The Valuation of Catastrophe Bonds With Exposure to Currency Exchange Risk

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6 June 2012

Abstract

We present a new model for valuing, by an arbitrage approach, catastrophic risk bonds (CAT bonds) that takes into account the sponsor's exposure to currency exchange risk as well as the risk of occurence of catastrophic events. Our approach extends the model of Poncet and Vaugirard [2001], which includes a hedging cost for the currency risk, by incorporating a jump-diffusion process for catastrophic events, and a three dimensional stochastic process for the exchange rate as well as the domestic and foreign interest rates. Our contributions include the derivation of a semi-explicit analytical formula for the CAT bond price, and an extension to three factors of the Monte Carlo simulation approach of Joshi and Leung [2007], using a Brownian bridge argument between catastrophic events.

Keywords: Finance, CAT bond valuation, catastrophic risk, exchange risk, jump-diffusion process, 3D Brownian motion, Monte Carlo simulation, preference sampling, Brownian bridge.

1 Introduction

Following hurricane Andrew that inflicted losses of seventeen billion dollars in 1992 to them, insurance and reinsurance companies looked for alternative sources of funding to indemnify natural catastrophes disasters by looking toward capital markets. Since then, we have

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witnessed a notable development of new financial products traded on the over-the-counter markets, the insurance-linked (or risk-linked) securities, which by way of securitization mechanisms, transfer the catastrophic risk to markets' investors. Indeed, according to Cummins [2008], "risk-linked securities are innovative financing devices that enable insurance risk to be sold in capital markets, raising funds that insurers and reinsurers can use to pay claims arising from mega-catastrophes and other loss events."

The catastrophic risk bond (or catastrophe bond, or CAT bond) is the most prominent type of risk-linked security, being the most popular and the most important in terms of volume. Issued by insurance or reinsurance companies to share the risk of losses due to natural disasters with other investors, catastrophe bonds are bonds in which the coupon and principal payments are stopped after the occurrence of a triggering event. The great majority of CAT bonds are denominated in U.S. dollars for liquidity reason (others are euro or yen-denominated, notably), exposing the issuer whose national currency is not the dollar to an exchange risk that he is obliged to assume. Naturally, this risk should thus be compensated and can be modelled in general by introducing a currency hedging cost in the bond price evaluation.

To the best of our knowledge, the only contribution on this topic is due to Poncet and Vaugirard [2001]. These authors take into account the exchange risk by considering that the sponsor sells options on his currency to hedge himself. They put themselves in a non catastrophic universe where the loss index is a diffusion process, and they derive an explicit formula. Their results indicate that the exchange risk has a negative effect on the CAT bond price, albeit weak compared to the natural risk.

The goal of the present paper is to develop a valuation model for the CAT bonds that takes into account the exchange risk in addition to the risk of catastrophe, by considering an universe that allow for the presence of catastrophic events ignored by these authors. Our approach is to model the catastrophic risk, in addition to the exchange risk, by means of a jump diffusion process for the catastrophe index. Such a process has previously been used by Vaugirard [2002] in his work, but without exchange risk. Moreover, by introducing a model with three factors as in Hilliard et al. [1991] for the exchange rate, we are able to depict its dynamics in a richer framework. Our contributions consists of the derivation of a partially explicit analytical formula for the CAT bond price, and an extension to three factors of the Monte Carlo simulation approach of Joshi and Leung [2007], using a Brownian bridge argument between catastrophic events.

The paper is organized as follows. A brief description of CAT bonds is given in §2, followed by a literature review in §3. Then, our valuation model and formulas are derived in §4, and the Monte Carlo simulation approach is described in §5. Next, numerical results regarding sensitivity to model parameters are presented in §6, with concluding remarks in §7. Finally, the CAT bonds structured finance characteristics are discussed in Appendix A.

2 Brief description of CAT bonds

A CAT bond takes the form of a standard bond that pays interests in the form of coupons at regular interval in exchange for a predefined initial amount. Its particularity stems from the fact that the cash flows promised to the investor are contingent upon the absence of natural catastrophe before the bond maturity; if an index of natural risk (or an index of losses of the insurance sector) reaches a predetermined threshold, the holder of the CAT bond loses either his coupons, or either the face value, sometimes both of these, all depending on the initial contract. There is a similarity between the risk of natural catastrophe and the default risk of a bond of speculative grade.

The CAT bond is issued by a *Special-Purpose Vehicle*, or SPV for short, created for that purpose and located in a fiscally favorable area, tax-shelter or haven, such as the Cayman Islands. This setup leads to a financial instrument that is purely linked to the catastrophes, because the sponsor's credit risk has been eliminated. The money received at the CAT bond's issuance in the form of face value then feeds a fund dedicated to indemnifying the sponsor should a catastrophic event happen, so he can honor his obligations, which consist in reimbursing the insured having suffered material losses. Although designed as products purely linked to the risk of natural catastrophes, CAT bonds are nonetheless subjected to other risks, including the risk of counterparty arising from the rate swap, the risk of interest rate inherent to all bonds, and the risk of exchange which is the subject of this paper. See Appendix A for more detailed explanations on the structure of CAT bonds and their markets.

3 Literature review

3.1 CAT bond valuation

Incompleteness of markets in the presence of natural catastrophes

Natural catastrophes are often modelled by introducing a jump process in the underlying process, whether it represents a parametric index or losses. Financial markets are incomplete in that situation; hence, it is not possible to implement a portfolio replication strategy to evaluate a financial instrument, and the equivalent martingale measure is not unique. Two approaches were developed to tackle this situation. The first one, due to Merton [1976], consists in supposing that the risk associated to the jumps can be entirely eliminated by diversification; it is not systematic or idiosyncratic. We then say that the investors are risk-neutralized toward the risk of natural catastrophe. Under this hypothesis, the equivalent martingale measure is unique so arbitrage pricing becomes legitimate. The second way to proceed is to resort to an equilibrium model. We must then postulate a utility function for the investor, and perform the valuation by maximizing expected utility under the real probability measure.

Risk-neutral CAT bond valuation

Merton's approach leads to existence and uniqueness of the equivalent martingale measure which entails absence of arbitrage opportunities. The first studies of this kind dedicated to CAT bonds ignore the catastrophic risk, as in the case of Poncet and Vaugirard [2002] who used a simple diffusion process for the catastrophe index under a stochastic interest rates regime. Albeit somewhat unrealistic, this model offers nonetheless an explicit formula for the CAT bond price. Vaugirard [2002] extends the model by adding the natural catastrophes by means of a jump diffusion process. Vaugirard [2003] achieves a similar purpose by replacing the diffusion process by a mean reverting process. Lee and Yu [2002] proposed an insurance model for evaluating the CAT bonds whose underlying asset is a loss dynamics represented by a compound Poisson process, and studied the impact on the CAT bond price of the default risk, the moral hazard and the basic risk. For the sake of realism, some studies assume the frequency of catastrophes to be stochastic, as in the case of Baryshnikov et al. [2001]. Dassios and Jang [2003] suggest to use a "doubly stochastic" Poisson process. Albrecher et al. [2004] provide an algorithm using quasi-Monte Carlo for solving this model. This idea is used again by Hainaut [2010] who introduced a seasonal component for the intensity.

An alternative methodology is suggested by Jarrow [2010]. Exploiting the analogy between the catastrophe risk and the default risk, he adapts to CAT bonds a simple model for pricing credit derivatives that is based on two input parameters, the probability of a catastrophe occurring per unit of time, and the percentage of loss given the catastrophe. Even though it has obvious analytical simplicity, this model suffers from its dependence on the empirical estimation of its two parameters, rendered difficult by the lack of reliable data.

Equilibrium models

To resolve the problem of market incompleteness in the presence of catastrophic risk, Cox and Pedersen [1995] proposed a model based on the "representative agent" paradigm, which consists in postulating a utility function for the investor, as well a consumption process. The price of the CAT bond is obtained by maximizing the expected utility. This methodology is revisited by Egami and Young [2007] to evaluate structured CAT bonds, that is, composed of two layers, *junior* and *senior*. Reshetar [2008] evaluates multi-catastrophe CAT bonds that consider in particular the risk of terrorist attacks.

3.2 Modelling the exchange risk

Exchange risk modelling is frequently discussed in the literature on valuation of currency options. Garman and Kohlhagen [1983] extend Black-Scholes' formula to options on currency, where the foreign interest rate plays the same role as a dividend rate. Grabbe [1983] developed a model where the prices of domestic and foreign obligations are stochastic, following a geometric Brownian motion, and obtained an explicit formula. Amin and Jarrow [1991] approached the same problem using the framework of Heath et al. [1992]. Hilliard et al. [1991] provided a generalization of the previous models for stochastic domestic and foreign interest rates correlated with the exchange rate. They showed that the model where interest rates are stochastic yields better performance than that with constant rates for estimating the price of options on currencies.

Other authors note the importance of introducing a stochastic volatility for the exchange rate. Hakala and Wystup [2002] adapted Heston's volatility model to options on the exchange rate. Haastrecht et al. [2009] and Grzelak and Oosterle [2010] considered a model with four factors with stochastic rate and volatility, but did not find an explicit formula for the options on currency.

4 Valuation model

4.1 Analytical framework

Dynamics of the economy

We choose to model the exchange rate S_t by a geometric Brownian motion and the domestic and foreign interest rates r_d and r_f each according to the model of Vašíček [1977]. We obtain a model with three state variables which, under the domestic equivalent martingale measure $\mathbb{Q}$, can be written as

$$\begin{cases} dS_t/S_t = (r_d - r_f)dt + \sigma_S dW_t^S \\ dr_d = \kappa_d(\theta_d - r_d)dt + \sigma_d dW_t^d \\ dr_f = \kappa_f(\theta_f - r_f)dt + \sigma_f dW_t^f \end{cases} \tag{1}$$

where κ_d and κ_f are the mean reversion rates, θ_d and θ_f are the long-term means, σ_S, σ_d and σ_f are the instantaneous volatilities, and W_t^S, W_t^d and W_t^f are three Brownian motions admitting the correlation matrix

$$\Gamma = \begin{pmatrix} 1 & \rho_{Sd} & \rho_{Sf} \\ . & 1 & \rho_{df} \\ . & . & 1 \end{pmatrix}.$$

In the model of Vašíček, the price at t of the zero-coupon domestic bond (here American, that is denominated in USD) earning \$1 at T is

$$P_d(t,\ T) = \exp\left\{A(\tau) - B(\tau)r_d\right\}, \tag{2}$$

where

$$B(\tau) = \frac{1 - e^{-\kappa_d \tau}}{\kappa_d}$$

and

$$A(\tau) = \left(\theta_d - \frac{\sigma_d^2}{2\kappa_d}\right)(B(\tau) - \tau) - \frac{\sigma_d^2}{4\kappa_d} B(\tau)^2,$$

denoting $\tau = T - t$ the time until maturity. The price of the zero-coupon foreign bond is expressed by a similar formula; it suffices to replace the index d by f. The exchange rate model we employ is more complex than the one of Grabbe [1983] used par Poncet and Vaugirard [2001], which is a model with two state variables, r_d and $P_d S_t$. It allow us to study the dynamics and interactions of a larger number of parameters.

Price of a call option on foreign currency

If we denote $\nu^2(\tau)$ the conditional variance of the forward exchange rate $F(t,\ T)$, and σ_{df}, σ_{Sd} and σ_{Sf} the covariances between the three state variables S_t, r_d and r_f, then Hilliard et al. [1991] propose the approximation

$$\nu^2(\tau) = \sigma_S^2 \tau + \frac{\tau}{3}(\sigma_d^2 + \sigma_f^2 - 2\sigma_{df}) + \tau^2(\sigma_{Sd} - \sigma_{Sf}) \tag{3}$$

and show that the price of the call option on the foreign currency in this model is

$$C(t,\ T) = P_d(t,\ T)[F(t,\ T)N(d_1) - KN(d_2)], \tag{4}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution,

$$d_1 = \frac{\ln(F(t,\ T)/K) + \frac{1}{2}\nu^2(\tau)}{\nu(\tau)}$$

and

$$d_2 = d_1 - \nu(\tau).$$

The spot and forward exchange rates are linked by the parity relation of the interest rates,

$$F(t,\ T) = \frac{P_d(t,\ T)}{P_f(t,\ T)} S_t. \tag{5}$$

The latter relation will be useful in the following sections and will be used in the simulation algorithm.

Natural catastrophe index

To model the loss index, we also use the jump diffusion process introduced par Merton [1976] for option pricing and used by Vaugirard [2002]:

$$dI_t/I_t^- = \mu dt + \sigma dW_t + (Y-1)dN_t, \tag{6}$$

where W_t is a Brownian motion, N_t is a standard Poisson process of constant intensity λ_N, and Y is a log-normal random variable, such that at a jump epoch t_n,

$$I_{t_n}^+ = Y I_{t_n}^-.$$

The choice of this process rather than a simple diffusion process constitutes the main innovation of the present research with respect to the contribution of Poncet and Vaugirard [2001], in the framework of pricing a CAT bond subject to exchange risk. According to Merton's hypothesis, the risk introduced by the Poisson process is assumed diversifiable, which justifies arbitrage pricing. It should be noted also that the index I_t is not traded on the market; a specific risk premium is thus assigned to it. Considering the $\mathbb{Q}$-dynamics of I_t, (6) becomes

$$dI_t/I_t^- = (\mu - \lambda\sigma)\, dt + \sigma dW_t + (Y-1)dN_t, \tag{7}$$

where λ is the risk premium due to the jumps.

4.2 Analytical evaluation

Cash flow structure

We define the instant η when the CAT bond is triggered as the first time when the index I_t reaches the trigger threshold H:

$$\eta = \min_{t\geq 0} I_t \geq H.$$

The promised cash flow of the CAT bond at maturity T is

$$X = V\mathbb{1}_{\eta>T} + (1-\omega)V\mathbb{1}_{\eta<T} = V\left(1-\omega\mathbb{1}_{\eta<T}\right),$$

whete $\mathbb{1}_{\eta<T}$ is the "indicator" random variable defined by

$$\mathbb{1}_{\eta<T} = \begin{cases} 1 & \text{if } \eta \leq T \\ 0 & \text{else.} \end{cases}$$

Indeed, if the trigger is not hit before maturity, the face value V is reimbursed entirely, but if the CAT bond is triggered before maturity, it is reduced by a proportion ω.

To take into account the exchange risk, we must add the pay-off of a hypothetical hedge that would be setup by the sponsor who would not want to assume this risk. He wishes to protect himself against an appreciation of his currency against the American dollar, that is, against an increase of S_t, in the event when a catastrophe happened that would force him to convert in his own currency the compensation that the SPV would pay him. As such, when the CAT bond is triggered, the grantor takes a long position on a call on his own currency to fix the exchange rate at T. The pay-off of this strategy is

$$(S_T - K)\mathbb{1}_{S_T>K}\mathbb{1}_{\eta<T}.$$

Adding this pay-off for the quantity $\omega V/K$ to the CAT bond's pay-off not subject to the exchange risk (negative pay-off for the investor who sold the option), we obtain the expression for the total cash flow

$$X_{\text{CAT}} = V - \omega V\mathbb{1}_{\eta<T} - \frac{\omega V}{K}(S_T - K)\mathbb{1}_{S_T>K}\mathbb{1}_{\eta<T}. \tag{8}$$

Forward martingale measure

The forward martingale measure $\mathbb{Q}^T$ is defined to be the equivalent measure when we choose as *numéraire* the risk-free, zero-coupon bond earning \$1 at T. The likelihood process is defined by

$$L_t = \left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t}. \tag{9}$$

If the price of a zero-coupon bond earning 1\$ at T has for dynamics under $\mathbb{Q}$ to follow

$$\frac{dP(t,\ T)}{P(t,\ T)} = r_t dt + \sigma(t,\ T)dW_t^{\mathbb{Q}},$$

then the dynamics of L_t is

$$dL_t/L_t = \sigma(t,\ T)dW_t^{\mathbb{Q}},$$

and the relation between a $\mathbb{Q}$-Brownian motion and a $\mathbb{Q}^T$-Brownian motion is, according to Girsanov's theorem,

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{Q}^T} + \sigma(t,\ T)dt. \tag{10}$$

The fundamental property of the forward martingale measure, established by Geman et al. [1995], is

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} X \middle| \mathcal{F}_t \right] = P(t,\ T)\mathbb{E}_{\mathbb{Q}^T}\left[X \middle| \mathcal{F}_t \right], \tag{11}$$

where $P(t,\ T)$ is the price at t of a risk-free, zero-coupon bond paying \$1 at T. Equation (11) holds for whatever the dependence relation that may exist between X and the discount factor; it is thus clear that the passage to $\mathbb{Q}^T$ is a natural tool whenever we evaluate a financial product whose cash flows are linked in any manner to the interest rate.

General formula for the price of a CAT bond

We denote the discount factor

$$D(t,\ T) = e^{-\int_t^T r_d(u)du}.$$

The price of the CAT bond is then

$$P_{\text{CAT}}(t,\ T) = \mathbb{E}_{\mathbb{Q}}\left[D(t,\ T) X_{\text{CAT}} \middle| \mathcal{F}_t \right],$$

that is, using equation (8),

$$\begin{aligned} P_{\text{CAT}}(t,\ T) =& V P_d(t,\ T) - \omega V \mathbb{E}_{\mathbb{Q}}\left[D(t,\ T) \mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right] \\ &- \frac{\omega V}{K} \mathbb{E}_{\mathbb{Q}}\left[D(t,\ T)(S_T - K) \mathbb{1}_{S_T>K} \mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right]. \end{aligned} \tag{12}$$

By passing to the forward martingale measure $\mathbb{Q}^T$, we can take $D(t,\ T)$ out of both conditional expectations:

$$\begin{aligned} P_{\text{CAT}}(t,\ T) =& V P_d(t,\ T) - \omega V P_d(t,\ T) \mathbb{E}_{\mathbb{Q}^T}\left[\mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right] \\ &- \frac{\omega V}{K} P_d(t,\ T) \mathbb{E}_{\mathbb{Q}^T}\left[(S_T - K) \mathbb{1}_{S_T>K} \mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right]. \end{aligned} \tag{13}$$

Finally, by factoring out $V P_d(t,\ T)$, we find the general formula for the price of the CAT bond with exchange risk:

$$\begin{aligned} P_{\text{CAT}}(t,\ T) =& V P_d(t,\ T) \left\{ 1 - \omega \mathbb{E}_{\mathbb{Q}^T}\left[\mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right] \right. \\ &\left. - \frac{\omega}{K} \mathbb{E}_{\mathbb{Q}^T}\left[(S_T - K) \mathbb{1}_{S_T>K} \mathbb{1}_{\eta<T} \middle| \mathcal{F}_t \right] \right\}. \end{aligned} \tag{14}$$

Poncet and Vaugirard [2001] derived a similar equation and then they obtained an explicit formula, because if the catastrophe index follows a diffusion process, the second conditional expectation admits an analytical expression, as was established by Heynen and Kat [1994]. However, with the jump-diffusion process used in the present work, such an expression does not exist; we must impose additional assumptions if we want to push further the calculations.

Independence between natural catastrophes and the rest of the economy

It is reasonable to suppose that the catastrophe index is independent of the rest of the economy, which allows us to simplify considerably equation (14). First, we can split the last term in two conditional expectations. Further, the dynamics of I_t remain unchanged when we go from $\mathbb{Q}$ to $\mathbb{Q}^T$ (it is given by equation (7)), which means that

$$\mathbb{E}_{\mathbb{Q}^T}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right].$$

The price formula becomes

$$\begin{aligned} P_{\text{CAT}}(t,\ T) =& V P_d(t,\ T)\Big\{1 - \omega\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right] \\ & - \frac{\omega}{K}\mathbb{E}_{\mathbb{Q}^T}\left[(S_T - K)\mathbb{1}_{S_T>K}\middle|\,\mathcal{F}_t\right]\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right]\Big\}. \end{aligned} \tag{15}$$

We obtain the price of the call option on the currency,

$$P_{\text{CAT}}(t,\ T) = V P_d(t,\ T)\left\{1 - \omega\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right] - \frac{\omega}{K}\frac{C(t,\ T)}{P_d(t,\ T)}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\eta<T}\middle|\,\mathcal{F}_t\right]\right\}. \tag{16}$$

We can also write the previous formula in a more compact form,

$$P_{\text{CAT}}(t,\ T) = V P_d(t,\ T)\left\{1 - \omega\left(1 + \frac{1}{K}\frac{C(t,\ T)}{P_d(t,\ T)}\right)\mathbb{Q}(\eta < T)\right\}. \tag{17}$$

Note that the trigger probability $\mathbb{Q}(\eta < T)$ is the only factor that does not admit an explicit formula. The formula for $P_d(t,\ T)$ is given in equation (2) and that for $C(t,\ T)$ is given in equation (4).

5 Monte Carlo simulation

5.1 Statement of the problem

To determine by numerical methods the instant when the CAT bond is triggered is tantamount to valuing barrier options in continuous time. It is a matter of detecting the moment the catastrophe index reaches a predetermined level, which is the trigger threshold of the CAT bond. If we just proceed naively by checking the level of the index at each point of the discrete time interval, we expose ourselves to the risk that the trigger threshold would be touched between two time steps so it would not be detected. Since one misses the trigger point, to proceed in this manner would give a biased result. Moreover, such a simulation is prohibitively costly; it would take a very large number of trajectories and a very small discretization step size.

We cannot afford to simulate this model by brute force. We must consider an unbiased alternative, which is sufficiently flexible to the specific aspects of the CAT bond problem and fairly fast. Such a method exists for evaluating barrier options. Metwally and Atiya [2002] designed an algorithm based on the Brownian bridge technique for evaluating such options in the model of Merton [1976]. It consists of exploiting the fact that between two jumps, the underlying asset follows a classical geometric Brownian motion. This method gives unbiased results and its speed largely exceeds that of a naive simulation. Joshi and Leung [2007] revisited this method, together with an importance sampling technique; this allows to save computation time on the trajectories that lead to a null pay-off when the barrier has been reached. These authors boast on the speed of their algorithm under weak jump frequencies. However, their method requires an explicit formula for the barrier option in the model without jumps.

In the present work, we opt for the method of Joshi and Leung [2007] which is more recent than that used by Vaugirard [2002] for CAT bond valuation. It is unbiased and, as long as we consider weak intensities, it it fast. In our model, the CAT bond yields an explicit formula when the catastrophe index is without jumps, which makes this method adequate for the problem under study. We must nonetheless adapt it to take into account the exchange rate and the two interest rates, domestic and foreign, that are stochastic.

5.2 Method of Joshi and Leung used for CAT bond valuation

General description

The idea of Metwally and Atiya is to first generate the instants when jumps occur, and to move between each jump by simulating a normal distribution for the logarithm of the underlying asset. As such, between two jumps, the dynamics of the underlying asset is a Brownian motion of which we know both the start and end values, so it is a Brownian bridge. The probability that a barrier be reached between two jumps is known, it is given by the Brownian bridge maximum formula (see, for instance, Karatzas and Shreve [1991]). Rather than testing each time whether the barrier is reached, Joshi and Leung use the method of importance sampling by modifying the probability measure to force each random variation of the underlying asset not to cross the barrier, and they correct the final pay-off with this probability. Further, they condition on the arrival time of the first jump, in order to use the explicit formula from Black-Scholes' model in the cases when there happens to be no jump before the maturity date. Simulation is performed only for those trajectories that contain at least one jump. Therefore, this method requires more computations per trajectory, but leads to better accuracy by using a much smaller number of trajectories.

Conditioning on the first jump

We decompose the price as a function of the arrival time of the first jump:

$$P_{\text{CAT}}(t,\ T) = \mathbb{P}\left(t_1 > T\right) P_{\text{CAT}}(t,\ T)\big|_{\text{no jump}} + \mathbb{P}\left(t_1 \leq T\right) P_{\text{CAT}}(t,\ T)\big|_{\text{jump}},$$

but the waiting time between each jump follows an exponential distribution with density $\lambda_N e^{-\lambda_N t}$, so the probability that no catastrophe happens is:

$$p = \mathbb{P}\left(t_1 > T\right) = e^{-\lambda_N T}.$$

We obtain

$$P_{\text{CAT}}(t,\ T) = pP_{\text{CAT}}(t,\ T)\big|_{\text{no jump}} + (1-p)P_{\text{CAT}}(t,\ T)\big|_{\text{jump}}. \tag{18}$$

If there are no jumps, the catastrophe index follows a simple geometric Brownian motion;

the CAT bond price then follows an explicit formula

$$P_{\text{CAT}}(t,\ T) = VP_d(t,\ T)\left\{1 - \omega\left(1 + \frac{1}{K}\frac{C(t,\ T)}{P_d(t,\ T)}\right)\mathbb{Q}(\eta < T)\right\},$$

where the trigger probability can be expressed as

$$\mathbb{Q}(\eta < T) = N(d_1) + \left(\frac{I_0}{H}\right)^{1-2\gamma/\sigma^2} N(d_2), \tag{19}$$

with

$$d_1 = \frac{\ln(I_0/H) + (\gamma - \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln(I_0/H) - (\gamma - \sigma^2/2)T}{\sigma\sqrt{T}},$$

and

$$\gamma = \mu - \lambda\sigma.$$

Next, what remains to estimate by simulation is the CAT bond price when there has been at least one jump before maturity.

Simulation of trajectories and preference sampling

The $\mathbb{Q}$-dynamics of I_t is

$$dI_t/I_t^- = (\mu - \lambda\sigma)dt + \sigma dW_t + (Y-1)dN_t.$$

Between two jumps, I_t follows a simple geometric Brownian motion wih drift $\mu - \lambda\sigma$:

$$dI_t/I_t^- = (\mu - \lambda\sigma)dt + \sigma dW_t$$

hence the dynamics of $\ln I_t$ is

$$d\ln I_t = (\mu - \lambda\sigma - \frac{1}{2}\sigma^2)dt + \sigma dW_t.$$

Let t_{n-1} and t_n be $(n-1)$st and nth jump instants, $I^+_{t_{n-1}}$ the value of the catastrophe index immediately after the $(n-1)$st jump, $I^-_{t_n}$ its value immediately preceding the nth jump, and $\tau_n = t_n - t_{n-1}$. We can write the law of passage of $\ln I_t$ to t_{n-1} and t_n:

$$\ln I^-_{t_n} \sim N\left(\ln I^+_{t_{n-1}} + (\mu - \lambda\sigma - \frac{1}{2}\sigma^2)\tau_n\ ;\ \sigma^2\tau_n\right). \tag{20}$$

The algorithm to construct a Gaussian variate $Z \sim N(\mu, \sigma^2)$ which is below a real number x is

$$Z = \phi^{-1}(\theta u,\ \mu,\ \sigma^2), \quad \theta = \mathbb{P}\left(Z < x\right),$$

where $\phi(z, a, b^2)$ is the cumulative distribution function of the normal with mean a and standard deviation b. The likelihood ratio in this case is then

$$\theta = \mathbb{Q}(\ln I_{t_n}^- < \ln H) = \phi\left(\ln H\ ;\ \ln I_{t_{n-1}}^+ + (\mu - \lambda\sigma - \frac{1}{2}\sigma^2)\tau_n\ ;\ \sigma^2\tau_n\right). \tag{21}$$

Y is a log-normal random variable; it can be written as

$$Y = m \exp\left(-\frac{1}{2}\sigma_{\text{jump}}^2 + \sigma_{\text{jump}}\varphi_n\right), \text{ where } \varphi_n \sim N(0,1),$$

and since $I_{t_n}^+ = Y I_{t_n}^-$, then $\ln I_{t_n}^+ = \ln I_{t_n}^- + \ln Y$, from where

$$\ln I_{t_n}^+ = \ln I_{t_n}^- + \ln(m) - \frac{1}{2}\sigma_{\text{jump}}^2 + \sigma_{\text{jump}}\varphi_n.$$

Finally, we obtain

$$\ln I_{t_n}^+ \sim N\left(\ln I_{t_n}^- + \ln(m) - \frac{1}{2}\sigma_{\text{jump}}^2\ ;\ \sigma_{\text{jump}}^2\right). \tag{22}$$

As we will constrain this variable not to cross the trigger threshold, the associated likelihood ratio is

$$\theta_2 = \mathbb{Q}(\ln I_{t_n}^+ < \ln H) = \phi\left(\ln H\ ;\ \ln I_{t_n}^- + \ln(m) - \frac{1}{2}\sigma_{\text{jump}}^2\ ;\ \sigma_{\text{jump}}^2\right). \tag{23}$$

Denoting by Z_t the Brownian bridge on the interval $[t_{n-1};\ t_n]$ defined by $Z_{t_{n-1}} = \ln(I_{t_{n-1}}^+)$ and $Z_{t_n} = \ln(I_{t_n}^-)$, we have :

$$\mathbb{P}\left(\max_{t_{n-1} \le t \le t_n} Z_t \ge \ln H \middle| I_{t_{n-1}}^+,\ I_{t_n}^-\right) = \exp\left\{-\frac{2\ln(I_{t_n}^-/H)\ln(I_{t_{n-1}}^+/H)}{\sigma_I^2 T}\right\}. \tag{24}$$

Hence, the probability that the CAT bond is not triggered during the interval $[t_{n-1};\ t_n]$ is

$$\mathbb{P}_n = 1 - \exp\left\{-\frac{2\ln(I_{t_n}^-/H)\ln(I_{t_{n-1}}^+/H)}{\sigma_I^2 T}\right\}. \tag{25}$$

Computing the price for a trajectory

Let t^* be the last jump instant of a given trajectory, and $\bar{\theta}$ the aggregate likelihood ratio from 0 to t^*. Then the price of the CAT bond for the trajectory is

$$\bar{\theta} P_d(t^*,\ T) P_{CAT}(t^*,\ T).$$

Indeed, when the trigger threshold has not been hit at the last jump, the price of the CAT bond at t^* is the price of a CAT bond in the model without jumps arriving to maturity at T. To compute $P_d(t^*,\ T)$ and $P_{CAT}(t^*,\ T)$, we need to simulate $r_d(t^*)$ and $F(t^*,\ T)$. The solution of Vašíček's SDE is

$$r_d(t^*) = r_d(0)\mathrm{e}^{-\kappa_d t^*} + \kappa_d\theta_d \int_0^{t^*} \mathrm{e}^{-\kappa_d t^*} ds + \sigma_d \int_0^{t^*} \mathrm{e}^{-\kappa_d t^*} dW_t^d.$$

It is clear that $r_d(t^*)$ is Gaussian, with

$$\mathbb{E}[r_d(t^*)] = r_d(0)\mathrm{e}^{-\kappa_d t^*} + \theta_d(1 - \mathrm{e}^{-\kappa_d t^*})$$

and

$$\mathrm{Var}[r_d(t^*)] = \frac{\sigma_d^2}{2\kappa_d}(1 - \mathrm{e}^{-2\kappa_d t^*}).$$

The distribution of $r_f(t^*)$ is similar. As for S_{t^*}, have the passage formula between 0 and t^*,

$$S_{t^*} = S_0 \exp\left(\int_0^{t^*} r_d(u)du - \int_0^{t^*} r_f(u)du + \sigma_S W_{t^*}^S\right).$$

Both integrals are Gaussian but to simulate them, we need to know their joint distribution respectively with $r_d(t^*)$ and $r_f(t^*)$. The integral in r_d is Gaussian and its parameters are (see, for instance, Glasserman [2004])

$$\mathbb{E}\left[\int_0^{t^*} r_d(u)du\right] = \frac{r_d(0) - \theta_d}{\kappa_d}(1 - \mathrm{e}^{-\kappa_d t^*}) + \theta_d t^*$$

and

$$\mathrm{Var}\left[\int_0^{t^*} r_d(u)du\right] = \frac{\sigma_d^2}{2\kappa_d^3}\left(2\kappa_d t^* - 3 + 4\mathrm{e}^{-\kappa_d t^*} - \mathrm{e}^{-2\kappa_d t^*}\right),$$

and the covariance between $r_d(t^*)$ and its stochastic integral is

$$\mathrm{Cov}\left(r_d(t^*),\ \int_0^{t^*} r_d(u)du\right) = \frac{\sigma_d^2}{2\kappa_d}(1 - 2\mathrm{e}^{-\kappa_d t^*} + \mathrm{e}^{-2\kappa_d t^*}).$$

Finally, $F(t^*,\ T)$ is obtained by the parity relation of the interest rates given by equation (5). We have thus explained in this section how to extend the method of Joshi and Leung to a foreign exchange rate model with three stochastic factors independent of I_t.

Simulation algorithm

For each trajectory,

1. Fix the likelihood ratio to 1;
2. Compute the first jump instant by simulating a Poisson random variable, with importance sampling to ensure that it is anterior to T;
3. Simulate all successive jump instants until T;
4. For each jump instant before maturity,
   (a) compute the Gaussian increment of the catastrophe index since the previous jump instant according to (20), with importance sampling to ensure that the trigger threshold is not crossed. Update the likelihood ratio by multiplying it by θ;
   (b) compute the probability that the trigger threshold be reached during this time step using the Brownian bridge maximum formula. Multiply the likelihood ratio by $\mathbb{P}_n$ given by equation (25);
   (c) compute the amplitude of the jump according to (22) with importance sampling to ensure that the trigger threshold is not crossed, and update the likelihood ratio by multiplying it by θ_2;
5. Simulate $r_d(t^*)$, $r_f(t^*)$ and S_{t^*} and compute $P_d(t^*,\ T)$ and $C(t^*,\ T)$;
6. Compute the CAT bond price in the model without jumps at t^*, $P_{CAT}(t_n,\ T)$, by the explicit formula (17);
7. Discount this price by $P_d(0,\ t_n)$ and multiply it by the likelihood ratio $\bar{\theta}$ accumulated during the trajectory. This obtains the CAT bond price for this trajectory.

Lastly, there remains to compute the mean of the prices obtained for all trajectories or paths, and to insert the result in equation (18). A numerical illustration of the simulation algorithm is given in Appendix B.

6 Numerical results

6.1 Choice of parameter values

For the parameters of the catastrophe index, we have kept the values that were used by Vaugirard [2002] in order to be able to compare the results given by our model with theirs, that is $\lambda = 0.1$, $\lambda_N = 0.5$, $\mu = 0.2$ $\sigma = 0.5$, $\omega = 0.9$, $\sigma_{\text{jump}} = 0.2$ and $m = 1.1$, and we have kept the proportion $I_0/H = 0.5$. Table 1 gives the reference values of the model parameters that were used in the simulation experiment.

Parameter	**Description**	**Value**
V	Face value of the CAT bond	1000
I_0	Initial level of the catastrophe index	100
H	Trigger threshold of the CAT bond	200
ω	Part of the face value exposed to catastrophic risk	0,9
λ	Risk premium for natural catastrophes	0.1
λ_N	Intensity of catastrophes	0.5
μ	Drift of the catastrophe index	0.2
σ	Volatility of the catastrophe index	0.5
m	Mean of the Poisson jumps	1.1
σ_{jump}	Standard deviation of the Poisson jumps	0.2
κ_d, κ_f	Mean reversion speeds of the interest rates	0.1
θ_d, θ_f	Long-term means of the interest rates	0.1
σ_S	Volatility of the exchange rate	0.1
σ_d	Volatility of the domestic interest rate	0.05
σ_f	Volatility of the foreign interest rate	0.05
ρ_{Sd}	Correlation between S_t and r_d	0.5
ρ_{Sf}	Correlation between S_t and r_f	-0.4
ρ_{df}	Correlation between r_d and r_f	0.25
S_0	Initial level of the exchange rate	0.0125
K	Strike price of the call option on currency	0.0125
$r_d(0)$	Initial level of the domestic interest rate	0.1
$r_f(0)$	Initial level of the foreign interest rate	0.1

Table 1: Reference values of the model parameter used in the simulation experiment

6.2 Impact of the catastrophe risk

We present here the price of a CAT bond as a function of the jump intensity (that is, the frequency of catastrophes) in the model with exchange rate that we have developed. We also give this price in the model of Vaugirard [2002] without exchange risk, with the same parameters for the catastrophe index. The results are shown in Table 2.

Jump intensity λ_N	Vaugirard [2002]	Our model
0	760.47	753.86
0.1	754.14	747.62
0.2	748.52	742.11
0.3	742.26	736.00
0.4	734.59	728.40
0.5	728.18	722.07
0.75	713.7	707.95
1	696.33	690.85
1.25	681.3	676.03
1.5	665.58	660.66
2	635.53	631.10

Table 2: Comparison of CAT bond prices as a function of jump intensity

We observe that the frequency of catastrophes and the exchange risk have a negative impact on the CAT bond price. Nonetheless, the risk of catastrophe is preponderant, in as much as the price difference between the models with and without exchange risk is of the order of 1% for the parameter values we used.

6.3 Sensitivity analysis of the economic parameters

Recall the equation for the conditional variance of the forward exchange rate,

$$\nu^2(\tau) = \sigma_S^2 \tau + \frac{\tau}{3}(\sigma_d^2 + \sigma_f^2 - 2\sigma_{df}) + \tau^2(\sigma_{Sd} - \sigma_{Sf}).$$

Since the price of the call option on currency is an increasing function of this conditional variance, an increase of ν^2 should have a negative impact on the price of the CAT bond because it would represent a raise of the hedging cost of the exchange risk. We expect that the parameters having a positive sign in this variance would have a negative relation with the price of the CAT bond. We must also verify whether the impact of these parameters is important or not.

σ_S	CAT bond price
0.01	726.73
0.025	726.02
0.05	724.74
0.075	723.41
0.1	722.07
0.15	719.33
0.2	716.58
0.3	710.97

Table 3: Impact of the exchange rate variance

Table 3 shows that the variance of the exchange rate has a negative impact on the price of the CAT bond, which is consistent with the previous conjecture. Table 4 shows that the variance of the interest rates has an almost null impact on the price of the CAT bond. Surprisingly, the variance of the domestic interest rate exhibits a positive relation with the price of the CAT bond.

σ_d	CAT bond price	σ_f	CAT bond price
0.02	722.14	0.02	722.17
0.03	722.07	0.03	722.07
0.05	721.98	0.05	721.83
0.075	721.97	0.075	721.48
0.1	722.09	0.1	721.08
0.15	722.74	0.15	720.19

Table 4: Impact of the variance of the interest rates

By varying the correlation structure between the three stochastic factors, we observe in Table 5 that only the correlations with the exchange rate have a real impact on the CAT bond price; by contrast, the variations of the correlation between the two interest rates have a negligible impact. The correlation between the exchange rate and the domestic interest rate exhibits a negative relation with the CAT bond price, while that between the exchange rate and the foreign interest rate admits a positive relation, which confirms the intuition provided by the equation for the variance of the forward rate.

ρ_{Sd}	CAT bond price	ρ_{Sf}	CAT bond price	ρ_{df}	CAT bond price
-0.8	722.84	-0.7	721.86	-0.8	722.08
-0.5	722.65	-0.5	721.99	-0.5	722.07
-0.2	722.46	-0.2	722.21	-0.2	722.06
0	722.34	0	722.36	0	722.06
0.2	722.23	0.2	722.51	0.2	722.06
0.5	722.07	0.5	722.73	0.5	722.07
0.75	721.94	0.8	722.97		

Table 5: Influence of the correlation structure

Table 6 shows that the initial level of the domestic interest rate has a strong negative impact on the price of the CAT bond, essentially due to its contribution in the discount factor: the higher the interest rate, the lower is the discount factor. The level of the foreign interest rate has instead a very weak impact, negative too.

$r_d(0)$	CAT bond price	$r_f(0)$	CAT bond price
0.08	735.52	0.08	723.01
0.09	728.78	0.09	722.56
0.1	722.07	0.1	722.07
0.11	715.37	0.11	721.53
0.12	708.7	0.12	720.96

Table 6: Influence of the initial level of the interest rates

The sensitivity analysis of the parameters that we just performed shows that, in addition to the catastrophic risk, the CAT bond price is mainly affected by the variance of the exchange rate and its correlations with the domestic and foreign interest rates. The other parameters show weak influence. The role of these parameters stems from their contribution to the conditional variance of the forward exchange rate, which is the main driver of the price of the call option on currency, and thus of the hedging cost of the exchange risk.

7 Conclusion

We proposed a no-arbitrage pricing model for CAT bonds subject to the risk of currency exchange. It extends the model of Poncet and Vaugirard [2001] by taking into account the catastrophic events by means of a jump-diffusion process, in an economy with three stochastic factors, the exchange rate as well as the domestic and foreign interest rates. The exchange risk was modelled by introducing a hypothetical hedge with a call option on the currency. A semi-explicit formula was derived, in which only the triggering instant of the CAT bond may not be computed directly. For the model to be tractable from a numerical point of view, we had to choose an exchange rate model that gives an explicit formula for the price of a call option on currency, provided in this case by Hilliard et al. [1991], and to suppose that the natural catastrophes are independent of the rest of the economy.

Numerical computations were performed by Monte Carlo simulation with the method of Joshi and Leung [2007], based on the Brownian bridge technique and importance sampling, which results in good accuracy for a much reduced number of trajectories. In terms of running time, it is much faster than a brute force simulation by naive discretization for the weak jump intensities. Prior to this, we had to derive an extension of the method to take into account the three stochastic factors of the economy, which required the joint distribution of these variables between two arbitrary instants.

Our numerical results show that the exchange risk remains weak compared to the risk of natural catastrophe. The sensitivity analysis of the parameters has revealed that the volatility of the exchange rate and its correlations with the interest rates are the main economic parameters affecting the price of the CAT bond, the first two having a negative impact, the last one having a positive impact. This can be explained analytically by noticing that these are the factors that contribute the most to the conditional variance of the forward exchange rate and thus to the price of the call option on currency used for hedging.

A Presentation of CAT bonds

A.1 Economic motivation

CAT bonds are designed to protect their issuer from rare events of such big severity that it is uncommon to cover them by reinsurance. A contract for such a risk would be particularly onerous because a reinsurer who intervenes in a sensitive geographical area is already much exposed. CAT bonds are less an alternative to traditional reinsurance than a complement. They also have the property to bring the catastrophe risk onto a territory where it is absent a priori, the financial markets. Hence they direct this risk toward the economic agents who wish to take it. These, in turn, find in them an interesting diversification tool for their portfolios, because the CAT bonds have yields that are weakly correlated with those of the market, if not uncorrelated (Litzenberger et al. [1996]). Hence CAT bonds are said to be "zero-beta" assets. The last major advantage of such an investment is that it promises strong yields, much juicier than those of classical bonds which are attributed the same grade by rating agencies.

A drawback of the CAT bonds is their high initial cost for the issuer, owing in particular to the creation of a SPV (Canter et al. [1996]). From the point of view of the investor, their disadvantage is the margin requirement, which corresponds to 100% of the initial amount.

A.2 Rigorous definition

Structure

A CAT bond takes the form of a traditional bond that pays interest in the form of coupons at regular interval in exchange for a predefined initial amount. Its singularity stems from the fact that the cash flows promised to the investor are contingent upon the absence of natural catastrophe before the product's maturity; if an index of natural risk (or an index of losses of the insurance sector) reaches a predetermined threshold, the holder of the CAT bond loses either his coupons, or either the face value, sometimes both of them, all depending on the initial contract. An analogy clearly appears between the risk of natural catastrophe and the default risk of a bond of a speculative category.

The CAT bond is issued by a *Special-Purpose Vehicle*, or SPV for short, created for that purpose and located in a tax-shelter or haven, such as the Cayman Islands. This setup leads to a financial instrument that is purely linked to the catastrophes, because it has been detached from the sponsor's credit risk. The money received at the CAT bond's issuance in the form of face value then feeds a fund dedicated to indemnizing the sponsor should a catastrophic event happen, so he can honor his obligations, which consist in reimbursing the insured having suffered material losses. Meanwhile, the SPV invests the money in the short term in risk-free assets, for example American Treasury bonds, and pay to the investors variable rate coupons, that is in the form of LIBOR + risk premium. CAT bond holders often take a position in a swap to guarantee their rate. The above structure is illustrated schematically in Figure 1.

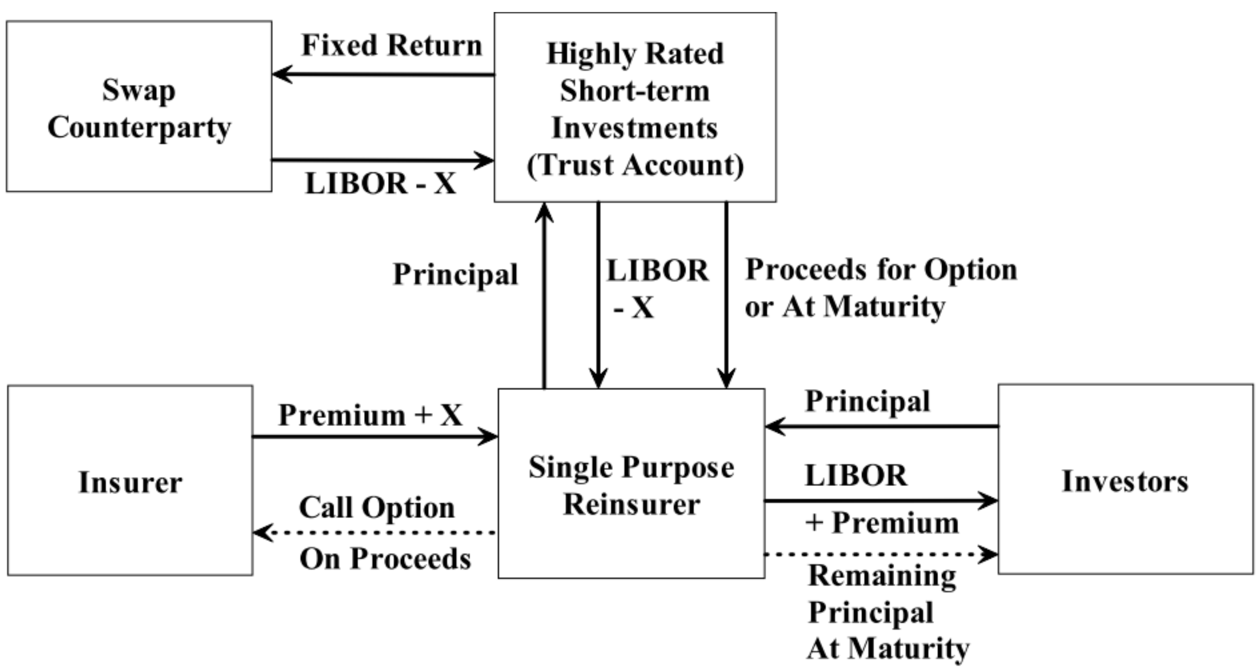


Figure 1: Typical structure of a CAT bond.

Although designed as products purely linked to the risk of natural catastrophes, CAT bonds are nonetheless subjected to other risks, including the risk of counterparty arising from the rate swap, the risk of interest rate inherent to all bonds, and the risk of exchange which is the object of this paper.

Triggers

It is fundamental to be able to determine in a formal, objective, and impartial manner the instant when we detect a natural catastrophe of sufficient amplitude to "trigger" the CAT bond, that is to suspend payment of future cash flows to the investor and indemnify the sponsor. This task is fulfilled by a parameter called the triggering event (or *trigger*). Cummins [2008] enumerates three types of triggers:

- indemnity trigger: the CAT bond is triggered according to the issuer's level of losses;
- index-based trigger: the cash flows depend on the level of a prespecified index;
- hybrid trigger: several triggers are used in a single CAT bond.

One detail deserves to be made precise; the triggering is arbitrated over a precise geographic area that is clearly defined in the contract. For example, the seismic event that happened off the coast of Japan on 11 March 2011 has not triggered any of the CAT bonds that concerned the Japanese Islands, because their span was restricted to the city of Tokyo and its suburbs, which have been spared.

There exists a great variety of index-based triggers. We enumerate three index categories:

- index of industry losses: triggered when the losses for the whole insurance sector provoked by a catastrophic event reach a certain threshold. Examples are the PCS index (Property Claim Services), NatCatSERVICE of Munich Re, or else PERILS;
- model based index: index computed using a model provided by a catastrophe modelling firm, such as Applied Insurance Research, Equecat or Risk Management Solution;
- parametric index: triggered by a specific physical measurement, such as wind speed for a CAT bond on hurricane, or a level on the Richter scale for an earthquake.

Moral hazard and basis risk

Which trigger to choose? The answer results from an equilibrium between moral hazard and basis risk (Doherty [1997]). The basis risk owes to an imperfect correlation between the losses accumulated by the transferor and the cash flows of the CAT bond. It is, more or less, the risk that the transferrer would have losses that would not be entirely covered by the payments of the CAT bond. Among the set of possible triggers, that of indemnity procures the strongest moral hazard but it has no basis risk. At the opposite, the parametric trigger is clear of any moral hazard and it carries a high basis risk.

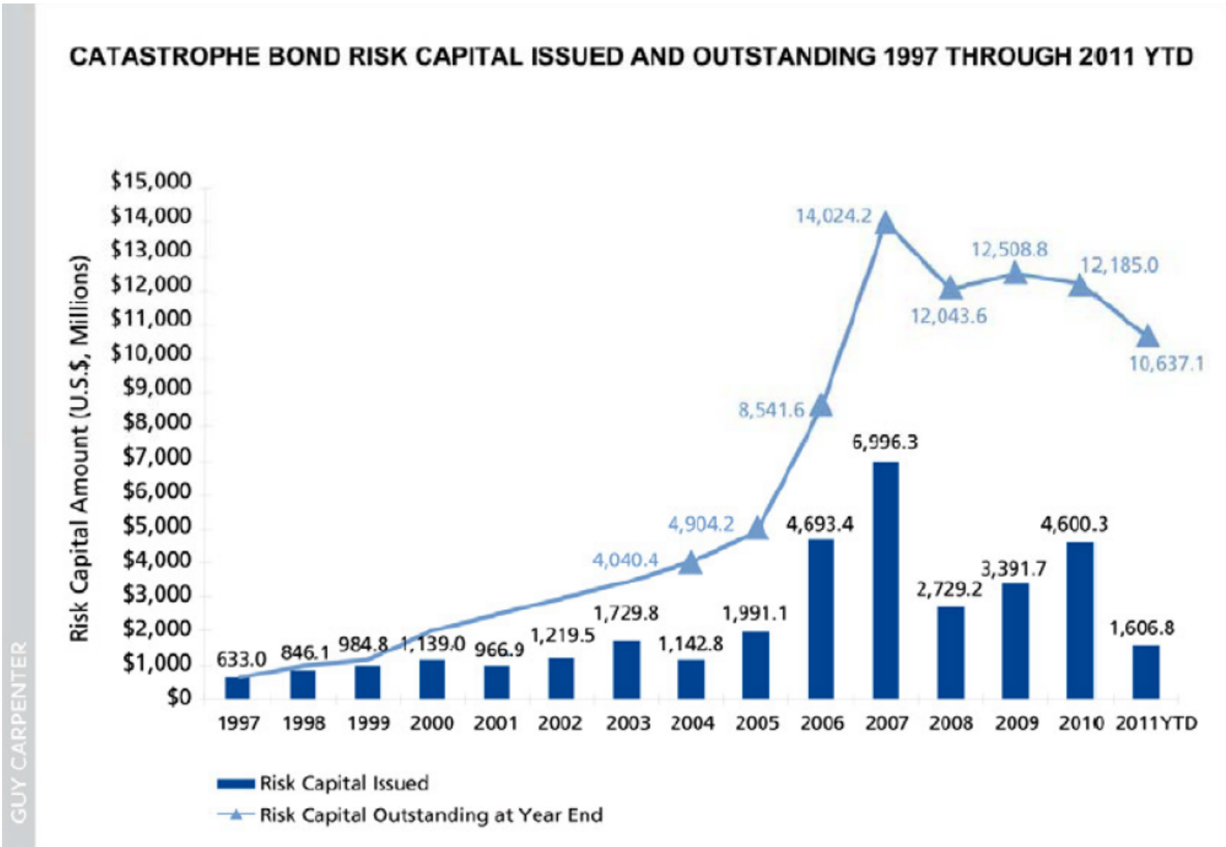
Logically, the indemnity trigger is preferred by the CAT bond 's issuer, who is concerned only with the basis risk, but it imposes on the investor some inconvenience due to the verification of the losses; the latter is performed by an independent supervisor and may take some time. As a consequence, a higher remuneration is often demanded of the CAT bond with an indemnity trigger. The investor, on the other hand, tends to favor a parametric trigger, because it is easy to verify its level, and it does not bring in big surprises. This confrontation between both parties could be resolved by using a double or multiple trigger.

A.3 A brief look at the CAT bond market

The first CAT bond was issued by Hannover Re in 1994 for an amount of 85 M$. After a timid debut in the years 1990, there was a strong acceleration of issuances in 2006 and 2007 by volume as by number of transactions, which is put in evidence in Figure 2. However the market suffered from the bankruptcy of the Lehman Brothers investment bank, which played the role of financial partner in four important contracts. The first CAT bond to have been triggered is the KAMP Re Ltd. issued by Zurich Financial in 2005 for an amount of 190 M$, following hurricane Katrina.

Issuers of CAT bonds are mostly insurers and reinsurers, but there is also emergence of institutional sponsors such as the Mexican state. The largest issuer is the reinsurer Swiss Re. CAT bonds are products sold to institutional investors. The first clients were insurance companies, mutual fonds and hedge funds, but market expansion has seen the appearance of funds that specialize in insurance-linked securities, who nowadays hold nearly half this market.

The first CAT bonds had a maturity of one to ten years, but recent evolution shows that the issuances of the last two years usually had a maturity of three years. They are also characterized by a secondary market that is barely developed and not quite liquid; the investors indeed tend to keep these products until maturity. There results a lack of liquidity



Source: GC Securities

Figure 2: Evolution of the CAT bond market.

on this market and high issuance spreads, as well as an opaque valuation.

Cummins and Weiss [2009] observe that for the period from 1997 to 2007, the demand represents 31.8% of total volume for the coverage of hurricanes in the United States, 29.6% for earthquakes in the USA, 15.2% for storms in Europe, 11% for earthquakes in Japan and 8% for typhoons in Japon. As for the proportions according to the type of trigger, CAT bonds with indemnity trigger occupy 30% of total volume, while parametric triggers represent 25.9%, industrial indexes 21.5%, hybrid indexes 14% and model indexes 8.5%.

B Numerical illustration of a trajectory simulation

To illustrate how the numerical simulation method of §5 works, we provide the details of the step-by-step simulation of a trajectory or path. Suppose that the index value at $t = 0$ is 100 and the trigger threshold is 200, with maturity in 1 year.

- We initialize the likelihood ratio at 1: $\theta = 1$.

- We first simulate the jump instants: two jumps occur at instants $t_1 = 0,1053$ and $t_2 = 0,1368$. $\log I_0 = 4,6051$.

- Between t_0 and t_1, $\log I_t$ follows a simple Brownian motion, $\log I_{t_1}$ follows a normal distribution. We simulate $\log I_{t_1}$ given I_0 and conditionally to $\log I_{t_1}^- < \log H$. We obtain $\log I_{t_1}^- = 4.6402$.

- The probability that $\log I_{t_1}$ stays below $\log H$ is $\theta_1 = 0,9999$. We update the likelyhood ratio: $\theta = 0.9999$.

- The probability that the catastrophe index has not crossed the trigger threshold between t_0 and t_1 is computed with (25). We obtain $\mathbb{P}_n = 0,9999$, we then update the likelihood ratio: $\theta = \theta * \mathbb{P}_n$.

- We simulate the jump that occurs at t_1 conditionally to the fact that the index value after the jump remains below the trigger threshold, and we obtain $\log I_{t_1}^+ = 4,7558$. The probability that the jump does not trigger the CAT bond is $\theta_2 = 0,9982$. We update the likelihood ratio: $\theta = \theta * \theta_2 = 0,9982$.

- We simulate $\log I_{t_2}$ given $\log I_{t_1}$ and conditionally to $\log I_{t_2} < \log H$. We obtain $\log I_{t_2}^- = 4.9064$. The probability that $\log I_{t_2}$ remains below $\log H$ is $\theta_1 = 0.9999$. We update the likelihood ratio: $\theta = \theta * \theta_1$.

- The probability that the CAT bond has not been triggered between t_1 and t_2 is $\mathbb{P}_n = 1$.

- We simulate the jump that occurs at t_2 conditionally to the fact that $\log I_{t_2}^+$ remains below $\log H$. We find $\log I_{t_2}^+ = 4.8033$. The probability there is no triggering by this jump is $\theta_2 = 0.9433$, which gives an accumulated likelihood ratio $\theta = \theta * \theta_2 = 0.9416$.

- To obtain the CAT bond price for this path, we compute the price in the model without jumps of a CAT bond that begins at t_2 and matures at $T = 1$, that we discount with

$P_d(t_2)$, and multiply by θ which is the cumulative probability of not crossing the trigger threshold between $t = 0$ and $t = T$. We obtain the price for the trajectory: $P = 740.97$.

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