jump size due to information emergence must be eliminated in the initial pricing to ensure an arbitrage-free condition. Therefore, when jump size gradually increases due to information emergence, the initial value for LIBOR will become smaller, as does the value of underlying assets. Since this is disadvantageous to the floater, the value of QRAIRS will be smaller.

In addition, from Table 2, Figure 3, and Figure 4, m_2 has greater impact on the swap rate of the QRAIRS than m_1 , because jump size $m_2 > m_1$.

Table 2. Jump size changes and the swap rate of the QRAIRS

| Changes in jump size m_1 | The swap rate of QRAIRS corresponding to m_1 | Changes in jump size m_2 | The swap rate of QRAIRS corresponding to m_2 |
|-------------------------------|---|-------------------------------|---|
| 0.0 | 0.0237 | 0.0 | 0.0237 |
| 0.1 | 0.0228 | 0.1 | 0.0221 |
| 0.2 | 0.0220 | 0.2 | 0.0207 |
| 0.3 | 0.0213 | 0.3 | 0.0195 |
| 0.4 | 0.0206 | 0.4 | 0.0184 |
| 0.5 | 0.0201 | 0.5 | 0.0174 |
| 0.6 | 0.0195 | 0.6 | 0.0166 |

Note: Initial jump sizes are $m_1 = 0.01$ and $m_2 = 0.5$.

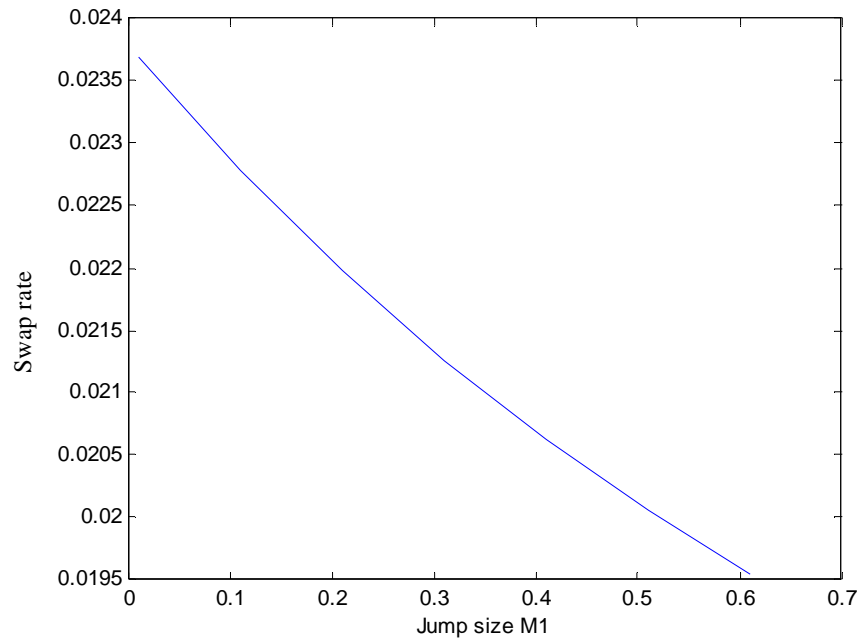


Figure 3. The relationship between jump size m_1 and the swap rate of the QRAIRS

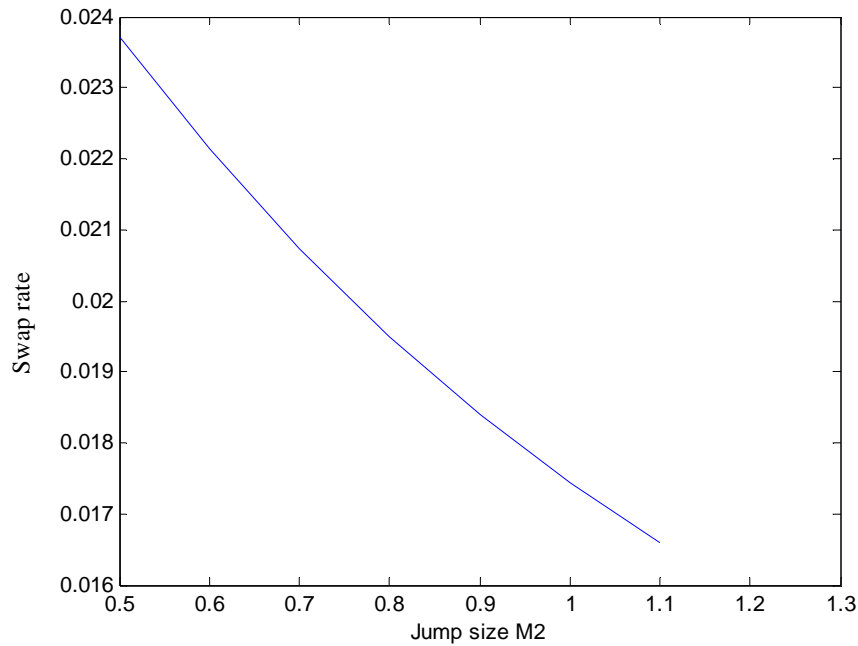


Figure 4. The relationship between jump size m_2 and the swap rate of the QRAIRS

4.2. The impact of jump volatility on the swap rate of QRAIRS

From Table 3, when jump volatilities s_1 and s_2 increase (within an interval from 0 to 0.3), the swap rate of QRAIRS also increases. This is because when jump volatility increases, the total volatility of QRAIRS follows suit. Hence when jump volatility increases, both parties of the QRAIRS must accept a higher swap rate because the floating rate payer is subsidized by the fixed rate payer for bearing risk. The higher the swap rate is, the higher the value of QRAIRS will be.

Similarly, from Table 3, Figure 5, and Figure 6, s_2 has greater impact on the swap rate of QRAIRS relative to s_1 , because $s_2 > s_1$. This result is consistent with our assumption described earlier in this section.

Table 3. Jump volatility changes and the swap rate of QRAIRS

| Changes in jump volatility s_1 | The swap rate of QRAIRS corresponding to s_1 | Changes in jump volatility s_2 | The swap rate of QRAIRS corresponding to s_2 |
|----------------------------------|--|----------------------------------|--|
| 0.00 | 0.0237 | 0.00 | 0.0237 |
| 0.05 | 0.0240 | 0.05 | 0.0245 |
| 0.10 | 0.0242 | 0.10 | 0.0252 |
| 0.15 | 0.0245 | 0.15 | 0.0259 |
| 0.20 | 0.0247 | 0.20 | 0.0265 |
| 0.25 | 0.0250 | 0.25 | 0.0271 |
| 0.30 | 0.0252 | 0.30 | 0.0276 |

Note: Initial jump volatilities are $s_1 = 0.15$, and $s_2 = 0.45$.

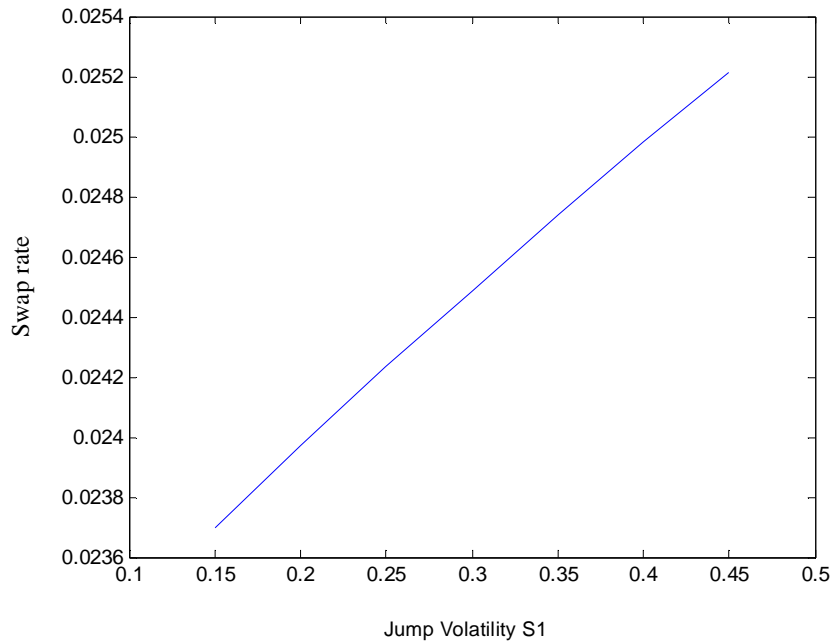


Figure 5. The relationship between jump volatility s_1 and the swap rate of QRAIRS

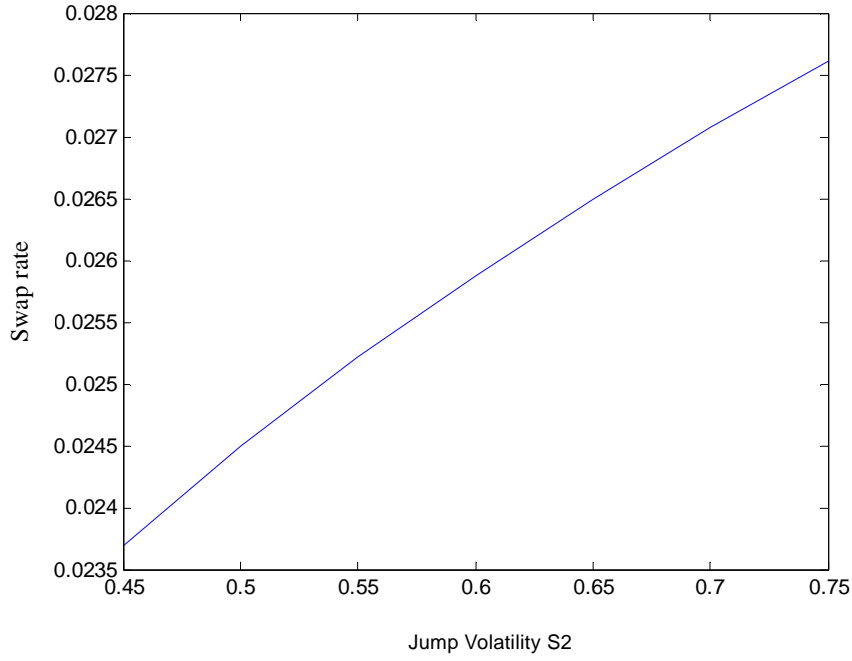


Figure 6. The relationship between jump volatility s_2 and the swap rate of QRAIRS

4.3. The impact of jump frequency on the swap rate of QRAIRS

From Table 4, when jump frequency (λ_1 and λ_2) increases (within an interval from 0 to 0.45), the swap rate of QRAIRS decreases. This result can be obtained from Eq. (29)—there exists a negative relation between jump frequency λ_1 and the swap rate of QRAIRS. From probability theory, we know that given the frequency of actual occurrence, the value of the probability decreases when the expected value of the Poisson distribution increases, which is consistent with our QRAIRS pricing model. From an investor’s viewpoint, the higher jump frequency of LIBOR is associated with higher uncertainty risk, hence a decline in the value of QRAIRS can be regarded as the additional market premium for bearing risk.

Similarly, from Table 4, Figure 7 and Figure 8, λ_2 has greater impact on the swap rate of QRAIRS relative to λ_1 because jump frequency $\lambda_2 > \lambda_1$.

Table 4. Jump frequency changes and the swap rate of QRAIRS

| Changes in jump frequency λ_1 | The swap rate of QRAIRS corresponds to λ_1 | Changes in jump frequency λ_2 | The swap rate of QRAIRS corresponds to λ_2 |
|---------------------------------------|--|---------------------------------------|--|
| 0.0 | 0.0237 | 0.0 | 0.0237 |
| 0.1 | 0.0235 | 0.1 | 0.0216 |
| 0.2 | 0.0234 | 0.2 | 0.0195 |
| 0.3 | 0.0232 | 0.3 | 0.0178 |
| 0.4 | 0.0230 | 0.4 | 0.0159 |
| 0.5 | 0.0228 | 0.5 | 0.0136 |
| 0.6 | 0.0226 | 0.6 | 0.0119 |

Note: Initial jump frequencies are $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$.

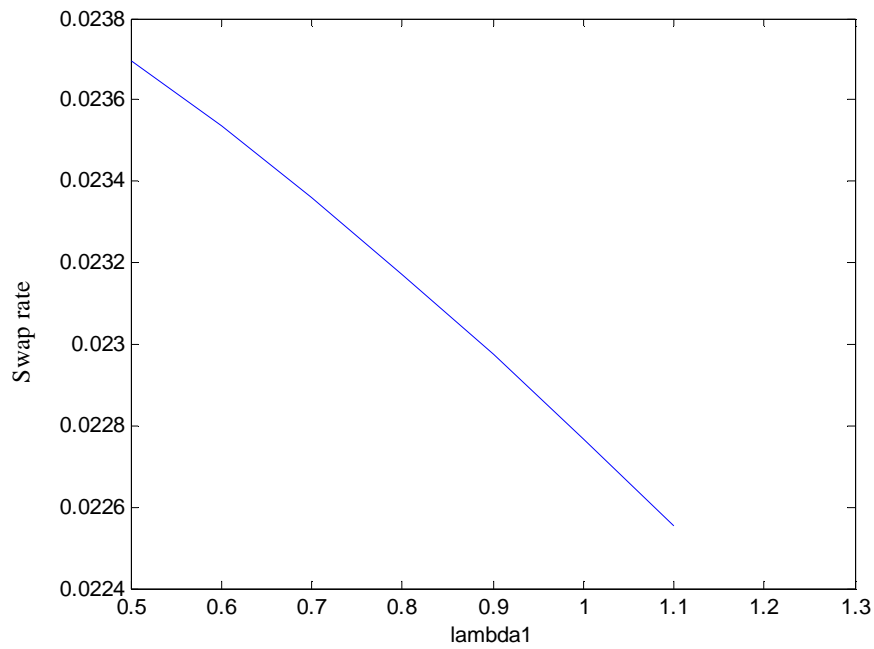


Figure 7. The relationship between jump frequency λ_1 and the swap rate of QRAIRS

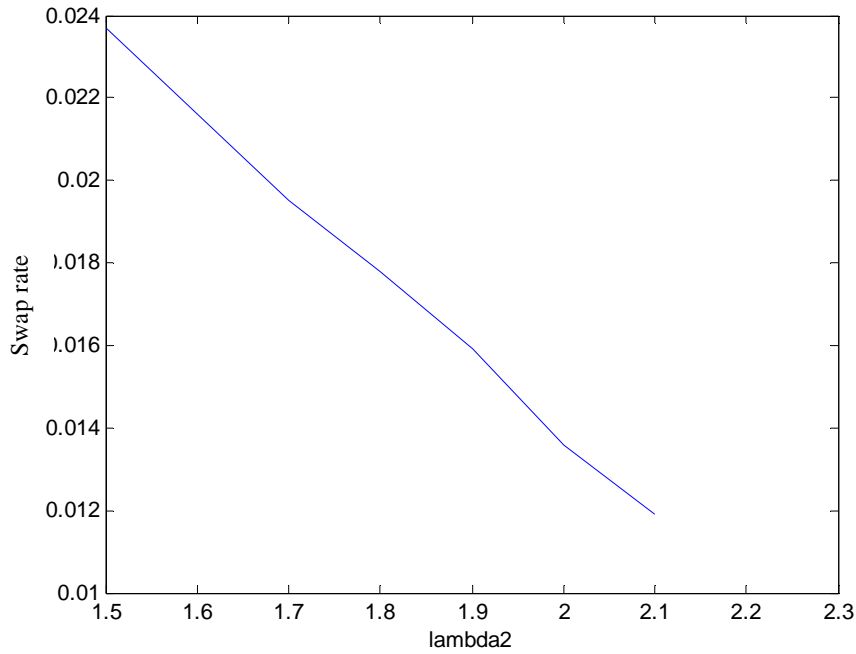


Figure 8. The relationship between jump frequency λ_2 and the swap rate of QRAIRS

5. Conclusions

Interest rates constitute an important economic index. Since the jump process of interest rates is discontinuous in the event of sudden information emergence, we derive the LIBOR market model with jump risk. Since interest rates often follow multiple independent jump processes, it would be more accurate to use our model to price the derivatives. To this end, we use numerical analysis to illustrate the influence of jump size, jump volatility, and jump frequency on the pricing of QRAIRS.

Numerical analysis shows that there is a negative relation between jump size, jump frequency, and the swap rate of QRAIRS. However, there is a positive relation between jump volatility and the swap rate of QRAIRS, meaning the larger the jump volatility, the higher the value of QRAIRS. Furthermore, the stronger the LIBOR-related information, the greater will

be the impact of jump risk on QRAIRS. From the numerical results, we know that increases in jump size will decrease the value of LIBOR in time $t = 0$. Since this is disadvantageous to the floater, the value of QRAIRS will decrease. Similarly, the higher the jump frequency of LIBOR, the greater the uncertainty risk, hence the value of QRAIRS declines because investors are paid with a premium to bear risk. On the other hand, when jump volatility increases, both parties must agree to a higher swap rate because the floating rate payer is subsidized by the fixed rate payer for bearing risk.

References

- Black, F., and M. Scholes, (1973), The pricing of options and corporate liabilities, *Journal of Political Economy*, 81(3), 637-654.
- Björk, T., Y. Kabanov and W. Runggaldier, 1997, Bond Market Structure in the Presence of Marked point process, *Mathematical Finance*, 7, 211-239.
- Brace, A., D. Gatarak and M. Musiela, 1997, The Market Model of Interest Rate Dynamics, *Mathematical Finance*, 7, 127-155.
- Chiarella, C. and T. D. To, 2003, The Jump Component of The Volatility Structure of Interest Rate Futures Markets, *The Journal of Futures Markets*, 23(12), 1125-1158.
- Cox, J. C., J. E. Ingersoll and S. A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica*, 53, 385-407.
- Elliott, R. J., 1982, *Stochastic Calculus and Application*, New York: Springer-Verlag.
- Glasserman, P., and S.G. Kou, 2003, The term structure of simple forward rates with jump risk, *Mathematical Finance*, 13, 383-410.
- Heath, D., R. Jarrow and A. Morton, 1992, Bond Price and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica*, 60, 77-105.
- Ho, T. S. Y. and S. B. Lee, 1986, Term Structure Movements and the Pricing of Interest Rate Contingent Claims, *The Journal of Finance*, 41, 1011-1029.
- Hull, J. and A. White, 1990, Pricing Interest Rate Derivative Securities, *The Review of Financial Studies*, 3, 573-592.
- Merton, R., 1976, Option Pricing when Underlying Stock Returns are Discontinuous, *Journal of Financial Economics*, 3, 125-144.
- Shikarawa, H., 1991, Interest Rate Option Pricing with Poisson-Gaussian Forward Rate Curve Process, *Mathematical Finance*, 1, 77-94.
- Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics*, 5, 177-188.