The α-Hypergeometric Stochastic Volatility Model

José Da Fonseca * Claude Martini †

June 5, 2014

Abstract

The aim of this work is to introduce a new stochastic volatility model for equity derivatives. To overcome some of the well-known problems of the Heston model, and more generally of the affine models, we define a new specification for the dynamics of the stock and its volatility. Within this framework we develop all the key elements to perform the pricing of vanilla European options as well as of volatility derivatives. We clarify the conditions under which the stock price is a martingale and illustrate how the model can be implemented.

Keywords: Equity stochastic volatility Models, Volatility derivatives, European option pricing.

*Auckland University of Technology, Business School, Department of Finance, Private Bag 92006, 1142 Auckland, New Zealand. Phone: ++64 9 9219999 extn 5063. Email: jose.dafonseca@aut.ac.nz

†Zeliade Systems, 56, Rue Jean-Jacques Rousseau, 75001 Paris, France. Phone: ++33 1 40 26 17 81. Email: cmartini@zeliade.com

1 Introduction

Positivity is an essential property in financial modelling. In the field of equity derivatives since the seminal work of Black and Scholes (1973) several extensions were proposed to handle the stochastic behaviour of the volatility. Among all the proposed models the Heston (1993) model is, certainly, the most analysed model essentially because of its analytical tractability. However, when calibrated on option prices one usually obtains that the Feller condition, ensuring that the process does not reach zero in finite time, is not satisfied (see Da Fonseca and Grasselli (2011) for calibration results on several indexes as well as extensions of the Heston model). The mean reverting parameter is problematic to estimate and uses to be small. Notice that this fact seems to be widely known among practitioners and is sometimes mentioned in academic works, see Henry-Labordère (2009) page 183. It seems to us that this problem is mainly related to the fact that option prices contain integrated volatility. One consequence is that the volatility remains "too close" to zero and contrast with its empirical distribution which is closer to a lognormal one. This partially motivates the model proposed in Gatheral (2008) which specifies a (double) lognormal dynamic for the volatility. In that case, a major drawback is that no closed-form solution is available for vanilla options turning the calibration a tedious exercise.

From an historical point of view it is well known that the first stochastic volatility model was proposed in Hull and White (1987) and define for the volatility dynamic a geometric Brownian motion which is therefore non stationary. A closed-form solution (for vanilla options) for this model was proposed much later in Leblanc (1996) but it was also shown in Jourdain (2004) that the spot loses its martingale property for some parameter values. From a modelling point of view the model of Chesney and Scott (1989) is certainly the most natural one as it specifies for the volatility the exponential of a stationary Ornstein-Uhlenbeck process. By construction the volatility is positive. Unfortunately, no closed-form solution for the characteristic function of the stock is available for this model. In Stein and Stein (1991) a closed-form solution is obtained for a stochastic volatility model whose volatility follows an Ornstein-Uhlenbeck process, hence Gaussian which is problematic regarding the aspect of positivity.

Our purpose is to develop a stochastic volatility model which is tractable, that is to say, for which most of the key ingredients to perform derivative pricing can be computed efficiently and has positive distribution for the volatility.

The structure of the paper is as follows. In a first section, we introduce the volatility dynamics and study both the volatility and spot properties. For the volatility, we perform a transformation of the process that allows us to heavily use the results of Donati-Martin et al. (2001) whereas for the stock we closely follow Jourdain (2004). In a second section, specifying further the volatility dynamic we analyse the volatility and compute the Mellin transform of the stock using an approach based on the resolvent. For this part, we were inspired by Pintoux and Privault (2010) and Pintoux and Privault (2011) and heavily used the surveys of Matsumoto and Yor (2005a) and Matsumoto and Yor (2005b). We postpone the discussion of related works to the third section and the last section conclude the paper.

2 The Model and its Properties

The dynamics is given is that model by

$$df_t = f_t e^{v_t} dw_{1,t} \quad (1)$$

$$dv_t = (a - b e^{\alpha v_t})dt + \sigma dw_{2,t} \quad (2)$$

with $\alpha > 0$ and $(w_{1,t}, w_{2,t})_{t\geq 0}$ a Brownian motion with $dw_{1,t}.dw_{2,t} = \rho dt$ under the risk neutral probability measure P. We denote by $\mathbb{E}$ the expectation under this probability (also, we may denote $\mathbb{E}^P$ whenever needed to avoid confusion). So v_t is the instantaneous log volatility and the instantaneous variance is given by $V_t = e^{2v_t}$. We assume $b > 0$ and $\sigma > 0$, yet there is no constraint on the sign of a.

2.1 Study of the Variance Process

2.1.1 The variance as a functional of w_2

Let v denote any solution of the SDE (2). Let us observe that $v_t - v_0 + b\int_0^t \exp\alpha v_s ds = at + \sigma w_{2,t}$. Introducing the integral $I(t) = \int_0^t \exp\alpha v_s ds$, we note that $\frac{dI(t)}{dt} = \exp\alpha v_t$ so that:

$$\ln\frac{dI(t)}{dt} + \alpha b I(t) = \alpha(v_0 + at + \sigma w_{2,t})$$

or yet

$$\frac{dI(t)}{dt}\exp\alpha b I(t) = \exp\alpha(v_0 + at + \sigma w_{2,t})$$

which gives in turn by integrating:

$$\exp\alpha b I(t) = 1 + \alpha b\int_0^t \exp\alpha(v_0 + as + \sigma w_{2,s})ds.$$

We get eventually:

$$I(t) = \frac{\ln\left(1 + \alpha b\int_0^t \exp\alpha(v_0 + as + \sigma w_{2,s})ds\right)}{\alpha b}$$

and by differentiating:

$$V_t = \exp 2v_t = \left(\frac{dI(t)}{dt}\right)^{\frac{2}{\alpha}} = \frac{V_0\exp 2at + 2\sigma w_{2,t}}{(1 + \alpha b V_0^{\frac{\alpha}{2}}\int_0^t \exp\alpha(as + \sigma w_{2,s})ds)^{\frac{2}{\alpha}}}.$$

Conversely, let v be defined by the preceding equation, i.e. $v_t = \frac{1}{2}\ln V_0 + at + \sigma w_{2,t} - \frac{\ln(1+\alpha b V_0^{\frac{\alpha}{2}}\int_0^t \exp\alpha(as+\sigma w_{2,s})ds)}{\alpha}$. Then $v_0 = \frac{1}{2}\ln V_0$ and:

$$\int_0^t \exp\alpha v_s ds = \int_0^t \frac{V_0^{\frac{\alpha}{2}}\exp\alpha(as + \sigma w_{2,s})}{1 + \alpha b V_0^{\frac{\alpha}{2}}\int_0^s \exp\alpha(au + \sigma w_{2,u})du}ds = \frac{\ln\left(1 + \alpha b\int_0^t V_0^{\frac{\alpha}{2}}\exp\alpha(as + \sigma w_{2,s})ds\right)}{\alpha b}$$

so that $v_t = v_0 + at + \sigma w_{2,t} - b\int_0^t \exp\alpha v_s ds$, which is the integrated form of the above SDE.

We have therefore proven that there is existence and pathwise uniqueness for the SDE defining the variance behaviour. Moreover, we have an explicit solution to this SDE in terms of the driving Brownian motion w_2. Lastly, observe also that in the limiting case $\alpha = 0$, one directly gets $v_t^0 = (a - b)t + w_{2,t}$. It is easily checked that $\lim_{\alpha\to 0} v_t^\alpha = v_t^0$ pathwise, so that there is no loss of continuity when α converges to zero.

2.1.2 Basic properties

Dependency on α: From the driving SDE it is easily seen by scaling that

$$\alpha v_{v_0,\alpha,a,b,\sigma} = v_{\alpha v_0,1,\alpha a,\alpha b,\alpha\sigma},\, V^\alpha_{V_0,\alpha,a,b,\sigma} = V_{V_0^\alpha,1,\alpha a,\alpha b,\alpha\sigma}$$

this can be checked also directly on the preceding formulas.

What happens for negative b: It follows that the SDE has a well defined solution when b *and* α are negative (????). If $b < 0$ and $\alpha > 0$, it follows form the expression of $I(t)$ that the solution is well defined up to the stopping time

$$T^* = \inf\left\{t|\int_0^t \exp\alpha(v_0 + as + \sigma w_{2,s})ds > -\frac{1}{\alpha b}\right\}$$

Noiseless limit: The above computations are valid when $\sigma = 0$. In this case the formula simplifies to:

$$I(t) = \frac{\ln\left(1 + \frac{b}{a} e^{\alpha v_0}(e^{\alpha a t} - 1)\right)}{\alpha b}$$

and by differentiating:

$$V_t = \frac{V_0 e^{2at}}{(1 + \frac{b}{a} V_0^{\frac{\alpha}{2}}(e^{\alpha a t} - 1))^{\frac{2}{\alpha}}}.$$

It follows in particular that $\frac{I(t)}{t} \to \frac{a}{b}$ when $t \to \infty$.

2.1.3 Connection with the Wong-Shyryaev process

So the driving process of the variance is

$$dv_t = (a - b e^{\alpha v_t})dt + \sigma dw_{2,t}$$

Consider now $Z_t = e^{-\alpha v_t} = V_t^{-\frac{\alpha}{2}}$.

$$d(e^{-\alpha v_t}) = -\alpha e^{-\alpha v_t}[(a - b e^{\alpha v_t})dt + \sigma dw_{2,t}] + \frac{\alpha^2 \sigma^2}{2} e^{-\alpha v_t} dt$$

so that

$$dZ_t = [\alpha b + (\frac{\alpha^2 \sigma^2}{2} - \alpha a) Z_t]dt + \alpha \sigma Z_t dC_t$$

with $Z_0 = e^{-\alpha v_0}$ where C is the Brownian motion $-w_{2,t}$. This process (modulo a convenient rescaling) has been studied in Donati-Martin et al. (2001) and in Peskir (2006) where it is called the Shyryaev process. It is also sometimes called the Wong process. We will mostly make use of Donati-Martin et al. (2001). To alleviate the notation, let us write it as:

$$dZ_t = (m + n Z_t)dt + p Z_t dC_t$$

with

$$m = \alpha b, n = (\frac{\alpha^2 \sigma^2}{2} - \alpha a), p = \alpha \sigma$$

This S.D.E. is affine in Z and thus easy to solve: introduce X the solution of the homogeneous SDE $dX_t = n X_t dt + p X_t dC_t$, so that $X_t = X_0 e^{(n - \frac{p^2}{2})t + p C_t}$, and look for solutions of the form $U_t X_t$ where U is a process with finite variation. Then $d(U_t X_t) = U_t(n X_t dt + p X_t dC_t) + X_t dU_t$, so that UX is a solution as soon as $dU_t = m X_t^{-1} dt$. Therefore:

$$Z_t = X_t (Z_0 + m \int_0^t X_s^{-1} ds)$$

with $X_s = e^{(n - \frac{p^2}{2})s + p C_s}$.
Now let us look for a scaling factor c such that $p C_{cu} = \sqrt{2} D_u$ for some Brownian motion D. Obviously $c = \frac{2}{p^2}$ and:

$$X_u = e^{(n - \frac{p^2}{2}) c \frac{u}{c} + \sqrt{2} D_{\frac{u}{c}}}$$

So, within the notations of Donati-Martin et al. (2001) we have:

$$\nu = (n - \frac{p^2}{2})c = -\alpha a c = -a \frac{2}{\alpha \sigma^2}$$

$$Z_t = X_{c\frac{t}{c}} (Z_0 + mc \int_0^{\frac{t}{c}} X_{cu}^{-1} du) = mc Y_{\frac{t}{c}}^{(\nu)} (\frac{Z_0}{mc})$$

where

$$mc = \alpha b \frac{2}{p^2} = \frac{2b}{\alpha\sigma^2}$$

and $Y_t^\nu(x)$ satisfying the SDE

$$Y_t = x + \sqrt{2}\int_0^t Y_u dB_u + \int_0^t (1 + (1+\nu)Y_u)du. \tag{3}$$

Proposition 1 *The instantaneous volatility V_t can be expressed as a function of the process $Y_t^{(\nu)}$ through the relation:*

$$e^{-\alpha v_t} = V_t^{-\frac{\alpha}{2}} = qY_{\frac{t}{c}}^{(\nu)}(\frac{V_0^{-\frac{\alpha}{2}}}{q})$$

with

$$\nu = -\frac{2a}{\alpha\sigma^2}, q = \frac{2b}{\alpha\sigma^2}, c = \frac{2}{\alpha\sigma^2}.$$

The Green function of V: The Green function $u_\lambda(x,y)$ associated to $Y^{(\nu)}(x)$ has been computed in Theorem 3.1 Donati-Martin et al. (2001), so we have the Green function of v or V:

$$u_\lambda(x,y) = \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(1+\mu)}(\frac{1}{y})^{1-\nu}e^{-\frac{1}{y}}[1(y \le x)\phi_1(x)\phi_2(y) + 1(x < y)\phi_2(x)\phi_1(y)]$$

where

$$\mu = \sqrt{\nu^2 + 4\lambda}$$

and

$$\phi_1(x) = \left(\frac{1}{x}\right)^{\frac{\mu+\nu}{2}} \Phi\left(\frac{\mu+\nu}{2}, 1+\mu; \frac{1}{x}\right)$$

$$\phi_2(x) = \left(\frac{1}{x}\right)^{\frac{\mu+\nu}{2}} \Psi\left(\frac{\mu+\nu}{2}, 1+\mu; \frac{1}{x}\right).$$

The function Φ is the confluent hypergeometric function of the first kind, which has the integral representation:

$$\Phi(\alpha,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 e^{zu}u^{\alpha-1}(1-u)^{\gamma-\alpha-1}du.$$

and Ψ is the confluent hypergeometric function of the second kind. We will use the following integral representation (DLMF (2010) 13.4.18):

$$\Psi(a-1,b;z) = \frac{z^{1-b}e^z}{2i\pi}\int_{-i\infty}^{i\infty} \frac{\Gamma(b-1+t)\Gamma(t)}{\Gamma(a-1+t)}z^{-t}dt$$

where the contour of integration passes on the right of the poles of the integrand.

The moments of $V^{-\frac{\alpha}{2}}$: It is also easy to compute the l^{th} moment $M^{(l)}$ of Z as we have:

$$dZ_t^l = lZ_t^{l-1}((m + nZ_t)dt + pZ_t dC_t) + l(l-1)Z_t^{l-2}p^2Z_t^2 dt$$

whence

$$dM_t^{(l)} = (mlM_t^{(l-1)} + (nl + l(l-1)p^2)M_t^{(l)})dt.$$

2.1.4 The pricing of Variance Swaps

We are interested in:

$$t\text{VS}(t) = \int_0^t \mathbb{E}[V_s]ds = \int_0^t \mathbb{E}\left[Z_s^{-\frac{2}{\alpha}}\right]ds = q^{-\frac{2}{\alpha}}\int_0^t \mathbb{E}[\frac{1}{Y_{\frac{s}{c}}^{(\nu)}(\frac{V_0^{-\frac{\alpha}{2}}}{q})^{\frac{2}{\alpha}}}]ds = cq^{-\frac{2}{\alpha}}\int_0^{\frac{t}{c}} \mathbb{E}[\frac{1}{Y_r^{(\nu)}(\frac{V_0^{-\frac{\alpha}{2}}}{q})^{\frac{2}{\alpha}}}]dr$$

The Laplace transform of vs(t) via the Green function: By using the standard algebraic operations on Laplace transforms we know that the Laplace transform of the integral is the Laplace transform of the integrand divided by λ. So the first step is to compute:

$$I(x) = \int_0^\infty e^{-\lambda r}\mathbb{E}[\frac{1}{Y_r^{(\nu)}(x)^{\frac{2}{\alpha}}}]dr = \int_0^\infty y^{-\frac{2}{\alpha}}u_\lambda(x,y)dy.$$

We have $\frac{\Gamma(1+\mu)}{\Gamma(\frac{\mu+\nu}{2})}I = \phi_1(x)I_2 + \phi_2(x)I_1$ with

$$I_1 = \int_x^\infty \left(\frac{1}{y}\right)^{\frac{2}{\alpha}+1-\nu} e^{-\frac{1}{y}}\phi_1(y)dy = \int_x^\infty \left(\frac{1}{y}\right)^{\frac{2}{\alpha}+1+\frac{\mu-\nu}{2}} e^{-\frac{1}{y}}\Phi\left(\frac{\mu+\nu}{2}, 1+\mu; \frac{1}{y}\right)dy,$$

$$I_2 = \int_0^x \left(\frac{1}{y}\right)^{\frac{2}{\alpha}+1-\nu} e^{-\frac{1}{y}}\phi_2(y)dy = \int_0^x \left(\frac{1}{y}\right)^{\frac{2}{\alpha}+1+\frac{\mu-\nu}{2}} e^{-\frac{1}{y}}\Psi\left(\frac{\mu+\nu}{2}, 1+\mu; \frac{1}{y}\right)dy.$$

Then

$$\mathrm{vs}(t) = \frac{c}{qt}\mathcal{L}^{-1}\left(\frac{1}{\lambda}I(x = \frac{1}{q}V_0^{-\frac{\alpha}{2}})\right)(\frac{t}{c})$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

2.1.5 Computation of I1

By the change of variable $z = \frac{1}{y}$, $I_1 = \int_0^{\frac{1}{x}} z^{\frac{\mu-\nu}{2}+\frac{2}{\alpha}-1}e^{-z}\Phi(\frac{\mu+\nu}{2}, 1+\mu; z)dz$. Let us introduce $a = 1+\frac{\mu+\nu}{2}$, $b = 1+\mu$, so that $b-a = \frac{\mu-\nu}{2}$. By Kummer's tranformation $e^{-z}\Phi(a-1,b;z) = \Phi(b-a+1,b;-z)$ and

$$I_1 = \int_0^{\frac{1}{x}} z^{b-a+\frac{2}{\alpha}-1}\Phi(b-a+1,b;-z)dz.$$

Therefore, by Fubini's theorem as in Love et al. (1982), $I_1 = \sum_{n=0}^\infty \frac{(b-a+1)_n}{(b)_n n!}(-1)^n \int_0^{\frac{1}{x}} z^{b-a+\frac{2}{\alpha}-1+n}dz$ which leads to

$$I_1 = x^{-b+a-\frac{2}{\alpha}}\sum_{n=0}^\infty \frac{(b-a+1)_n}{(b-a+\frac{2}{\alpha}+n)(b)_n n!}(-1)^n x^{-n}$$

Note that $(b-a+\frac{2}{\alpha}+n) = \frac{(b-a+\frac{2}{\alpha}+1)_n(b-a+\frac{2}{\alpha})}{(b-a+\frac{2}{\alpha})_n}$ so that eventually:

$$\begin{aligned} I_1 &= \frac{x^{-b+a-\frac{2}{\alpha}}}{(b-a+\frac{2}{\alpha})}\sum_{n=0}^\infty \frac{(b-a+1)_n(b-a+\frac{2}{\alpha})_n}{(b-a+\frac{2}{\alpha}+1)_n(b)_n n!}(-1)^n x^{-n} \\ &= \frac{x^{-b+a-\frac{2}{\alpha}}}{(b-a+\frac{2}{\alpha})}H\left(\left[b-a+1, b-a+\frac{2}{\alpha}\right], \left[b, b-a+1+\frac{2}{\alpha}\right], -\frac{1}{x}\right) \end{aligned}$$

where H is the *generalized hypergeometric* function.

2.1.6 Computation of I2

I2 as a complex integral We have in the same way $I_2 = \int_{\frac{1}{x}}^\infty z^{b-a+\frac{2}{\alpha}-1}e^{-z}\Psi(a-1,b;z)dz$. Recall the Barnes integral representation (DLMF (2010) 13.4.18):

$$\Psi(a-1,b;z) = \frac{z^{1-b}e^z}{2i\pi}\int_{-i\infty}^{i\infty}\frac{\Gamma(b-1+t)\Gamma(t)}{\Gamma(a-1+t)}z^{-t}dt$$

where the contour of integration passes on the right the poles of the integrand. Then

$$e^{-z}z^{b-a+\frac{2}{\alpha}-1}\Psi(a-1,b;z) = \frac{1}{2i\pi}\int_{-i\infty}^{i\infty}\frac{\Gamma(b-1+t)\Gamma(t)}{\Gamma(a-1+t)}z^{\frac{1}{\alpha}-(a+t)}dt$$

Moreover we know that the integral converges locally uniformly in z, so that we can apply Fubini's theorem and permute the integrals. Observe now that $a = 1 + \frac{\mu+\nu}{2} = 1 + \frac{\sqrt{\nu^2+4\lambda}-|\nu|}{2}$, so that $1 - a < 0$. and if λ is large enough so that $1 + \frac{2}{\alpha} - a < 0$, then the inner integral is finite and

$$I_2 = -\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} \frac{\Gamma(b-1+t)\Gamma(t)}{\Gamma(a-1+t)(1+\frac{2}{\alpha}-(a+t))}(\frac{1}{x})^{1+\frac{2}{\alpha}-(a+t)} dt.$$

This formula is valid as soon as $a > 1 + \frac{2}{\alpha}$, which amounts after a simple computation to

$$\lambda > \lambda^* = \frac{4}{\alpha^2} + \frac{2\,|\nu|}{\alpha}$$

Writing $(1 + \frac{2}{\alpha} - (a+t)) = \frac{\Gamma(2+\frac{2}{\alpha}-(a+t))}{\Gamma(1+\frac{2}{\alpha}-(a+t))} = \frac{\Gamma(1-(a-\frac{2}{\alpha}-1)-t)}{\Gamma(1-(a-\frac{2}{\alpha})-t)}$ and recalling the definition of Meijer G function, I_2 looks like

$$-(\frac{1}{x})^{1+\frac{2}{\alpha}-a} G\left([[a-\frac{2}{\alpha}],[a-1]],[[0,b-1],[a-\frac{2}{\alpha}-1]],\frac{1}{x}\right).$$

Nevertheless, the paths of integration are not the same for the two formulas, since the defining path in the Meijer G function is not on the right of all the poles of the integrand.

An explicit hypergeometric series for I2: At this stage the natural step is to apply the theorem of residues to get a series from the above complex integrals. The poles of the integrand are located:

- at $t = 1 + \frac{2}{\alpha} - a(< 0)$, with residue $-\frac{\Gamma(b-1+t)\Gamma(t)}{\Gamma(a-1+t)}$
- at $t = -n, n \in N$, with residue $\frac{\Gamma(b-1+t)}{\Gamma(a-1+t)(1+\frac{2}{\alpha}-(a+t))n!}(-1)^n y^{1+\frac{2}{\alpha}-(a+t)}$
- at $t = -n+1-b, n \in N$, with residue $\frac{\Gamma(t)}{\Gamma(a-1+t)(1+\frac{2}{\alpha}-(a+t))n!}(-1)^n y^{1+\frac{2}{\alpha}-(a+t)}$

so that by Cauchy's residue theorem we get:

$$\begin{aligned} I_2 &= \Gamma(1+b-a)\Gamma(1+\frac{2}{\alpha}-a) \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} y^{1+\frac{2}{\alpha}+n-a}}{n!}\left(\frac{\Gamma(b-1-n)}{\Gamma(a-1-n)(1+\frac{2}{\alpha}+n-a)} + \frac{y^{b-1}\Gamma(1-b-n)}{\Gamma(a-b-n)(\frac{2}{\alpha}+b-a+n)}\right) \end{aligned} \quad (4)$$

I_2 can be computed easily by making explicit the recurrences between successive terms of the 2 series. Calls to the Γ function are only required for the constant and index zero terms.

2.1.7 Final formula for I

Since $I(x) = \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(1+\mu)}(\phi_1(x)I_2(x) + \phi_2(x)I_1(x))$ we get the final formula for I given by:

Proposition 2 *For any $\lambda > \lambda^*$ where $\lambda^* = \frac{4}{\alpha^2} + \frac{2|\nu|}{\alpha}$,*

$$I = \frac{\Gamma(a-1)}{\Gamma(b)}(y^{a-1} I_2 \Phi(a-1,b,y) + y^{b+\frac{2}{\alpha}-1}\frac{\Psi(a-1,b,y)}{(b-a+\frac{2}{\alpha})}h)$$

where $a = 1 + \frac{\mu+\nu}{2}, b = 1+\mu$ and $y = \frac{1}{x}$ and

$$h = H([b-a+1, b-a+\frac{2}{\alpha}],[b, b-a+1+\frac{2}{\alpha}],-y)$$

$$\begin{aligned} I_2 &= \Gamma(1+b-a)\Gamma(1+\frac{2}{\alpha}-a) \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} y^{1+\frac{2}{\alpha}+n-a}}{n!}\left(\frac{\Gamma(b-1-n)}{\Gamma(a-1-n)(1+\frac{2}{\alpha}+n-a)} + \frac{y^{b-1}\Gamma(1-b-n)}{\Gamma(a-b-n)(\frac{2}{\alpha}+b-a+n)}\right) \end{aligned} \quad (5)$$

2.1.8 Short term behaviour

We now analyse the short term behaviour of the instantaneous volatility. We start from the formula:

$$V_t = \frac{V_0 \exp 2at + 2\sigma w_{2,t}}{(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha(as + \sigma w_{2,s}) ds)^{\frac{2}{\alpha}}}.$$

By introducing the exponential martingale $e^{2\sigma w_{2,t} - \frac{(2\sigma)^2}{2} t}$ we get by Girsanov's theorem:

$$\mathbb{E}[V_t] = V_0 e^{(2a + \frac{(2\sigma)^2}{2})t} \mathbb{E}^Q \left[\left(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha((a + \sigma^2)s + \sigma \tilde{w}_{2,s}) ds \right)^{\frac{-2}{\alpha}} \right]$$

with $\tilde{w}_{2,t} = w_{2,t} - 2\sigma t$ a Brownian motion under Q. For a given t, the set of paths such that the time integral is larger than an arbitrary small level becomes exponentially small in probability so that

$$\mathbb{E}[V_t] \sim V_0 e^{(2a + \frac{(2\sigma)^2}{2})t} \mathbb{E}^Q \left[1 - 2bV_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha((a + \sigma^2)s + \sigma \tilde{w}_{2,s}) ds \right].$$

Now $\mathbb{E}^Q[\int_0^t \exp \alpha((a + \sigma^2)s + \sigma \tilde{w}_{2,s}) ds] = \frac{e^{(\alpha(a+\sigma^2) + \frac{\alpha^2 \sigma^2}{2})t} - 1}{\alpha(a+\sigma^2) + \frac{\alpha^2\sigma^2}{2}}$. Therefore, in the following proposition the second statement results from the first one by integration:

Proposition 3 *As $t \to 0$*

- $\mathbb{E}[V_t] \sim V_0(1 + (2a + \frac{(2\sigma)^2}{2} - 2bV_0^{\frac{\alpha}{2}})t)$
- $\mathrm{vs}(t) \sim V_0(1 + (2a + \frac{(2\sigma)^2}{2} - 2bV_0^{\frac{\alpha}{2}})\frac{t}{2})$

Short term behaviour when $\alpha = 2$: There is an easy majorization in case $\alpha = 2$, which also provides an excellent approximation for short term maturities: by using the concavity of the logarithm and Jensen's inequality:

$$t\mathrm{vs}(t) = \mathbb{E}^P[\frac{1}{2b} \ln(1 + 2bV_0 A_t)] < \frac{1}{2b} \ln(1 + 2bV_0 \mathbb{E}^P[A_t])$$

where $\mathbb{E}^P[A_t] = \frac{e^{(2a+2\sigma^2)t} - 1}{(2a+2\sigma^2)}$. This will yield an excellent short term approximation because $\ln 1 + x \sim x$ near 0 and A_t is small in probability for small t.

Proposition 4 *($\alpha = 2$) Let $f(t) = \frac{1}{2bt} \ln(1 + 2bV_0 \frac{e^{(2a+2\sigma^2)t} - 1}{(2a+2\sigma^2)})$. Then $\mathrm{vs}(t) < f(t)$ for every $t > 0$. Moreover as $t \sim 0$,*

$$\mathrm{vs}(t) \sim f(t) \sim V_0(1 + (a + \sigma^2 - bV_0)t).$$

The last approximation is useful for practical purposes.

2.1.9 Long term behaviour when $a > 0$

We start also from the formula:

$$V_t = \frac{V_0 \exp 2at + 2\sigma w_{2,t}}{(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha(as + \sigma w_{2,s}) ds)^{\frac{2}{\alpha}}}.$$

Since $a > 0$, the behaviour of the average $\int_0^t e^{\alpha(as + \sigma w_{2,s})} ds$ will go very fast to infinity as $t \to \infty$. It is clear in particular that $\int_0^t e^{\alpha(as+\sigma w_{2,s})} ds$ will become much larger than 1 so that

$$V_t \sim \frac{V_0 e^{2at + 2\sigma w_{2,t}}}{(\alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha(as + \sigma w_{2,s}) ds)^{\frac{2}{\alpha}}}.$$

Now this simplifies to $\frac{1}{(\alpha b)^{\frac{2}{\alpha}}} \frac{1}{(\int_0^t \exp \alpha(a(s-t) + \sigma(w_{2,s} - w_{2,t})) ds)^{\frac{2}{\alpha}}}$, and the whole point is to observe that by time-reversal we will get an average with a *negative* drift, whose behaviour at infinity converges to the inverse of a Gamma law: by scaling $\int_0^t e^{\alpha(a(s-t)+\sigma(w_{2,s}-w_{2,t})}ds = \frac{4}{\alpha^2\sigma^2}\int_0^{t\alpha^2\frac{\sigma^2}{4}} e^{2(-\frac{2a}{\alpha\sigma^2}u+B_u)}du$ for some Brownian motion B. With the notations of Dufresne (1998) with a drift $\mu = \frac{2a}{\alpha\sigma^2}$ we have therefore

$$V_t \sim (\alpha b)^{-\frac{2}{\alpha}}(\frac{2}{\alpha^2\sigma^2})^{-\frac{2}{\alpha}}(2A^{(-\mu)}_{t\alpha^2\frac{\sigma^2}{4}})^{-\frac{2}{\alpha}}$$

which entails

$$V_t \to (\alpha b)^{-\frac{2}{\alpha}}(\frac{2}{\alpha^2\sigma^2})^{-\frac{2}{\alpha}}(\mathrm{Gamma}(\mu,1))^{\frac{2}{\alpha}}.$$

Observing that the expectations will converge too thanks to the monotone convergence theorem, we get:

Proposition 5 *Assume $a > 0$. As $t \to \infty$,*

$$\mathbb{E}[V_t], \mathrm{vs}(t) \to (\frac{2b}{\alpha\sigma^2})^{-\frac{2}{\alpha}}\mathbb{E}[(Gamma(\mu,1))^{\frac{2}{\alpha}}]$$

where $\mu = \frac{2a}{\alpha\sigma^2}$. In particular,

- *For $\alpha = 2$, $\mathbb{E}[V_t], \mathrm{vs}(t) \to \frac{a}{b}$.*
- *For $\alpha = 1$, $\mathbb{E}[V_t], \mathrm{vs}(t) \to (\frac{\sigma^2}{2b})^2\frac{2a}{\sigma^2}(1+\frac{2a}{\sigma^2})$.*

Note that this is consistent with the large time behaviour of the noiseless limit obtained in 2.1.2 for the case $\alpha = 2$. The noiseless limit in the above formula for $\alpha = 1$ is $(\frac{a}{b})^2$, which is not that of $I(t)$ which is $\frac{a}{b}$ irrespective of α: just note that $I(t)$ is *not* the integrated variance when $\alpha \neq 2$, so there is no contradiction or mysterious lack of continuity behaviour.

2.2 Study of the Spot Process

2.2.1 A full-blown martingale

Consider now the dynamic of the forward f_t, it is defined by the stochastic exponential of the local martingale $L_t = \int_0^t e^{v_s}dw_{1,s}$. Then $< L >_t = \int_0^t e^{2v_s}ds$ and Novikov's criterion tells us that f_t is a uniformly integrable martingale if $\mathbb{E}[\exp\frac{<L>_t}{2}] < \infty$.

Case $\alpha = 2$: In this case:

$$\exp\frac{< L >_t}{2} = \exp\frac{1}{2}\int_0^t e^{2v_s}ds = \exp\frac{I(t)}{2} = (1+2b\int_0^t \exp 2(v_0+as+\sigma w_{2,s})ds)^{\frac{1}{4b}}.$$

Therefore, assuming $b > 0$, $\exp\frac{<L>_t}{2} < (1+\frac{b}{|a|}V_0\exp 2\,|a|\,t\exp 2\sigma w^*_{2,t})^{\frac{1}{4b}}$ where w^*_2 denotes the running maximum of the Brownian motion w_2.
Now $\exp 2\sigma w^*_{2,t} \geq 1$ and this is less than $\max(1,\frac{b}{a}V_0\exp 2at)^{\frac{1}{4b}}\exp\frac{2\sigma w^*_{2,t}}{4b}$. Since $w^*_{2,t}$ has the same law as $|w_{2,t}|$, this is integrable and

$$\mathbb{E}[\exp\frac{< L >_t}{2}] < \infty.$$

Case $\alpha > 2$: In this case we have $< L >_t = \int_0^t e^{2v_s}ds \leq t^{1-\frac{2}{\alpha}}(\int_0^t e^{\alpha v_s}ds)^{\frac{2}{\alpha}}$ by Holder's inequality. Now

$$\int_0^t e^{\alpha v_s}ds = \frac{\ln(1+\alpha b\int_0^t V_0^{\frac{\alpha}{2}}\exp\alpha(as+\sigma w_{2,s})ds)}{\alpha b}.$$

To conclude note that since $\alpha > 2$, $(\int_0^t e^{\alpha v_s}ds)^{\frac{2}{\alpha}} \leq \max(1,\int_0^t e^{\alpha v_s}ds)^{\frac{2}{\alpha}} \leq \max(1,\int_0^t e^{\alpha v_s}ds)$ and the equality

$$\max(1,z) = z + (1-z)1(z<1)$$

tells us that $e^{\max(1,\int_0^t e^{\alpha v_s}ds)} \leq e^{\int_0^t e^{\alpha v_s}ds}e^1$ and we can conclude as above.

Case $\alpha < 2$: In this case, the mean reversion force is weaker and we expect that the log volatility may become large, and therefore also the forward f in case of positive correlation.

We follow step-by-step the reasoning of Jourdain (2004). First note that $\mathbb{E}[f_t] = f_0 \mathbb{E}[\mathcal{E}(\rho \int_0^t \exp v_s dw_{2,s})]$. Since f_t is a positive local martingale, it is a supermartingale and the map $t \to \mathbb{E}[f_t]$ is non-increasing. Therefore, f_t is a martingale if and only if it is constantly equal to f_0, i.e. $\frac{\mathbb{E}[f_t]}{f_0} = 1$.

$$\mathbb{E}[\mathcal{E}(\rho \int_0^t \exp v_s dw_{2,s})]$$

turns out to be the probability of non explosion of a Markovian SDE associated to the initial one by means of the Girsanov theorem, and Feller criterion for explosion provides then an explicit necessary and sufficient condition for this probability to be one.
Adopting for a while the notations of Jourdain (2004), we denote w_2 by B. Introduce the probability Q under which $d\tilde{B}_t = dB_t - \frac{a}{\sigma}dt + \frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \exp \alpha\sigma B_t dt$ is a Brownian motion. By Girsanov's theorem, $\frac{dQ}{dP} = \mathcal{E}(L_T)$ with $L_t = \frac{a}{\sigma}B_t - \frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha\sigma B_s dB_s$. By the Yamada-Watanabe theorem, the law of (v, B) under P is the same as the law of $(v_0 + \sigma B, \tilde{B})$ under Q, and

$$\mathbb{E}^P[\mathcal{E}(\rho \int_0^t \exp v_s dB_s)] = \mathbb{E}^Q[\mathcal{E}(\rho\sqrt{V_0} \int_0^t \exp \sigma B_s d\tilde{B}_s)].$$

This is equal to:

$$\mathbb{E}^P[\mathcal{E}(\rho\sqrt{V_0} \int_0^t \exp \sigma B_s (dB_s + (\frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \exp \alpha\sigma B_s - \frac{a}{\sigma})ds))\mathcal{E}(\frac{a}{\sigma}B_t - \frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \int_0^t \exp \alpha\sigma B_s dB_s)]$$

which rewrites as:

$$\mathbb{E}^P[\mathcal{E}(\int_0^t (\rho\sqrt{V_0} \exp \sigma B_s - \frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \exp \alpha\sigma B_s + \frac{a}{\sigma})dB_s)] = \mathbb{E}^P[\mathcal{E}(\int_0^t b(B_s)dB_s)]$$

with

$$b(z) = \rho\sqrt{V_0} \exp \sigma z - \frac{b}{\sigma}V_0^{\frac{\alpha}{2}} \exp \alpha\sigma z + \frac{a}{\sigma}.$$

The next step is to observe that $\mathbb{E}^P[\mathcal{E}(\int_0^t b(B_s)dB_s)] = P(\tau_\infty > t)$ where τ_∞ is the explosion time of the SDE

$$dZ_s = b(Z_s)ds + dB_s.$$

We can now apply the Feller criterion for explosions, which tells us that $P(\tau_\infty = \infty) = 1$ if and only if

$$a(-\infty) = a(\infty) = \infty$$

where $a(z) = \int_0^z p'(x) \int_0^x \frac{2}{p'(y)} dy dx$ where p is any scale function of the Z SDE.
Now the function $p(x) = \int_0^x \exp -2 \int_0^y b(z)dz dy$ is a scale function, and we are left with explicit computations.
Observe that $\int_0^y b(z)dz = \frac{\rho\sqrt{V_0}}{\sigma}(\exp \sigma y - 1) - \frac{bV_0^{\frac{\alpha}{2}}}{\alpha\sigma^2}(\exp \alpha\sigma y - 1) + \frac{a}{\sigma}y$ so that

$$p'(x) = C \exp\left(-2\frac{\rho\sqrt{V_0}}{\sigma} \exp \sigma x + 2\frac{bV_0^{\frac{\alpha}{2}}}{\alpha\sigma^2} \exp \alpha\sigma x - 2\frac{a}{\sigma}x\right)$$

for some positive constant C.

Behaviour at $-\infty$: $p'(x) \sim C \exp -2\frac{a}{\sigma}x$ with also $\int_x^0 \frac{2}{p'(y)}dy$ which is positive and increasing as $x \to -\infty$, so that $a(-\infty) = \infty$ when $a > 0$. This argument is still valid when $a = 0$. When $a < 0$, then $\int_x^0 \frac{2}{p'(y)}dy \sim C^{-1} \exp 2\frac{a}{\sigma}x$ so that the integrand converges to the constant 1 and the integral diverges.

Behaviour at $+\infty$: There again, $\int_x^0 \frac{2}{p'(y)}dy$ is positive and increasing as $x \to \infty$. The behaviour is driven by the terms in the outer exponential:

- When $\rho \leq 0$, the exponential terms will dominate the linear one, and $a(\infty) = \infty$.
- When $\rho > 0$ and $\alpha > 1$, the second positive exponential will dominate the first negative one, and $a(\infty) = \infty$.
- When $\alpha = 1$, $p'(x)$ writes
$$C \exp\left(\frac{2\sqrt{V_0}}{\sigma^2}(b - \rho\sigma)\exp\sigma x - \frac{2a}{\sigma}x\right)$$
so that if $b > \rho\sigma$, $a(\infty) = \infty$. A straightforward computation shows that this also holds when $b = \rho\sigma$.

The other cases require a little more work. So let's assume $\alpha \leq 1$ and $\rho > 0$. Then $p'(x)$ rewrites
$$\exp A \exp \alpha x - B \exp x - Dx$$
for positive A, B and D has the sign of a.

As a result, $a(z) = \int_0^z \int_0^x \exp\left(A(\exp\alpha x - \exp\alpha y) - B(\exp x - \exp y) - D(x-y)\right)dxdy$. It follows that $\frac{\partial a}{\partial\alpha}(z) = \int_0^z \int_0^x A(x\exp\alpha x - y\exp\alpha y)\exp\left(A(\exp\alpha x - \exp\alpha y) - B(\exp x - \exp y) - D(x-y)\right)dxdy > 0$, so that if we show that $a(\infty) < \infty$ for $\alpha = 1$, it will also hold for $\alpha < 1$. By the last bullet above this can only happen if $b < \rho\sigma$, that is $A < B$.
With $C = B - A$ we are led to consider the integral
$$\int_0^z \exp\left(-Ce^x - Dx\right)\int_0^x \exp\left(Ce^y + Dy\right)dydx$$
setting $u = Ce^x, v = Ce^y$ we get a constant times $\int_C^{Ce^z} u^{-D-1}e^{-u}\int_C^u v^{D-1}e^v dvdu$. By Fubini's theorem this is equal to $\int_C^{Ce^z} v^{D-1}e^v \int_v^{Ce^z} u^{-D-1}e^{-u}dudv$. This in turn is less than $\int_C^{Ce^z} v^{D-1}e^v \int_v^{\infty} u^{-D-1}e^{-u}dudv = \int_C^{Ce^z} v^{D-1}e^v\Gamma(-D, v)$ where Γ is the upper incomplete Gamma function. Now $\Gamma(-D, v) \sim_{v\to\infty} v^{-D-1}e^{-v}$, so that the integrand behaves like v^{-2} at infinity and the integral is finite.

All in all, the sole remaining case is $\alpha < 1, b \geq \rho\sigma > 0$. Let us operate the change of variable $u = 2\frac{\rho\sqrt{V_0}}{\sigma}\exp\sigma x$, we are led to the integral:
$$\int_A^Z \exp\left(-u + cu^\alpha\right)u^{-\frac{2a}{\sigma^2}-1}\int_A^u \exp\left(v - cv^\alpha\right)v^{\frac{2a}{\sigma^2}-1}dvdu$$
with a positive c (by hypothesis $\alpha < 1$). We proceed as above: by Fubini's theorem and letting the inner integral go to infinity, this is less than:
$$\int_A^Z \exp\left(v - cv^\alpha\right)v^{\frac{2a}{\sigma^2}-1}\int_v^\infty \exp\left(-u + cu^\alpha\right)u^{-\frac{2a}{\sigma^2}-1}dudv.$$

We claim that $I = \int_v^\infty \exp\left(-u + cu^\alpha\right)u^{-\frac{2a}{\sigma^2}-1}du \sim \exp\left(-v + cv^\alpha\right)v^{-\frac{2a}{\sigma^2}-1}$, and we can conclude as in the case of the incomplete Gamma function above that $a(\infty) < \infty$. Indeed, by first scaling through the change of variable $u = vz$, $I = \int_1^\infty \exp\left(-vz + cv^\alpha z^\alpha\right)v^{-\frac{2a}{\sigma^2}}z^{-\frac{2a}{\sigma^2}-1}dz$. Hence
$$I = v^{-\frac{2a}{\sigma^2}}\exp\left(-v + cv^\alpha\right)\int_1^\infty \exp\left(-v(z-1) + cv^\alpha(z^\alpha - 1)\right)z^{-\frac{2a}{\sigma^2}-1}dz.$$
Setting $z - 1 = t$ and $r = vt$ we get:
$$I = v^{-\frac{2a}{\sigma^2}-1}\exp\left(-v + cv^\alpha\right)\int_0^\infty \exp\left(-r + cv^\alpha\left(\left(1 + \frac{r}{v}\right)^\alpha - 1\right)\right)\left(1 + \frac{r}{v}\right)^{\frac{2a}{\sigma^2}-1}dr.$$

As $v \to \infty$ the integrand goes pointwise to $\exp(-r)$. Since $\int_0^\infty \exp(-r)dr = 1$, the last point to check is that we can apply the dominated convergence theorem. This is indeed the case since, on one hand, one always has $(1+\frac{r}{v})^{\frac{2a}{\sigma^2}-1} < (1+r)^{\max\left(\frac{2a}{\sigma^2}-1,0\right)}$ for $v > 1$ and on another hand by concavity $cv^\alpha((1+\frac{r}{v})^\alpha - 1) < cv^{\alpha-1}\alpha r$ with $\alpha < 1$, so that for v large enough $cv^{\alpha-1}\alpha < 1-\epsilon$ with $\epsilon > 0$ and the integrand is less than $e^{-\epsilon r}(1+r)^{\max\left(\frac{2a}{\sigma^2}-1,0\right)}$.

We have therefore proven:

Proposition 6 *f is a martingale if and only $\alpha \geq 2$, or $\alpha < 2$ and either:*

- $\rho \leq 0$
- $\alpha > 1$
- $\alpha = 1$ *and* $b \geq \rho\sigma$

2.2.2 Inversion

Since f is a true martingale, we can look at the dynamic of $\frac{1}{f}$ under the change of measure induced by the martingale $\frac{f_T}{f_0}$. By Ito's formula,

$$d\frac{1}{f_t} = -\frac{1}{f_t^2}df_t + \frac{1}{f_t^3}d < f >_t$$

Now $df_t = \sqrt{V_t}f_t dw_{1,t}$ and $d < f >_t = V_t f_t^2 dt$ so that with $g_t = \frac{1}{f_t}$,

$$dg_t = -\sqrt{V_t}g_t dw_{1,t} + V_t g_t dt$$

Under the probability $Q = \frac{f_T}{f_0}P$, $\tilde{w}_{1,t} = w_{1,t} - \int_0^t \sqrt{V_s}ds$ is a martingale, and even a Brownian motion by Lévy characterization theorem. So:

$$dg_t = \sqrt{V_t}g_t(-dw_{1,t} + \sqrt{V_t}dt) = -\sqrt{V_t}g_t d\tilde{w}_{1,t}$$

.

What happens to the variance SDE? Under Q, $\tilde{w}_{2,t} = w_{2,t} - \rho\int_0^t \sqrt{V_s}ds$ is a Brownian motion, so that

$$dv_t = (a - be^{\alpha v_t})dt + \sigma(dw_{2,t} - \rho\sqrt{V_t}dt) + \sigma\rho\sqrt{V_t}dt = (a - be^{\alpha v_t})dt + \sigma d\tilde{w}_{2,t} + \sigma\rho e^{v_t}dt$$

so it will belong to the same family if and only if $\rho = 0$, in which case the inverted model is the initial one, or $\alpha = 1$, in which case the mean reversion parameter of the inverted model is given by $b - \rho\sigma$. In particular, if $b = \rho\sigma$, $v_t = a + \sigma\tilde{w}_{2,t}$ under Q.

3 The Hypergeometric Model for $\alpha = 1$ and its Morse Potential Representation

We further specify the dynamic of the volatility by taking $\alpha = 1$ so that we are able to compute the Mellin transform of the forward price which is the essential ingredient to price vanilla options. In that case, the dynamic is given by:

$$df_t = f_t e^{v_t} dw_{1,t} \quad (6)$$
$$dv_t = (a - be^{v_t})dt + \sigma dw_{2,t} \quad (7)$$

with as in the previous case $dw_{1,t}.dw_{2,t} = \rho dt$.

3.1 Volatility analysis

We want to compute $\mathbb{E}[e^{\theta v_t}]$. Define a probability Q under which $\tilde{w}_{2,t} = w_{2,t} + \int_0^t \frac{a - be^{v_s}}{\sigma} ds$ is a Brownian motion, we deduce after replacing $\sigma \int_0^t e^{v_u} d\tilde{w}_{2,u} = e^{v_t} - e^{v_0} - \frac{\sigma^2}{2}\int_0^t e^{v_u} du$ that

$$\mathbb{E}[e^{\theta v_t}] = e^{-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}e^{v_0}} e^{-\frac{a^2 t}{2\sigma^2}} \mathbb{E}^Q \left[\exp\left(\left(\theta + \frac{a}{\sigma^2}\right) v_t - \frac{b}{\sigma^2} e^{v_t} \right) \exp\left(\beta_1 \int_0^t e^{v_u} du - \frac{\beta_2^2}{2} \int_0^t e^{2v_u} du \right) \right]$$

with $\beta_1 = \frac{ab}{\sigma^2} + \frac{b}{2}$, $\beta_2^2 = \frac{b^2}{\sigma^2}$ and $dv_t = \sigma d\tilde{w}_{2,t}$. Denote by $F(t,v)$ the expectation then it solves, thanks to Feynman-Kac theorem, the partial differential equation

$$\begin{aligned} \partial_t F &= \frac{\sigma^2}{2}\frac{d^2F}{dv^2} - \frac{\beta_2^2}{2} e^{2v} F + \beta_1 e^v F \\ F(0,v) &= e^{\left(\theta + \frac{a}{\sigma^2}\right)v - \frac{b}{\sigma^2}e^v}. \end{aligned}$$

Denote by $g(\sigma^2 t, v) = F(t,v)$ then is solves the partial differential equation

$$\begin{aligned} \partial_t g &= -Hg \\ g(0,v) &= e^{\left(\theta + \frac{a}{\sigma^2}\right)v - \frac{b}{\sigma^2}e^v} \end{aligned}$$

with $H = -\frac{1}{2}\frac{d^2}{dv^2} + \frac{\nu_2^2}{2} e^{2v} - \nu_1 e^v$ with $\nu_1 = \frac{\beta_1}{\sigma^2}$ and $\nu_2^2 = \frac{\beta_2^2}{\sigma^2}$. The operator H involves a Morse potential, see Grosche (1988), page 228 in Grosche and Steiner (1998), Ikeda and Matsumoto (1999) and the surveys Matsumoto and Yor (2005a) and Matsumoto and Yor (2005b).

We denote by $q(t,v,y)$ the heat kernel associated with e^{-tH} then we have

$$F(t, v_0) = \int_{-\infty}^{+\infty} q(\sigma^2 t, v_0, y) F(0,y) dy.$$

The Green function associated we the Laplace transform of the heat kernel is given

$$G(v, y; s^2/2) = \int_0^{+\infty} e^{-\frac{s^2}{2}t} q(t,v,y) dt. \tag{8}$$

Taking the Laplace transform of $\mathbb{E}[e^{\theta v_t}]$ we deduce

$$\begin{aligned} \int_0^{+\infty} e^{-\frac{s^2}{2}t} \mathbb{E}\left[e^{\theta v_t}\right] dt &= e^{-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}e^{v_0}} \int_0^{+\infty} e^{-\left(\frac{a^2}{\sigma^2} + s^2\right)t/2} \int_{-\infty}^{+\infty} q(\sigma^2 t, v_0, y) F(0,y) dy dt \qquad (9) \\ &= \frac{1}{\sigma^2} e^{-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}e^{v_0}} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-\frac{\eta^2}{2}t} q(t, v_0, y) dt F(0,y) dy \\ &= \frac{1}{\sigma^2} e^{-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}e^{v_0}} \int_{-\infty}^{+\infty} G(v_0, y; \eta^2/2) F(0,y) dy \end{aligned}$$

with $\eta^2 = \frac{a^2}{\sigma^4} + \frac{s^2}{\sigma^2}$. We know from Matsumoto and Yor (2005a) pages 341-342 or Matsumoto and Yor (2005b) page 360 that

$$G(v, y; \eta^2/2) = \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1 + 2\eta)} e^{-(v+y)/2} W_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y_>}\right) M_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y_<}\right)$$

with $y_> = \max(v,y)$ and $y_< = \min(v,y)$ while $W_{\kappa,\eta}$ and $M_{\kappa,\eta}$ are the Whittaker functions related to the confluent hypergeometric functions by the relations

$$W_{\kappa,\eta}(z) = z^{\eta+\frac{1}{2}}e^{-z/2}\Psi\left(\eta-\kappa+\frac{1}{2},1+2\eta;z\right)$$
$$M_{\kappa,\eta}(z) = z^{\eta+\frac{1}{2}}e^{-z/2}\Phi\left(\eta-\kappa+\frac{1}{2},1+2\eta;z\right).$$

It is known that the heat kernel is given by:

$$q(t,v,y) = \int_0^{+\infty}\frac{e^{2\frac{\nu_1}{\nu_2}u}}{2\sinh(u)}e^{-\frac{\nu_1}{\nu_2}(e^v+e^y)\coth(u)}\theta\left(2\frac{\nu_1}{\nu_2}e^{(v+y)/2}/\sinh(u),t\right)du, \tag{10}$$
$$\theta(r,t) = \frac{r}{(2\pi^3t)^{\frac{1}{2}}}e^{\pi^2/(2t)}\int_0^{+\infty}e^{-u^2/(2t)}e^{-r\cosh(u)}\sinh(u)\sin\left(\frac{u\pi}{t}\right)du. \tag{11}$$

We wish to compute:

$$\int_{-\infty}^{+\infty}G(v_0,y;\eta^2/2)F(0,y)dy = \int_{-\infty}^{v_0}G(v_0,y;\eta^2/2)F(0,y)dy+\int_{v_0}^{+\infty}G(v_0,y;\eta^2/2)F(0,y)dy$$
$$= J_1+J_2. \tag{12}$$

We have:

$$J_1 = \frac{\Gamma\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)}{\nu_2\Gamma(1+2\eta)}e^{-v_0/2}W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{v_0}\right)\int_{-\infty}^{v_0}e^{-y/2}M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{y}\right)F(0,y)dy$$
$$= \frac{\Gamma\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)}{\nu_2\Gamma(1+2\eta)}e^{-v_0/2}W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{v_0}\right)(2\nu_2)^{\frac{1}{2}-n-\frac{a}{\sigma^2}}\int_0^{z_0}z^{\eta-1+\theta+\frac{a}{\sigma^2}}e^{-z}\Phi\left(\eta-\frac{a}{\sigma^2},1+2\eta;z\right)dz$$

where $z_0 = 2\nu_2e^{v_0}$, $\frac{\nu_1}{\nu_2} = \frac{a}{\sigma^2}+\frac{1}{2}$ and we used the representation for the Whittaker function $M_{\kappa,\eta}(z)$. Similarly, the representation for the Whittaker function $W_{\kappa,\eta}(z)$ leads to:

$$J_2 = \frac{\Gamma\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)}{\nu_2\Gamma(1+2\eta)}e^{-v_0/2}M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{v_0}\right)\int_{v_0}^{+\infty}e^{-y/2}W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{y}\right)F(0,y)dy$$
$$= \frac{\Gamma\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)}{\nu_2\Gamma(1+2\eta)}e^{-v_0/2}M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2e^{v_0}\right)(2\nu_2)^{\frac{1}{2}-n-\frac{a}{\sigma^2}}\int_{z_0}^{+\infty}z^{\eta-1+\theta+\frac{a}{\sigma^2}}e^{-z}\Psi\left(\eta-\frac{a}{\sigma^2},1+2\eta;z\right)dz.$$

To connect these results to the previous ones we just need:

Remark 7 *If we denote $a_1-1=\eta-\frac{a}{\sigma^2}$ and $b_1=1+2\eta$ then the two integrals above (i.e. involved in J_1 and J_2) can be rewritten as:*

$$\int_{z_0}^{+\infty}z^{b_1-a_1+\theta-1}e^{-z}\Psi\left(a_1-1,b_1;z\right)dz,$$
$$\int_0^{z_0}z^{b_1-a_1+\theta-1}e^{-z}\Phi\left(a_1-1,b_1;z\right)dz$$

which can be computed thanks to the expressions obtained for I_1 and I_2.

3.1.1 The variance swaps revisited

The variance swap is given by:

$$t\mathrm{VS}(t) = \int_0^t\mathbb{E}\left[e^{2v_u}\right]du$$

and its Laplace transform is:

$$\int_0^{+\infty} e^{-s^2t/2} t\text{VS}(t)dt = \frac{2}{s^2}\int_0^{+\infty} e^{-s^2t/2}\mathbb{E}\left[e^{2v_t}\right]dt. \tag{13}$$

The equation (13) is the left hand side of equation (9) and leads to the integrals J_1 and J_2 given above and thanks to Remark 7 the series representations for I_1 and I_2 enable the efficient computation of the variance swap.

Remark 8 *To check that*

$$\int_0^{+\infty} e^{-\frac{s^2}{2}t}\mathbb{E}\left[e^{\theta v_t}\right]dt < +\infty$$

we need to verify that:

$$\int_{v_0}^{+\infty} e^{-y/2} W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 e^y\right)\exp\left\{\left(\theta+\frac{a}{\sigma^2}\right)y - \frac{b}{\sigma^2}e^y\right\}dy,$$
$$\int_{-\infty}^{v_0} e^{-y/2} M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 e^y\right)\exp\left\{\left(\theta+\frac{a}{\sigma^2}\right)y - \frac{b}{\sigma^2}e^y\right\}dy$$

are finite. As the Whittaker $W_{\kappa,\eta}$ function is related to the confluent hypergeometric function Ψ and using relation 6.2.2 of Beals and Wong (2010) which is:

$$\Psi(\alpha,\beta;z) \sim z^{-\alpha} \quad \textit{if} \quad \Re(z)\to+\infty \quad \Re(\alpha)>0.$$

we conclude that because $\Re(\nu_2)>0$ and $\Re\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)>0$ for s large enough (η depends on s) so the integrand of the first integral behaves like:

$$e^{y\left(\frac{2a}{\sigma^2}+\theta\right)}\exp\left\{\frac{-2b}{\sigma^2}e^y\right\} \quad \textit{as} \quad y\to+\infty$$

and therefore the integral will be finite for all values of θ (b is positive). For the second integral replacing the Whittaker function $M_{\kappa,\eta}$ by its expression and using the property 13.2.13 of DLMF (2010), that is $\Phi(\alpha,\beta;z)\sim 1$ for $z\sim 0$, we deduce that the integrand behaves like:

$$e^{y\left(\eta+\theta+\frac{a}{\sigma^2}\right)} \quad \textit{as} \quad y\to-\infty$$

for all values of θ there exists a value of s such that $\eta+\theta+\frac{a}{\sigma^2}>0$ so the integral is finite. Notice also that to the extent that $\Re(\nu_2)>0$ we can have a potential with complex coefficients and the integrals remain finite.

3.2 The Mellin transform of the spot

In order to perform the pricing of vanilla options we need to compute the Mellin transform of the spot. We have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^\lambda\right] &= \mathbb{E}\left[\exp\left(-\frac{\lambda}{2}\int_0^t e^{2v_u}du + \lambda\int_0^t e^{2v_u}dw_{1,u}\right)\right]\\
&= \mathbb{E}\left[\exp\left(-\frac{\lambda}{2}\int_0^t e^{2v_u}du + \lambda\rho\int_0^t e^{2v_u}dw_{2,u} + \lambda\sqrt{1-\rho^2}\int_0^t e^{2v_u}dw_{2,u}^{\perp}\right)\right]\\
&= \mathbb{E}\left[\exp\left(\left(-\frac{\lambda}{2}+\frac{\lambda^2(1-\rho^2)}{2}\right)\int_0^t e^{2v_u}du + \lambda\rho\int_0^t e^{2v_u}dw_{2,u}\right)\right]
\end{aligned}$$

where we used the standard Brownian motion $(w_{2,t}, w_{2,t}^{\perp})$. Furthermore, the relation:

$$\sigma\int_0^t e^{v_u}dw_{2,u} = e^{v_t} - e^{v_0} - \int_0^t e^{v_u}(a-be^{v_u})du - \frac{\sigma^2}{2}\int_0^t e^{v_u}du \tag{14}$$

leads to:

$$\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^{\lambda}\right] = e^{-\frac{\lambda\rho}{\sigma}e^{v_0}}\mathbb{E}\left[\exp\left(\alpha_0 e^{v_t} + \alpha_1\int_0^t e^{v_s}ds - \frac{\alpha_2^2}{2}\int_0^t e^{2v_s}ds\right)\right]$$

with

$$\begin{aligned}
\alpha_0 &= \frac{\lambda\rho}{\sigma},\\
\alpha_1 &= -\frac{\lambda\rho}{\sigma}\left(a + \frac{\sigma^2}{2}\right),\\
\alpha_2^2 &= -\lambda^2(1-\rho^2) - \frac{2b\rho\lambda}{\sigma} + \lambda.
\end{aligned}$$

Using Girsanov's theorem we deduce that:

$$\begin{aligned}
J &= \mathbb{E}\left[\exp\left(\alpha_0 e^{v_t} + \alpha_1\int_0^t e^{v_s}ds - \frac{\alpha_2^2}{2}\int_0^t e^{2v_s}ds\right)\right]\\
&= \mathbb{E}^Q\left[\exp\left(\alpha_0 e^{v_t} + \alpha_1\int_0^t e^{v_s}ds - \frac{\alpha_2^2}{2}\int_0^t e^{2v_s}ds\right)\exp\left(\int_0^t \frac{a-be^{v_u}}{\sigma}d\tilde{w}_s - \frac{1}{2}\int_0^t \frac{(a-be^{v_u})^2}{\sigma^2}ds\right)\right]
\end{aligned}$$

with $dv_t = \sigma d\tilde{w}_t$ and $\tilde{w}_t = w_{2,t} + \int_0^t \frac{a-be^{v_u}}{\sigma}du$ a Brownian motion under Q. Using again the equality (14) (with the convenient parameters) we deduce that:

$$\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^{\lambda}\right] = e^{-\frac{a}{\sigma^2}v_0 + (\frac{b}{\sigma^2} - \frac{\lambda\rho}{\sigma})e^{v_0}}e^{-\frac{a^2t}{2\sigma^2}}\mathbb{E}^Q\left[\exp\left(\frac{av_t}{\sigma^2} + \beta_0 e^{v_t} + \beta_1\int_0^t e^{v_s}ds - \frac{\beta_2^2}{2}\int_0^t e^{2v_s}ds\right)\right] \quad (15)$$

with

$$\begin{aligned}
\beta_0 &= \alpha_0 - \frac{b}{\sigma^2} = \frac{\lambda\rho\sigma - b}{\sigma^2},\\
\beta_1 &= \alpha_1 + b\left(\frac{a}{\sigma^2} + \frac{b}{2}\right) = (b - \lambda\rho\sigma)\left(\frac{a}{\sigma^2} + \frac{1}{2}\right),\\
\beta_2^2 &= \alpha_2^2 + \frac{b^2}{\sigma^2} = -\lambda^2(1-\rho^2) + \lambda\left(1 - \frac{2b\rho}{\sigma}\right) + \frac{b^2}{\sigma^2}.
\end{aligned}$$

As above, introduce:

$$\nu_1 = \frac{\beta_1}{\sigma^2}, \quad \nu_2^2 = \frac{\beta_2^2}{\sigma^2}$$

and $F(0,v) = \exp\left(\frac{av}{\sigma^2} + \beta_0 e^v\right)$. We denote be $F(t,v)$ the expectation in (15) then thanks to Feynman-Kac's formula it solves the partial differential equation:

$$\begin{aligned}
\partial_t F &= \frac{\sigma^2}{2}\frac{d^2F}{dv^2} - \frac{\beta_2^2}{2}e^{2v}F + \beta_1 e^v F,\\
F(0,v) &= e^{\frac{av}{\sigma^2} + \beta_0 e^v}.
\end{aligned}$$

Proceeding as above we obtain:

$$F(t,v_0) = \int_{-\infty}^{+\infty} q(\sigma^2 t, v_0, y)F(0,y)dy$$

which requires the kernel q, known from (10), but is hard to exploit. We can also use the Green function given by (8) as follows, we compute the Laplace transform:

$$\int_0^{+\infty} e^{-\frac{s^2}{2}t} e^{-\frac{a^2 t}{2\sigma^2}} F(t, v_0) dt \quad = \quad \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} G(v_0, y; \eta^2/2) F(0, y) dy \tag{16}$$

with $\eta^2 = \frac{a^2}{\sigma^4} + \frac{s^2}{\sigma^2}$ and proceed as in the previous example and write the integral appearing in the r.h.s of (16) as in (12) (as a sum of two integrals denoted J_1 and J_2 given below). Taking into account the particular form of $F(0, v)$ we are led to the computation of:

$$J_1 \quad = \quad \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1 + 2\eta)} e^{-v_0/2} W_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{v_0}\right) \int_{-\infty}^{v_0} e^{-y/2} M_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y}\right) F(0, y) dy, \tag{17}$$

$$J_2 \quad = \quad \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1 + 2\eta)} e^{-v_0/2} M_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{v_0}\right) \int_{v_0}^{+\infty} e^{-y/2} W_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y}\right) F(0, y) dy \tag{18}$$

where $z_0 = 2\nu_2 e^{v_0}$. Using the representation for the Whittaker functions $W_{\kappa,\eta}(z)$ and $M_{\kappa,\eta}(z)$ the two integrals above can be transform into

$$\int_{-\infty}^{v_0} e^{-y/2} M_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y}\right) F(0, y) dy = (2\nu_2)^{\frac{1}{2} - \frac{a}{\sigma^2}} I_1$$

and

$$\int_{v_0}^{\infty} e^{-y/2} W_{\frac{\nu_1}{\nu_2}, \eta}\left(2\nu_2 e^{y}\right) F(0, y) dy = (2\nu_2)^{\frac{1}{2} - \frac{a}{\sigma^2}} I_2$$

with $z_0 = 2\nu_2 e^{v_0}$,

$$I_1 = \int_0^{z_0} z^{\eta - 1 + \frac{a}{\sigma^2}} e^{(-\frac{1}{2} + \frac{\beta_0}{2\nu_2})z} \Phi\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) dz \tag{19}$$

and

$$I_2 = \int_{z_0}^{\infty} z^{\eta - 1 + \frac{a}{\sigma^2}} e^{(-\frac{1}{2} + \frac{\beta_0}{2\nu_2})z} \Psi\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) dz. \tag{20}$$

The behaviour of these integrals will be driven by the quantity $-\frac{1}{2} + \frac{\beta_0}{2\nu_2}$, so let us investigate it:

Lemma 9 *Let $\delta(\lambda) = -\frac{1}{2} + \frac{\beta_0(\lambda)}{2\nu_2(\lambda)}$. Denote by λ_-, λ_+ the roots of the polynomial $(\lambda\rho\sigma - b)^2 + \sigma^2\lambda(1-\lambda)$. Then:*

- *$\delta(\lambda)$ is defined for $\lambda \in]\lambda_-, \lambda_+[$, with $\lambda_- < 0 < 1 < \lambda_+$*
- *$\delta(0) = -1$*
- *$\delta(1) = 0$ if $b < \rho\sigma$, $\delta(1) = -1$ if $b > \rho\sigma$*
- *$\delta(\lambda) < 0$ for $\lambda \in]0, 1[$*
- *When $\rho < 0$, $\delta(\lambda) < 0$ for $\lambda \in]\lambda_-, \lambda_+[$*

Proof. Observe that

$$\sigma^2 \beta_2^2 = (\lambda\rho\sigma - b)^2 + \sigma^2\lambda(1 - \lambda)$$

so that with $\nu_2 = \frac{\beta_2}{\sigma}$

$$\frac{\beta_0}{2\nu_2} = \frac{1}{2} \frac{(\lambda\rho\sigma - b)}{\sqrt{(\lambda\rho\sigma - b)^2 + \sigma^2\lambda(1 - \lambda)}}.$$

In particular, $\frac{\beta_0}{2\nu_2}(\lambda = 0) = -\frac{1}{2}$ and $\frac{\beta_0}{2\nu_2}(\lambda = 1) = sgn(\rho\sigma - b)\frac{1}{2}$. Note that $-\frac{1}{2} + \frac{\beta_0}{2\nu_2} = \frac{1}{2}(\frac{(\lambda\rho\sigma - b)}{\sqrt{(\lambda\rho\sigma - b)^2 + \sigma^2\lambda(1-\lambda)}} - 1) < 0$ as soon as $\lambda \in [0, 1[$ or $\lambda = 1$ and $b > \rho\sigma$. If $\rho < 0$ then $\frac{\beta_0}{2\nu_2}(\lambda = 1) = -\frac{1}{2}$ and the maximum m of $\lambda \to \frac{\beta_0}{2\nu_2}(\lambda)$ is attained between 0 and 1 with $-\frac{1}{2} < m < 0$. Moreover β_2 is well defined as long

as $\lambda_- \le \lambda \le \lambda_+$ with $\lambda_- < 0 < 1 < \lambda_+$. The last constraint to check is $\beta_0 < 0$. Assuming $\rho < 0$, this amounts to $\lambda > \frac{b}{\rho\sigma}$ which is negative and even smaller than λ_- since it cancels the first squared monomial in the expression of β_2^2. It follows that all the range λ_-, λ_+ is allowed, with the exponent $\frac{\beta_0}{2\nu_2}$ living between $-\frac{1}{2}$ and its maximum m for $\lambda \in [0,1]$ and decreasing to $-\infty$ close to the λ_- or λ_+.
■

Computation of I_1: Because $\Phi\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1+2\eta; 0\right) = 1$, I_1 is well defined if and only if $\eta + \frac{a}{\sigma^2} > 0$, which is always true since $\eta = \sqrt{\frac{a^2}{\sigma^4} + \frac{s^2}{\sigma^2}}$. By Fubini's theorem, $I_1 = \sum_{n=0}^{\infty} \frac{(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2})_n}{(1+2\eta)_n n!} \int_0^{z_0} z^{\eta - 1 + \frac{a}{\sigma^2} + n} e^{\delta(\lambda) z} dz$. Let $i_n = \int_0^{z_0} z^{\eta-1+\frac{a}{\sigma^2}+n} e^{\delta(\lambda)z} dz$.

When $\delta(\lambda) < 0$: Then $i_n = (-\delta(\lambda))^{-\eta - \frac{a}{\sigma^2} - n} \gamma\left(\eta + \frac{a}{\sigma^2} + n, -\delta(\lambda) z_0\right)$ with γ the lower incomplete Gamma function. Then by integration by parts we have

$$\delta(\lambda) i_{n+1} = z_0^{\eta + \frac{a}{\sigma^2} + n} e^{\delta(\lambda) z_0} - \left(\eta + \frac{a}{\sigma^2} + n\right) i_n$$

So there is a straightforward recurrence to compute the term of the series of I_1.

When $\delta(\lambda) = 0$: Then $i_n = \frac{z_0^{\eta + \frac{a}{\sigma^2} + n}}{\eta + \frac{a}{\sigma^2} + n}$.

Computation of I_2: Let us investigate first the key coefficients in I_2.
Note that

$$\frac{\nu_1}{\nu_2} = -\left(\frac{a}{\sigma^2} + \frac{1}{2}\right) 2 \frac{\beta_0}{2\nu_2}$$

We have:

Lemma 10 *$I_2(\lambda = 1)$ is finite*

Proof. We know that $\Psi\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1+2\eta; z\right) \sim z^{\frac{\nu_1}{\nu_2} - \eta - \frac{1}{2}}$, so the integrand behaves like

$$z^{\frac{\nu_1}{\nu_2} + \frac{a}{\sigma^2} - \frac{3}{2}} e^{\frac{(-1 + sgn(\rho\sigma - b))z}{2}}$$

at infinity. Therefore I_2 will be finite if $b \ge \rho\sigma$. If $b < \rho\sigma$ then it will be finite if and only if $\frac{\nu_1}{\nu_2} + \frac{a}{\sigma^2} < \frac{1}{2}$ which is true since $\frac{\nu_1}{\nu_2} = -\left(\frac{a}{\sigma^2} + \frac{1}{2}\right)$. ■

When $\delta(\lambda) < -1$ (in particular, when $\lambda \in]1, \lambda_+[$): We have

$$z^{\eta - 1 + \frac{a}{\sigma^2}} e^{\delta(\lambda) z} \Psi\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \frac{\Gamma(bb - 1 + t)\Gamma(t)}{\Gamma(aa - 1 + t)} z^{-\eta - 1 + \frac{a}{\sigma^2} - t} e^{(\delta(\lambda)+1)z} dt$$

with $aa = \eta - \frac{\nu_1}{\nu_2} + \frac{3}{2}$ and $bb = 1 + 2\eta$.

Moreover, we know that the integral converges locally uniformly in z, so that we can apply Fubini's theorem and obtain

$$I_2 = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \frac{\Gamma(bb-1+t)\Gamma(t)}{\Gamma(aa-1+t)} \int_{z_0}^{\infty} z^{-\eta - 1 + \frac{a}{\sigma^2} - t} e^{(\delta(\lambda)+1)z} dz dt$$

where the inner integral is finite since $\delta(\lambda) + 1 < 0$ by assumption.
Now $j(t) = \int_{z_0}^{\infty} z^{-\eta-1+\frac{a}{\sigma^2}-t} e^{(\frac{1}{2} + \frac{\beta_0}{2\nu_2})z} dz = \|\frac{1}{2} + \frac{\beta_0}{2\nu_2}\|^{\eta - \frac{a}{\sigma^2} + t} \Gamma(-\eta + \frac{a}{\sigma^2} - t, \frac{z_0}{2}\|1 + \frac{\beta_0}{\nu_2}\|)$ where $\Gamma(,)$ denotes the upper incomplete Gamma function. Since $z_0 \neq 0$ we know (DLMF (2010) 8.2, (ii)) that $\Gamma(a, z)$ is an entire function of a, and will not contribute to the poles of the integrand.

An explicit hypergeometric series for I2: We apply the theorem of residues to get a series from the above complex integral. The poles of the integrand are located:

- at $t = -n, n \in N$, with residue $\frac{\Gamma(bb-1-n)}{\Gamma(aa-1-n)n!}(-1)^n j(-n)$
- at $t = -n+1-bb, n \in N$, with residue $\frac{\Gamma(-n+1-bb)}{\Gamma(aa-n-bb)n!}(-1)^n j(-n+1-bb)$

so that by Cauchy's residue theorem we get:

$$I_2 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!}\left(\frac{\Gamma(bb-1-n)j(-n)}{\Gamma(aa-1-n)} + \frac{\Gamma(1-bb-n)j(-n+1-bb)}{\Gamma(aa-bb-n)}\right)$$

I_2 can be computed easily by writing down the explicit recurrences between successive terms of the two series. Calls to the Γ and $\Gamma(,)$ functions are only required for the constant and index zero terms. We have the following recurrence relations for a generic $j^*(t) = \int_{z_0}^{\infty} z^{\alpha-t}e^{-\beta z}dz$:

$$j^*(-(n+1)) = \frac{e^{-\beta z_0} z_0^{\alpha+n+1}}{\beta} + \frac{\alpha+n+1}{\beta} j^*(-n)$$

Complete expression of the double transform

$$\int_0^{\infty} e^{-\frac{s^2}{2}t}\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^{\lambda}\right] dt = \frac{1}{\sigma^2} e^{-\frac{a}{\sigma^2}v_0 + (\frac{b}{\sigma^2} - \frac{\lambda\rho}{\sigma})e^{v_0}} \int_{-\infty}^{\infty} G(v_0, y, \frac{\eta^2}{2})F(0,y)dy$$

where $F(0,y) = e^{\frac{a}{\sigma^2}y + \beta_0 e^y}$, $\eta^2 = \frac{a^2}{\sigma^4} + \frac{s^2}{\sigma^2}$ and

$$G(v,y,\frac{\eta^2}{2}) = \frac{2^{2\eta+1}\nu_2^{2\eta}\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\Gamma(1+2\eta)} e^{\eta(v+y)} e^{-\nu_2(e^v+e^y)}[1(y>v)\Psi(;2\nu_2 e^y)\Phi(;2\nu_2 e^v) + 1(y<v)\Psi(;2\nu_2 e^v)\Phi(;2\nu_2 e^y)]$$

where the 1st and 2nd arguments of the Kummer functions are $\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1+2\eta$. This leads to

$$\int_0^{\infty} e^{-\frac{s^2}{2}t}\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^{\lambda}\right] dt = \frac{2^{\eta+1-\frac{a}{\sigma^2}}\nu_2^{2\eta-\frac{a}{\sigma^2}}\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\sigma^2\Gamma(1+2\eta)} e^{(\eta-\frac{a}{\sigma^2})v_0} e^{(\frac{b}{\sigma^2} - \frac{\lambda\rho}{\sigma} - \nu_2)e^{v_0}} [\Phi(;z_0)I_2 + \Psi(;z_0)I_1]. \tag{21}$$

3.2.1 A purely analytical approach to the martingale property

Since f is a positive super martingale, $\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)\right] \leq 1$ and it follows that f is a martingale if and only if for some $s > 0$

$$\int_0^{\infty} e^{-\frac{s^2}{2}t}\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^{\lambda}\right] dt = \frac{2}{s^2}$$

with λ set to 1. Therefore there is some hope to re-find the results of the "full-blown" section (in case $\alpha = 1$) in the previous calculations when $\lambda = 1$. Note also that when $\lambda = 0$ this should hold irrespective of the model parameters.

We start by the following:

Lemma 11 *If $\lambda = 0$, then:*

- $\frac{\nu_1}{\nu_2} = \frac{a}{\sigma^2} + \frac{1}{2}$
- $\frac{b}{\sigma^2} - \frac{\lambda\rho}{\sigma} - \nu_2 = 0$

This also holds when $\lambda = 1$ if and only if $b \geq \rho\sigma$.

Observe now that:

Lemma 12 *Under the conditions of Lemma 11,*

- $I_1 = \frac{z_0^{\eta+\frac{a}{\sigma^2}}}{\eta+\frac{a}{\sigma^2}} e^{-z_0}\Phi(\eta - \frac{a}{\sigma^2}+1, 1+2\eta; z_0)$
- $I_2 = z_0^{\eta+\frac{a}{\sigma^2}} e^{-z_0}\Psi(\eta - \frac{a}{\sigma^2}+1, 1+2\eta; z_0)$

Proof. Combining lemmas 9 and 11 we get that I_1 writes:

$$I_1 = \int_0^{z_0} z^{\eta-1+\frac{a}{\sigma^2}} e^{-z}\Phi\left(\eta - \frac{a}{\sigma^2}, 1+2\eta; z\right) dz.$$

With $aa = \eta - \frac{a}{\sigma^2}+1, bb = 1+2\eta$ the integrand is $z^{bb-aa-1}e^{-z}\Phi\left(aa-1, bb; z\right)$ and by (DLMF (2010), 13.3.19) this is also

$$\frac{1}{bb-aa}\frac{d}{dz}z^{bb-aa}e^{-z}\Phi\left(aa, bb; z\right).$$

Since $bb > aa$ and $\Phi\left(aa, bb; 0\right) = 1$ the first assertion follows. I_2 rewrites

$$I_2 = \int_{z_0}^{\infty} z^{\eta-1+\frac{a}{\sigma^2}} e^{-z}\Psi\left(\eta - \frac{a}{\sigma^2}, 1+2\eta; z\right) dz,$$

and by (DLMF (2010), 13.3.26) this is also

$$-\frac{d}{dz}z^{bb-aa}e^{-z}\Psi\left(aa, bb; z\right).$$

Since $\Psi\left(aa, bb; z\right) \sim z^{-aa}$ as $z \to \infty$ the result follows. ■
To conclude, let us first assemble the pieces together:

In case $\lambda = 0$ or $\lambda = 1$ with $b \geq \rho\sigma$ Combining the expression (21) with lemmas 11 and 12 and using $z_0 = 2\nu_2 e^{v_0}$we get, with $A(z_0) = \int_0^\infty e^{-\frac{s^2}{2}t}\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^\lambda\right] dt$,

$$A(z_0) = \frac{2\Gamma\left(\eta - \frac{a}{\sigma^2}\right)}{\sigma^2\Gamma(1+2\eta)} z_0^{2\eta}e^{-z_0}[\Phi(;z_0)\Psi(\eta - \frac{a}{\sigma^2}+1, 1+2\eta; z_0) + \Psi(;z_0)\frac{\Phi(\eta - \frac{a}{\sigma^2}+1, 1+2\eta; z_0)}{\eta+\frac{a}{\sigma^2}}].$$

We know that in the case $\lambda = 0$, this expression is the Laplace transform with respect to time of 1. Since it is equal to the Laplace transform with respect to time of of $\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)\right]$ when $\lambda = 1$ with $b \geq \rho\sigma$ we have proven the martingale property in that case.

Working out the identity with Kummer functions: We know that this expression should be equal to $\frac{2}{s^2}$ for any s. Can we show this?

When $z_0 \to 0$ Then $\Phi(;z_0) \to 1$ and at least when $\eta > \frac{1}{2}$, $\Psi(aa, bb, ; z) \sim \frac{\Gamma(bb-1)}{\Gamma(aa)}z^{1-bb}$ with $bb = 1+2\eta$ so that

$$A(z_0) \sim \frac{2\Gamma\left(\eta - \frac{a}{\sigma^2}\right)}{\sigma^2\Gamma(1+2\eta)}[\frac{\Gamma(2\eta)}{\Gamma(\eta - \frac{a}{\sigma^2}+1)} + \frac{\Gamma(2\eta)}{(\eta+\frac{a}{\sigma^2})\Gamma(\eta - \frac{a}{\sigma^2})}]$$

which is equal to $\frac{1}{\sigma^2\eta}[\frac{1}{\eta-\frac{a}{\sigma^2}} + \frac{1}{\eta+\frac{a}{\sigma^2}}] = \frac{2}{s^2}$

When $z_0 \to \infty$ Then $\Phi(aa, bb; z_0) \sim \frac{\Gamma(bb)}{\Gamma(aa)} e^{z_0} z_0^{a-bb}$ and $\Psi(aa, bb; z) \sim z_0^{-aa}$ so that:

$$A(z_0) \sim \frac{2\Gamma\left(\eta - \frac{a}{\sigma^2}\right)}{\sigma^2} \frac{1}{z_0} [\frac{1}{z_0 \Gamma(\eta - \frac{a}{\sigma^2})} + \frac{z_0}{(\eta + \frac{a}{\sigma^2})\Gamma(\eta - \frac{a}{\sigma^2} + 1)}]$$

which tends to $\frac{2\Gamma\left(\eta - \frac{a}{\sigma^2}\right)}{\sigma^2 (\eta + \frac{a}{\sigma^2})\Gamma(\eta - \frac{a}{\sigma^2} + 1)} = \frac{2}{s^2}$.
The last piece is the following:

Lemma 13 *Let* a, b *such that* $a > 0$ *and* $b > 1$. *Then* $\forall z$

$$\Phi(a, b; z)\Psi(a+1, b; z)(b - a - 1) + \Phi(a+1, b; z)\Psi(a, b; z) = \frac{\Gamma(b)}{a\Gamma(a)} z^{1-b} e^z$$

Proof. To alleviate the notations let $\Phi \equiv \Phi(a, b; z)$ and $\Psi \equiv \Psi(a, b; z)$ and let us drop the dependency in z. We know that the Wronskian $\Phi\Psi' - \Psi\Phi'$ is given by $-\frac{\Gamma(b)}{a\Gamma(a)} z^{-b} e^z$. By substituting the expressions of the derivatives we get

$$\frac{\Gamma(b)}{a\Gamma(a)} z^{1-b} e^z = z(\frac{\Psi}{b}\Phi(a+1, b+1) + \Phi\Psi(a+1, b+1))$$

so we want to prove the identity

$$z(\Psi\Phi(a+1, b+1) + b\Phi\Psi(a+1, b+1)) = b(\Phi\Psi(a+1, b)(b - a - 1) + \Psi\Phi(a+1, b))$$

which in turn amounts to $\Psi(z\Phi(a+1, b+1) - b\Phi(a+1, b)) = b\Phi((b-a-1)\Psi(a+1, b) - z\Psi(a+1, b+1))$.
Now by (DLMF (2010), 13.3.4) we have:

$$z\Phi(a+1, b+1) - b\Phi(a+1, b) = -b\Phi$$

and by (DLMF (2010), 13.3.10)

$$(b - a - 1)\Psi(a+1, b) - z\Psi(a+1, b+1) = -\Psi$$

and the result follows. ■

3.3 Pricing vanilla option

We want to compute

$$c(t, f_0) = e^{-rt}\mathbb{E}\left[(f_t - k)_+\right].$$

We take the Mellin transform with respect to the strike as in Jeanblanc et al. (2009) (see also Panini and Srivastav (2004)) to obtain

$$\mathcal{M}(c(t, f_0), \lambda) = e^{-rt} \int_0^{+\infty} k^{\lambda-2}\mathbb{E}\left[(f_t - k)_+\right] dk = \frac{e^{-rt}}{\lambda(\lambda - 1)}\mathbb{E}\left[f_t^\lambda\right]. \quad (22)$$

It involves the Mellin transform of the spot that was previously determined.

In fact, if we denote $h(s)$ the right hand side of (21) then

$$\mathbb{E}\left[f_t^\lambda\right] = \frac{f_0^\lambda}{2\pi i} \int_{\tilde{\gamma}-i\infty}^{\tilde{\gamma}+i\infty} e^{st} h(s) ds. \quad (23)$$

and the option price is given by the inverse Mellin transform

$$c(t, f_0) = \frac{e^{-rt}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}(\lambda) k^{-(\lambda-1)} d\lambda. \quad (24)$$

4 Related Works

Our work contributes to the literature aiming at overcoming the issues faced when implementing the affine model. The model proposed here is also presented in Henry-Labordère (2009) page 281 where it is called the Geometric Brownian. The techniques used in Henry-Labordère (2009) are different from those used here (certainly they can be connected). Also, it seems to us that the problem of martingale property of the stock is not analysed for that particular model. Lastly, we don't know whether the formulas developed in this book lead to a reasonable numerical implementation. To illustrate the problem at stake and underline the usefulness of the series representation for I1 and I2 we just need to mention the fact the use of equations (10) and (11) (this function being the Hartman-Waston density) often lead to tedious numerical problems, see for example Barrieu et al. (2004).

We were able to obtain an explicit solution for the case $\alpha = 1$ but we also established that the martingale property in that case depends on the parameter values. Extending the results to a general α is certainly of interest. If we understand Henry-Labordère (2009) in this general case the model might not be solvable.

Another work to which we are related is Itkin (2013) who studied a stochastic volatility model using Lie group analysis. He obtains a closed-form solution for the transition probability for the volatility process involving confluent hypergeometric functions. Our model coincides with his when $\alpha = 1$ but the author mainly focuses on volatility derivatives and the techniques used to derive his results are different form ours.

5 Conclusion

We propose a new stochastic volatility model for which we develop the key elements to perform equity and volatility derivatives pricing. We found the conditions on the parameters ensuring the martingale property of the stock. For a particular set of parameter (i.e. $\alpha = 1$) we compute the Mellin transform of the stock which enables the pricing of vanilla options. The model has, by construction, a volatility which is positive and therefore solve a major drawback of the traditional square root process, used for example in the Heston (1993) model, which imposes a constraint on the parameters (i.e., the Feller condition) that is not satisfied in practice.

References

P. Barrieu, A. Rouault, and M. Yor. A study of the Hartman-Watson distribution motivated by numerical problems related to the pricing of Asian options. *Journal of Applied Probability*, 41(4):939–1254, 2004.

R. Beals and R. Wong. *Special Functions*. Cambridge University Press, Cambridge, 2010.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, May-June 1973.

M. Chesney and L. Scott. Pricing European Currency Options: A comparison of the modified Black-Scholes model and a random variance model. *Journal of Financial and Quantitative Analysis*, 24:267–284, 1989.

J. Da Fonseca and M. Grasselli. Riding on the smiles. *Quantitative Finance*, 11(11):1609–1632, 2011.

DLMF. Digital library of mathematical functions. *National Institute of Standards and Technology,dlmf.nist.gov*, 2010.

C. Donati-Martin, R. Ghomrasni, and M. Yor. On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options. *Revista Matemática Iberoamericana*, 17:179–193, 2001.

D. Dufresne. Laguerre series for Asian and other options. *Working Paper*, 1998.

J. Gatheral. Consistent Modeling of SPX and VIX options. In: The Fifth World Congress of the Bachelier Finance Society. 2008.

C. Grosche. The path integral on the Poincaré upper half-plane with a magnetic field and for the Morse potential. *Annals of Physics*, 187:110–134, 1988.

C. Grosche and F. Steiner. *Handbook of Feynman Path Integrals*. Springer-Verlag, Berlin Heidelberg New York, 1 edition, 1998.

P. Henry-Labordère. *Analysis, Geometry, and Modeling in Finance*. Chapman & Hall/CRC, Boca Raton, 1 edition, 2009.

S. L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6(2):327–343, 1993.

J. Hull and A. White. The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance*, 42(2):281–300, 1987.

N. Ikeda and H. Matsumoto. Brownian motion on the hyperbolic plane and Selberg trace formula. *Journal of Functional Analysis*, 163:63–110, 1999.

A. Itkin. New solvable stochastic volatility models for pricing volatility derivatives. *Review of Derivatives Research*, 16: 111–134, 2013.

M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical Methods for Financial Markets*. Springer-Verlag, Berlin Heidelberg New York, 1 edition, 2009.

B. Jourdain. Loss of martingality in asset price models with lognormal stochastic volatility. *Working paper CERMICS*, 2004.

B. Leblanc. Une approche unifiée pour une forme exacte du prix d'une option dans les différents modèles à volatilité stochastique. *Stochastics and Stochastic Reports*, 57:1–35, 1996.

E. Love, T. Prabhakar, and N. Kashyap. A confluent hypergeometric integral equation. *Glasgow Mathematical Journal*, 23 (1):31–40, 1982.

H. Matsumoto and M. Yor. Exponential functionals of brownian motion, I: Probability laws at fixed time. *Probability Surveys*, 2:312–347, 2005a.

H. Matsumoto and M. Yor. Exponential functionals of brownian motion, II: Some related diffusion processes. *Probability Surveys*, 2:348–384, 2005b.

R. Panini and P. Srivastav. Option pricing with Mellin transforms. *Mathematical and Computer Modelling*, 40:43–56, 2004.

G. Peskir. On the fundamental solution of the Kolmogorov-Shiryaev equation. In Y. Kabanov, R. Liptser, and J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance, The Shiryaev Festschrift*, pages 535–546, Berlin Heidelberg New York, 2006. Springer-Verlag.

C. Pintoux and N. Privault. A direct solution to the Fokker-Planck equation for exponential brownian functionals. *Analysis and Applications*, 8(3):287–304, 2010. doi: 10.1142/S0219530510001655.

C. Pintoux and N. Privault. The Dothan pricing model revisited. *Mathematical Finance*, 21(2):355–363, 2011.

E. M. Stein and J. C. Stein. Stock Price Distribution with Stochastic Volatility : An Analytic Approach. *Review of Financial Studies*, 4:727–752, 1991.