A Unified Model: Arbitrage-free Term Structure Movements of Flow Risks

By

Thomas S. Y. Ho

President

Thomas Ho Company, Ltd

55 Liberty Street 4B

New York

NY 10005

Tom.ho@thomasho.com

and

Sang Bin Lee

Professor in Finance

Hanyang University

Seoul, Korea

leesb@hanyang.ac.kr

The authors would like to thank the participants at Financial Innovation Conference, Owen School, Vanderbilt University 2008 for their comments. We would also like to thank Eric A Anderson and Charles Fenner for their insights in energy contracts. We are responsible for any errors in the paper.

A Unified Model: Arbitrage-free Term Structure Movements of Flow Risks

Abstract

This paper first dichotomizes risk drivers into "stock" or "flow" attributes. Stock risk drivers are prices of tradable securities and flow risk drivers are rates represented by the stochastic movements of a term structure of securities. This paper then shows that the Black Scholes model is the relative valuation model for the stock risk drivers while the proposed unified model is for the flow risk drivers.

The unified model can be described in the Ho-Lee model framework. We apply this model to five different flow risk drivers: interest rate, credit risk, liquidity risk, energy risk, and inflation risk. We then show that the unified model provides an analytical framework for securities that are subjected to several of these flow risk drivers, offering many applications.

For example, the 2008 financial crisis clearly shows the importance of the use of a unified model in enterprise risk management. The crisis demonstrates that risk management should not take a silo approach to manage each flow risk driver, such as interest rate risk and credit risk. We propose an integrated approach to manage risks using the unified model.

I. Introduction

Economic quantities are classified by stocks and flows. The former are values at a point in time and the latter are quantities measured per unit time. This classification of economic quantities is widely used in accounting and econometric models, where balance sheet items and capital are considered stocks and income statement items and transactions are flows. The concept of stocks and flows is also applicable to the understanding of risks in the financial markets, where risk drivers can also be dichotomized into stocks and flows.

Stock risk drivers are exemplified by equities, commodities and the firm values. They are the underlying risk drivers to many financial instruments, such as equity options and commodity futures. At any time, stock risk drivers have observable values that follow some stochastic processes which affect the values of their contingent claims. Flow risk drivers such as the interest rates and inflation rates also have their stochastic movements. However, their value is measured per unit of time. And therefore their contingent claims are generally affected by the accumulated flow risks over a time period. Therefore, a flow risk driver must be related to a term structure of risks, such as the term structure of interest rates, and the stochastic movements of the term structure, and not just the stochastic movements of the rate, are important to the understanding of flow risks.

The Black-Scholes model (1973) is a relative valuation model that values contingent claims on stock risk drivers. It assumes that in a perfect capital market the stochastic movements of a derivative and its underlying security are instantaneously, perfectly correlated. This characteristic of a derivative enables the market participants to form dynamic hedging strategies to construct a risk free portfolio of derivatives and their underlying securities and that in turn leads to the risk neutral valuation of the derivative relative to the observed underlying security price. Furthermore, the model suggests that benchmark derivative prices can be used to determine the implied volatility of the underlying security returns, and the implied volatility can be used to price other derivatives based on the same underlying securities. That is, the relative valuation model can be defined by the underlying security and a set of benchmark contingent claims or options on the security. This fundamental insight has been elaborated in the continuous time model context (See Harrison and Krepp, 1979).

The Ho-Lee model (1986) is also a relative valuation model based on the Black-Scholes paradigm. It is a valuation model of contingent claims on flow risk drivers, as opposed to the stock risk drivers. The model specifies arbitrage-free movements of flow risk drivers to value a broad range of financial instruments. The term "arbitrage-free" is defined as the absence of an arbitrage opportunity among the contingent claims and the observed term structure of securities. This definition is consistent with that of the Ho-Lee model paradigm. Dynamic hedging between the term structure of risks and its derivatives is used to derive the risk neutral valuation of the derivatives, and the benchmark derivatives with embedded options are used to determine the implied volatilities of the term structure movements. Details of these concepts are described in a continuous time model context and can be found in Rebonato (1998).

The model has generated extensive research in alternative specifications of term structure movements of interest rates and applications in capital markets. The purpose of this paper is to present the Ho-Lee model as a unified model, in the sense that, (1) it is a contingent claim valuation model on flow risk drivers, not just a model confined to interest rates, and, (2) it is a model that combines multiple risk drivers to determine the contingent claims value, not just an analysis of contingent claims on a particular risk driver, isolated from other risk drivers.

The unified model is an important contribution to financial research because the valuation framework is applicable to financial instruments that are prevalent in the capital markets; they can be found in trading positions, investment portfolios, balance sheets of financial institutions, energy sectors, inflation-linked financial products in pension plans and other financial sectors. The unified model is also important in providing a framework to value and hence to analyze financial instruments with multiple sources of risks, a topic particularly important in enterprise risk management where risk managers have the daunting task of rolling up market risks, credit risks, and liquidity risks into a coherent framework.

The financial crisis of 2008 clearly shows the importance of the applications of a unified model. First, the risks of securities such as asset-backed securities and collateralized bond obligations and the credit derivative swaps of super senior CDO tranches are option embedded contingent claims on credit and market risks. Much research has been devoted to analyzing interest rate options embedded in financial instruments. However, much less research has been applied to valuing credit options and liquidity options, and scant attention has been paid to the combined effects of these three major risk factors on financial instruments. For example, the observed increased default rate, lack of liquidity in the mortgage loan and mortgage-backed securities markets, slowing of refinancing of mortgages are not isolated events, but are consequences of the combined effect of credit, interest rate and liquidity risks on the mortgage value and on the financial markets. Standard approaches in analyzing these securities would significantly measure the risks erroneously resulting in significant losses in the complex securities.

Second, risks of enterprises should not be analyzed in isolation. Basel II has advocated that risks should be analyzed by separate departments. When risks are managed in silos, enterprise risk management, regulatory monitoring and supervision processes would fail to provide a coherent framework to manage the risks in the financial system. Financial institutions fail to manage the combined effect of the flow risks in the financial crisis. Third, the economic capital of a financial institution should be recognized as a contingent claim of market, credit and liquidity risk and it has significant embedded option on these risk drivers. It cannot be measured simply as the asset value net of the liability value as it is commonly reported. The option embedded in the economic capital may lead to unanticipated consequences in "deleveraging" of a balance sheet. This paper will relate the unified model to these issues, which were significant causes of the financial crisis.

The paper proceeds as follows. In section B we identify the differences between a stock risk driver and a flow risk driver. As well, we present the Ho-Lee model (1986) as a unified model. Section C provides five examples of contingent claims on flows risk drivers. Section D describes the unified model in valuing contingent claims on multiple flow risk drivers, where the term structure of the risk drivers has multiple movements. The section also describes the modeling of these movements. Section E describes the implications of the unified model in enterprise risk management. Finally, Section F contains the conclusion.

II. The Black Scholes Model and the Ho Lee Model

In this section, we use a binomial lattice model to illustrate the Black Scholes model in a way comparable to the Ho-Lee model. Let $s(n,i)$ and $C(n,i)$ be the underlying security and the call option respectively at time n and state i. Let the option time to expiration and strike be T and X respectively. Let $S(n)$ be the forward price of the security, where

$$S(n) = Se^{nr} \tag{S1}$$

The stochastic price of $s(n,i)$ is defined by equation (S1) with the binomial probability of 0.5 and with a given risk free rate r:

$$s(n,i) = S(n)(2/(1+\delta))^n \delta^i \tag{S2}$$

Then, the Black Scholes model can be expressed as a set of recursive equations

$$C(n-1,i) = 0.5e^{-r}(C(n,i+1)+C(n,i)) \text{ for } i = 0,\cdots,n-1 \tag{S3}$$

with the terminal condition for a call option, as an illustration

$$C(T,i) = \max(S(T,i)-X,0) \text{ for } i = 0,\ 1,\ 2,\cdots,T \tag{S4}$$

The value of the call option is $C = C(0,0)$. The binomial movement of the security S ensures that the option price is the observed stock price S, when the strike price is zero, satisfying the arbitrage-free condition. δ determines the local volatility of the underlying security. When C is observable in the market, then the option price can be used to determine δ, and thus specifying the implied volatility of the underlying security, which in turn can be used to value other derivatives of S. The underlying security S is a stock risk driver.

δ determines the volatility of the stock σ . Indeed, we have $\delta = Exp[-2\frac{\sigma}{\sqrt{n}}]$, where n is the number of steps in one year. This can be proved as follows. The stock price at time n and state i is $Se^{nr}(\frac{2}{1+\delta})^n \delta^i$. The logarithm of the stock return is $Log[\frac{Se^{nr}(\frac{2}{1+\delta})^n \delta^i}{S}] = nr + n\ Log\frac{2}{1+\delta} + i\ Log\delta$ and the variance is $(\log\delta)^2 \frac{1}{2}(1-\frac{1}{2})n = \sigma^2$.

Proposition 1. The binomial model defined by S(1)-S(4) converges to the continuous time Black Scholes model.

Proof:

First we show that the binomial distribution approaches the lognormal distribution as $n \to \infty$.[1]

We know that

$$\log(\frac{S_T}{S_0}) = \mu\tau + \sigma\sqrt{\tau}Z \text{ , where } Z \text{ is a standard normal distribution.} \tag{C1}$$

From (S2),

$$\log(\frac{S_T}{S_0})_n = r\tau + n\ \log(\frac{2}{1+\delta}) + j\log(\delta) \tag{C2}$$

From the DeMoivre-Laplace theorem, $\lim_{n\to\infty} \text{Prob}\left[a \le \frac{j-nq}{\sqrt{nq(1-q)}} \le b\right] = N(b) - N(a)$

Since q is 0.5, $j \approx (\sqrt{n}/2)Z + n/2$ where the notation $\approx$ means converge in distribution.

Substituting into (p2) gives $\log(\frac{S_T}{S_0})_n = r\tau + n\ \log(\frac{2}{1+\delta}) + ((\sqrt{n}/2)Z + n/2)\log(\delta)$

Let $\mu\tau = r\tau + n\ \log(\frac{2}{1+\delta}) + n/2 \cdot \log(\delta)$ and $\sigma\sqrt{\tau} = \sqrt{n}/2 \cdot \log(\delta)$

then

[1] We assume that τ is 1 for simplicity.

$$N(b)-N(a)=\lim_{n\to\infty}\text{Prob}\left[a\le\frac{j-nq}{\sqrt{nq(1-q)}}\le b\right]=\lim_{n\to\infty}\text{Prob}\left[a\le\frac{j-\frac{n}{2}}{\sqrt{n/4}}\le b\right]$$

$$=\lim_{n\to\infty}\text{Prob}\left[a\le\frac{r\tau+n\ \log(\frac{2}{1+\delta})-r\tau-n\ \log(\frac{2}{1+\delta})+\log(\delta)j-\log(\delta)\cdot n/2}{\log(\delta)\cdot\sqrt{n/4}}\le b\right]$$

$$=\lim_{n\to\infty}\text{Prob}\left[a\le\frac{\log(\frac{S_T}{S_0})_n\text{-}\mu\tau}{\sigma\sqrt{\tau}}\le b\right]$$

Therefore, $(S_T)_n$ converges in distribution to S_T.

Second, what we have to show is that

$c=e^{-r\tau}E[Max[0,(S_T)_n-K]]->e^{-r\tau}E[Max[S_T-k]]\ as\ n->\infty$

If the binomial probability distribution converges to the lognormal, then its moments must also converge to the moments of the lognormal.

Therefore,

$$c=e^{-r\tau}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{2}\right)^n Max\left[0,Se^{r\tau}\left(\frac{2}{1+\delta}\right)^n\delta^j-K\right]\to SN(d_1)-Ke^{-r\tau}N(d_1-\sigma\sqrt{\tau})\ \text{as}\ n\to\infty.$$

Proposition 2. The stochastic price $S(n,i)$ satisfies the no-arbitrage condition. That is: the risk neutral present value of the expected value of $S(n,i)$ is the observed price S and the following relationship holds $s(n,i)=0.5e^{-r}(s(n+1,i+1)+s(n+1,i))$

Proof:

The expected value of the stock at period n is $\sum_{i=0}^{n}Se^{rn}(\frac{2}{1+\delta})^n\delta^i\frac{n!}{(n-i)!i!}\frac{1}{2^n}=Se^{rn}$. Therefore, the discounted value of Se^{rn} as of time 0 is S.

Substituting $S(n+1)(2/(1+\delta))^{n+1}\delta^{i+1}$ ($S(n+1)(2/(1+\delta))^{n+1}\delta^{i}$) into $S(n+1,i+1)$ $(S(n+1,i))$ and simplifying, we can show that the arbitrage-free condition $s(n,i)=0.5e^{-r}(s(n+1,i+1)+s(n+1,i))$ holds.[2] QED

The Ho-Lee model (1986) extends the Black Scholes derivative valuation model in the perfect capital market context to value contingent claims on a term structure of flow risk drivers. A risk

---

[2] It is known that the discrete CRR model converges to the continuous BS model as the number of partitions grows. By the same token, equations (S1-S4) also converge to the continuous BS model at the same speed.

factor $u(n,i)$ is a flow risk driver that follows a stochastic process. The concept of "flow", where a per unit time construct is necessary in describing $u(n,i)$. The symbol "u" is used to denote the unified model that applies a broad range of flow risk drivers. A rate index $\rho(n,i)$ is defined as the proportional change of a stochastic quantity over a binomial period, and it is related to the flow risk driver by:

$$u(n,i) = \exp(-\rho(n,i)) \tag{u1a}$$

A binomial lattice with time n and state i represents the stochastic movements of the risk factor $u(n,i)$ or the rate $\rho(n,i)$. At the initial time (n=0), the market can form the forward factor per unit time period $u(n,i)$ or the forward rate per unit time period $\rho(n)$ where the time to the delivery date is n. As time passes, a path in the binomial lattice is used to represent the realization of events. The forward factor or rate would converge to the realized risk factor and the rate at time n. The forward risk factor and the rate are also related by:

$$U(n) = \exp(-\rho(n)) \tag{u1b}$$

The term structure of a flow risk driver is defined as a term structure of marketable forward contracts on the flow risk driver. Initially, at time 0, the forward values are $u(n)$ for $n = 1, 2, \ldots, T$. Then, the Ho-Lee model shows that the stochastic movement of the flow risk driver is given by:

$$u(n,i) = U(n)(2/(1+\delta^n))\delta^i \tag{u2}$$

The contingent claim on the flow risk driver is given by:

$$C(n-1,i) = 0.5\, u(n,i)(C(n,i+1) + C(n,i)) \; for \; i = 0, \cdots, n-1 \tag{u3}$$

with the terminal condition for a call option, as an illustration,

$$C(T,i) = \max(u(T,i) - X, 0) \; for \; i = 0, 1, 2, \cdots, T \tag{u4}$$

Analogous to the Black-Scholes model, when the strike price X is zero and time to expiration is T, in applying the recursive equations, we can retrieve the option value which equals $e^{-\sum_{i=0}^{T} U(i)}$, the forward price.[3]

Lemma: Let the forward rate per unit time period be given by $\rho(n)$ for the time to delivery, n>0. The primitive security value is given by

[3] If we roll back from period T where the value at time T and state i is $u(T,i)$, then we have $e^{-\sum_{i=0}^{T} U(i)}$ equals the forward price

$$A(n,i)=\frac{Exp[-\sum_{t=0}^{n-1}\rho(t)]\,f(n,i)}{\prod_{t=0}^{n-1}(1+\delta^t)} \text{ for some real value } 0<\delta<1 \tag{1}$$

where $f(n,i)$ is a function of integer n and I and it is uniquely determined by the recursive relation

$$f(m,k)=\delta^{k-1}f(m-1,k-1)+\delta^k f(m-1,k) \tag{2}$$

with boundary conditions

$$f(m,m)=\delta^{\frac{m(m-1)}{2}},\ f(m,0)=1,\ f(m,1)=\sum_{t=1}^{m}\delta^{m-t}, \tag{3}$$

and

$$f(m,m-1)=(\sum_{t=1}^{m}\delta^{m-t})\delta^{\frac{m(m-1)}{2}}. \tag{4}$$

Proof: We prove the lemma by induction. First, we show that the Equation (1) holds for $n=1$ and $i=0,\ 1$. For $n=1$ and $i=0$, $A(1,0)=\frac{Exp[-\rho(0)]}{2}$ by Equation (1). But by Equation (u3), we note that $A(1,0)=\frac{Exp[-\rho(0)]}{2}$. Similarly, we can show that Equation (1) holds for $n=1$ and $i=1$. Next, we assume that Equation (1) is correct for all periods up to n-1 and for all the states in these periods. The induction proof requires us to show that, given assumption, Equation (1) also holds for time n and all the states at time n. Consider the pricing of $A(n,i)$. By definition, at time n, the only payoff is at state i and it is \$1. Then at time $n-1$, there are two states that have positive payoffs. At state i and time $n-1$, the payoff is $\frac{1}{2}Exp[-\rho(n-1)]\frac{2}{(1+\delta^{n-1})}\delta^i$. Similarly, at state $i-1$ and time $n-1$, the payoff is $\frac{1}{2}Exp[-\rho(n-1)]\frac{2}{(1+\delta^{n-1})}\delta^{i-1}$. Finally we use induction hypothesis. Using primitive securities to price an asset, we have

$$\frac{1}{2}Exp[-\rho(n-1)]\frac{2}{(1+\delta^{n-1})}\delta^i A(n-1,i)+\frac{1}{2}Exp[-\rho(n-1)]\frac{2}{(1+\delta^{n-1})}\delta^{i-1}A(n-1,i-1) \tag{5}$$

Substituting into $A(n-1,i)$ and $A(n-1,i-1)$ by the induction hypothesis, we confirm that Equation (1) holds.[4] Q.E.D.

[4] For a general form of Arrow-Debreu primitive securities, see Appendix A.

Proposition 3. The model $u(1)-u(4)$ provides an arbitrage-free valuation model for a flow risk driver, given the observed term structure of forward values $U(n)$.

Proof: We can show that $\sum_{i=0}^{n} A(n,i) = Exp[-\sum_{t=0}^{n-1}\rho(t)]$, which says that the sum of the primitive securities over the states at time n is equal to the security with \$1 at time n. Specifically, the sum of Arrow-Debreu primitive securities is 1 at time 1 if we assume that the interest rates at every period is zero for simplicity (i.e., $\rho(t)=0 \text{ for } t \geq 0$). Now, we assume that the sum of Arrow-Debreu primitive securities at time n is 1, which means that the no-arbitrage condition holds at time n. Consider a zero-coupon bond which matures at time n+1. The value of the zero-coupon bond at time n is

$$\sum_{i=0}^{n} \frac{2\delta^i}{(1+\delta^n)} \frac{f(n,i)}{\prod_{t=0}^{n-1}(1+\delta^t)} = \sum_{i=0}^{n} \frac{2\delta^i f(n,i)}{\prod_{t=0}^{n}(1+\delta^t)} = 1 \tag{6}$$

To avoid the arbitrage opportunities, the value of the zero coupon bond at time 0 is 1. The sum of the Arrow-Debreu primitive securities at time n+1 is

$$\sum_{i=0}^{n+1} \frac{f(n+1,i)}{\prod_{t=0}^{n}(1+\delta^t)} = \sum_{i=0}^{n} \frac{\delta^i f(n,i)}{\prod_{t=0}^{n}(1+\delta^t)} + \sum_{i=1}^{n+1} \frac{\delta^{i-1} f(n,i-1)}{\prod_{t=0}^{n}(1+\delta^t)} = \sum_{i=0}^{n} \frac{2\delta^i f(n,i)}{\prod_{t=0}^{n}(1+\delta^t)} \tag{7}$$

Equation (4) is equal to 1 by Equation (3).Q.E.D.

In comparing equations (S1-S4) and (u1–u4), we can see the similarity between the Black Scholes model and the Ho-Lee model, particularly in their determination of the contingent claim values. In comparing equations S2 and U2, the functional form of the two models is also similar in that it is a product of three factors. The main difference lies on the specification of the forward contracts. The Black-Scholes model specifies the forward price of a stock while the Ho-Lee model specifies the forward contract on a flow entity, where the time unit is defined by the binomial period. Also, compare S3 and u3, the risk free rate is used in the roll back for the stock risk model and the risk driver is used for the flow risk model. Expressed in the continuous models, a flow risk driver has to be expressed as an "instantaneous rate entity", such as a money market account.

III. Examples of Term Structure of Flow Risk Drivers

A. Interest Rate Model

An example of the Ho-Lee model is the interest rate model. The discount index $u(n,i)$ is the discount factor for a one binomial period at time n and state i. Specifically, it is the present value of \$1 paid at time $n+1$ at the binomial outcomes in the states i and $i+1$. In the interest rate model, we use the observed yield curve, or the discount function $P(n)$ as input to the model. Given the discount function $P(n)$, we can derive the forward values:

$$p(n) = P(n+1)/P(n) \tag{P1}$$

Then, the stochastic movements of the discount factor are given by:

$$p(n,i) = p(n)(2/(1+\delta^n))\delta^i \tag{P2}$$

Heath, Jarrow and Morton (1992) provide an analogous continuous time framework to equation (P2). The δ represents the volatilities and factor $2/(1+\delta^n)$ is called the convexity term that ensures the yield curve movements are arbitrage-free. The valuation of interest rate contingent claims is given by

$$C(n-1,i) = 0.5\, p(n,i)(C(n,i+1)+C(n,i)) \;\mathit{for}\; i = 0,\cdots,n-1 \tag{P3}$$

with the terminal condition for a call option, as an illustration, given by

$$C(T,i) = \max(p(T,i)-X,0) \;\mathit{for}\; i = 0,1,2,\cdots,T \tag{P4}$$

To show that the interest rate model is arbitrage free, we show that the zero coupon bond price based on the initial forward rates equals that based on the backward substitution methodology. We use the method of induction for the proof. We assume that the initial forward rates $\rho(n)$ *where* $n = 0,1,2\cdots,T$ are given. $\rho(n)$ is the initial forward rate which is applicable over the period of n to n+1. Consider a zero coupon bond. The face value is assumed to be 1. The price of a zero coupon bond which matures at period 1 is $e^{-\rho(0)}$. The price of the same bond by backward substitution on the binomial lattice is also $e^{-\rho(0)}$. The price of a zero coupon bond which matures at period 2 is $e^{-\rho(0)}e^{-\rho(1)}$. The price of the same bond by backward substitution on the binomial lattice is $\frac{1}{2}e^{-\rho(0)}\left(e^{-\rho(1)}\frac{2}{1+\delta}\delta + e^{-\rho(1)}\frac{2}{1+\delta}\right) = e^{-\rho(0)}e^{-\rho(1)}$. Now we assume that the price of a zero coupon bond which matures at period T is the same regardless of whether we achieve by backward substitution or from the initial forward rates. The price is $e^{-\rho(0)}e^{-\rho(1)}e^{-\rho(2)}\cdots e^{-\rho(T-1)}$. The price of a zero coupon bond which matures at period $T+1$ is $e^{-\rho(0)}e^{-\rho(1)}e^{-\rho(2)}\cdots e^{-\rho(T-1)}e^{-\rho(T)}$ from the initial forward rates. The price of the same bond by backward substitution is also $e^{-\rho(0)}e^{-\rho(1)}e^{-\rho(2)}\cdots e^{-\rho(T-1)}e^{-\rho(T)}$, because the price of a zero coupon bond which matures at $T+1$ is equal to $e^{-\rho(T)}\frac{2\delta^i}{1+\delta^{T-1}}A(T,i)$ where $A(T,i)$ is Arrow-Debreu primitive securities at time T and state i. Adding over i. and simplifying, we get $e^{-\rho(0)}e^{-\rho(1)}e^{-\rho(2)}\cdots e^{-\rho(T-1)}e^{-\rho(T)}$.

Swaptions can be used to calibrate the interest rate volatility, specified by the parameter δ. The Brace, Gatarek and Musiela (1997) interest rate model is called a market model as the specification of the model fits both the swap curve and a set of swaption prices exactly. The prevalent use of swaptions in specifying the interest rate model shows the importance of the use of implied volatilities in contingent claims pricing. By way of contrast, Cox, Ingersoll and Ross (1985) and Vasicek (1977) interest rate models are not relative valuation model in the sense of the Black-Scholes model. These models specify the interest rate risk process to derive the yield curve and they do not take the yield curve as given, using equation (P1).

B. Credit Valuation Model

The unified model can be applied to the credit valuation model. Details of a credit model are described in Ho-Lee (2009a, 2009b). The marginal survival factor $s(n,i)$ is the flow risk driver for the credit valuation for a one binomial period at time n and state i. Specifically, it is the probability for a bond at time n and state i to survive another period, conditional on the bond has survived till time n. In the credit valuation model, we use the observed marginal default rate curve, or the survival function $S(n)$ as input to the model. Given the survival function $S(n)$, we can derive the forward values:

$$s(n) = S(n)/S(n-1) \tag{S1}$$

Then, the stochastic movements of the survival factor are given by:

$$s(n,i) = s(n)(2/(1+\delta^n))\delta^i \tag{S2}$$

The valuation of credit risk contingent claims is given by

$$C(n-1,i) = 0.5\, s(n,i)(C(n,i+1)+C(n,i)) \; for\ i = 0,\cdots,n-1 \tag{S3}$$

with the terminal condition for a call option, as an illustration

$$C(T,i) = \max(s(T,i)-X,0) \; for\ i = 0,1,2,\cdots,T \tag{S4}$$

However, unlike the interest rate model, the survival function and the option of the survival rates of bond is not directly observable in the capital markets. We need to relate the survival function to tradable financial instruments. Credit derivative swaps (CDS) can be used for this purpose. These swaps are actively traded in the market with a term structure of CDS premium observable to the marketplace for many corporate bonds, and even sovereign, entities.

The CDS spread of $c(T)$ with a T period tenor is the solution to the set of recursive equations

$$V(n-1) = p(n)[s(n)V(n)-(1-s(n))(1-R)+c(T)] \quad for\ n = 1,\cdots,T \tag{S5}$$

with the terminal conditions $V(T) = V(0) = 0$, where R is the recovery rate, p(n) the time value discount factor for one period at time n. S5 says that the seller of the CDS would receive the credit premium of the period (n-1), but would have to pay the loss at default in case the default event occurs at during this period. The calculation is conditional to the bond survives till time (n-1). Therefore, given the CDS premiums $c(n)$ for n= 1, 2, …, N, we can determine the $s(n)$.

In this example, CDSs are used to determine the credit premiums in defining the implied credit premium, forward looking estimates of credit risk, as opposed to using the historical defaults experience. However, the method can be used for non-option embedded bonds to estimate the credit premium if the liquidity premium can be appropriately isolated from the credit premium.

The stochastic movements of the CDS curve are different from those of the interest rate curve. While the survival function, like the discount function, should be monotonically downward sloping and positive, its movement may follow a different mean reversion process. The specification of the binomial lattice should also reflect such differences. For example, the CDS curve can rise rapidly.

There are papers proposing arbitrage free valuation methodologies of credit contingent claims such as Duffie (2005), Das et al. (2006) and Longstaff and Rajan (2008). These models assume that the hazard rate follows a mean-reversion process similar to that of the Cox, Ingersoll and Ross model. Furthermore, the model is then calibrated to the observed CDS curve. This approach must necessarily calibrate the hazard rate movement model confined to the mean reversion process. These models cannot separate the calibration of the CDS curve and its volatilities. By way of contrast, this paper extends the generalized Ho-Lee model to the CDS curve movements that ensures the hazard rate movement is arbitrage-free for any given CDS curve. This feature separates the specification of volatilities of the hazard rate from the fitting of the model to the CDS curve. This separation enables the model to have several advantages over other models.

C. Liquidity Valuation Model

Liquidity is broadly defined as a measure of the performance of the market, the ability to transact any time without moving the market price. A fixed income security may not have a ready market at any time and a premium is added to the discounting of the cash flows to compensate for such illiquidity. Liquidity premiums vary with time in a stochastic way and they affect the values of fixed income securities. We have defined credit spread of a bond by its credit derivative swap curve. However, liquidity spread is inseparable from the credit spreads. Lower credit corporate papers tend to be less liquid. For this reason, we can define the liquidity spread to be the discount rate of a bond net of the interest rate and the credit spread. Liquidity spread within the context of our model, is defined as the catch all term, the option adjusted returns net of time value and credit risks for a credit paper. In our relative valuation approach, we are not concerned with the modeling of the economics of liquidity. Specifically, we denote the liquidity forward price and the liquidity function to be $l(n)$ and $L(n)$ respectively, then $l(n)$ can be derived from a zero coupon bond price $V(T)$ with maturity T by the following model:

$$V(T) = P(T)S(T)L(T) \quad for\ T = 1, 2, \cdots \tag{l1a}$$

We assume that there is no recovery rate for clarity of the exposition and $P(T)$ and $S(T)$ are the discount function and the survival function respectively. By definition of the liquidity function, analogous to the discount function in the interest rate model, we have:

$$l(n) = L(n+1) / L(n) \tag{l1b}$$

Then, the stochastic movements of the liquidity factor are given by:

$$l(n,i) = l(n)(2/(1+\delta^n))\delta^i \tag{l2}$$

The valuation of liquidity contingent claims is given by

$$C(n-1,i) = 0.5\ l(n,i)(C(n,i+1) + C(n,i))\ for\ i = 0, \cdots, n-1 \tag{l3}$$

with the terminal condition for a call option, as an illustration

$$C(T,i) = \max(l(T,i) - X, 0)\ for\ i = 0, 1, 2, \cdots, T \tag{l4}$$

Recent research shows that liquidity risk is an important factor in explaining the changes in the yield spread of corporate bonds. For example, Chen et al (2007) empirically show that the liquidity risk is a significant factor to determine the yield spread variations of corporate bonds using more than 4,000 corporate bonds. Their sample consists of both the investment grade and high yield sectors. Covitz and Downing (2007) show that liquidity is important in the yield spread even for the commercial paper of less than 35 days to maturity. There should be a correlation of the liquidity risk and the credit risk since lower credit often relates to lower liquidity, as noted before. Modeling the joint stochastic movements will be discussed in the following section.

D. Energy Valuation Model

There are many energy markets with a range of forward contracts and option contracts. For example, there are the natural gas, oil, and electricity markets (See Eydeland and Wolyniec (2003) for an extensive discussion of the energy spot and derivative markets.). To illustrate the application of the unified model to this broad financial sector, we will discuss only the electricity contracts here. Electricity is not a storable commodity, which has to be consumed when produced. The electricity futures are priced based on the volume bought over a time period at a delivery date. In comparing the forward price and the spot price, the "cost of carry" can be positive or negative depend on the month of the year, and therefore the forward prices can be higher or lower than the spot price. Electricity is a good example to use because it gives the contrast to the contingent claims pricing on financial instruments. Since, electricity is not storable, the "cost of carry" cannot be explained by any dynamic trading strategies using electricity spot prices. However, for financial instruments, continuous arbitrage arguments can be used.

In applying the unified model, we assume that the electricity price is based on a fixed volume over one binomial period. The flow risk driver is the proportion of the electricity price at time n state i to the initial electricity price, denoted by $e(n,i)$, called the normalized forward price. Then, the stochastic movements of the electricity price factor are given by:

$$e(n,i) = e(n)(2/(1+\delta^n))\delta^i \tag{e2}$$

The valuation of interest rate contingent claims is given by

$$C(n-1,i) = 0.5\ e(n,i)(C(n,i+1)+C(n,i))\ \mathit{for}\ i = 0,\cdots,n-1 \tag{e3}$$

with the terminal condition for a call option, as an illustration:

$$C(T,i) = \max(e(T,i)-X,0)\ \mathit{for}\ i = 0,1,2,\cdots,T \tag{e4}$$

Electricity options on forward contracts can be used to calibrate the electricity price volatility, specified by the parameter δ. The importance of the energy model is to show that the Ho-Lee model does not apply only to the "rate risk" but to any flow risks, such as the cost of energy per unit time.

Energy contingent claims are growing in the market. With the gradual deregulation of the energy markets, where the energy prices are increasingly determined by the market, the use of forward contracts and energy options is growing. For example, a utility company may buy gas to generate electricity. However, this conversion depends of the prevailing relative prices of gas to electricity. Since electricity is not storable, often forward contracts and options on energy prices are used to ensure availability of electricity at a certain period under stochastic spot prices and demands. Prices of gas vary stochastically depending on the suppliers. Utility companies may seek to buy the cheapest electricity among the suppliers. Contingent claims are used to provide options to the utility companies to seek the cheapest sources of gas.

E. Inflation Contingent Claims Valuation Model

The unified model can be applied to inflation contingent claims valuation. Let the inflation discount index be $\eta(n,i)$ a one binomial period at time n and state i. Let the forward inflation discount factor be $\eta(n)$. Then the stochastic movements of the inflation discount index are given by:

$$\eta(n,i) = \eta(n)(2/(1+\delta^n))\delta^i \tag{η 2}$$

The valuation of inflation rate contingent claims is given by

$$C(n-1,i) = 0.5\ \eta(n,i)(C(n,i+1)+C(n,i))\ \mathit{for}\ i = 0,\cdots,n-1 \tag{η 3}$$

with the boundary condition

$$C(T,i) = \max(\eta(T,i) - X, 0) \; for \; i = 0,1,2,\cdots,T \qquad (\eta\ 4)$$

However, similar to the credit valuation model, the forward inflation index and the option of the inflation index are not directly observable in the capital markets. We need to relate the forward inflation index to tradable financial instruments.

Treasury Inflation Protection Securities (TIPS) are US Treasury bonds which have the principal adjusted to the inflation rates. For clarity of exposition, we make several simplifying assumptions here on the specification of TIPS. We assume that the bond has a coupon rate k for each binomial period, with an initial the principal and maturity of $1 and T periods respectively. Furthermore, we assume that the convexity effect of TIPS is negligible such that a static inflation model can be used.

Then the value of TIPS is given by

$$V(T) = c(1/\eta(1)P(1) + 1/(\eta(1)\eta(2))P(2) + \cdots 1/(\eta(1)\cdots\eta(T))P(T)) + 1/(\eta(1)\cdots\eta(T))P(T) \qquad (\eta\ 5)$$

where $P(n)$ is the discount function. Given the values $V(n)$ for $\mathrm{n} = 0,1,2,\cdots,$, we can determine the η(n).

The implied term structure of inflation rates can be estimated from the TIPS as many TIPS have been issued since their introduction in 1997. DePrince Jr. (2003) derives the term structure of inflation rates using TIPS from February 1997 to September 2002 and the paper shows that the implied inflation has economic content. Since inflation rate forecasts using surveys are only short term, the implied inflation rates can be useful for market participants. There are many contingent claims on inflation rates. Many insurance and pension products are indexed on inflation rates. Therefore a unified model incorporating inflation rates has many applications.

IV. Modeling the Stochastic Process of the Unified Model

The main extension of the Ho-Lee model from the Black-Scholes model is based on the definition of the “underlying securities”. The Ho-Lee model does not require the underlying risk driver to be represented by a tradable security. Instead, the Ho-Lee model determines the stochastic movements of the forward contracts on the flow risk driver based on the market observable forward prices of the risk factor.

Based on the risk neutral measure, a security drifts at a risk free rate in the Black-Scholes framework. However, this is not a requirement for any risk factors in the Ho-Lee framework. For this reason, much of the interest rate model research focuses on modeling the stochastic movements the risk driver when the Ho-Lee framework is used. We summarize some of the results in modeling the risk driver movements in this section.

The Ho-Lee (1986) unified model can be extended to a generalized model (Ho-Lee (2007)) that allows the risk driver to exhibit a broad range of behavior. We summarize some of the characteristics below. The model is presented in Appendix B.

1. *State-time dependent volatilities*. In section B, we use the Ho-Lee model (1986) to illustrate the unified model. To capture a broader range of stochastic movements of the risk factor, δ in equation (u2) does not have to be constant. In fact, it can be state and time dependent, $\delta = \delta(n,i)$. Therefore, the model can be calibrated to an implied volatility function. In this case, the stochastic risk index can be shown to have the following functional form below:

$$u(n,i) = u(n)F(n,i)\delta(n,i) \tag{8}$$

The function $F(n,i)$ is determined by the specification of the $\delta(n,i)$. An implied volatility function that declines with time n can induce a mean reversion behavior of the risk factor. This observation is made in the Black, Derman and Toy model (1990).

2. *Regime switching*: The model avoids explosively high rates exhibited by the lognormal models on the one hand and negative rates exhibited by the normal models on the other. Since we can construct δ to be dependent on the state, the forward volatilities of each factor can be proportional to the rate level. When the rates are low, the process exhibits a lognormal behavior. When the rates are high, the process exhibits a normal distribution behavior. Cheyette (1997) shows that interest rate movements tend to change the stochastic process regime depending on the interest rate level. Ho and Mudavanhu (2007) show empirically that the specification of the change in regime in the interest rate distribution can be implied from the swaption markets.

3. *Multi-movement model:* The movements of a risk factor can have multiple risk drivers. At each node, each factor can take a binary movement. Let *e* be a movement over one step, represented by 0 or 1; and there are four possible states emulating from each node for a two factor model represented by $v = (e1, e2)$. Let e* represent the opposite state of e such that $e^* = 0$ and 1 when $e = 1$ and 0 respectively. Then, the two consecutive movements of $(e1, e2)$ followed by (e^*1, e^*2) must meet at the same node. This is the recombining property that reduces the possible future scenarios for analysis significantly without a comparable loss of accuracy of the model.

For clarity of exposition, we have presented the two-factor model which is $u(n,i,j) = u(n)F(n,i)F(n,j)\delta(n,i)\delta(n,j)$. The generalization of the two-factor model to the multifactor model for equation u2 is straightforward. For an m-factor model, (u2) is represented by

$$u(n, i_1, \ldots, i_m) = u(n)C(n,m)\Lambda(i_1, \ldots, i_m) \tag{9}$$

where $C(n,m) = \left[\frac{2}{1+\delta_1{}^n}\right] \ldots \left[\frac{2}{1+\delta_m{}^n}\right]$ and $\Lambda(i_1, \ldots, i_m) = \delta_1^{i_1} \ldots \delta_m^{i_m}$

This shows that $u(n, i_1, \ldots, i_m)$ has three components. First, $u(n)$ represents the forward price based on the initial term structure. Second, $C(n,m)$ is the convexity term given by $\left[\frac{2}{1+\delta_1{}^n}\right] \ldots \left[\frac{2}{1+\delta_m{}^n}\right]$. This term adjusts the price such that the additional returns derived from the convexity are countered by this adjustment to assure the arbitrage-free condition. Third, the stochastic movement term $\Lambda(i_1, ..., i_m)$ is given by $\delta_1^{i_1} \ldots \delta_m^{i_m}$

The arbitrage-free condition for the model in equation (9) is proved in Ho-Lee (2007). The model for multiple risk drivers can be defined analogously. For the term structure of interest rates, Litterman and Schienkman (1991) have shown that historically, the yield curve exhibits three principal movements. Ho and Mudavanhu (2007) describes a multi-movement model that is consistent with this empirical observation where the arbitrage-free yield curve movements can be parallel movements or steepening movements, and the movements are implied from the swaption market using multiple implied volatility functions.

Multi–factor model. We use the analogous multi-dimensional recombining lattice described in subsection 3. Combining multiple risk factors is also relatively straight forward. Let the risk factor be the combined of m risk factors, $u^1(n,i) \ldots\ u^{i_m}(n, i_m)$, then, the model is:

$$u(n, i_1, \ldots i_m) = u^{i_1}(n) \ldots u^{i_m}(n) F^{i_1}(n, i_1) \ldots F^{i_m}(n, i_m) \delta^{i_1}(n, i_1) \ldots \delta^{i_m}(n, i_m) \tag{10}$$

Therefore, the unified model enables us to value securities subjected to several types of risks. Ho and Lee (2008 b) provides a more detail description of the multi-factor model. When the risk factors are uncorrelated, then all the binomial probabilities are the same. Otherwise, appropriate binomial probabilities give the desired correlations (see Ho-Lee (2004a) pp 249). Note that when there are correlations between two factors, then a non-option embedded security in one factor may have embedded options from another factor. For example, when there is a correlation between the credit risk and interest rate risk, a CDS may have embedded interest rate options. The option is induced by the recovery payments at default viewed as interest rate risk "prepayment option" when default rate changes with the interest rate level. However, such an option value is typically very small.

Also, this observation does not mean that the methodology presented is inappropriate. We just need to take this observation into account in the calibration process.

4. *Arbitrage-free condition*. The model takes the forward discount factors and their associated term structure volatilities as inputs to the valuation. The projected discount rate of a fixed income instrument is given as the combination of the stochastic shifts from the sum of the forward curves of the factors with an adjustment for the convexity effect to ensure the arbitrage-free condition held. The value derived from the roll back valuation algorithm using the lattice model is always consistent with these input parameters.

5. *Term Structure of Rates and the Term Structure Movements:* in Section Ⅲ, we have focused on the stochastic movements of the discount indices by specifying the risk driver at each binomial node (n,i). In fact, using the roll back process, the entire term structure of the discount index can be determined at each node. For example, for the interest rate model, the term structure of interest rate can be determined at each node. For the credit model, the credit derivative swap curve can also be specified at each node of the binomial lattice. Therefore, the contingent claims can be specified for the entire term structure and not confined only to the one period risk driver. Furthermore, for a multi-factor model, at each node $(i, j, k \cdot\cdot)$, multiple movements of the term structures can be modeled.

V. Implications of the Unified Model to Risk Management

A. Relevance of the Unified Model to the Financial Crisis 2008

A tranche of a CDO has an embedded option within a CDO package, which in turn is pooled from asset backed security (ABS) tranches, themselves have embedded options. Each ABS is a pool of mortgage loans or bonds, which have also embedded options. These options of driven by multiple risk drivers. For this reason, the actuarial credit risk measure would not be inappropriate to identify the risks of these securities that led to the subprime crisis.

Managing credit, market, liquidity risks separately would not be able to provide a coherent valuation and analytical framework. For example, Basel II requirements suggest entirely different risk measurement approaches to market risk, credit risk and liquidity risks, and the Basel requirements provide no guidance in rolling up the risks to the enterprise level. As a result, financial institutions have separate market risk and credit risk departments. Such separation of risk analyses can easily fail to identify properly the risk of structured products such as the asset backed securities and collateralized debt obligations which have complicated embedded credit and interest rate options.

A coherent way in isolating the credit risk from interest rate risk is important. For example, an internal measure of the credit guarantee cost is a useful way to bifurcate the credit option value from the interest rate option in a bond. One of the Federal Home Loan Bank (FHLB) lending programs to their network of banks uses this approach. This approach allows their affiliated banks to retain the credit risk while selling the interest rate options to the FHLBs. The unified model can be used to identify these values.

The application of the unified model goes beyond the valuation of securities. Enterprises can be modeled as contingent claims of multiple flow risk drivers. For a financial institution, often the economic capital is defined as the asset net of the liability. However, in many instances, the liability cannot be valued separately from the assets. For example, the "run on a bank" behavior of the depositors can be viewed as the depositors' American put option on the banks' asset value with a strike price of the account value. The put option adds additional value to the deposit liability and lowering the appropriate measure of the economic capital. Another example is the margin requirements on some liabilities. Often, financial institutions' funding requires margins, which increase with the value of the assets net of the face value of liability. As a result of this requirement, the financial institution has to sell assets to generate cash as the asset liquidity spread widens. This process is called "deleveraging" which has shown to be very significant in the 2008 financial crisis. The descriptions of these implications of these embedded options in the economic capital are beyond the scope of this paper. However, we have highlighted the importance of using contingent claims analysis to the analysis on economic capital and risk management of a financial institution.

Financial research has studied corporate liabilities in terms of the Merton model, where equity is viewed as a call option on the firm value, the total capitalization, with the strike price equaling the face value of a bond. As the 2008 financial crisis shows, distressed financial institutions have value based more on their economic capital value than on the total capitalization. As noted above, the economic capital values are contingent claims on multiple flow risk drivers, where a unified model may be more appropriate for analysis than the Merton model that is based on the stock risk drivers.

B. Benchmark securities and model calibration

A portfolio of these fixed income securities with similar liquidity risk and credit risk that have market prices is called a benchmark portfolio. The term structures of volatilities of interest rates, default rates, and liquidity rates can be calibrated from the prices of the fixed income securities in the benchmark portfolio. The calibrated model can then be used to value other financial instruments that have similar credit risk and liquidity risk. Any discrepancy between the model value and the market price is called the "cheap/rich" value. Therefore, the financial instrument is relatively valued to the benchmark portfolio and that the model determines the risk neutral measures for the valuation purpose.

The unified model is calibrated to the appropriate benchmark portfolio that is chosen to solve the problem at hand. This approach recognizes that there are often other factors affecting the quoted prices, such as the bid-ask spreads, commissions, and other microstructure factors. The benchmark portfolio should contain securities relevant to the instruments to be relatively valued. For example, CDSs are not all liquid. The choice of CDSs used in the benchmark portfolio should take liquidity into account. In this sense, the unified model should be viewed as a generalized relative valuation model. The model provides the value of a security relative to a benchmark portfolio.

The calibrated term structure of default rates and liquidity rates should be upward sloping in normal market conditions. This is because the default risk and liquidity cost both increases with time to maturity. As a result, the forward default rates and liquidity rates must also increase in calendar time in projected into the future, as the maturity of the fixed income security shortens. However, when a fixed income security maturity shortens, we expect that both the credit spread and the liquidity spread tighten, not rising as predicted by the model under risk neutral measure, as opposed to the market measure. By the same token, the normal yield curve is upward sloping. The arbitrage-free interest rate models would consistently "predict" that the short term rates would rise in the future, and clearly this prediction is inconsistent with the historical experience. This apparent contradiction does not affect the robustness of the model which is based on the risk neutral measure.

C. Combining interest rate risk, credit risk and liquidity risk in enterprise risk management

Historically, from the deregulation of interest rates in the late 1970s till the mid 1990s, the US experienced significant interest rate risks fluctuating between 6% and 12%. However, in the 2000s, interest rates have fallen, to less than 4% in recent months. In this low interest rate regime, credit risk and liquidity risk have become significant contributors to the value of many financial instruments. That is, credit risk and liquidity risk factors have to be considered along with interest rate factors. The Ho-Lee framework is therefore proposed here to deal with this current market condition.

Let us consider the valuation of a callable bond to illustrate the economics of the unified model. Let the bond has a maturity of N years and is callable at par any time after M year, with M <N. The bond has a coupon rate such that the bond has a value at par. The bond is subject to interest rate, credit and liquidity risks.

The bond has an embedded call option. Interest rate models would consider the option subject only to the interest rate risk. The term structure of volatilities is estimated from the swaption market prices. However, the bond would be called if the credit risk premium or the liquidity premium has been tightened. The unified model would take the combined effect into consideration.

Valuation of fixed income securities, for example corporate bonds with call or put options, can be determined by the standard roll back procedure using the lattice model. The arbitrage-free condition would ensure that when the bond's call option is stripped off, the bond value calculated by the roll back method is identical to the standard cash flow bond model. Therefore, all arbitrage-free conditions would apply. For example, the put-call parity of options is automatically held. Since the arbitrage-free condition is satisfied at each node point, the roll back method using the Bellman's optimization can be used.

Thus far, we have described the valuation on a contingent claim on the term structures of interest rate, default rates and liquidity rates. We now extend this discussion to describe the valuation of a portfolio of credit and liquidity risks. Arbitrage-free interest rate models can extend from a single security valuation to that for a portfolio simply because the identical yield curve applies to all the securities. Such is not the case with credit and liquidity risks. The term structure of default rates and liquidity rates are not the same, not even perfectly correlated, across securities. A model of a portfolio of risk factors has to be described.

For simplicity, let us consider a portfolio of two bonds. To determine VaR, we assume that the risk sources are the discount function, the survival function and the liquidity function. Consider the log of these three functions. We assume that they are uniformly distributed and are constant in time. The associated term structures of volatility are flat for simplicity.

To determine the individual VaR, such as market, credit and liquidity VaR, we perturb each risk source while the other two risk sources are kept constant. To determine the interest rate VaR, we keep the other survival and liquidity functions fixed. Since the discount function is uniformly distributed, we generate the profit/loss distribution from the discount function disturbances.[5] Once we generate the distribution of the profit/loss, we can determine the maximum possible

[5] The three factor unified Ho and Lee model is given in Appendix B.

loss given the significance level, which is the Market VaR. Similarly, we can get credit VaR and liquidity VaR of an individual bond.

Since the three risk sources determine the bond value and the discount function is applicable across the bonds, we have altogether five random variables, which are uniformly distributed by assumption. We use the Gaussian Copula method to generate the joint distribution of five random variables. Specifically, we map the uniform distribution into the normal distribution on a percentile-to-percentile basis. Then, we generate multi-variate normal distribution with correlation coefficients being historical correlations for simplicity. We determine the portfolio VaR from the joint cumulative distribution, given the significance level. The diversification effect is the difference between the portfolio VaR and the sum of individual VaR. The diversification effect depends on the correlation structure among the risk sources.

The component VaR is defined by the contribution part of the portfolio VaR from each risk source. To determine the component VaR from a risk source, we eliminate the risk source while we keep the other risk sources. The decrease in the portfolio VaR can be regarded as the contribution of the risk source to the portfolio VaR. Then, we eliminate another risk source to determine the component VaR of the second risk source. Repeating the same procedure until we eliminate the last risk source, we can determine the component VaR of each risk sources.

D. Financial Engineering Considerations

A unified model must necessarily require extensive computational resources. Therefore, a specification of the unified model must be related to issues in financial engineering, as well as the principles in financial theories. Note that the unified model is constructed based on binomial periods, but the model does not require the binomial model to have constant calendar time period. For example, the binomial steps do not have to be one month for all periods. They can vary and therefore the unified model can be adapted to a variable step size construction quite simply.

Consider the callable bond in this illustration. Since the call protection period is only one year, the model must measure this one year short dated option accurately. The unified model with variable step size would use a small lattice step size in the early years. For the latter year, a longer step size is used to save the computing time without consequential loss of accuracy.

Since a recombining lattice enumerates all possible risk scenarios with the assigned risk neutral measures, within the context of the model, we can assign probabilities to each risk scenario. Furthermore, based on any measure of distance between two risk scenarios, we can form equivalent classes in this risk space with their representative risk scenarios. See Ho (1992) for such a methodology. As a result, we can determine the set of representative scenarios with their assigned probability weights for the purpose of valuation when scenario paths have to be used, as in the case of valuation a mortgage loan using a prepayment model. The structured sampling approach can enhance the computational efficiency significantly.

VI. Conclusions

This paper shows that risk drivers can be dichotomized into "stock" or "flow" attributes. The Black-Scholes model is a relative valuation model for the stock risk drivers and the Ho-Lee a model for the flow risk drivers. A unified model that values contingent claims on multiple flow risk drivers is presented here.

The model is then applied to five different flow risk drivers: interest rate, credit risk, liquidity risk, energy risk, and inflation risk. We show how the unified model can be adjusted for each type of flow risk drivers and how the benchmark securities are used to calibrate the models. Furthermore, we show that the unified model provides valuation of contingent claims on multiple flow risk drivers. Hence, the unified model provides a model and a framework to analyze the combined effect of multiple flow risk driver on securities and on the economic capital of a financial institution.

The 2008 financial crisis illustrates the importance of the applications of the unified model. For example, CDO and ABS are structured products that have significant credit and interest rate embedded options. This example shows that risk management should not take a silo approach to manage market risk, credit risk and liquidity risk in isolation. This is because the impact of the sum of these risks is not the same as that of the combined risks. The options embedded in the security do not offer a simple separation of the effects of the risk drivers.

Furthermore, we show that the economic capital of a financial institution is a contingent claim on multiple risk drivers and therefore the economic capital cannot be measured simply as the asset net of the liability. The unified model should be used to analyze a distressed financial institution along with the Merton model which is a relative valuation model on the stock risk driver and not modeling the economic capital as a contingent claim on flow risk drivers. The unified model can provide a coherent framework to manage the enterprise risk. In particular, we show how the model is used to generate the value-at-risk measures for the combined market, credit and liquidity risks.

Appendix A: A General form of the Arrow-Debreu Primitive Securities

The Arrow-Debreu primitive securities at time n are equal to a vector with (n+1) elements, which we can calculate by multiplying the following n matrixes.

$$\underset{1\times 2}{\begin{bmatrix} \frac{e^{-\rho(0)}}{1+\delta^0} & \frac{e^{-\rho(0)}}{1+\delta^0} \end{bmatrix}} \times \underset{2\times 3}{\begin{bmatrix} \frac{e^{-\rho(1)}\delta}{1+\delta^1} & \frac{e^{-\rho(1)}\delta}{1+\delta^1} & 0 \\ 0 & \frac{e^{-\rho(1)}}{1+\delta^1} & \frac{e^{-\rho(1)}}{1+\delta^1} \end{bmatrix}} \times \underset{3\times 4}{\begin{bmatrix} \frac{e^{-\rho(2)}\delta^2}{1+\delta^2} & \frac{e^{-\rho(2)}\delta^2}{1+\delta^2} & 0 & 0 \\ 0 & \frac{e^{-\rho(2)}\delta}{1+\delta^2} & \frac{e^{-\rho(2)}\delta}{1+\delta^2} & 0 \\ 0 & 0 & \frac{e^{-\rho(2)}}{1+\delta^2} & \frac{e^{-\rho(2)}}{1+\delta^2} \end{bmatrix}}$$

$$\times \underset{4\times 5}{\begin{bmatrix} \frac{e^{-\rho(3)}\delta^3}{1+\delta^3} & \frac{e^{-\rho(3)}\delta^3}{1+\delta^3} & 0 & 0 & 0 \\ 0 & \frac{e^{-\rho(3)}\delta^2}{1+\delta^3} & \frac{e^{-\rho(3)}\delta^2}{1+\delta^3} & 0 & 0 \\ 0 & 0 & \frac{e^{-\rho(3)}\delta^1}{1+\delta^3} & \frac{e^{-\rho(3)}\delta^1}{1+\delta^3} & 0 \\ 0 & 0 & 0 & \frac{e^{-\rho(3)}}{1+\delta^3} & \frac{e^{-\rho(3)}}{1+\delta^3} \end{bmatrix}} \cdots\cdots\cdots$$

$$\cdots\cdots\cdots \times \underset{n\times(n+1)}{\begin{bmatrix} \frac{e^{-\rho(n-1)}\delta^{n-1}}{1+\delta^{n-1}} & \frac{e^{-\rho(n-1)}\delta^{n-1}}{1+\delta^{n-1}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{e^{-\rho(n-1)}\delta^{n-2}}{1+\delta^{n-1}} & \frac{e^{-\rho(n-1)}\delta^{n-1}}{1+\delta^{n-1}} & \cdots & 0 & 0 \\ \vdots & \vdots & \frac{e^{-\rho(n-1)}\delta^{n-3}}{1+\delta^{n-1}} & \frac{e^{-\rho(n-1)}\delta^{n-3}}{1+\delta^{n-1}} & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{e^{-\rho(n-1)}}{1+\delta^{n-1}} & \frac{e^{-\rho(n-1)}}{1+\delta^{n-1}} \end{bmatrix}}$$

For example, when n is equal to 1, the Arrow-Debreu primitive securities at time 1 is the first matrix such that $\mathrm{A}(1,1)$ is equal to $\frac{e^{-\rho(0)}}{1+\delta^0}$ and $\mathrm{A}(1,0)$ is equal to $\frac{e^{-\rho(0)}}{1+\delta^0}$. When n is equal to 2, we can generate the Arrow-Debreu primitive securities at times 2 by multiplying the first two matrixes such that

$$\underset{1\times 2}{\begin{bmatrix} \frac{e^{-\rho(0)}}{1+\delta^0} & \frac{e^{-\rho(0)}}{1+\delta^0} \end{bmatrix}} \times \underset{2\times 3}{\begin{bmatrix} \frac{e^{-\rho(1)}\delta}{1+\delta^1} & \frac{e^{-\rho(1)}\delta}{1+\delta^1} & 0 \\ 0 & \frac{e^{-\rho(1)}}{1+\delta^1} & \frac{e^{-\rho(1)}}{1+\delta^1} \end{bmatrix}} = \underset{1\times 3}{\begin{bmatrix} \frac{\sum_{t=0}^{1} -\rho(t) \quad \delta}{\prod_{t=1}^{1} 1+\delta^t} & \frac{\sum_{t=0}^{1} -\rho(t) \quad (1+\delta)}{\prod_{t=1}^{1} 1+\delta^t} & \frac{\sum_{t=0}^{1} -\rho(t) \quad 1}{\prod_{t=1}^{1} 1+\delta^t} \end{bmatrix}}$$

The first element of the vector is $A(2,2)$, the second element is $A(2,1)$, and the last element is $A(2,0)$. Similarly, we can generate the Arrow-Debreu primitive securities at the following periods and determine them accordingly. The intuition behind the general form of the Arrow-Debreu primitive securities is the recursive relationship among $A(n,i)$, $A(n\text{-}1,i)$ and $A(n\text{-}1,i\text{-}1)$ for any n and i such as

$$A(n,i) = \frac{e^{-r(n-1)}\delta^i}{(1+\delta^{n\text{-}1})} A(n-1,i) + \frac{e^{-r(n-1)}\delta^{i-1}}{(1+\delta^{n\text{-}1})} A(n-1,i-1)$$

Appendix B: Three Factor Unified Model

The model is provided in Ho Lee (2007). For completeness, we specify the model in this appendix. Let $B_{i,j,l}^{n}(T)$ be the T year bond price at time *n*, at state (i,j,k) Then $B_{i,j,l}^{n}(T)$ is specified by combining three one-factor models. Specifically, we have

$$
\begin{aligned}
B_{i,j,l}^{n}(T) &= \frac{P(n+T)}{P(n)}\prod_{k=1}^{n}\left(\frac{1+\delta_{0,r}^{k-1}(n-k)}{1+\delta_{0,r}^{k-1}(n-k+T)}\right)\prod_{k=0}^{i-1}\delta_{k,r}^{n-1}(T) \\
&\times \frac{S(n-1+T)}{S(n-1)}\prod_{k=1}^{n}\left(\frac{1+\delta_{0,h}^{k-1}(n-k)}{1+\delta_{0,h}^{k-1}(n-k+T)}\right)\prod_{k=0}^{j-1}\delta_{k,h}^{n-1}(T) \\
&\times \frac{L(n+T)}{L(n)}\prod_{k=1}^{n}\left(\frac{1+\delta_{0,l}^{k-1}(n-k)}{1+\delta_{0,l}^{k-1}(n-k+T)}\right)\prod_{k=0}^{l-1}\delta_{k,l}^{n-1}(T)
\end{aligned}
\tag{B.1}
$$

where

$$
\begin{aligned}
\delta_{i,r}^{n}(T) &= \delta_{i,r}^{n}\delta_{i,r}^{n+1}(T-1)\left(\frac{1+\delta_{i+1,r}^{n+1}(T-1)}{1+\delta_{i,r}^{n+1}(T-1)}\right) \\
\delta_{i,h}^{n}(T) &= \delta_{i,h}^{n}\delta_{i,h}^{n+1}(T-1)\left(\frac{1+\delta_{i+1,h}^{n+1}(T-1)}{1+\delta_{i,h}^{n+1}(T-1)}\right) \\
\delta_{i,l}^{n}(T) &= \delta_{i,l}^{n}\delta_{i,l}^{n+1}(T-1)\left(\frac{1+\delta_{i+1,l}^{n+1}(T-1)}{1+\delta_{i,l}^{n+1}(T-1)}\right)
\end{aligned}
\tag{B.2}
$$

and the one period forward volatilities are given by definition,

$$
\begin{aligned}
\delta_{i,r}^{m}(1) = \delta_{i,r}^{m} &= \exp\left(-2\cdot\sigma_{r}(m)\min\left(h_{i,r}^{m},H_{r}\right)\Delta t^{3/2}\right) \\
\delta_{i,h}^{m}(1) = \delta_{i,h}^{m} &= \exp\left(-2\cdot\sigma_{h}(m)\min\left(h_{i,h}^{m},H_{h}\right)\Delta t^{3/2}\right) \\
\delta_{i,l}^{m}(1) = \delta_{i,l}^{m} &= \exp\left(-2\cdot\sigma_{l}(m)\min\left(h_{i,l}^{m},H_{l}\right)\Delta t^{3/2}\right)
\end{aligned}
\tag{B.3}
$$

Where the functions $\sigma_{m}(n) = (a+bn)\exp(-cn)+d \; where \; m = r,h,l$ are specified by the parameters *a*, *b*, *c*, and *d*, which can be obtained from the calibration to the market price of an option on CDS, etc. This specification of the implied volatility function allows for a broad range of shapes including downward sloping or dumped shape.

Using the direct extension, we can specify the one period hazard rates for the two-factor model for any future period *m* and state *i*, and $h_{i,1}^{m}$ and $h_{i,2}^{m}$ are defined by

$$
\begin{aligned}
h_{i,r}^{m}\Delta t &= -\log\left(\frac{P(m+1)}{P(m)}\right) - \sum_{k=1}^{m}\log\left(\frac{1+\delta_{0,r}^{k-1}(m-k)}{1+\delta_{0,r}^{k-1}(m-k+1)}\right) - \sum_{k=0}^{i-1}\log\left(\delta_{k,r}^{m-1}(1)\right)\\
h_{i,h}^{m}\Delta t &= -\log\left(\frac{S(m)}{S(m-1)}\right) - \sum_{k=1}^{m}\log\left(\frac{1+\delta_{0,h}^{k-1}(m-k)}{1+\delta_{0,h}^{k-1}(m-k+1)}\right) - \sum_{k=0}^{i-1}\log\left(\delta_{k,h}^{m-1}(1)\right)\\
h_{i,l}^{m}\Delta t &= -\log\left(\frac{L(m)}{L(m-1)}\right) - \sum_{k=1}^{m}\log\left(\frac{1+\delta_{0,l}^{k-1}(m-k)}{1+\delta_{0,l}^{k-1}(m-k+1)}\right) - \sum_{k=0}^{i-1}\log\left(\delta_{k,l}^{m-1}(1)\right)
\end{aligned} \qquad \text{(B.4)}
$$

REFERENCES

Bandreddi, S., S. Das, and R. Fan, 2007, Correlated Default Modeling with a Forest of Binomial Trees, *Journal of Fixed Income* 17, pp.38-56.

Black, F., E. Derman, and W. Toy, 1990, A One-factor Model of Interest Rates and Its Application to Treasury Bond Options, *Financial Analysts Journal* 46, pp.33-39.

Black, F., and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, Journal of Political Economy 81, pp. 637-654.

Brace, A., D. Gatarek, and M. Musiela, 1997, The Market Model of Interest Rate Dynamics, *Mathematical Finance* 7, pp. 127-155.

Chen, L., D. Lesmond, and J. Wei, 2007, Corporate Yield Spreads and Bond Liquidity, Journal of Finance 62, pp. 119-149.

Cheyette, O, 1997, *Interest rate models* In Frank J. Fabozzi, ed., Advances in Fixed Income Valuation, Modeling, and Risk Management, New Hope, PA: Frank J. Fabozzi Associates.

Cox, J., J. Ingersoll, and S. Ross, 1985, A theory of term structure of interest rates, *Econometrica* 53, pp. 363-384.

Covitz, D. and C. Downing, 2007, Liquidity or Credit Risk? The Determinant of Very Short-Term Corporate Yield Spreads, Journal of Finance 62, pp. 2303-2328.

Das, S., L. Freed, G. Geng, and N. Kapadia, 2006, Correlated Default Risk, Journal of Fixed Income 16, pp. 7-32.

DePrince Jr. A, 2003, Assessing the Term Structure of Expected Inflation Using Treasury Inflation-Protected Securities, Business Economics 38, pp. 46-54.

Duffie, D, 2005, Credit Risk Modeling with Affine Processes, Journal of Banking and Finance 29, pp. 2751-2802.

Eydeland, A., and K. Wolyniec, 2003, Energy and Power Risk Management: New Developments in Modeling, Pricing, and Hedging. John Wiley & Sons.

Harrison, J. and D. Kreps, 1979, Martingales and Arbitrage in Multi-Period Securities Markets, *Journal of Economic Theory* 20, pp. 381-408.

Heath, D., R. Jarrow, and A. Morton, 1992, Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica* 60, pp. 77-105.

Ho, T. S. Y., and S. B. Lee, 1986, Term structure movements and pricing interest rate contingent claims, *The Journal of Finance* 41, pp. 1011-1029.

Ho, T. S. Y., and S. B. Lee, 2004a, *The Oxford Guide to Financial Modeling.* New York: The Oxford University Press.

Ho, T. S. Y., and S. B. Lee, 2004b, A Closed-Form Multifactor Binomial Interest Rate Model, *Journal of Fixed Income* 14, pp. 8-16.

Ho, T. S. Y., and S. B. Lee, 2007, Generalized Ho-Lee Model: A Multi-factor State-Time Dependent Implied Volatility Function Approach, *Journal of Fixed Income* 17, pp. 18-37.

Ho, T. S. Y., and S. B. Lee, 2009a, A Unified Credit and Interest Rate Arbitrage-free Contingent Claim Model, Journal of Fixed Income 18, pp.5-17.

Ho, T. S. Y., and S. B. Lee, 2009b, Valuation of Credit Contingent Claims: An Arbitrage-free Credit Model, Journal of Investment Management 7, pp.49-65.

Ho, T. S. Y., and B. Mudavanu, 2007, Interest Rate Models' Implied Volatility Function Stochastic Movements, *Journal of Investment Management* 5, pp. 1-22.

Litterman, R., and J. A. Scheinkman, 1991, Common Factors Affecting Bond Returns, *Journal of Fixed Income* 1, pp. 54-61.

Longstaff, F., and A. Rajan, 2008, An Empirical Analysis of the Pricing of Collateralized Debt Obligation, Journal of Finance 63, pp. 529-563.

Rebonato, R, 1998, Interest Rate Option Models 2nd ed. Chichester: John Wiley.

Vasicek, O, 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, pp. 177-188.