Cointegration and stochastic correlation models for commodity derivatives

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Abstract

Cointegration and stochastic correlations, including stochastic volatilities, are statistically significant for the spot prices of crude oil and gasoline. As these commodities are not traded on exchange, their futures prices provide us with strong empirical support that cointegration contributes significantly to the stochastic movements of their convenience yields in addition to their storage costs. We develop continuous-time cointegrated asset dynamics with a stochastic covariance matrix to simultaneously capture the effects of cointegration and stochastic correlations. Our proposed model allows us to super-calibrate the cointegration parameters by fitting to the observed term structure of futures prices. We demonstrate the model's use in valuing options on a single asset and on multiple assets using Fourier transform techniques.

JEL classification: G12; G13

Key words: cointegration, stochastic correlation, stochastic convenience yield.

1. Introduction

A number of stylized facts about commodity prices have been uncovered in the literature. Whereas Bessembinder et al. (1995) and Schwartz (1997) observe the mean reversion of commodity spot prices, other researchers have identified a long-term equilibrium relationship between multiple commodities, termed cointegration; cf., Serletis (1994), Duan and Theriault (2007), Lardica and Mignona (2008), Maslyuka and Smyth (2009), and Westgaarda et al. (2011). Trolle and Schwartz (2009) recognize the unspanned stochastic volatility in commodity markets, particularly that in the crude oil market.

Cointegration and stochastic volatility are important ingredients in derivatives pricing and hedging. Alexander (1999) points out that the concept of cointegration is essential in hedging with futures. Duan and Pliska (2004) consider the valuation of crack spread options on two cointegrated assets. They employ an error correction model with GARCH to value the spread option, but assume the correlation between the two assets to be a constant value. Dempster et al. (2008) develop spread option pricing models for the situation in which the two underlying commodity prices of the spread are cointegrated. They model the spread process directly using two latent factors, as the correlation between two asset returns is notoriously difficult to model.

In this paper, we propose a tractable continuous-time model that simultaneously captures cointegration, stochastic volatilities, and stochastic correlations. More specifically, the cointegration structure is reflected by the diffusion limit of the discrete error correction model, whereas the stochastic correlations are subsumed in the stochastic matrix dynamics following the Wishart affine stochastic correlation (WASC) model. Our continuous-time cointegration dynamics are consistent with those in Duan and Pliska (2004) and Chiu and Wong (2011). The WASC process proposed by Bru (1991) generalizes the Heston (1993) model to allow for stochastic conditional correlation. The use of WASC processes for multivariate stochastic volatilities is exploited by Gourieroux and Sufana (2010), and Gourieroux et al. (2009) provide thorough analysis of the properties of these processes in both discrete and continuous-time settings. Buraschi et al. (2010) investigate stochastic correlation risk in asset allocation using a WASC process. We derive explicit solution to futures prices and an analytical expression for option prices under our proposed model.

In addition to the valuation of commodity derivatives, this paper contributes to the finance literature by providing insights into several interesting questions. First, if cointegration exists among commodities, does it affect the convenience yields of commodity futures? If the underlying assets are traded, then the locally risk-neutral (LRN) valuation adopted by Duan and Pliska (2004), which assumes a martingale condition for the discounted asset prices, causes cointegration to have no impact on convenience yields. However, crude oil and gasoline are not traded on exchange, and hedging with spot prices is impossible in the oil market. Using a time-deterministic market price of risk, our model permits the cointegration of non-tradable underlying assets under the market-implied pricing measure. We test this hypothesis using the spot and futures prices of crude oil and gasoline. Although our empirical result reconfirms the theory of storage in Routledge et al. (2000), it also strongly supports the hypothesis that cointegration is an important contributing factor to convenience yields. Our empirical analysis also supports the formula of convenience yield derived with the proposed model.

Second, Hilliard and Reis (1998) and Casassus and Collin-Dufresne (2005) assume exogenous mean-reverting convenience yield model to improve the valuation of commodity derivatives. As cointegrating factors form a mean-reverting system and are factors of convenience yield, our model offers an endogenous explanation for the stochastic mean-reverting convenience yield. When we contrast our model with that in Hilliard and Reis (1998), we find very similar behavior in the yield dynamics. In addition, as the cointegrating factor is a linear combination of log-spot-prices, and spot prices exhibit stochastic correlations and volatilities, our model predicts that the volatility of the convenience yield is stochastic. This prediction coincides exactly with Broadie et al. (2000), who use a non-parametric approach to show that putting the stochastic dividend and volatility together improves empirical option prediction.

Third, how can risk-neutral cointegration parameters be calibrated to the observed market prices? We render a super-calibration of the risk-neutral characteristic function to the term structure of futures contracts. The risk-neutral cointegration matrix can then be filtered by examining the elasticity of futures prices to spot prices. The calibrated characteristic function is then applied to the valuation of options on a single asset and to options on multiple assets using fast Fourier transform (FFT) techniques.

The remainder of this paper is organized as follows. Section 2 presents the proposed model under the physical measure, provides a method for identifying the pricing measure implied by the market, and derives the joint characteristic function of the underlying assets. Section 3 offers comprehensive empirical analysis using the spot and futures prices of crude oil and gasoline. We show that these two commodities are cointegrated and have a stochastic correlation coefficient under the physical probability measure. Then, we further test our hypothesis that cointegration affects the convenience yields of commodity futures. Section 4 links the characteristic function to the Fourier transform of option prices and employs several numerical examples to demonstrate the application of the framework. The super-calibration of the model to the term structure of futures prices is also discussed. Section 5 concludes the paper.

2. The Model

This section introduces and discusses derivative pricing models with cointegration and stochastic volatilities. After reviewing the discrete-time cointegrated assets dynamics with GARCH investigated in Duan and Pliska (2004), we explain the need to generalize their concept to the commodity derivatives market, in which the locally risk-neutral (LRN) valuation is inadequate to explain stochastic convenience yields when the commodities are not traded on exchange but their futures contracts are. A pricing measure corresponding to a simple mean-shifting approach is proposed and illustrated with the continuous-time cointegrated assets dynamics with WASC. Our approach retains cointegration under the market-implied risk-neutral measure.

2.1. Review of a discrete-time model

In the discrete-time model of Duan and Pliska (2004), with respect to the data generating probability measure $\mathbb{P}$ (or physical probability measure), an error correction dynamic for the n-component asset price time series with k $(1 \leq k \leq n)$ cointegrating factors is defined as follows.

$$\begin{aligned} \ln S_{i,t} - \ln S_{i,t-1} &= r - \frac{1}{2}\sigma_{i,t}^2 + \lambda\sigma_{i,t} + \sum_{j=1}^{k} \delta_{ij} z_{j,t-1} + \sigma_{i,t}\epsilon_{i,t}, \quad i = 1, \ldots, n, \\ \sigma_{i,t}^2 &= \beta_{i0} + \beta_{i1}\sigma_{i,t-1}^2 + \beta_{i2}\sigma_{i,t-1}^2(\epsilon_{i,t-1} - \theta_i)^2, \\ z_{j,t} &= a_j + b_j t + \sum_{i=1}^{n} c_{ij} \ln S_{i,t} \text{ for } j = 1, \ldots, k, \end{aligned} \tag{1}$$

where $(c_{1j}, \cdots, c_{nj})$ are linearly independent vectors for $j = 1, \ldots, k$; β_{ij} are constants for $i = 1, \ldots, n$ and $j = 0, 1, 2$; $[\theta_1, \cdots, \theta_n]$ is a constant vector; and the random vector $[\epsilon_{1,t}, \cdots, \epsilon_{n,t}]$ follows a multivariate normal distribution with mean zero and a constant correlation coefficient matrix. In the error correction model, the vector of $[z_{1,t}, \cdots, z_{k,t}]$

should be a stationary time series, such that each $z_{j,t}$ has bounded variance at all time points.

As the physical dynamics in (1) are not suitable for derivatives valuation, Duan and Pliska (2004) adopt the equilibrium pricing measure $\widehat{\mathbb{Q}}$, which satisfies the following assumptions.

1. $\widehat{\mathbb{Q}}$ and $\mathbb{P}$ are mutually absolute continuous;
2. conditional on the information up to $t-1$, $\mathcal{F}_{t-1}$, $S_{i,t}/S_{i,t-1}$ for all $i = 1, \ldots, n$ has a multivariate lognormal distribution under $\widehat{\mathbb{Q}}$.
3. $\mathbb{E}^{\widehat{\mathbb{Q}}}[S_{i,t}/S_{i,t-1}|\mathcal{F}_{t-1}] = e^r$ for all i; and
4. $Cov^{\widehat{\mathbb{Q}}}(\ln \frac{S_{i,t}}{S_{i,t-1}}, \ln \frac{S_{j,t}}{S_{j,t-1}}|\mathcal{F}_{t-1}) = Cov^{\mathbb{P}}(\ln \frac{S_{i,t}}{S_{i,t-1}}, \ln \frac{S_{j,t}}{S_{j,t-1}}|\mathcal{F}_{t-1})$ for all i and j, almost surely with respect to $\mathbb{P}$.

Then, the LRN dynamics take the form

$$\begin{aligned}
&\ln S_{i,t} - \ln S_{i,t-1} = r + \sigma_{i,t}\xi_{i,t}, \quad i = 1, \ldots, n, \\
&\sigma_{i,t}^2 = \beta_{i0} + \beta_{i1}\sigma_{i,t-1}^2 + \beta_{i2}\sigma_{i,t-1}^2 \left(\xi_{i,t-1} - \theta_i - \lambda_i - \sum_{j=1}^{k} \delta_{ij} \frac{z_{j,t-2}}{\sigma_{i,t=1}} \right)^2, \\
&z_{j,t} = a_j + b_j t + \sum_{i=1}^{n} c_{ij} \ln S_{i,t} \text{ for } j = 1, \ldots, k,
\end{aligned} \tag{2}$$

where $[\xi_{1,t}, \ldots, \xi_{n,t}]$ follows a multivariate normal distribution with a zero mean vector under $\widehat{\mathbb{Q}}$. Although it is easy to derive the diffusion limit of (1), that of (2) is much less obvious and is not explicitly given in Duan and Pliska (2004). It can be recognized from (2) that the cointegrating factors, $[z_{1,t}, \ldots, z_{k,t}]$, disappear in the conditional asset price process but appear in the volatility dynamics in (2).

Assumption 3 for deriving (2) is referred to as the martingale condition. When the underlying assets are liquidly traded, a derivative security can be hedged using its underlying asset at time t, at which the physical process (1) has locally deterministic volatility. The hedging procedure is equivalent to adjusting the drift of the process to the risk-free rate. If (2) is the correct process under the pricing measure, then the T-futures price of the i-th asset quoted at t is $S_{i,t}e^{r(T-t)}$. However, futures contracts in the commodity market usually observe (stochastic) convenience yields, cf., e.g., Fama and French (1987), Schwartz (1997), Casassus and Collin-Dufresne (2005), and Liu and Tang (2011). The theory of storage implies that the convenience yield depends mainly on the commodity inventories, and predicts a negative relationship between the convenience yield and inventories. The convenience yield can then be partially explained in terms of the interest forgone in storing the commodity. Regression approaches analogous to Fama and French (1987) and Bailey and Ng (1991) are adopted here to explain the change in the convenience yield. In hypothesis I, the interest rate term reflects the time value of the storage cost, and the term $(T-t)^{-1}$ is used to model the Samuelson effect, where $T-t$ is the time-to-maturity of the futures contract.

Hypothesis I: $F_{i,T}(t) = \mathbb{E}^{\widehat{\mathbb{Q}}}[S_{i,T}|\mathcal{F}_t] = S_{i,t}e^{(r-\widehat{q}_{i,T}(t))(T-t)}$ for $i = 1, \ldots, n$, where $F_{i,T}(t)$ is the T-futures price of $S_{i,t}$ quoted at time t and

$$\Delta\widehat{q}_{i,T}(t) = \beta^I_{i0} + \beta^I_{i1}\Delta r_t + \beta^I_{i2}\Delta(T-t)^{-1},$$

where $\Delta\widehat{q}_{i,T}(t) = \widehat{q}_{i,T}(t) - \widehat{q}_{i,T}(t-\Delta t)$, $\Delta r_t = r_t - r_{t-\Delta t}$, and $\Delta(T-t)^{-1} = \frac{1}{T-t} - \frac{1}{T-(t-\Delta t)}$.

In hypothesis I, the convenience yield is assumed to be independent of the underlying spot price of the futures contract. The theory of storage in Fama and French (1987) expects the regression coefficient of the interest rate term to be positive. In the following hypothesis II, we follow Bailey and Ng (1991) and use the underlying spot price as the proxy for the inventory level contributing to the convenience yield. In other words, for the i-th commodity, the convenience yield $\widehat{q}_{i,T}(t)$ depends on the underlying spot price $S_{i,t}$ and the interest rate, as in the model of Casassus and Collin-Dufresne (2005).

Hypothesis II: $F_{i,T}(t) = \mathbb{E}^{\widehat{\mathbb{Q}}}[S_{i,T}|\mathcal{F}_t] = S_{i,t}e^{(r-\widehat{q}_{i,T}(t))(T-t)}$ for $i = 1, \ldots, n$, and

$$\Delta\widehat{q}_{i,T}(t) = \beta_{i0}^{II} + \beta_{i1}^{II}\Delta r_t + \beta_{i2}^{II}\Delta\frac{1}{T-t} + \widehat{\beta}_i^{II}\Delta X_{i,t} + \widetilde{\beta}_i^{II}\Delta\frac{X_{i,t}}{T-t},$$

where $\Delta X_{i,t} = X_{i,t} - X_{i,t-\Delta t}$, and $\Delta\frac{X_{i,t}}{T-t} = \frac{X_{i,t}}{T-t} - \frac{X_{i,t-\Delta t}}{T-(t-\Delta t)}$.

Hypotheses I and II both come from the traditional theory of storage in which cointegration has no impact on the convenience yield. An alternative hypothesis, however, is that, in addition to the conventional explanatory variables, cointegrating factors are also important contributors to the stochastic convenience yields once the underlying assets are not traded on exchange. The martingale condition with respect to the spot prices may not hold under the market-implied pricing measure, but it is expected to hold with respect to the futures contracts. This alternative hypothesis will be detailed shortly.

2.2. The proposed model

The proposed model extends the physical process (1) to cope with stochastic volatilities and correlations. However, we base our model on a diffusion limit generalizing that of (1).

Consider n commodity spot prices whose values at time t are represented in an n-dimensional vector S_t. The $n \times n$ positive definite stochastic covariance matrix is denoted by V_t. We postulate the joint dynamics of logarithmic spot prices $X_t = \ln S_t$ and V_t under the physical measure $\mathbb{P}$ as

$$\mathrm{d}X_t = [\theta(t) - \kappa X_t + (\mathrm{Tr}(D_1V_t), ..., \mathrm{Tr}(D_nV_t))']\mathrm{d}t + \sqrt{V_t}\mathrm{d}Z_t^{\mathbb{P}}, \tag{3}$$

$$\mathrm{d}V_t = (\beta Q'Q + MV_t + V_t M')\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathbb{P}} Q + Q'(\mathrm{d}W_t^{\mathbb{P}})'\sqrt{V_t}, \tag{4}$$

where $Z_t^{\mathbb{P}} \in \mathbb{R}^n$ and $W_t^{\mathbb{P}} \in \mathbb{R}^{n\times n}$ are a vector Brownian motion and matrix Brownian motion under the physical measure $\mathbb{P}$, respectively. κ is an $n \times n$ constant matrix of cointegration coefficients. If κ is a positive diagonal matrix, then all of the individual assets are stationary and exhibit mean-reversion. $\theta(t)$ is an n-dimensional deterministic function. β is a scalar, such that $\beta > n - 1$, and M and Q are $n \times n$ constant real square matrices. $\mathrm{Tr}(\cdot)$ denotes the trace operator. Q' is the (unconjugated) transpose of matrix Q. $\sqrt{V_t}$, the square root of V_t, is defined as the unique symmetric positive definite matrix, such that $\sqrt{V_t}\sqrt{V_t} = V_t$. The continuous-time process of the stochastic matrices in (4) is essentially the continuous time Wishart process introduced by Bru (1991). The Wishart process ensures that, at any point in time, the stochastic variance matrix is always a positive definite matrix. $\mathrm{Tr}(D_i V_t)$ in (3) represents the risk premium of V_t.

In our model, X_t and V_t are correlated. Following Da Fonseca et al. (2007), we require that $Z_t^{\mathbb{P}}$ in (3) and $W_t^{\mathbb{P}}$ in (4) are linearly correlated in the following way.

$$\mathrm{d}Z_t^{\mathbb{P}} = \mathrm{d}W_t^{\mathbb{P}}\rho + \sqrt{1-\rho'\rho}\,\mathrm{d}B_t^{\mathbb{P}}, \tag{5}$$

where $\mathrm{d}B_t^{\mathbb{P}}$ is an n-dimensional Brownian motion independent of $\mathrm{d}W_t^{\mathbb{P}}$, and ρ is a vector of correlations, such that $\rho_i \in [-1, 1]$ and $\rho'\rho < 1$. The market model introduced by Gourieroux and Sufana (2010) is also based on (5) and further assumes that $\rho_i = 0$, for which the Brownian motions of asset returns are independent of those driving the covariance matrix, which leads to a symmetric volatility smile for vanilla options. Buraschi et al. (2010) assume that the Brownian motion driving the asset returns is generated by the Brownian motion matrix of the covariance process.

The multivariate cointegration dynamics with WASC in (3) and (4) strike a balance between tractability and the ability to fit empirical stylized facts. In particular, it enables the characteristic function of log-asset-values to take an exponential affine form. Consider the joint characteristic function of X_T evolving as (3) and (4) conditional on $\mathcal{F}_t$. Denote the characteristic function under $\mathbb{P}$ as

$$f^{\mathbb{P}}(x, v, t; u) = \mathbb{E}^{\mathbb{P}}[e^{\mathbf{i}uX_T} | X_t = x, V_t = v], \tag{6}$$

where $T \geq t$, $\mathbf{i} = \sqrt{-1}$, and $u = (u_1, ..., u_n)$ is a row vector of size n.

Lemma 1. *If X_t follows the dynamics in (3) and (4), then the joint characteristic function defined in (6) is given by*

$$f^{\mathbb{P}}(x, v, t; u) = \exp\left[\mathrm{Tr}(A(\tau; u)v) + B(\tau; u)'x + C(\tau; u)\right], \tag{7}$$

where $\tau = T - t$, $A(\tau; u) = H(\tau; u)^{-1}G(\tau; u)$, $B(\tau; u) = \mathbf{i}e^{-\kappa'\tau}u'$, $C(\tau; u) = \mathbf{i}u\int_0^\tau e^{-\kappa s}\theta(t-s)\mathrm{d}s - \frac{\beta}{2}\mathrm{Tr}\left[\ln H(\tau; u) + M'\tau - \mathbf{i}\int_0^\tau e^{-\kappa' s}\mathrm{d}s u'\rho' Q\right]$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} G(\tau; u)' \\ H(\tau; u)' \end{pmatrix} = \begin{pmatrix} M + Q'\rho B(\tau; u)' & -2Q'Q \\ \frac{B(\tau;u)B(\tau;u)'}{2} + \sum_{j=1}^n U_j(1)B(\tau; u)D_j & -(M' + B(\tau; u)\rho' Q) \end{pmatrix}' \begin{pmatrix} G(\tau; u)' \\ H(\tau; u)' \end{pmatrix},$$

with $H(0; u) = \mathbf{I}_n$, $G(0; u) = \mathbf{0}_n$, $\mathbf{I}_n$ and $\mathbf{0}_n$ being the identity and zero matrices of order n, respectively. $U_j(y)$ denotes a row vector with the j-th element equal to y and the remaining elements equal to 0.

Proof. See Appendix A. $\square$

The matrix exponential $e^{-\kappa\tau}$ can be computed easily based on the Cayley-Hamilton theorem by using a polynomial of $-\kappa$, as detailed in Appendix B.

2.3. Market price of risk specifications

For the purpose of derivatives valuation, it is necessary to discuss the joint process (X_t, V_t) under the pricing measure implied by the market, or the so-called risk-neutral measure $\mathbb{Q}$, and the linkage between the physical dynamics and the risk-neutral dynamics. In addition, as commodities are not traded on exchange, but their futures contracts are, the martingale condition should hold with respect to futures prices instead of spot prices. Here, we adopt the extended affine market price of risk specification suggested by Cheredito et al. (2007) and Trolle and Schwartz (2008) to preserve cointegration under $\mathbb{Q}$. This is the most flexible market price of risk specification that preserves the affine structure of the state vector under the change of measure. In our setting, the extended affine specification is given by

$$\Lambda_{X,t} = (\sqrt{V_t})^{-1}(\lambda_X^0 + \lambda_X^1 X_t + (\mathrm{Tr}(D_1^\lambda V_t), ..., \mathrm{Tr}(D_n^\lambda V_t))'), \tag{8}$$

$$\Lambda_{V,t} = (\sqrt{V_t})^{-1}(\Lambda_V^0 + V_t \Lambda_V^1), \tag{9}$$

from which Brownian motions under $\mathbb{P}$ and $\mathbb{Q}$ are linked through

$$\mathrm{d}Z_t^{\mathbb{P}} = \mathrm{d}Z_t^{\mathbb{Q}} - \Lambda_{X,t}\mathrm{d}t, \quad \mathrm{d}W_t^{\mathbb{P}} = \mathrm{d}W_t^{\mathbb{Q}} - \Lambda_{V,t}\mathrm{d}t. \tag{10}$$

Hence, the joint dynamics of X_t and V_t under $\mathbb{Q}$ are given as follows.

$$\mathrm{d}X_t = [\theta^{\mathbb{Q}}(t) - \kappa^{\mathbb{Q}} X_t + (\mathrm{Tr}(D_1^{\mathbb{Q}} V_t), ..., \mathrm{Tr}(D_n^{\mathbb{Q}} V_t))']\mathrm{d}t + \sqrt{V_t}\mathrm{d}Z_t^{\mathbb{Q}}, \tag{11}$$

$$\mathrm{d}V_t = ((\beta - 2\gamma)Q'Q + M^{\mathbb{Q}} V_t + V_t (M^{\mathbb{Q}})')\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathbb{Q}} Q + Q'(\mathrm{d}W_t^{\mathbb{Q}})'\sqrt{V_t}, \tag{12}$$

which is related to their $\mathbb{P}$-processes (3) and (4) by reparametrization,

$$\begin{aligned}
\theta^{\mathbb{Q}}(t) &= \theta(t) - \lambda_X^0,\ \kappa^{\mathbb{Q}} = \kappa + \lambda_X^1,\ D_j^{\mathbb{Q}} = D_j - D_j^{\lambda}, \\
M^{\mathbb{Q}} &= M - Q'(\Lambda_V^1)',\ \mathrm{d}Z_t^{\mathbb{Q}} = \mathrm{d}W_t^{\mathbb{Q}}\rho + \sqrt{1-\rho'\rho}\mathrm{d}B_t^{\mathbb{Q}}, \qquad (13) \\
\mathrm{d}B_t^{\mathbb{Q}} &= \mathrm{d}B_t^{\mathbb{P}} + (\sqrt{1-\rho'\rho})^{-1}(\Lambda_{X,t}\mathrm{d}t - \Lambda_{V,t}\rho\mathrm{d}t).
\end{aligned}$$

To preserve the type of distribution under $\mathbb{P}$ and $\mathbb{Q}$, we require that $\Lambda_V^0 = \gamma Q'$ for a scalar γ, such that $\beta - 2\gamma > n - 1$.

The advantage of the extended affine specification is that the cointegration coefficients and the long-run level of the covariance processes can be adjusted independently when changing the measure. This approach is valid if V_t does not attain its boundary value of zero under both $\mathbb{Q}$ and $\mathbb{P}$, as shown by Cheredito et al. (2007). The condition $\beta - 2\gamma > n - 1$ ensures the positivity of V_t. We derive the joint characteristic function of X_T conditional on $\mathcal{F}_t$ under $\mathbb{Q}$ as follows.

Lemma 2. *If X_t follows the dynamics in (11) and (12), then the joint characteristic function for X_T is given by*

$$\begin{aligned}
f^{\mathbb{Q}}(x, v, t; u) &= \mathbb{E}^{\mathbb{Q}}[e^{\mathbf{i}uX_T} | X_t = x, V_t = v] \\
&= \exp\left[\mathrm{Tr}(A^{\mathbb{Q}}(\tau; u)v) + B^{\mathbb{Q}}(\tau; u)'x + C^{\mathbb{Q}}(\tau; u)\right], \qquad (14)
\end{aligned}$$

where $A^{\mathbb{Q}}(\tau; u)$, $B^{\mathbb{Q}}(\tau; u)$, and $C^{\mathbb{Q}}(\tau; u)$ are, respectively, the $\mathbb{Q}$-counterparts of $A(\tau; u)$, $B(\tau; u)$, and $C(\tau; u)$ in Lemma 1 with the reparametrization (13).

As the $\mathbb{Q}$-characteristic function for X_T is exponentially affine in the state variables, the marginal $\mathbb{Q}$-characteristic function of $X_{j,T}$ conditional on $\mathcal{F}_t$ for $j = 1, \ldots, n$, can be extracted from Lemma 2 as follows.

$$f_j^{\mathbb{Q}}(x,v,t;\phi) = \mathbb{E}^{\mathbb{Q}}[e^{\mathbf{i}\phi X_{j,T}}|X_t = x, V_t = v] = f^{\mathbb{Q}}(x,v,t;u = U_j(\phi)), \qquad (15)$$

where $U_j(\phi)$ denotes a row vector with the j-th element equal to ϕ and the remaining elements equal to 0.

The characteristic functions are not only useful in describing the distributional properties of the model, but also in deriving formulas for standard derivative products. An obvious application is to derive a closed-form solution of futures prices.

Corollary 1. *Under the $\mathbb{Q}$-dynamics (11) and (12), the futures price of the j-th commodity with maturity T is given by*

$$F_{j,T}(t) = \mathbb{E}_t^{\mathbb{Q}}[S_{j,T}] = f_j^{\mathbb{Q}}(x,v,t;\phi = -\mathbf{i}), \quad \text{for } j = 1,...,n, \qquad (16)$$

where $f_j^{\mathbb{Q}}(x,v,t;\phi)$ is defined in (15).

This corollary implies that the futures price of the j-th commodity depends not only on its underlying spot price but also on other commodities with a cointegrating relationship. It thus deduces the third hypothesis of the convenience yield from our model, in which the cointegrating assets are used as explanatory variables to explain the variations in the convenience yield. We further provide an explicit formulation of the convenience yield in the empirical analysis in the next section to validate the specification of hypothesis III.

Hypothesis III: The T-futures price of commodity i, $F_{i,T}(t) = \mathbb{E}^{\widehat{\mathbb{Q}}}[S_{i,T}|\mathcal{F}_t] = S_{i,t}e^{(r-\widehat{q}_{i,T}(t))(T-t)}$

for $i = 1, \ldots, n$, where

$$\Delta\widehat{q}_{i,T}(t) = \beta_{i0}^{III} + \beta_{i1}^{III}\Delta r_t + \beta_{i2}^{III}\Delta\frac{1}{T-t} + \sum_{j=1}^{n}\widehat{\beta}_{ij}^{III}\Delta X_{j,t} + \sum_{j=1}^{n}\widetilde{\beta}_{ij}^{III}\Delta\frac{X_{j,t}}{T-t},$$

in which $\Delta\widehat{q}_{i,T}(t)$ reflects the change in the convenience yield i from t to T.

3. Empirical Analysis

This section is devoted to an empirical examination of the proposed model and its consequences. We start by investigating the existence of cointegration and stochastic correlation in the spot prices of crude oil and gasoline. Then, their futures price data are used to test hypothesis I, II, and III.

3.1. Data description

The data set obtained from Bloomberg consists of the NYMEX daily spot prices of WTI crude oil and gasoline from January 03, 2008 to July 15, 2011. We also collect data on their futures prices with different maturities over the same period of time. The summary statistics of the futures contracts are exhibited in Table 1.

Table 1: Summary statistics of futures contracts on crude oil and gasoline

Maturity	Number of Observations	Maturity	Number of Observations
Jan 2011	746	Jul 2011	748
Feb 2011	746	Aug 2011	743
Mar 2011	748	Sep 2011	722
Apr 2011	748	Oct 2011	701
May 2011	746	Nov 2011	678
Jun 2011	747	Dec 2011	659

3.2. Unit root and cointegration tests

An investigation of the joint dynamics of this pair of spot prices shows that they are correlated and cointegrated. Figure 1 plots the daily closing prices against time. The

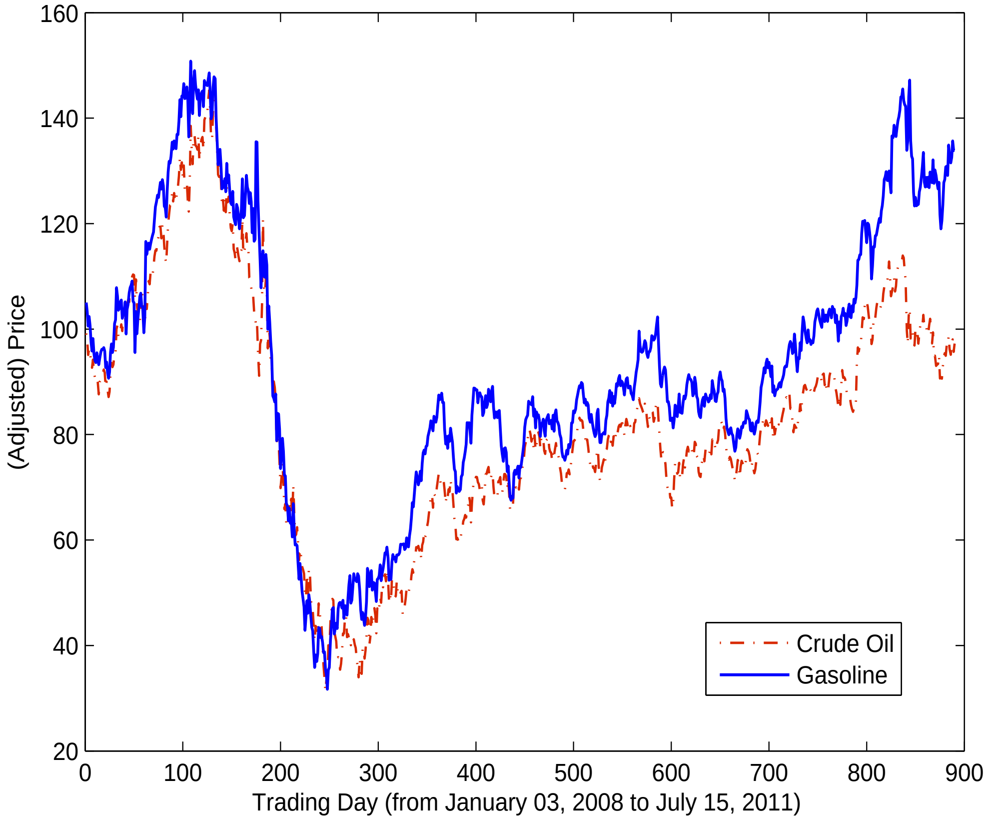

Figure 1: Time series of crude oil price and adjusted gasoline price

prices of gasoline are multiplied by 42 because crude oil is quoted as price per barrel, whereas gasoline is quoted as price per gallon (42 gallons = 1 barrel). The figure shows a high degree of correlation because shocks to one series are accompanied by shocks to the paired series. The figure also shows that the series follow each other through time, and the spreads between prices appear to be mean reverting. These properties usually occur in a pair of cointegrated time series.

Table 2 formalizes this observation by providing the correlations of the log-spot prices and the results of three cointegration tests. The sample correlation coefficient between crude oil and gasoline was 0.9599. The significant correlations in the table indicate that spot prices are related by a positive long-run relationship and react to shocks in the

Table 2: Bivariate time-series properties of log-spot prices of crude oil and gasoline

Correlation	ADF test	Phillips-Ouliaris cointegration test	Phillips-Perron unit root test
0.9599	test statistic=-4.2225 p-value = 0.01	test statistic = -26.0649 p-value = 0.0175	test statistic = -42.8071 p-value = 0.01

same direction on a short-run basis. Treating such series as two correlated processes is inappropriate and can lead to seriously erroneous conclusions. For the model allowing cointegration, we need to identify the cointegration relationship. Because there are only two series, one cointegration relationship at most is considered. We employ the Engle-Granger two-step method to obtain the cointegrating vector. A time trend variable is included to take into account the growth in the underlying variables. The cointegration regression gives rise to the following stationary series.

$$Z_t = 0.6801 + 0.00016t + X_{1,t} - 0.957599X_{2,t}, \tag{17}$$

where $X_{1,t}$ and $X_{2,t}$ denote the log-spot price of crude oil and gasoline, respectively. The p-value of the augmented Dickey-Fuller (ADF) test on Z_t is 0.01, which suggests that cointegration exists. We also check the Phillips-Ouliaris cointegration test and the Phillips-Perron unit root test statistics. Both also show a significant cointegration relationship. In other words, equation (17) is not due to the spurious regression effect. The pair of commodities contribute to the cointegrating factor in the opposite direction.

As our cointegration dynamics permit volatilities and correlations to be stochastic, we investigate the market behavior of correlation further. Instead of taking the average over the entire sample, we can use rolling window estimates of correlation. Figure 2 plots the estimated correlations with different rolling window lengths. Descriptive statistics of the rolling window correlations are reported in Table 3. When using this rolling-window approach to calculate time-varying correlations, the length of the window determines the smoothness of the temporal movements of the data. In general, a shorter window

Table 3: Rolling window correlation estimates between crude oil and gasoline

	30 days rolling	100 days rolling	250 days rolling
Mean	0.7742	0.8449	0.8976
Median	0.8796	0.9064	0.9263
Minimum	-0.3822	0.3557	0.6927
Maximum	0.9929	0.9877	0.9898
Standard Deviation	0.2463	0.1476	0.0797
No. of observations	861	791	641

will produce a more erratic time series of sample correlations but will give a better representation of the contemporaneous correlation. Here, we set the lengths of the rolling window to 30 days, 100 days and 250 days. It can be seen from Figure 2 that the correlation is not constant over time. The proposed dynamics in (3) and (4) are appealing when cointegration and stochastic covariances are incorporated simultaneously.

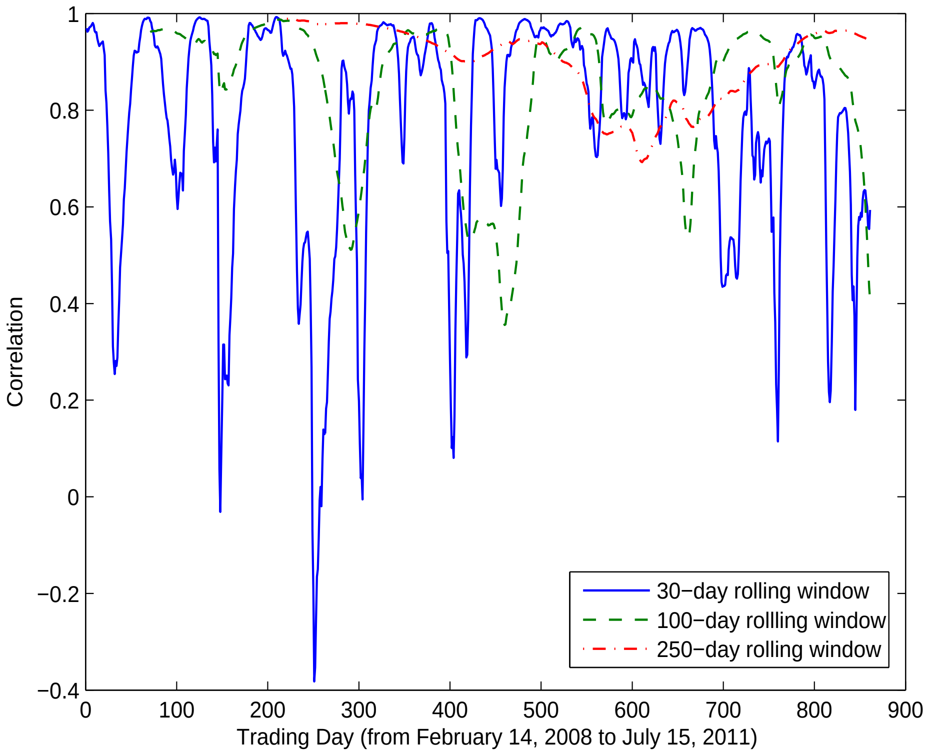

Figure 2: Rolling window correlations between crude oil and gasoline

3.3. Cointegration under the risk-neutral measure

Using a GARCH equilibrium-based option pricing approach, Duan and Pliska (2004) apply the LRN valuation to show that the usual Black-Scholes results are recovered when volatilities are deterministic. When cointegration is combined with stochastic volatilities, the cointegration parameters explicitly affect option value because it shows up in the volatility dynamics through the measure transformation. However, there is no case in which asset prices can be cointegrated under a risk-neutral pricing measure; otherwise, the discounted asset prices would not be martingales. This paper instead considers a flexible market price of risk specifications that preserves the cointegration structure under the change of measure in Section 2.

By making use of the term structure of the futures prices of crude oil and gasoline described in Section 3.1, we here test whether investors anticipate cointegration in spot asset prices under the risk-neutral measure. The futures price is a martingale under this measure. The convenience yield is the benefit that is obtained from holding the spot commodity but not the futures contract. Some evidence suggests the convenience yield should be specified by a stochastic process. Rather than examining evidence of ex post reversion using time series of futures prices, we use the implied convenience yield from futures contracts with varying delivery horizons. This approach offers two advantages. First, there is little ambiguity as to the source of any cointegration detected using our method. Subject only to the maintained assumption that the no-arbitrage cost-of-carry condition holds, our test detects the cointegration that is expected to occur in equilibrium, but has no power to detect that which from noise or inefficiencies. Second, our approach links the stochastic convenience yield and cointegration models. In the stochastic convenience yield model of Casassus and Collin-Dufresne (2005), the three state variables are not directly observed, and hence they choose to fit the principal components of the futures curves. In our model, the cointegrated assets are endogenous stochastic factors of the convenience yields.

Consider the situation of two cointegrated commodities and a constant interest rate. Corollary 1 asserts that the futures price of S_1 depends on the current values of both S_1 and S_2. The explicit formula is:

$$\begin{aligned} F_{1,T}(t) &= \exp\left[\mathrm{Tr}(A^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))v) + B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x + C^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))\right] \\ &= S_{1,t}\exp\left[r\tau - q_1(t,T)\tau\right], \end{aligned} \tag{18}$$

where

$$q_1(t,T) = r + \frac{x_1}{\tau} - \frac{\mathrm{Tr}(A^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))v)}{\tau} - \frac{B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x}{\tau} - \frac{C^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))}{\tau},$$

in which

$$\frac{B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x}{\tau} = \left(\frac{\alpha_0^{\mathbb{Q}}(\tau) - \kappa_{11}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau)}{\tau}\right)x_1 - \frac{\kappa_{12}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau)}{\tau}x_2, \tag{19}$$

$[\kappa_{11}^{\mathbb{Q}}\ \kappa_{12}^{\mathbb{Q}}]$ is the first row of $\kappa^{\mathbb{Q}}$, and $\alpha_i^{\mathbb{Q}}(\tau) = \gamma_{i0} + \sum_{j=1}^{n}\gamma_{ij}e^{\lambda_j\tau}$, as shown in Appendix B by the Cayley-Hamilton theorem. By Taylor's theorem, we have $\alpha_i^{\mathbb{Q}}(\tau) = \widehat{\gamma}_{i0} + \widehat{\gamma}_{i1}\tau + R_1(\tau)$, where $R_1(\tau)$ denotes the remainder term. Substituting the Taylor expansion of $\alpha_i^{\mathbb{Q}}(\tau)$, $i = 0, 1$, into (19) gives us

$$\frac{B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x}{\tau} \simeq (\widehat{\gamma}_{00} - \kappa_{11}^{\mathbb{Q}}\widehat{\gamma}_{10})\frac{x_1}{\tau} + (\widehat{\gamma}_{01} - \kappa_{11}^{\mathbb{Q}}\widehat{\gamma}_{11})x_1 - \kappa_{12}^{\mathbb{Q}}\widehat{\gamma}_{10}\frac{x_2}{\tau} - \kappa_{12}^{\mathbb{Q}}\widehat{\gamma}_{11}x_2.$$

Therefore, we obtain the following relation of the convenience yield of S_1.

$$\Delta q_1(t,T) = \Delta\widehat{q}_{1,T}(t) + \text{Error Term}, \tag{20}$$

where the function

$$\Delta\widehat{q}_{1,T}(t) = \beta_{i0}^{III} + \beta_{i1}^{III}\Delta r_t + \beta_{i2}^{III}\Delta\frac{1}{T-t} + \sum_{j=1}^{n}\widehat{\beta}_{ij}^{III}\Delta X_{j,t} + \sum_{j=1}^{n}\widetilde{\beta}_{ij}^{III}\Delta\frac{X_{j,t}}{T-t}$$

is specified in hypothesis III. Allowing the convenience yield to depend on the log spot price and the risk-free rate, Casassus and Collin-Dufresne (2005) document that the misspecification of the convenience yield can have a significant impact on option valuation and risk management. Correctly modeling the convenience yield process thus constitutes a significant step toward modeling commodity-related contingent claims. The proposed model not only accommodates the dependence of the convenience yield on the underlying spot price and risk-free rate, but also allows the convenience yield to have stochastic volatility and correlation with the underlying spot prices.

3.4. The regression model and results

To estimate the convenience yield, we propose a simple regression model, as specified in our three hypotheses. To illustrate, the variance-covariance matrix is assumed to be constant. In hypothesis I, we regress the changes in the convenience yield on the changes in the set of state variables $(\Delta r_t, \Delta 1/(T-t))$. For the risk-free rate r_t, we use the daily 3-month constant maturity Treasury yield obtained from the U.S. Department of the Treasury. The estimates of the regression coefficients, and their standard errors (SE) and p-values, are given in Table 4, in addition to the explanatory power of the regressions, as measured by adjusted R^2. The interest rate term and the Samuelson effect are far from sufficient to explain the variations in the convenience yield. In hypothesis II, we add the underlying spot price as a regressor to reflect the level dependence feature of the convenience yield. More specifically, we regress the changes in the convenience yield $\widehat{q}_{i,T}(t)$ for $i = 1, 2$ on the changes in the set of state variables $(\Delta r_t, \Delta 1/(T-t), \Delta X_{i,t})$ and

Table 4: Parameter estimates of the regression model (hypothesis I)

Explanatory Variables	Crude Oil ($\Delta\widehat{q}_{1,T}(t)$)	Gasoline ($\Delta\widehat{q}_{2,T}(t)$)
Intercept	$2.951e^{-6}$	0.00019
(SE)	($1.253e^{-4}$)	(0.0002157)
[p-value]	[0.981]	[0.377]
$\Delta 1/(T-t)$	$5.371e^{-4}$	-0.00205
(SE)	($4.1374e^{-4}$)	(0.0016996)
[p-value]	[0.194]	[0.226]
Δr_t	2.183^{**}	2.384^{**}
(SE)	(0.2015)	(0.3444)
[p-value]	[$< 2e^{-16}$]	[$4.76e^{-12}$]
Adjusted R^2	0.0133	0.0053

Note: $*$ and $**$ denote significance at the 5% and 1% levels, respectively.

the interaction term $\Delta X_{i,t}/\tau$. The regression results are given in Table 5. All coefficients of $\Delta X_{i,t}$ are significant in our regression model. The adjusted R^2 is improved significantly compared to the results in Table 4, which implies that a realistic model of commodity prices should allow the convenience yield to depend on the underlying spot prices and the interest rate. This is consistent with the findings of Casassus and Collin-Dufresne (2005) on level dependence in the convenience yield, which leads to mean reversion in the spot prices under the risk-neutral measure. As the spot price rises and inventories decline, the convenience yield also rises. Thus, the estimate of the regression coefficients of $\Delta X_{i,t}$ should be positive. Our regression result incorporates this feature as well.

In hypothesis III, we include the pair of cointegrating assets as explanatory variables. The regression results are presented in Table 6. To explain $\Delta\widehat{q}_{i,T}(t)$, we further add $\Delta X_{j,t}$, $j \neq i$, as an explanatory variable, and all of the coefficients of $\Delta X_{j,t}$ are significant. The adjusted R^2 is improved by about 6% in the case of crude oil and by about 20% in the case of gasoline. The regression coefficients of $\Delta X_{i,t}$ remain positive, reflecting the relationship between inventories and spot prices. In addition, the regression coefficients of $\Delta X_{1,t}$ and $\Delta X_{2,t}$ are of different signs, which is consistent with the specification of the cointegrating factor Z_t in (17). These findings imply that the market anticipates a cointegration relationship under the risk-neutral measure. The proposed change of measure preserves the feature of cointegration and thus has greater flexibility to reflect

Table 5: Parameter estimates of the regression model (hypothesis II)

Explanatory Variable	Crude Oil ($\Delta\widehat{q}_{1,T}(t)$)		Gasoline ($\Delta\widehat{q}_{2,T}(t)$)	
Intercept	$-9.498e^{-6}$	$-2.694e^{-6}$	$9.845e^{-5}$	$8.709e^{-5}$
(SE)	($9.292e^{-5}$)	($9.294e^{-5}$)	($2.004e^{-4}$)	($1.83e^{-4}$)
[p-value]	[0.9185]	[0.9768]	[0.623]	[0.6341]
$\Delta 1/(T-t)$	$2.874e^{-5}$	$-2.02e^{-2*}$	$-2.401e^{-3}$	-1.013^{**}
(SE)	($3.068e^{-4}$)	($8.21e^{-3}$)	($1.579e^{-3}$)	($2.425e^{-2}$)
[p-value]	[0.9253]	[0.0138]	[0.128]	[$< 2e^{-16}$]
Δr_t	0.0507^{**}	0.0518^{**}	0.0592	1.093^{**}
(SE)	(0.1507)	(0.1507)	(0.3235)	(0.2956)
[p-value]	[0.00075]	[0.00059]	[0.067]	[0.00022]
$\Delta X_{1,t}$	**0.2489****	**0.2456****		
(SE)	($2.943e^{-3}$)	($3.237e^{-3}$)		
[p-value]	[$< 2e^{-16}$]	[$< 2e^{-16}$]		
$\Delta X_{1,t}/\tau$		$4.384e^{-3*}$		
(SE)		($1.778e^{-3}$)		
[p-value]		[0.01367]		
$\Delta X_{2,t}$			**0.2494****	**0.1071****
(SE)			($6.687e^{-3}$)	($6.994e^{-3}$)
[p-value]			[$< 2e^{-16}$]	[$< 2e^{-16}$]
$\Delta X_{2,t}/\tau$				0.1793^{**}
(SE)				($4.298e^{-3}$)
[p-value]				[$< 2e^{-16}$]
Adjusted R^2	0.4577	0.458	0.142	0.2847

Note: $*$ and $**$ denote significance at the 5% and 1% levels, respectively.

market phenomena. Although the residual of the regression analysis may come from the stochastic volatility or from the default premium, which are absent in the model, the adjusted R^2 is over 47% for both commodities. $X_{1,t}$ and $X_{2,t}$ are highly correlated, and the correlation of $\Delta X_{1,t}$ and $\Delta X_{2,t}$ is 0.7452. The improvements in the adjusted R^2 in Table 6 relative to Table 5 shows that the explanatory power of the cointegrating asset is due to the cointegration relationship rather than the correlation relationship.

4. Application to the Valuation of Derivatives

The $\mathbb{Q}$-characteristic function of the log-spot prices is useful in computing European options through Fourier inversion. An advantage of the proposed model is that it incorporates the empirical features of cointegration and the stochastic covariance matrix in the commodity market, as shown in the previous section, and renders the $\mathbb{Q}$-characteristic function as a tractable exponential affine form.

For practical purposes, the $\mathbb{Q}$-characteristic function can be super-calibrated to the

Table 6: Parameter estimates of the regression model (hypothesis III)

Explanatory Variables	Crude Oil ($\Delta\widehat{q}_{1,T}(t)$)		Gasoline ($\Delta\widehat{q}_{2,T}(t)$)	
Intercept	$-4.483e^{-5}$	$4.925e^{-5}$	$2.468e^{-5}$	-0.0002455
(SE)	($8.838e^{-5}$)	($8.842e^{-5}$)	($1.743e^{-4}$)	(0.0001570)
[p-value]	[0.612]	[0.5775]	[0.887]	[0.118]
$\Delta 1/(T-t)$	$-2.409e^{-2**}$	$-1.637e^{-2}$	-0.9783^{**}	-0.6847^{**}
(SE)	($7.807e^{-3}$)	($9.409e^{-3}$)	($2.312e^{-2}$)	(0.0218)
[p-value]	[0.002]	[0.0818]	[$< 2e^{-16}$]	[$< 2e^{-16}$]
Δr_t	0.8545^{**}	0.8504^{**}	1.342^{**}	1.209^{**}
(SE)	(0.1437)	(0.1437)	(0.2816)	(0.2535)
[p-value]	[$2.85e^{-9}$]	[$3.4e^{-9}$]	[$1.92e^{-6}$]	[$1.87e^{-6}$]
$\Delta X_{1,t}$	0.3388^{**}	0.3372^{**}	$\mathbf{-0.2443}^{**}$	$\mathbf{-0.0745}^{**}$
(SE)	($4.341e^{-3}$)	($4.486e^{-3}$)	($8.161e^{-3}$)	(0.00825)
[p-value]	[$< 2e^{-16}$]	[$< 2e^{-16}$]	[$< 2e^{-16}$]	[$< 2e^{-16}$]
$\Delta X_{2,t}$	$\mathbf{-0.1345}^{**}$	$\mathbf{-0.1319}^{**}$	0.3038^{**}	0.1955^{**}
(SE)	($4.416e^{-3}$)	($4.762e^{-3}$)	($9.356e^{-3}$)	(0.0087)
[p-value]	[$< 2e^{-16}$]	[$< 2e^{-16}$]	[$< 2e^{-16}$]	[$< 2e^{-16}$]
$\Delta X_{1,t}/\tau$	$5.201e^{-3**}$	$7.004e^{-3**}$		-0.2272^{**}
(SE)	($1.69e^{-3}$)	($2.09e^{-3}$)		(0.005)
[p-value]	[0.0021]	[0.0008]		[$< 2e^{-16}$]
$\Delta X_{2,t}/\tau$		$-2.858e^{-3}$	0.1734^{**}	0.3078^{**}
(SE)		($1.947e^{-3}$)	($4.098e^{-3}$)	(0.0047)
[p-value]		[0.1421]	[$< 2e^{-16}$]	[$< 2e^{-16}$]
Adjusted R^2	0.51	0.5101	0.3512	0.4743

Note: $*$ and $**$ denote significance at the 5% and 1% levels, respectively.

observed term-structures of commodity futures.

Proposition 1. *If the underlying asset follows the WASC model with cointegration in (11-12), then the joint characteristic function calibrated to the term structure of futures prices is given by*

$$f^{\mathbb{Q}}(x, v, t; u, F_T(t)) = \prod_{j=1}^{n} [F_{j,T}(t)]^{\mathbf{i}u_j} \exp[\mathrm{Tr}(\Delta A(\tau; u)v) + \Delta\widetilde{C}(\tau; u)], \tag{21}$$

where

$$\Delta A(\tau; u) = A^{\mathbb{Q}}(\tau; u) - \mathbf{i}\sum_{j=1}^{n} u_j A^{\mathbb{Q}}(\tau; u = U_j(-\mathbf{i})),$$

$$\Delta\widetilde{C}(\tau; u) = \widetilde{C}(\tau; u) - \mathbf{i}\sum_{j=1}^{n} u_j \widetilde{C}(\tau; u = U_j(-\mathbf{i})),$$

$$\widetilde{C}(\tau; u) = -\frac{\beta - 2\gamma}{2}\mathrm{Tr}\left[\ln H^{\mathbb{Q}}(\tau; u) + (M^{\mathbb{Q}})'\tau - \mathbf{i}\int_0^{\tau} e^{-(\kappa^{\mathbb{Q}})'\tau} u'\rho' Q\right].$$

Proof. See Appendix C. □

The market-implied risk-neutral parameter $\theta^{\mathbb{Q}}(t)$ in (11) and (12) can be used to fit the term structure of futures prices exactly. The effects of the seasonality of commodity prices are fully reflected in and captured by the futures prices. Hence, there is no longer any need to estimate this deterministic function.

Based on the elasticity of distant futures prices with respect to spot prices, we can further calibrate the market-implied matrix of cointegration coefficients $\kappa^{\mathbb{Q}}(t)$. The explicit formulae for futures prices are

$$\begin{aligned}
F_{1,T}(t) &= \exp\left[\mathrm{Tr}(A^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))v) + B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x + C^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))\right] \\
F_{2,T}(t) &= \exp\left[\mathrm{Tr}(A^{\mathbb{Q}}(\tau; U_2(-\mathbf{i}))v) + B^{\mathbb{Q}}(\tau; U_2(-\mathbf{i}))'x + C^{\mathbb{Q}}(\tau; U_2(-\mathbf{i}))\right],
\end{aligned}$$

where

$$\begin{aligned}
B^{\mathbb{Q}}(\tau; U_1(-\mathbf{i}))'x &= (\alpha_0^{\mathbb{Q}}(\tau) - \kappa_{11}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau))x_1 - \kappa_{12}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau)x_2, \\
B^{\mathbb{Q}}(\tau; U_2(-\mathbf{i}))'x &= (\alpha_0^{\mathbb{Q}}(\tau) - \kappa_{22}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau))x_2 - \kappa_{21}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau)x_1.
\end{aligned}$$

The elasticity of the date t futures prices maturing at T with respect to the date t spot prices is

$$\begin{aligned}
\frac{\partial F_{1,T}(t)}{\partial S_1(t)}\frac{S_1(t)}{F_{1,T}(t)} &= \frac{\partial \ln F_{1,T}(t)}{\partial x_1(t)} = \alpha_0^{\mathbb{Q}}(\tau) - \kappa_{11}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau) \\
&= \frac{1}{2\phi}e^{-\frac{1}{2}(\kappa_{11}^{\mathbb{Q}}+\kappa_{22}^{\mathbb{Q}}+\phi)\tau}\left(\kappa_{11}^{\mathbb{Q}} - e^{\phi\tau}\kappa_{11}^{\mathbb{Q}} + \left(-1+e^{\phi\tau}\right)\kappa_{22}^{\mathbb{Q}} + \left(1+e^{\phi\tau}\right)\phi\right),
\end{aligned}$$

where $\phi = \sqrt{(\kappa_{11}^{\mathbb{Q}})^2 + 4\kappa_{12}^{\mathbb{Q}}\kappa_{21}^{\mathbb{Q}} - 2\kappa_{11}^{\mathbb{Q}}\kappa_{22}^{\mathbb{Q}} + (\kappa_{22}^{\mathbb{Q}})^2}$. Similarly, we have

$$\begin{aligned}
\frac{\partial \ln F_{2,T}(t)}{\partial x_2(t)} &= \alpha_0^{\mathbb{Q}}(\tau) - \kappa_{22}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau) \\
&= \frac{e^{-\frac{1}{2}(\kappa_{11}^{\mathbb{Q}}+\kappa_{22}^{\mathbb{Q}}+\phi)\tau}\left(\left(-1+e^{\phi\tau}\right)\kappa_{11}^{\mathbb{Q}} + \kappa_{22}^{\mathbb{Q}} - e^{\phi\tau}\kappa_{22}^{\mathbb{Q}} + \left(1+e^{\phi\tau}\right)\phi\right)}{2\phi},
\end{aligned}$$

$$\frac{\partial \ln F_{1,T}(t)}{\partial x_2(t)} = -\kappa_{12}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau) = -\frac{e^{-\frac{1}{2}(\kappa_{11}^{\mathbb{Q}}+\kappa_{22}^{\mathbb{Q}}+\phi)\tau}\left(-1+e^{\phi\tau}\right)\kappa_{12}^{\mathbb{Q}}}{\phi},$$

$$\frac{\partial \ln F_{2,T}(t)}{\partial x_1(t)} = -\kappa_{21}^{\mathbb{Q}}\alpha_1^{\mathbb{Q}}(\tau) = -\frac{e^{-\frac{1}{2}(\kappa_{11}^{\mathbb{Q}}+\kappa_{22}^{\mathbb{Q}}+\phi)\tau}\left(-1+e^{\phi\tau}\right)\kappa_{21}^{\mathbb{Q}}}{\phi}.$$

To estimate the elasticities of the futures prices with respect to the contemporaneous spot prices, we regress the first difference of the logarithms of distant futures prices on the first difference of the logarithms of spot prices or near futures prices, which are used as proxies for the spot prices. Then, the elements in the market-implied matrix of cointegration coefficients $\kappa^{\mathbb{Q}}$ can be estimated separately by fitting the market-observed elasticities of the futures prices.

4.1. European vanilla option on a single asset

We first discuss the valuation of what is probably the best known option, the European vanilla call on a single asset. Consider S_1 to be the underlying asset of the plain vanilla call option whose payoff is given by

$$\max(S_{1,T} - K, 0),$$

where K is the strike price and T is the option's maturity.

Proposition 2. *(Carr and Madan, 1998) Suppose that the interest rate r is constant and the log-asset values X_t follow the risk-neutral dynamics (11-12). Then, the European call option pricing formula on S_1 at time 0 is given by*

$$C(S_{1,0}, K, T) \quad = \quad S_{1,0}e^{-rT-\alpha\kappa}\mathcal{F}_{k,\xi}^{-1}\left[\frac{f_1^{\mathbb{Q}}(x, v, t; \xi - (\alpha+1)\mathbf{i})}{\alpha^2 + \alpha - \xi^2 + \mathbf{i}(2\alpha+1)\xi}\right], \tag{22}$$

where $k = \log K$, $f_1^{\mathbb{Q}}(x, v, t; \phi)$ is defined in (15) and $\mathcal{F}^{-1}_{k,\xi}[\cdot]$ represents the inverse Fourier transform:

$$\mathcal{F}^{-1}_{k,\xi}\left[g(\xi)\right] = \frac{1}{\pi}\int_0^\infty e^{-\mathrm{i}\xi k} g(\xi) d\xi. \tag{23}$$

In addition, the inverse Fourier transform in (22) converges for some $\alpha > 0$.

Carr and Madan (1998) advocate the use of the FFT to implement the Fourier inversion numerically, which simultaneously computes the option values for a set of log strikes. Chourdakis (2004) shows significant improvement using the fractional FFT (FRFT), which is a linear transformation that generalizes the Fourier transform.

Similar to the case of futures prices, the vanilla call option pricing formula for the first asset depends on the current value of the second asset through the cointegrating factor. This can be recognized from the marginal characteristic function of $X_{1,T}$ or from its process. If we look at the process of $X_{1,t}$ alone, then we see that the cointegrating component $U_1(1)\kappa X_t$ can be viewed as a dividend yield. Hilliard and Reis (1998) use a latent Cox-Ingersoll-Ross (CIR) type model to describe the yield, which leads to a non-central χ^2-distributed dividend yield. In our case, the dividend yield process can be discovered from the cointegration relationship, which provides us with a reason to take a closer look at the cointegration feature to incorporate the randomness of the convenience yield.

Example 4.1. *We first contrast our futures price formula (18) with the futures price under stochastic convenience yield derived by Hilliard and Reis (1998). We base our computation on (11-12) with these parameters: $r = 0.05$, $T = 1$, $\beta - 2\gamma = 7$, $\rho =$*

$(0.6 \quad -0.6)'$, $\theta^{\mathbb{Q}}(t) = (0.1 \quad 0.05)'$, $S_0 = (1 \quad 1.1)$,

$$\kappa^{\mathbb{Q}} = \frac{1}{3}\begin{pmatrix} 1.1 & 2 \\ 2 & 1 \end{pmatrix}, \quad D_1^{\mathbb{Q}} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \quad D_2^{\mathbb{Q}} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$

$$M^{\mathbb{Q}} = \begin{pmatrix} -2.5 & -1.5 \\ -1.5 & -2.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.11 & 0.04 \\ 0.14 & 0.21 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0.09 & -0.036 \\ -0.036 & 0.09 \end{pmatrix}.$$

The upper part of Figure 3 plots the futures price against the spot convenience yield using the Hilliard-Reis model, and the lower part plots the futures price against the log-value of the current asset price of S_2. As we fix $S_{1,t} = 1$ in all of our computations, the change in $\log S_2$ corresponds to the change in the cointegrating factor of our model. Similar increasing trends are seen with the spot value of the convenience yield. Hence, we have further numerical evidence that the cointegrating factor contributes to futures prices as a stochastic convenience yield.

Example 4.2. *It would also be interesting to determine the performance of the FRFT applied to vanilla options under the proposed model. We examine the pricing error of the solution of Proposition 2 and Lemma 2 using the FRFT technique. The efficient simulation procedure of the Wishart process proposed by Gauthier and Possamaï (2009) is adopted. By regarding Monte Carlo (MC) simulation as the benchmark, we compute the percentage difference between the two methods. The simulation uses* $10,000$ *pairs of sample paths, and the time-step is 1/300. The FRFT employs a different N, i.e., $N = 8, 16, 32$. The damping coefficient α is set at* 3. *Table 7 shows that the numerical method is highly accurate because all of the errors are less than 1%. For the options with a maturity of one year, the 16-FRFT takes less than half a second to produce 16 option prices corresponding to different strike prices, whereas the simulation requires more than seven minutes. This numerical example verifies that our analytical solution is correct*

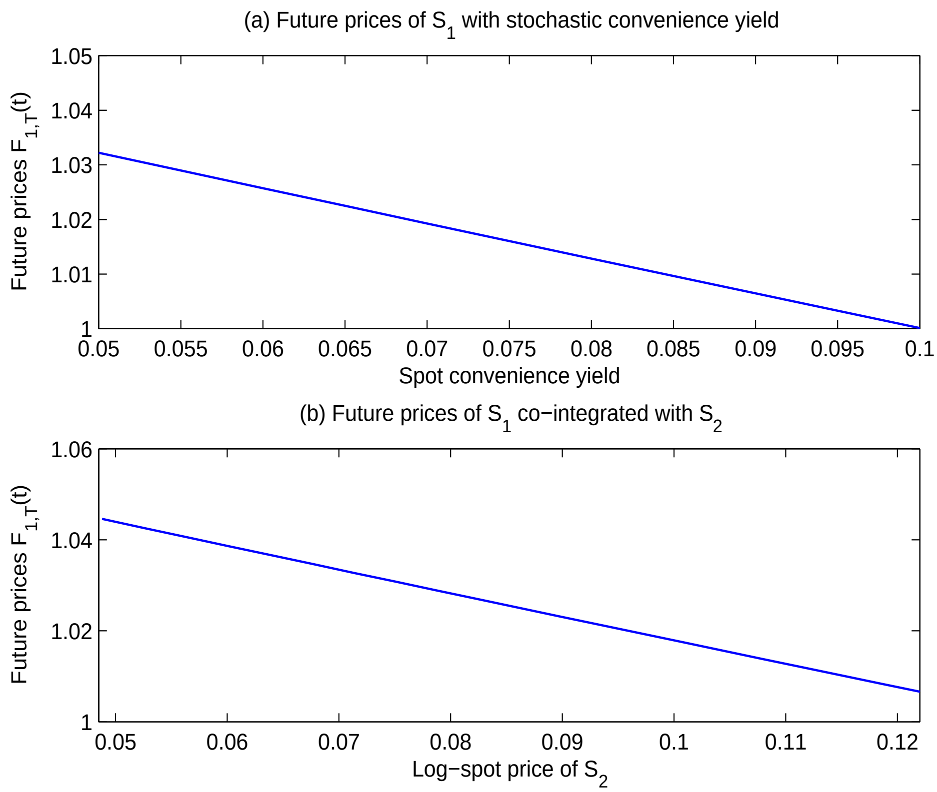

Figure 3: Futures prices with cointegration versus futures prices with stochastic convenience yield

and the FRFT is accurate and efficient.

4.2. Multivariate derivatives

Many options, such as spread, maximum, minimum, and basket options, are defined in terms of two or more underlying price processes. The underlying price system is typically modeled as a multivariate geometric Brownian motion whose volatility matrix is constant, in which case the option valuation problem is straightforward. However, such a model is unrealistic for capturing market phenomena. The proposed model accommodates the cointegration relationship and stochastic correlation, and thus provides greater flexibility in pricing multivariate derivatives. To illustrate this, consider the European

Table 7: Call option prices on a single asset

	T=0.5				T=1			
Strike Price	8-FRFT	16-FRFT	32-FRFT	MC	8-FRFT	16-FRFT	32-FRFT	MC
0.85	0.1652	0.1650	0.1649	0.1651	0.1851	0.1849	0.1848	0.1849
(%difference)	(0.05%)	(-0.06%)	(-0.12%)		(0.11%)	(0.00%)	(-0.05%)	
0.9	0.1283	0.1283	0.1282	0.1285	0.1518	0.1516	0.1516	0.1511
(%difference)	(-0.21%)	(-0.21%)	(-0.28%)		(0.46%)	(0.33%)	(0.33%)	
0.95	0.0969	0.0969	0.0969	0.0971	0.1228	0.1227	0.1227	0.1224
(%difference)	(-0.29%)	(-0.29%)	(-0.29%)		(0.32%)	(0.24%)	(0.24%)	
1	0.0714	0.0713	0.0713	0.0713	0.0982	0.0981	0.0981	0.098
(%difference)	(0.14%)	(0.00%)	(0.00%)		(0.20%)	(0.10%)	(0.10%)	
1.05	0.0514	0.0512	0.0512	0.0513	0.0778	0.0777	0.0777	0.0777
(%difference)	(0.19%)	(-0.19%)	(-0.19%)		(0.12%)	(0.00%)	(0.00%)	
1.1	0.0363	0.036	0.036	0.0362	0.0613	0.0611	0.061	0.061
(%difference)	(0.27%)	(-0.55%)	(-0.55%)		(0.49%)	(0.16%)	(0.00%)	
1.15	0.0252	0.0249	0.0249	0.0251	0.0479	0.0477	0.0476	0.0477
(%difference)	(0.39%)	(-0.79%)	(-0.79%)		(0.41%)	(-0.00%)	(-0.21%)	
CPU Time	0.1024s	0.2176s	0.4049s	216.393s	0.2144s	0.4609s	0.8321s	433.02s

option on two assets with payoff function $\Pi(X_T)$.[1] Let the corresponding valuation formula be $P(t, X_t, T)$. The following proposition links the option pricing formula to the joint characteristic function.

Proposition 3. *Suppose that the interest rate is constant. The present value of the option is given by*

$$P(0, X_0, T) = e^{-rT} \mathcal{F}^{-1}_{X_0,u} \left\{ \widehat{f}(X_0, V_0; T, u) \widehat{\Pi}(u) \right\}, \tag{24}$$

where $\mathcal{F}^{-1}_{X_0,u}$ is the 2-D inverse Fourier transform, and $\widehat{\Pi}(u)$ is the 2-D Fourier transform of the payoff function:

$$\begin{aligned} \widehat{\Pi}(u) &= \mathcal{F}_{X_T,u} \left\{ \Pi(X_T) \right\}, \\ \widehat{f}(X_0, V_0; T, u) &= \frac{f(X_0, V_0; T, u)}{e^{\mathbf{i} X_0 u'}}, \end{aligned} \tag{25}$$

in which $f(X_0, V_0; T, u)$ is defined in Lemma 2.

[1] The extension to options on three or more assets is straightforward but suffers from the curse of dimensionality.

Proof. See Appendix D. □

Given the characteristic function, all that remains is to derive the Fourier transform on different option payoffs. The crack spread option of the oil market has the payoff: $\max(S_{1,T} - S_{2,T} - K, 0)$. Hurd and Zhou (2010) have derived the Fourier transform of this payoff. Another popular option contingent on multiple assets is the maximum option or the best-of option of which the payoff is $\max(\max(S_{1,T}, S_{2,T}) - K, 0)$. We derive the corresponding Fourier transform of this option's payoff function.

Proposition 4. *Suppose that* $\Pi(X) = \max(\max(S_{1,T}, S_{2,T}) - K, 0)$. *For any real numbers* $\epsilon = (\epsilon_1, \epsilon_2)$ *with* $\epsilon_2 > 0$ *and* $\epsilon_1 + \epsilon_2 < -1$,

$$\widehat{\Pi}(u) = -\frac{\mathbf{i}K^{1-\mathbf{i}(u_1+u_2)}}{u_1 u_2(\mathbf{i} + u_1 + u_2)}. \tag{26}$$

Proof. See Appendix E. □

Example 4.3. *For options on several assets, Hurd and Zhou (2010) have already suggested a numerical method for pricing spread options in general. The derived characteristic function enables their numerical solution to work using FFT. To avoid repeating their procedures, we provide only numerical examples for the best-of option in Table 8. With an option maturity of 12 months, the FFT method produces option prices with around a* 1% *pricing difference compared to the simulation.*

Table 8: Best-of option prices

	$N = 32$				$N = 64$			
$S_{2,t}$	$K = 0.9$	$K = 1.0$	$K = 1.1$	$K = 1.2$	$K = 0.9$	$K = 1.0$	$K = 1.1$	$K = 1.2$
1	0.3006	0.2136	0.1408	0.0868	0.3004	0.2145	0.1412	0.0872
(%difference)	(-1.47%)	(-1.74%)	(-1.67%)	(-1.69%)	(-1.54%)	(-1.33%)	(-1.39%)	(-1.25%)
1.1	0.3304	0.2411	0.1615	0.1035	0.3325	0.2424	0.1627	0.1055
(%difference)	(-1.28%)	(-1.75%)	(-1.10%)	(-1.61%)	(-0.66%)	(-1.22%)	(-0.36%)	(0.28%)

5. Conclusion

We propose a dynamic model for cointegrated assets with stochastic variance-covariance, which captures most of the known stylized facts associated with financial markets, including leverage and asymmetric correlation effects. This model enables us to derive the solution for the joint characteristic function of asset returns in an elegant exponential affine form. As derivative pricing with the proposed model generally works with an incomplete market, there are infinitely many equivalent martingale measures. We consider an extended affine market price of risk specification to preserve cointegration after changing the measure. The pricing formula for futures contracts reveals that the cointegrating factor could be an important contributing factor to the convenience yield. Thus, we link the stochastic convenience yield and cointegration models. The proposed option pricing model with cointegration can be thought of as an option pricing model with stochastic dividend yield and volatility, but the stochastic dividend yield process is endogenously derived from the cointegration model. Although the convenience yield is not directly observable, the existence of a cointegration relationship may facilitate its analysis based on the observable cointegrated assets.

Based on empirical data of the spot prices and futures contracts of crude oil and gasoline, the model-free regression results strongly indicate that a cointegration relationship exists not only under the physical measure but is also anticipated in the market under the risk-neutral measure. Thus, the extended affine market price of risk specification turns out to be more flexible in the presence of cointegration. We further propose a super-calibration method that forces the risk-neutral discount factor to fit the observed term structure of futures prices exactly. This method is especially useful for commodity markets, in which spot prices are not available for most commodities.

Appendix A: Proof of Lemma 1

After reversing the time by $\tau = T - t$, the Feynman-Kac formula gives a partial differential equation (PDE) for the characteristic function

$$\mathcal{L}_{\{X,V\}} f^{\mathbb{P}} = \frac{\partial f^{\mathbb{P}}}{\partial \tau}, \tag{27}$$

where $\mathcal{L}_{\{X,V\}}$ is the joint infinitesimal generator of the couple $\{X_t, V_t\}$ under the physical measure. We have

$$\begin{aligned}
\mathcal{L}_{\{X,V\}} f^{\mathbb{P}} &= (\mathcal{L}_X + \mathcal{L}_V + \mathcal{L}_{X,V}) f^{\mathbb{P}}, \\
\mathcal{L}_X f^{\mathbb{P}} &= \frac{\partial f^{\mathbb{P}}}{\partial X'} [\theta(t) - \kappa X + (\mathrm{Tr}(D_1 V), ..., \mathrm{Tr}(D_n V))'] + \frac{1}{2} \mathrm{Tr} \left[\frac{\partial^2 f^{\mathbb{P}}}{\partial X \partial X'} \right], \\
\mathcal{L}_V f^{\mathbb{P}} &= \mathrm{Tr}[(\beta Q' Q + MV + VM') \mathcal{D} f^{\mathbb{P}} + 2V \mathcal{D} Q' Q \mathcal{D} f^{\mathbb{P}}], \\
\mathcal{L}_X f^{\mathbb{P}} &= 2\mathrm{Tr} \left[\left(\mathcal{D} Q' \rho \frac{\partial}{\partial X'} \right) f^{\mathbb{P}} V \right],
\end{aligned}$$

where $\mathcal{D} = \left[\frac{\partial}{\partial V^{ij}} \right] \big|_{i,j=1,\ldots,n}$ is a matrix differential operator. Because the Wishart process is a matrix affine process, the characteristic function of X_T is exponentially affine in the state variables given by

$$\begin{aligned}
f^{\mathbb{P}}(x, v, t; u) &= \exp\left[\mathrm{Tr}(A(\tau; u) v) + [B(\tau; u)]' x + C(\tau; u) \right], \\
f^{\mathbb{P}}(x, v, t = T; u) &= \exp(\mathbf{i} u x),
\end{aligned}$$

where $\tau = T - t$, $A(\tau = 0; u) = \mathbf{0}_n$, $B(\tau = 0; u) = \mathbf{i} u$, and $C(\tau = 0; u) = \mathbf{0}_n$. Substituting it into (27) yields

$$\mathrm{Tr} \left[\frac{\mathrm{d} A(\tau; u)}{\mathrm{d}\tau} v \right] + \frac{\mathrm{d} B(\tau; u)'}{\mathrm{d}\tau} x + \frac{\mathrm{d} C(\tau; u)}{\mathrm{d}\tau}$$

$$
\begin{aligned}
&= B(\tau;u)(\theta(t) - \kappa x + (\mathrm{Tr}(D_1 v), ..., \mathrm{Tr}(D_n v))') + \frac{1}{2}\mathrm{Tr}[B(\tau;u)B(\tau;u)'v] \\
&\quad +\mathrm{Tr}[(\beta Q'Q + Mv + vM')A(\tau;u) + 2vA(\tau;u)Q'QA(\tau;u) + 2A(\tau;u)Q'\rho B(\tau;u)'v].
\end{aligned}
$$

Matching the coefficients leads to the following system of ordinary differential equations (ODEs).

$$
\begin{aligned}
\frac{\mathrm{d}B(\tau;u)'}{\mathrm{d}\tau} &= -B(\tau;u)'\kappa, && (28)\\
\frac{\mathrm{d}C(\tau;u)}{\mathrm{d}\tau} &= B(\tau;u)'\theta(t) + \beta\mathrm{Tr}(Q'QA(\tau;u)), && (29)\\
\frac{\mathrm{d}A(\tau;u)}{\mathrm{d}\tau} &= \frac{1}{2}B(\tau;u)B(\tau;u)' + A(\tau;u)(M + Q'\rho B(\tau;u)') + (M' + B(\tau;u)\rho'Q)A(\tau;u) \\
&\quad +2A(\tau;u)Q'QA(\tau;u) + \sum_{j=1}^{n} U_j(1)B(\tau;u)'D_j, && (30)
\end{aligned}
$$

where $U_j(y)$ denotes a row vector with the j-th element equal to y and the remaining elements equal to 0. It is clear that from ODE (28) and $B(\tau = 0;u) = \mathbf{i}u'$, we have

$$
B(\tau;u) = \mathbf{i}e^{-\kappa'\tau}u'. \tag{31}
$$

By Radon's lemma (Freiling, 2002), ODE (30) can be linearized with the following procedure. Let

$$
G(\tau;u) = H(\tau;u)A(\tau;u), \tag{32}
$$

with $H(\tau;u)$ being invertible. Differentiating both sides of (32) with respect to τ yields

$$
\begin{aligned}
&\frac{\mathrm{d}G(\tau;u)}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}H(\tau;u)}{\mathrm{d}\tau}A(\tau;u) + H(\tau;u)\frac{\mathrm{d}A(\tau;u)}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}H(\tau;u)}{\mathrm{d}\tau}A(\tau;u) + +H(\tau;u)\left[\frac{1}{2}B(\tau;u)B(\tau;u)' + A(\tau;u)(M + Q'\rho B(\tau;u)')\right.
\end{aligned}
$$

$$+(M' + B(\tau;u)\rho' Q)A(\tau;u) + +2A(\tau;u)Q'QA(\tau;u) + \sum_{j=1}^{n} U_j(1)B(\tau;u)'D_j \Bigg] .$$

Matching both sides gives

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}\tau}(G(\tau;u) \quad H(\tau;u)) \\
= \quad & (G(\tau;u) \quad H(\tau;u)) \begin{pmatrix} M + Q'\rho B(\tau;u)' & -2Q'Q \\ \frac{1}{2}B(\tau;u)B(\tau;u)' + \sum_{j=1}^{n} U_j(1)B(\tau;u)'D_j & -(M' + B(\tau;u)\rho' Q) \end{pmatrix},
\end{aligned}$$

with $H(\tau;u) = \mathbf{I}_n$ and $G(\tau;u) = \mathbf{0}_n$. This system of first-order linear ODEs can be solved very efficiently using standard numerical schemes. We use the fourth-order Runge-Rutta method (Burden and Faires, 2011) in the calculation, which is strongly stable. Therefore, $A(\tau;u)$ is solved accordingly. Finally, consider

$$\begin{aligned}
& \frac{\mathrm{d}C(\tau;u)}{\mathrm{d}\tau} \\
= \quad & B(\tau;u)'\theta(T-\tau) + \beta \mathrm{Tr}(Q'QA(\tau;u)) \\
= \quad & \mathbf{i}ue^{-\kappa\tau}\theta(T-\tau) - \frac{\beta}{2}\mathrm{Tr}\left(H(\tau;u)^{-1}\frac{\mathrm{d}H(\tau;u)}{\mathrm{d}\tau} + M' + \mathbf{i}e^{-\kappa'\tau}u'\rho' Q\right),
\end{aligned}$$

whose solution can be obtained by directly integrating from 0 to τ with the zero initial condition as follows.

$$C(\tau;u) = \mathbf{i}u\int_0^\tau e^{-\kappa s}\theta(t-s)\mathrm{d}s - \frac{\beta}{2}\mathrm{Tr}\left[\ln H(\tau;u) + M'\tau - \mathbf{i}\int_0^\tau e^{-\kappa' s}\mathrm{d}s u'\rho' Q\right].$$

Appendix B: Computation of Matrix Exponential

According to the Cayley-Hamilton theorem, when $-\kappa$ is an $n \times n$ matrix, the matrix exponential $e^{-\kappa\tau}$ can be computed by a polynomial of $-\kappa$:

$$e^{-\kappa\tau} = \alpha_0(\tau)\mathbf{I}_n + \alpha_1(\tau)(-\kappa) + \cdots + \alpha_{n-1}(\tau)(-\kappa)^{n-1},$$

where $\alpha_k(t)$, $k = 1, \cdots, n$, are governed by the system of linear equations

$$\begin{pmatrix} e^{\lambda_1\tau} \\ e^{\lambda_2\tau} \\ \vdots \\ e^{\lambda_n\tau} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0(\tau) \\ \alpha_1(\tau) \\ \vdots \\ \alpha_{n-1}(\tau) \end{pmatrix},$$

and λ_i, $i = 1, \cdots, n$, are eigenvalues of the matrix $-\kappa$. Thus, $\alpha_i(\tau)$ can be represented by a linear combination in terms of $e^{\lambda_j\tau}$, $j = 1, \cdots, n$, i.e.,

$$\alpha_i(\tau) = f_i(e^{\lambda_1\tau}, \cdots, e^{\lambda_n\tau}) = \gamma_{i0} + \sum_{j=1}^{n} \gamma_{ij} e^{\lambda_j\tau}.$$

Appendix C: Proof of Lemma 1

Corollary 1 gives us

$$\begin{aligned} & F_{j,T}(t) = f^{\mathbb{Q}}(x, v, t; U_j(-\mathbf{i})) \\ = \; & \exp[\mathrm{Tr}(A^{\mathbb{Q}}(\tau; U_j(-\mathbf{i}))v) + B^{\mathbb{Q}}(\tau; U_j(-\mathbf{i}))'x \\ & +\mathbf{i}U_j(-\mathbf{i}) \int_0^{\tau} e^{-\kappa^{\mathbb{Q}}s} \theta^{\mathbb{Q}}(t-s)\mathrm{d}s + \widetilde{C}(\tau; U_j(-\mathbf{i}))]. \end{aligned}$$

We then have $\int_0^\tau e^{-\kappa^{\mathbb{Q}}s}\theta^{\mathbb{Q}}(t-s)\mathrm{d}s$ being an $n\times 1$ vector, of which the j-th element is

$$\ln F_{j,T}(t) - \mathrm{Tr}(A^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))v) - B^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))x - \widetilde{C}(\tau;U_j(-\mathbf{i})).$$

Thus,

$$\begin{aligned}
& \mathbf{i}u\int_0^\tau e^{-\kappa^{\mathbb{Q}}s}\theta^{\mathbb{Q}}(t-s)\mathrm{d}s \\
= \;& \mathbf{i}\sum_{j=1}^n u_j\left[\ln F_{j,T}(t) - \mathrm{Tr}(A^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))v) - B^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))x - \widetilde{C}(\tau;U_j(-\mathbf{i}))\right].
\end{aligned}$$

Because

$$B^{\mathbb{Q}}(\tau;u)'x - \mathbf{i}\sum_{j=1}^n u_j B^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))'x = \mathbf{i}ue^{-\kappa^{\mathbb{Q}}\tau}x - \mathbf{i}\sum_{j=1}^n u_j\mathbf{i}U_j(-\mathbf{i})e^{-\kappa^{\mathbb{Q}}\tau}x = 0,$$

we have

$$\begin{aligned}
& f^{\mathbb{Q}}(x,v,t;u,F_T(t)) \\
= \;& \exp[\mathrm{Tr}(A^{\mathbb{Q}}(\tau;u)v) + \widetilde{C}(\tau;u) + \mathbf{i}\sum_{j=1}^n u_j[\ln F_{j,T}(t) - \mathrm{Tr}(A^{\mathbb{Q}}(\tau;U_j(-\mathbf{i}))v) - \widetilde{C}(\tau;U_j(-\mathbf{i}))] \\
= \;& \prod_{j=1}^n [F_{j,T}(t)]^{\mathbf{i}u_j}\exp[\mathrm{Tr}(\Delta A(\tau;u)v) + \Delta\widetilde{C}(\tau;u)].
\end{aligned}$$

Appendix D: Proof of Lemma 3

$$\begin{aligned}
P(0,X_0,T) &= e^{-rT}\mathbb{E}\left[\Pi(X_T)|\,X_0,V_0\right] = e^{-rT}\mathbb{E}\left[\mathcal{F}^{-1}_{X_T,u}\left\{\widehat{\Pi}(u)\right\}\Big|\,X_0,V_0\right] \\
&= e^{-rT}\mathbb{E}\left[(2\pi)^{-2}\int\int_{\mathbb{R}^2+\mathbf{i}\epsilon} e^{\mathbf{i}X_T u'}\widehat{\Pi}(u)du\Big|\,X_0,V_0\right] \\
&= e^{-rT}(2\pi)^{-2}\int\int_{\mathbb{R}^2+\mathbf{i}\epsilon}\mathbb{E}\left[e^{\mathbf{i}X_T u'}\Big|\,X_0,V_0\right]\widehat{\Pi}(u)du
\end{aligned}$$

$$
\begin{aligned}
&= e^{-rT}(2\pi)^{-2}\int\int_{\mathbb{R}^2+\mathbf{i}\epsilon} e^{\mathbf{i}X_0u'}\widehat{f}(X_0,V_0;T,u)\widehat{\Pi}(u)du \\
&= e^{-rT}\mathcal{F}^{-1}_{X_0,u}\left\{\widehat{f}(X_0,V_0;T,u)\widehat{\Pi}(u)\right\}.
\end{aligned}
$$

Appendix E: Proof of Lemma 4

Suppose that $\epsilon_2 > 0$ and $\epsilon_1 + \epsilon_2 < -1$; application of the Fourier inversion theorem to $e^{\epsilon x}\Pi(x)$, $\epsilon = (\epsilon_1, \epsilon_2)$ implies that

$$
\Pi(x) = (2\pi)^{-2}\int\int_{\mathbb{R}^2+\mathbf{i}\epsilon} e^{\mathbf{i}ux}\widehat{\Pi}(u)\mathrm{d}^2u,
$$

where

$$
\begin{aligned}
\widehat{\Pi}(u) &= \int\int_{\mathbb{R}^2} e^{-\mathbf{i}ux}\Pi(x)\mathrm{d}^2x \\
&= \int_{\log K}^{\infty} e^{-\mathbf{i}u_1x_1}(e^{x_1}-K)\int_{-\infty}^{x_1} e^{-\mathbf{i}u_2x_2}\mathrm{d}x_2\mathrm{d}x_1 \\
&\quad + \int_{\log K}^{\infty} e^{-\mathbf{i}u_2x_2}(e^{x_2}-K)\int_{-\infty}^{x_2} e^{-\mathbf{i}u_1x_1}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_{\log K}^{\infty} e^{-\mathbf{i}u_1x_1}(e^{x_1}-K)\frac{\mathbf{i}e^{-\mathbf{i}u_2x_1}}{u_2}\mathrm{d}x_1 + \int_{\log K}^{\infty} e^{-\mathbf{i}u_2x_2}(e^{x_2}-K)\frac{\mathbf{i}e^{-\mathbf{i}u_1x_2}}{u_1}\mathrm{d}x_2 \\
&= -\frac{\mathbf{i}K^{-\mathbf{i}(\mathbf{i}+u_1+u_2)}}{u_1(u_1+u_2)(\mathbf{i}+u_1+u_2)} - \frac{\mathbf{i}K^{-\mathbf{i}(\mathbf{i}+u_1+u_2)}}{u_2(u_1+u_2)(\mathbf{i}+u_1+u_2)} \\
&= -\frac{\mathbf{i}K^{1-\mathbf{i}(u_1+u_2)}}{u_1u_2(\mathbf{i}+u_1+u_2)}.
\end{aligned}
$$

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