Pricing and Hedging Crude Oil Futures Options with Term Structure Models

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Abstract
Consistent with the cost-of-carry argument, term structure of futures prices is viewed as a structure and modeled by term structure models to price and hedge crude oil futures options. After examining several competing models, two-factor models in conjunction with specification of time to maturity in volatility function consistently outperform one-factor models in in-sample, out-of-sample, and hedging. While more-parameter models tend to provide better in-sample fitting than less-parameter models, they tend to overfit to option prices across strikes. Confirmed by the close relationship between futures price/volatility and time to maturity, all term structure models with various volatility functions perform better than the Black's model, but they tend to miss-specify to a certain scale due to the presence of mispricing.

Keyword: term structure model; futures options; crude oil option; volatility function
JEL Classification code : G12, G13

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1. Introduction

In the last three decades, option pricing has witnessed an explosion of new models and each relax some of the restrictive Black-Scholes assumptions (See the discussion of Bates, 1996; Bakshi et al., 1997; survey of option pricing model of Bates, 2003). Most works focused on how to improve to be quantitative consistent with the time series property of the underlying asset and option prices. One category of works considers stochastic nature of option variables (such as stochastic volatility).[1] The other category looks at deterministic variable, but attempts to relax the geometric Brownian motion assumption of Black-Scholes.[2] While both categories of models have their own strengths and weaknesses, it is important to ask whether it is necessary to have more realistic features with additional costs of complexity and implementing difficulties.

Even if implementation issues are ignored, the model with realistic feature is not free of problems. For instance, Bates (2003) states that standard stochastic volatility models with plausible parameters cannot easily match observed volatility smiles and smirks, and thud Bakshi et al. (1997) identified that extremely high levels of volatility-return correlation and volatility variation are required. Although incorporating a random jump into stochastic volatility model improves pricing S&P500 option across strikes, Bakshi et al. (1997) found that the incorporation does not seem to improve the stochastic volatility model's hedging performance further. Further, the results of Bakshi et al. (1997) show that even a model with stochastic volatility and jump is still highly misspecified in terms of internal consistency between model's implied parameters and (a) the S&P 500 returns, (b) the (implied) volatility, and (c) the spot interest rate. Jarrow et al. (2007) found that stochastic volatility and jump model cannot remove volatility smile when using interest rate cap.

Due to the limitation and problems reported by past literature, this study considers the model with deterministic volatility. In fact, we examine term structure models in pricing and hedging futures options. Under cost of carry argument, futures price and volatility with different time to maturity should contain some relationship. To model the relationship, term structure model (or interest rate model), originally used to describe evolution of interest rates, may be used. Carr and Jarrow (1994) show that

[1] The stochastic-volatility models include Heston (1993), Hull and White (1987), Melino and Turnbull (1990, 1995), Scott (1987), Stein and Stein (1991), and Wiggins (1987).

[2] Those models include the jump-diffusion pure jump models of Bates (1991), Madan and Chang (1996), and Merton (1976).

virtually any spot or futures options, such as stock/stock index, currency, commodity futures options, can be priced by term structure models with stochastic underlying asset price and stochastic interest rates. Unlike stochastic volatility model, option valuation generally requires a market price of risk parameter, which among other things, is difficult to estimate. While the volatility under term structure model is non stochastic, risk-neutral valuation of contingent claims can be maintained.

There are several strengths for term structure models. First, volatility functions, described the term structure of volatility, can be freely specified, and thus provide a tremendous opportunities for examining its validity. Second, to accommodate more diverse shape of volatility function and reconcile the recent empirical finding supporting multifactor models[3], the models can be extended to models with several factors. Third, under term structure models, futures options can be priced with stochastic futures prices and stochastic interest rates. However, the underlying asset price process of term structure model is non-Makov, option valuation with a lattice approach will have a non-recombined tree. Past empirical examination of interest rate options show relative few steps of tree can generate a converged value.[4]

This study compares alternative term structure models, nested in a general volatility function and the models are extended to multifactor framework. These models are evaluated with their time-series performance in terms of in-sample fitting, out-of-sample prediction, and hedging crude oil options across strikes and maturiies. Models are judged based on which model can produce the consistently smallest pricing or hedging errors. For comparison, the Cox, Ross, and Rubinstein (1979), a binomial version of Black's model is considered, and as the benchmark model.

The paper is organized as follow. The second section illustrates the term structures of futures price and volatility of crude oil options. Models and data are introduced and explained. Next, data and volatility parameters estimated are provided. The third section provides in-sample, out-of-sample, and hedging results. Finally, conclusion is provided.

2. Term structure of futures price and volatility

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[3] See Driessen, et al. (2003); Gupta and Subrahmanyam (2005); and Jarrow et al. (2007), Christoffersen, et al. (2009).

[4] For example, Amin and Morton (1994) found that to price Eurodollar futures options using Heath, Jarrow, Morton model, 10 steps of non-recombined tree is needed for one-factor model. Similar finding is documented by Kuo and Paxson (2006) who found that 6 steps of tree is necessary for obtaining a convergence value for two-factor Heath, Jarrow, and Morton model.

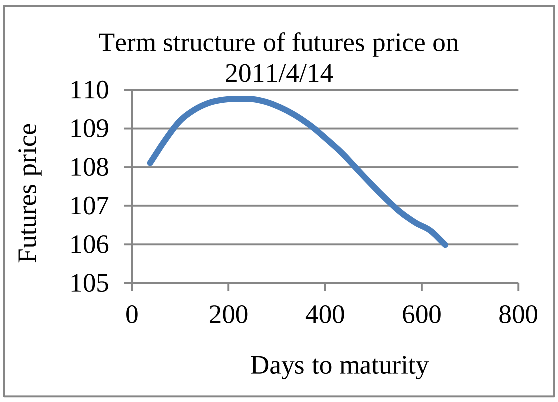
Options are products with payoffs and prices dependent upon the stochastic evolution of the asset prices and associated underlying financial variables. Commodity futures options with various maturities thus are subject to term structure of futures prices and volatilities. The characteristics are displayed in Figure 1.

Figure 1 offers 4 graphs of term structure of crude oil future price and volatility. The crude oil futures traded in CME contain a rich term structure seen in Panel A. The hump-shape curve observed on 14 April 2011, reflects that convenience yield becomes important when holding spot assets whereas long term crude oil futures provide no benefit for immediate uses. The time series of futures prices shown in Panel B is based on 60 days obtained from interpolation of adjacent prices. The crude oil futures reach to the peak of nearly $150 per barrel in June 2008, and decline sharply subsequently due to global financial crises.

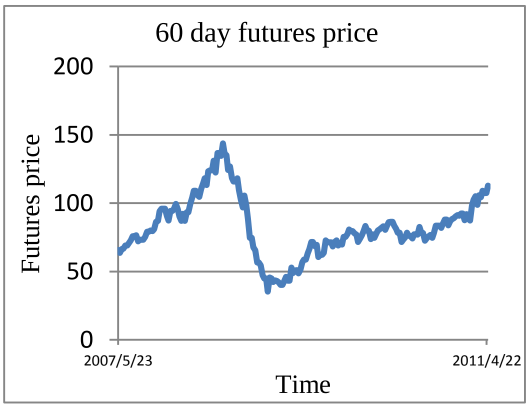
We also see the upward curve of the term structure of future prices of volatility shown in Panel C, consistent to the prediction of the cost of carry theory. The term structure of implied volatility is obtained from ATM call options across the spectrum of maturity on 17 November 2010, using Black's model. We see the upward shape of the structure in which the long-term futures price is more volatile than short-term counterparts. Panel D displays volatility smile for calls and puts. Call smile appears to show a U-shape where ITM volatility is greater than OTM volatility, but it is not asymmetric around zero. Nevertheless, for put smile, an opposite pattern is found.

Figure 1 Term Structure of futures price and volatility

Panel A

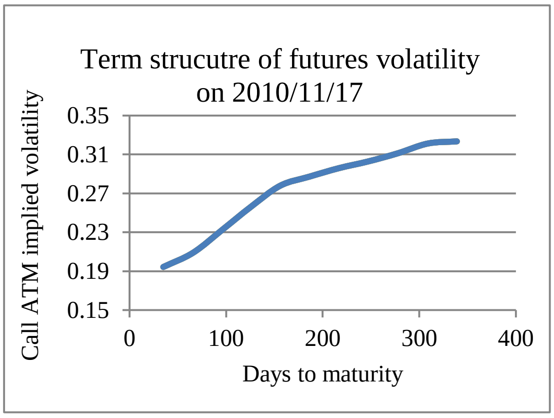


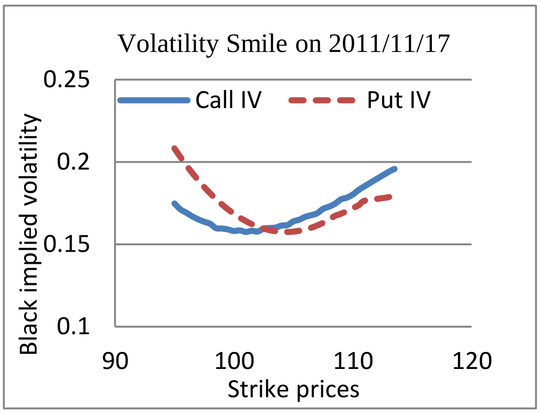
Panel B



Panel C

Panel D





A number of option valuation models are capable of explaining the behavior documented in Figures 1. The stochastic volatility of Heston (1993), for example, can explain this behavior when the asset price and volatility are negatively correlated. The negative correlation is what produces the asymmetric volatility smile observed in Figure 1, not the stochastic feature itself (Dumas, et al. 1998). Similarly, the jump model of Bates (1996) can generate these patterns if the average jump is negative. It is simplest to consider the term structure model with various volatility functions because apart from volatility parameters embedded in volatility functions, there are no additional unobservable parameters that need to be estimated.

2.2 Model

Interest rate models are used to model the term structure of interest rates and are used to price and hedge interest rate contingent claims. Commodity futures, such as crude oil futures prices containing a rich term structure of futures prices, can also apple this concept to describe term structure of futures price. To model futures price, the Heath, Jarrow, and Morton (1990, 1992) equation (hereafter called term structure model) can be specified as,

$$dF(t,T) = \mu(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T,F(t,T))dZ \tag{1}$$

where F(t,T) is the futures price for the contract maturing at T and traded on date t. $\mu(t,T)$ is the drift of instantaneous futures price and $\sigma(t,T,F(t,T))$ represents the instantaneous standard deviation of the futures price with maturity T at date t, which can be chosen rather arbitrarily, dz follows a one-dimension Brownian motion. The expression i=1,…, n indicates the number of factor.

When a number of regularity conditions and a standard no-arbitrage condition are satisfied, then the drift of the futures price under the risk-neutral measure is unique determined by the volatility functions,

$$\mu(t,T)dt = \sum_{i=1}^{n} \sigma_i\big(t,T,F(t,T)\big) \int_t^T \sigma_i\big(t,T,F(t,\mu)\big)\, du \tag{2}$$

Thus, the futures price processes are completely governed by their volatility function, where the volatility function can be chosen freely. Equation (2) provides the general volatility functions, which contains special cases for most existing option valuation models. If the volatility is specified with a constant number, it turns out to be the case of Black's model. If the volatility function is specified with a parameter and a portion of square root of futures price, then it is the case of Cox, et al. (1985).

Since equations (1) and (2) are originally specified with interest rates, the futures option price can be determined by stochastic futures price and stochastic interest rates. Valuation of futures options using term structure models implies that futures price and interest rates are stochastic. To implement this term structure model, one can use binomial tree method with a number of steps to determine option value. Since the term structure model is path dependent, the recombined tree cannot be obtained. Carr and Jarrow (1994) indicate that discrete version of this continuous version process within the framework of one-factor model can be described as the log price relatives that is:

$$F(t+\Delta,T) = \begin{cases} F(t,T)e^{\mu(t,T)\Delta+\sigma(t,T,F(t,T))\sqrt{\Delta}} & if \;\; 'up' \\ F(t,T)e^{\mu(t,T)\Delta-\sigma(t,T,F(t,T))\sqrt{\Delta}} & if\; 'down' \end{cases} \tag{3}$$

$\mu(t,T)$ has the interpretation as the instantaneous expected growth rate in F(t,T) and σ(t,T) has the interpretation as the instantaneous expected volatility of relative change in *F(t,T)*. Δ refers as the length of each time step. The probability of the up move is

$$\pi_t = \frac{1-exp(\mu(t,T))\Delta-\sigma(t,T,F(t,T))\sqrt{\Delta}}{exp(\mu(t,T)\Delta+\sigma\big(t,T,F(t,T)\big)\sqrt{\Delta}-exp(\mu(t,T)\Delta-\sigma(t,T,F(t,T))\sqrt{\Delta}} \tag{4}$$

The probability of down movement is $1-\pi_t$.

When $\mu(t,T) = -\frac{\sigma(t,T,F(t,T))^2}{2}\Delta$, π_t can be set to be 1/2.

Similarly, the equations (3) and (4) can be modified to multifactor models. Under the two-factor model, there will be three directions at a subsequent step with 3^n nodes,[5] and four directions with 4^n nodes for three-factor models for n steps.

2.2 Principal Component Analysis (PCA)

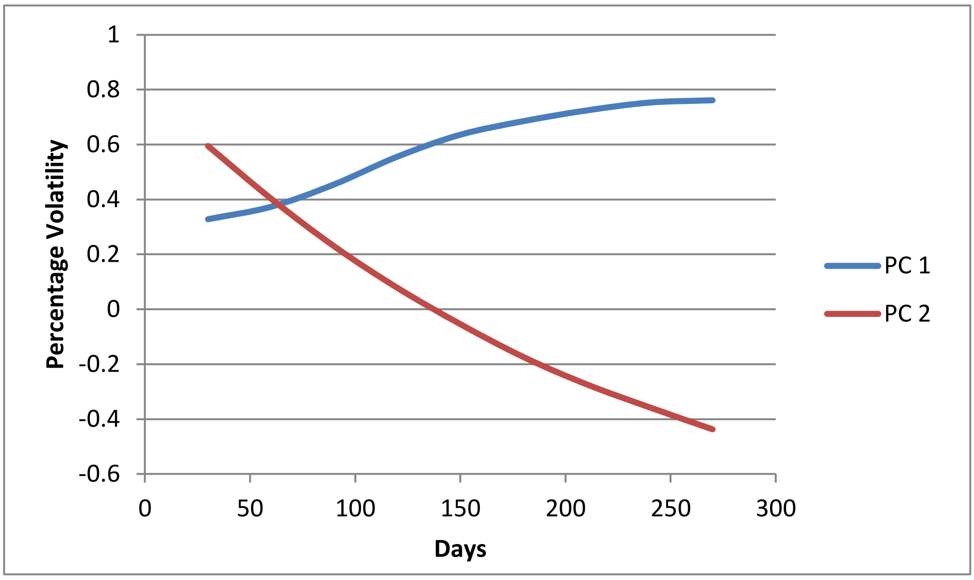
Under the model in equations (1) and (2), the change in futures price is completely determined by the volatility function where the function can be freely chosen. Jarrow (2002) and Driessen et al. (2003) suggest that one can use PCA to extract volatility function of the underlying asset returns. Christoffersen et al. (2009) apply PCA to obtain the number of factors, which governs the volatility smile. Hence, we run PCA from the return of historical underlying prices and option prices across strike prices to determine the volatility function and the number of factors. Figure 2 plots the average of the volatility function from PCA calculated based on weekly return of crude oil futures prices with the maturity from 30 to 270 days within the sample period in this study. Return of each futures prices series is computed with two steps. First, each day interpolation from two nearest maturity of futures prices is applied to obtain the desired maturity of futures price. In this step, we obtain 9 time series of futures prices with an increment of 30 days of maturity. Second, futures price return is computed from the log difference of futures price.

The result show that the first component explains 88.45% of the variation of the data, and that the first two components together explain over 98% of the variation in the data. The results therefore seem to suggest that a two-factor model may be a good representation of the data. The first factor explains the major volatility function, which governs the remaining time to maturity for futures prices whereas the second factor explains the decaying structure of volatility. Thus, the first factor may be called a level factor and the second factor may relate to the curvature. Since we use a rolling horizon strategy to determine futures price, the estimated volatility functions change weekly, but the shapes of these volatility functions turn out to be quite constant over time. Hence, it is unnecessary to price options by switching one volatility function to the next.

Figure 2 Principal Component Analysis

The principal component is calculated based on the return of weekly Crude oil futures prices from 10 January 2007 to 27 April 2011 where the return is obtained from the logarithm of price difference.

[5] See chapter 15 in Jarrow (2002) for the procedure of the two-factor model.



Following Christoffersen et al. (2009), PCA is implemented again from Black's implied volatilities across moneyness where moneyness is defined as futures price over strike price for calls and strike price over futures price for puts. To do that, each Wednesday we fit into a linear regression with the regressor of moneyness, square of moneyness, and day to maturity. Second, given the coefficients, then we calculate implied volatilities across moneyness (8 series) between 0.8 and 1.2 with 30-day maturity of option. In total we have 204 weekly time series from our sample period.

Figure 3 Principal Component Analysis according Moneyness

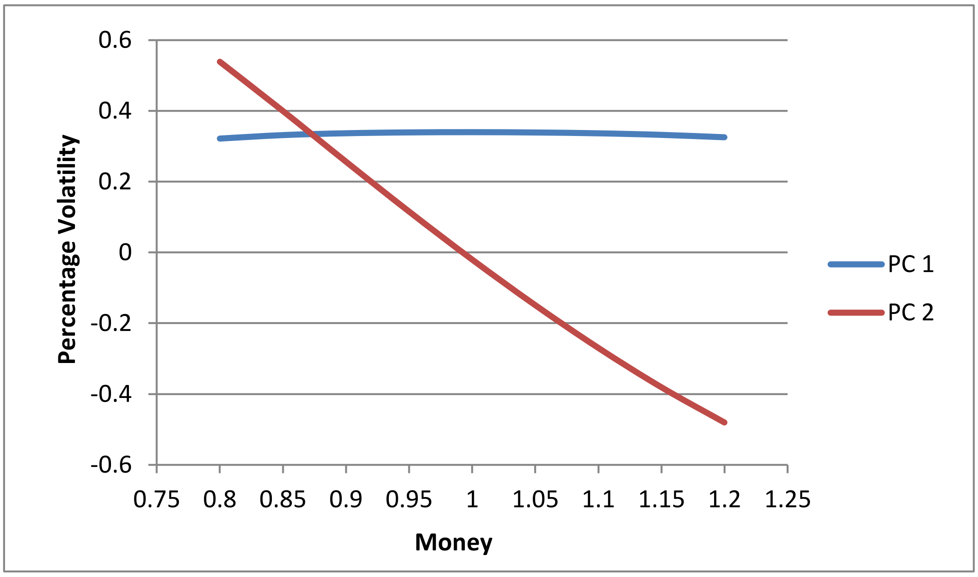


Figure 3 depicts the pattern for the average of volatility function based on volatility across moneyness. The volatility function turns out to be quite similar as that produced by futures price return. The results confirm that there are two factors explaining the volatility smile of crude oil futures options, which control 99% of variation of volatility smile, with the first factor explaining about 95% and the second factor explains the rest. In summary, to capture the stylized facts of term structure of futures price and options across moneyness, models with two factors should be needed.

2.3 Specification of Volatility Function

Since the change of futures prices in term structure model is completely governed by volatility function, the specification of the function matters performance of term structure model. One approach is to compare several competing functions guided by the PCA. Table 1 expresses the specification of volatility function selected in this paper. Model 1 is the simplest model, making volatility of futures prices to be constant across the spectrum of time to maturity. Model 2 allows the volatility function to be proportional to the futures prices. Since the first factor obtained from PCA is affected by time of maturity of futures prices, Model 3 and 4 allow volatility to be linearly or nonlinear affected by time to maturity of futures price. Model 5 and 6 incorporate a second factor to capture more diverse shapes of term structure of futures price volatility where the correlation between first and second factor is assumed to be independent.

Specifically, we consider the general form of volatility function as follows:

$$\sigma(t,T) = \{\sigma_1 F(t,T) + \sigma_2 (T-t) \times exp[-\lambda(T-t)]\} \quad (5)$$

Table 1 Models Used in This Study

The volatility function $\sigma(t,T,F(t,T))$ is simplified as $\sigma(t,T)$ where t is the time for futures contract traded on date t and matured on date T.

Model	Specification
Model 1 (Absolute):	$\sigma(t,T) = \sigma_1$
Model 2 (Proportional):	$\sigma(t,T) = \sigma_1 \times F(t,T)$
Model 3 (Exponential):	$\sigma(t,T) = \sigma_1 \times \exp^{-\lambda\times(T-t)}$
Model 4 (linear proportional)	$\sigma(t,T) = \sigma_1 + \sigma_2 \times (T-t)$
Model 5	$\sigma_1(t,T)=\sigma_1$
	$\sigma_2(t,T)=\sigma_2 + \sigma_3 \times (T-t)$
Model 6	$\sigma_1(t,T)=\sigma_1$
	$\sigma_2(t,T)=\sigma_2 \times \exp^{-\lambda\times(T-t)}$

3 Data and Methodology

3.1 Data

To price and hedge commodity options using term structure models, we need to select those options which are actively traded. Light sweet crude oil options are traded in New York Mercantile exchange which is currently listed in Chicago Mercantile Exchange. The advantage of using crude oil options is that the underlying asset is futures contract rather than spot assets and thus convenience yield can be ignored. More importantly, crude oil futures contain a rich term structure and their option prices critically depends upon the term structure of volatility, which provides a good opportunity to test ability of term structure models.

In this paper, light sweet crude oil options between 23 May 2007 and 28 April 2011 are selected. Three exclusionary criteria are imposed for selecting the observations. First, we exclude the contracts with no trading volume for the concern of price discreteness. Second, those options with less than 6 and more than 180 days are excluded in the sample. Contracts beyond 180 days are infrequently traded and less than 6 days that may contain microstructure effect (such as expiry effect). Third, we exclude options with moneyness outside the range of 0.8 and 1.2 where moneyness is defined as futures price divide strike price for calls and strike price divide futures price for puts. Finally, those option prices below 0.4 are excluded since they are infrequently traded and these options may incorrectly magnify our percentage of pricing or hedging errors.

Table 2 offers a summary of the characteristics of the transactions contained in the 205-week sample period. Of the 21,472 transactions, 11,415 were call option transactions and 10,057 were puts. The at-the-money options appear to have been the most active, with 50 percent of the call option trades and 51 percent of the put optionOut-of-the-money options were more active than in-the-money options, which have 40 percent and 10 percent of total trades for calls and 45 percent to 5 percent for puts, respectively. Over 60 percent of the transactions were on options with maturities han 90 days, verifying that most of the trading activity was in the nearest contract month.

The yield on the U.S. Treasury bill maturing on the contract month expiration day was used to proxy for the riskless rate on interest. The yields were computed weekly on the basis of the average of the T-bill's bid and ask discounts reported in the Wall Street Journal.

Table 2 Data Description

Weekly light sweet crude oil futures options between 23 May 2007 and 28 April 2011 are selected using four criteria given in the text. ITM, ATM, and OTM denote in-the-money options, at-the-money option, and out-of-the-money option, respectively. No. Obs. is number of observation. Tra. Vol is the trading volume and OI is open interest of options.

	Maturity		0-90 day			90-180 day		
	Money	Indicator	No.Obs	Tra. Vol.	OI	No.Obs	Tra. Vol.	OI
Call	1.1<F/K≦1.2	ITM1	340	247	4720	160	213	3724
	1.05<F/K≦1.1	ITM2	520	180	5176	170	128	4243
	1<F/K≦1.05	ATM1	1662	382	4166	906	270	3180
	0.95<F/K≦1	ATM2	2031	572	4660	1130	304	4169
	0.9<F/K≦0.95	OTM1	1283	532	5168	928	323	5045
	0.8<F/K≦0.9	OTM2	840	635	5900	1445	395	5922
Put	1.1<K/F≦1.2	ITM1	145	193	5037	41	93	5251
	1.05<K/F≦1.1	ITM2	251	191	4487	64	170	4437
	1<K/F≦1.05	ATM1	1207	337	3423	567	271	568
	0.95<K/F≦1	ATM2	2097	581	4196	1173	353	3604
	0.9<K/F≦0.95	OTM1	1321	680	6251	799	426	5885
	0.8<K/F≦0.9	OTM2	992	713	7590	1400	541	5874
Futures				8927	12690		2881	107994

3.2 Volatility Parameters Estimation

To price and hedge crude oil futures options, one has to estimate volatility parameters embedded in each volatility function. One can obtain volatility parameters from historical approach based on the underlying futures price or from option prices. The latter approach is adopted since implied volatility of option prices represents average market assessments of future volatility during the life of options.

Two stages are used to estimate volatility parameters. First, we estimate initial term structure of futures prices, which can be obtained from crude oil futures prices with different maturities. Second, given all relevant variables, we estimate implied volatility parameters by minimizing the sum of squared difference between market and model prices.

Specifically, each Wednesday we take initial term structure of futures prices from traded futures prices. Let $\varphi = [F(t,T_1), F(t,T_2), \ldots.., F(t,T_w)]$ be the initial term structure of futures prices where $T_1, T_2, \ldots\ldots T_w$ are the maturity date for various futures contracts. Then, given an initial guess of volatility parameters, a futures price tree is built forward using equation (3). Second, we build an American option price tree calculating backward. Let $C_i = (C_1, C_1, \ldots, C_q)$ be the number of options traded in the day and $\sigma_j = (\sigma_1, \sigma_1, \ldots, \sigma_m)$ for representing the number of volatility parameters in the volatility function. For each exercise price K and maturity T, the market and model option prices are denoted $C_i(\sigma_j, K, T)$ and $C_i(\widetilde{\sigma_j}, K, T)$, respectively. An iterative procedure is conducted to produce updated volatility parameters until the sum of squared differences between market and model prices are minimized, which are

$$min \sum_{i=1}^{q} \left[C_i(\sigma_j, K, T) - C_i(\widetilde{\sigma_j}, K, T)\right]^2 \tag{6}$$

This procedure is repeated for each model and on each Wednesday until the end of sample period.

For computing option price with stochastic interest rates, we need to estimate initial interest rate and volatility function required for HJM model where the rate is obtained from Treasury bill rate and volatility function is given from the function of the model. Then, equations (3) and (4) and (6) are applied, where *F(t,T)* is replaced by the forward rate *f(t,T)*, to determine the discount rate required for computing option value. Under stochastic interest rates, three lattice trees are needed, namely futures price tree, forward rate tree, and option price tree.

3.3 Parameter Stability

Table 3 shows value, the change, and autocorrelation of volatility parameters for Model 6. Since volatility parameters in a model are embedded in a volatility function, volatility parameters between each other in a model are likely to be highly correlated. It is not possible to precisely examine stability of each parameter purely from their change in parameter value. However, we may see that sensitivity of each parameter to the general volatility level. Except Model 1 and Model 2, there is only parameter in a volatility function.

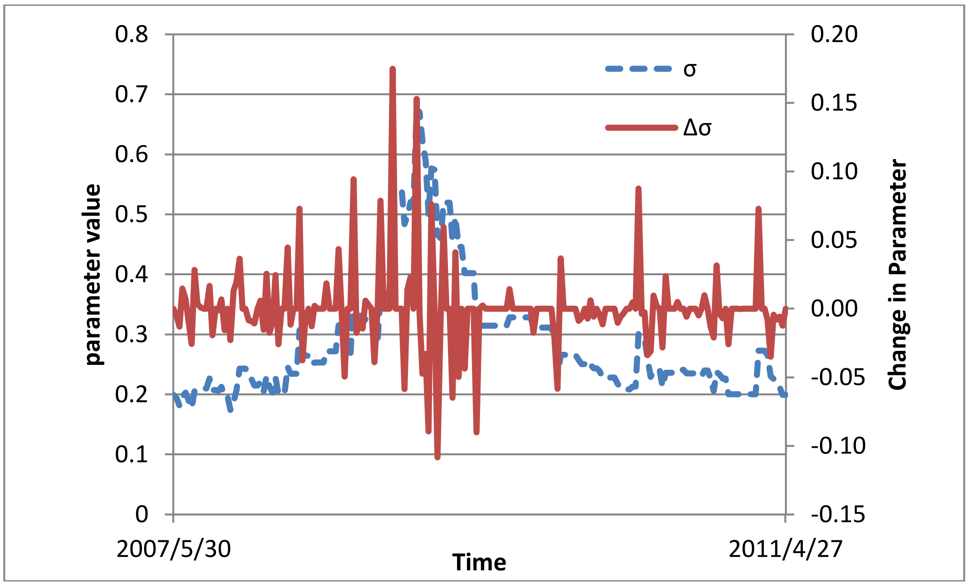
We see different scales of sensitivity of volatility parameters for Model 6 in Table 3. According standard deviation and their change in parameter value, the decay parameter, λ, the most sensitive parameters, and σ_2 is the most stable one. The autocorrelation value generally appears to decline when the lag increases, indicating that parameters on date t are more related to the value on previous day, but less related to the value of distant trading day.

Table 3 Parameter Estimation for Model 6

	Mean	Stdev	ρ_1	ρ_2	ρ_3	ρ_{10}	ρ_{50}	ρ_{100}
σ_1	0.077	0.086	0.69	0.59	0.58	0.34	-0.25	-0.01
$\Delta\sigma_1$	-2.1E-05	0.067	-0.33	-0.16	-0.22	-0.24	-0.06	-0.13
σ_2	0.171	0.076	0.70	0.54	0.58	0.46	-0.19	-0.12
$\Delta\sigma_2$	1.4E-05	0.059	-0.24	-0.33	-0.16	-0.15	-0.18	0.14
λ	-3.003	1.238	0.23	-0.17	-0.01	-0.03	-0.19	0.15
$\Delta\lambda$	5.0E-05	1.538	-0.24	-0.37	-0.19	-0.2	-0.23	0.22

To explicitly observe the time variation of parameter value, Figure 4 sows the value and its change in parameter for Model 1. From one week to the next, the volatility parameter changes across time, reaching to the peak at 0.6735 on 17 Dec 2008 during the period of the bankruptcy of the Lehman Brother. The change in value also exhibits different scales, particularly very large during the American subprime crisis during the second half of 2007 and 2008.

Figure 4 Parameter value across time for Model 1



4 Empirical Results

4.1 In-Sample Estimation

The purpose of this study is to examine whether term structure models can able to capture crude oil futures options prices across maturities and strikes. For comparison, the binomial tree model proposed by Cox, Ross, and Rubinstein (1979) is used to fit into weekly option prices as those used in term structure models. Each Wednesday, volatility parameters are obtained by using option across strikes and maturities to fit into each model until the sum of squared differences between market and model prices are minimized. This procedure is repeated each day and each model until the sample period is terminated. The difference between market and model price then is the in-sample error.

The advantage of term structure models is that not only the underlying futures prices are stochastic, but also the interest rates are stochastic. Table 4 shows in-sample performance of all models where Panel A displays results with stochastic interest rates and Panel B give results without stochastic interest rates. Three measures are used to evaluate in-sample performance of models, which are mean, absolute mean, and relative absolute. Mean is the average error between market and model price, absolute mean error is obtained from the absolute difference between the market and model price, and the relative absolute error is the difference of market and model price over market price.

The term structure model with or without stochastic interest rates gives some impacts on option prices. The stochastic interest rates are generated by Heath, Jarrow, and Morton (1990, 1992) model[6], which affect the discounting process of option price when calculating option price forward using the lattice approach. If the term structure model contains no stochastic interest rates, then the binomial tree of option price is discounted with constant interest rates, proxied by Treasury bill rates.[7] We see that the results with stochastic interest rates in Panel A generally give lower absolute and percentage error than those without stochastic interest rates in Panel B. The reduction of in-sample error varies between 1 and 3 percent across all models. Similar results are confirmed by Melino and Turnbull (1995) who found that pricing short-term options without stochastic interest rates is not serious problem. Amin and Bodhurtha (1991) who present evidence that long-term option prices can be significantly affected

[6] To implement Heath, Jarrow, and Morton model to estimate appropriate discount rate for option price calculation, initial interest rates and volatility function should be provided. The initial interest rates (forward rates) are obtained from Treasury bill rates. To simplify the estimation, the volatility function is assumed to be consistent with the function of each competing model. For example, Model 2 is the model with volatility function governed by proportion of futures price. We also use this function as the volatility function for interest rate model. The volatility parameter is given from the estimation of term structure models, using Equation (6).

[7] To match the remaining time of options, Treasury bill rates are linearly interpolated to obtain the desired period of days.

by stochastic rates.

Fitting performance with and without stochastic interest rates shows that the volatility function of oil options critically depends upon time to maturity of options contracts. We see that Model 1, specified with constant volatility, which gives a fixed volatility for different maturity of futures. In other words, 3-month futures and 1-year futures contracts are priced with the same maturity. The unrealistic nature of the model produces greater error, which shows the relative absolute error for all options being about 29% (Panel A). Model 2 also demonstrates that even the volatility function is proportional with futures prices, which cannot improve the fitting performance. The mean error is 0.179, which is the greatest among all models.

In contrast, those models with the volatility function relate to time to maturity, which experience greater fitting performance. Model 3 specified with exponential function of time to maturity significantly improves the fitting performance by 10 percent compared with Model 1 and Model 2. With linear time to maturity in volatility function in Model 4, 5, and 6, we experience a further reduction of 1% to 2%. The in-sample results confirm that the volatility function for oil futures options is better written with the structure of time to maturity of options.

Next, we also see two distinctive patterns from in-sample results. First, call absolute errors estimated from all models appear to be smaller than puts (except Model 2), but different results exist when the error is measured with relative error. The relative error may be biased since they are in a percentage form and may be magnified with smaller option prices. Second, when we compare all term structure models to CRR model, which has not considered the term structure effect of volatility function. According all measures, all models perform much lower fitting error than binomial tree model. Finally, adding an additional factor into one-factor model shows marginal improvement of fitting errors. For example, Model 5 reduces the relative absolute error by 1% (or 0.02 in absolute error) compared to Model 4, and Model 6 improves by 0.5% (or 0.04 in absolute error) over Model 3.

Table 4 In-Sample Performance

This Table has two panels where Panel A shows in-sample error for all term structure models containing stochastic interest rates and Panel B gives the results without considering stochastic interest rates. The implementation procedure for term structure models with/without stochastic interest rate can be found in the text. Three measures are used to calculate in-sample error. Mean is the average fitting error where fitting error is defined as the difference between market and model price. Abs is the average absolute fitting error. Rel Abs is the average relative absolute error calculated by the absolute fitting error divided by market option price.

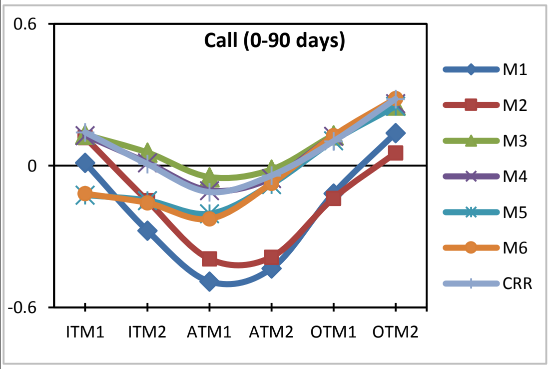
Panel A With stochastic interest rates

	All			Call			Put		
Model	Mean	Abs	Rel Abs	Mean	Abs	Rel Abs	Mean	Abs	Rel Abs
Model 1	0.120	0.582	0.290	0.125	0.597	0.274	0.114	0.565	0.308
Model 2	0.179	0.598	0.293	0.128	0.578	0.256	0.237	0.621	0.334
Model 3	0.074	0.371	0.184	0.085	0.393	0.170	0.063	0.347	0.201
Model 4	0.044	0.341	0.163	0.057	0.366	0.152	0.029	0.312	0.175
Model 5	-0.055	0.364	0.152	-0.245	0.411	0.152	0.160	0.312	0.151
Model 6	0.060	0.410	0.179	-0.187	0.435	0.172	0.212	0.382	0.186
CRR	0.458	0.730	0.303	0.431	0.711	0.275	0.489	0.752	0.334
Panel B No Stochastic interest rate									
Model 1	0.115	0.616	0.312	0.135	0.612	0.296	0.121	0.596	0.311
Model 2	0.185	0.639	0.332	0.136	0.602	0.275	0.235	0.633	0.342
Model 3	0.065	0.391	0.196	0.096	0.422	0.178	0.074	0.349	0.242
Model 4	0.051	0.361	0.172	0.069	0.385	0.159	0.036	0.327	0.182
Model 5	-0.057	0.375	0.163	-0.266	0.423	0.168	0.016	0.329	0.162
Model 6	-0.001	0.436	0.196	-0.275	0.462	0.191	0.236	0.395	0.196

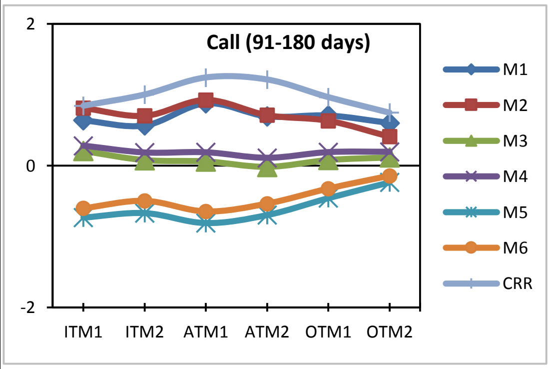
Evaluation of term structure models should be looked at time series performance of options, but also be looked at their fitting performance of cross sections of options. For this reason, Figure 5 display fitting performance across moneyness separately with 2 maturities categories. The range of moneyness is based on the definition in the section of data.

Figure 5 In-Sample Performance based on Category of Moneyness

Panel A

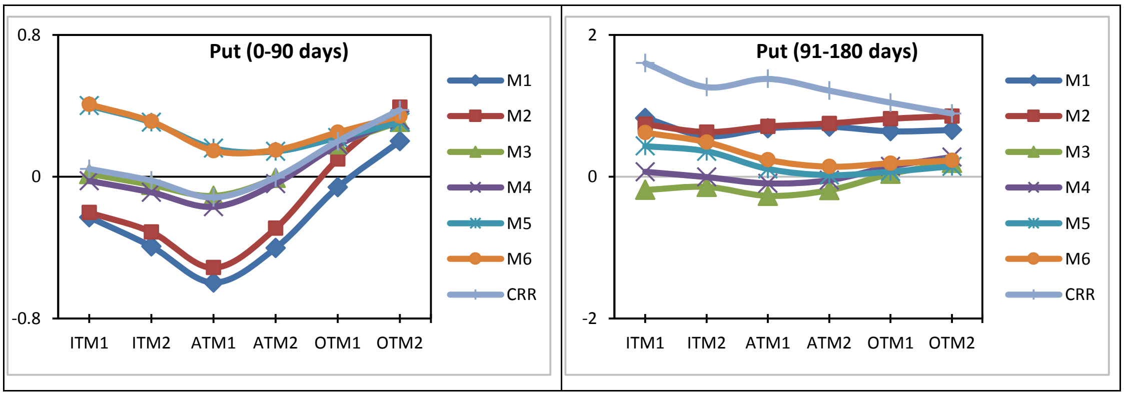


Panel B



Panel C

Panel D



Surprisingly, all term structure models cannot fully capture oil option prices across strikes. For the category of 0-90 days (short-term), as shown in Panel A (calls) and B (puts) most models tend to undervalue ITM and OTM options relative to market prices, but overvalue ATM options relative to market prices. This suggests that volatility should be increased for pricing short-term ITM and OTM options, but the volatility should be decreased for pricing ATM options. For the 91-180 day category, a different story appears in which call options series are uniformly undervalued by one-factor models, but overvalued by two-factor models (See Panel C). In the case of puts in Panel D undervalued options relative to market prices are systematically produced by most models. These results suggest that to fit options prices across moneyness, models should incorporate a variable that allow volatility to be various across strikes. Such as the volatility function suggested by Dumas et al. (1998) who incorporate strike prices (or moneyness) into the volatility or allowing return in futures price to have a random jump.

4.2 Pricing Options with Lagged Volatility

In this section, we examine out-of-sample performance of all term structure models. This approach is implemented by using implied volatility obtained from current week as an input to price options next week. There are two objectives to run this test. First, this approach helps examine time series stability of each model. The model could price better than the other model on current day, but becomes worse on the other day. The instability of model performance could arise since the model can overfit into options across strikes and maturity. An overfitting problem may be detected when the model goes out of sample. If the model consistently provides stable performance, it is like to be adopted by researchers and practitioners. Second, we use this approach to investigate the predictability of each model. If we believe that the implied volatility of futures options follows a Markov process, then current implied volatility is the best prediction of future volatility. It is can be true that market assessments of future volatility changes over time and thus current volatility is the best forecast of future volatility.

To fulfill these objectives, each model is fitted into option prices across strikes and maturities traded on current Wednesday until the termination of sample period. The volatility, obtained from the minimization of sum of squared differences between market and model prices at the first stage, then is used to price crude oil option traded on next Wednesday. Table 5 shows the pricing performance with lagged implied volatility. Using all measures as in in-sample pricing, the general results are that pricing with a lagged volatility for all models produces greater errors than pricing with a current volatility. A consistent result reveals that while instability of volatility produces a strong impact on model pricing performance, a model performs well in the sample that also performs well out of sample.

When using volatility on date t to price volatility on date t+7, absolute out-of-sample error increases from 0.1 to 0.3 and relative error increases from 2% to 7% on average for all options. Model 1 and Model 2 are the most stable models showing that relative error increases from 29% and 29.3% to 31.1 and 33%, respectively. However, Model 4, 5 and Model 6 are the most unstable models, where the relative error increases by 5.3%, 7%, and 6.4%, respectively. The increase of out-of-sample error from in-sample error shows that the best fitting models, such as Model 4 and 5, are more affected by change in volatility. In contrast, the poor fitting models, such as Model 1 and Model 2 are more stable than the better fitting models. It may be suspicious to say that Model 4 and Model 5 may be overfitted.

Table 5 Pricing Performance with lagged volatility

This table shows average pricing errors for all competing term structure models with stochastic interest rate using lagged volatility. Pricing error on current Wednesday is defined as the difference between market and model prices where volatility for the model price is given on previous Wednesday. The three measures, mean, abs, and rel abs are described in Table 4.

	All			Call			Put		
Model	Mean	Abs	Rel Abs	Mean	Abs	Rel Abs	Mean	Abs	Rel Abs
Model 1	0.634	0.632	0.311	0.296	0.646	0.296	0.125	0.620	0.329
Model 2	0.680	0.468	0.330	0.305	0.669	0.305	0.237	0.693	0.358
Model 3	0.469	0.448	0.229	0.216	0.489	0.216	0.069	0.447	0.244
Model 4	0.449	0.678	0.216	0.207	0.474	0.207	0.031	0.421	0.226
Model 5	0.478	0.476	0.221	0.226	0.525	0.226	0.157	0.425	0.216
Model 6	0.517	0.515	0.242	0.239	0.546	0.239	0.213	0.485	0.246
CRR	0.811	0.808	0.352	0.323	0.791	0.323	0.481	0.832	0.386

Although the well-performing model in the in-sample measure suffers more from the change in volatility, they remain the better performing model in out-of-sample pricing. Models 4 and 5 still provide lowest pricing errors for all three measures when all options are estimated on average, or are estimated separately by calls or puts.

Two findings may be worthy to note. First, the binomial model, which does not take term structure of futures prices and volatility into account, remains the worst performing model in out-of-sample pricing. Second, incorporating an additional factor into a one-factor model does not improve out-of-sample error. For example, adding a second factor into Model 4 increases the out-of-sample error from 21.6% to 22.1% (Model 5). In addition, when including an extra factor into Model 3, the error for Model 6 turns to 24.2% from 22.9%, suggesting that two-factor model may suffer from instability of volatility parameters more than one-factor models.

4.3 Testing equal conditional predictive ability

To formally assess the statistical significance of the superior out-of-sample performance of the competing option pricing models over the CRR model, we employ the equal conditional predictive ability test of Giacomini and White (2006) and report the testing results in Table 6. The test of Giacomini and White (2006) mainly improves Diebold and Mariano (1995) that has been in widespread use in predictive evaluation by several aspects. First, their test can exist in an environment where the sample is finite. Second, their model accommodates conditional predictive evaluation, in the way that we can predict which forecast was more accurate *at a specific future day*. In other words, it nests the unconditional predictive evaluation that only predicts which forecast was more accurate on average. Third, it captures the effect of estimation uncertainty on relative forecast performance.

To be specific, for a given loss function calculated with out-of-sample absolute error, the null hypothesis of equal conditional predictive ability of forecast function *f* and *g* for the target date $t+\tau$ can be:

$$\mathrm{H}_0\colon \mathrm{E}\left[L_{t+\tau}\left(Y_{t+\tau},\hat{f}_t\right)-L_{t+\tau}(Y_{t+\tau},\hat{g}_t)|I_t\right]\equiv \mathrm{E}[\Delta L_{t+\tau}|I_t]=0$$

where $Y_{t+\tau}$ is the variable of interest, here it should be market price $C_i(\sigma_j,K,T)$. $\hat{f}_t$ and $\hat{g}_t$ can be anyone of competing term structural model $C_i(\widetilde{\sigma_j},K,T)$. For a given chosen test function h_t that is $q\times 1$ vector, a Wald-type test statistic responding to the null hypothesis is:[8]

$$T = n\left(n^{-1}\sum_{t=1}^{T-1} h_t\,\Delta L_{t+1}\right)\hat{\Omega}_n^{-1}\left(n^{-1}\sum_{t=1}^{T-1} h_t\,\Delta L_{t+1}\right)$$
$$= \mathrm{n}\overline{\boldsymbol{Z}}'\hat{\Omega}_n^{-1}\overline{\boldsymbol{Z}}$$
(7)

where $\overline{\boldsymbol{Z}} = n^{-1}\sum_{t=1}^{T-1} Z_{t+1}$, $Z_{t+1} = h_t \Delta L_{t+1}$, and $\hat{\Omega}_n = n^{-1}\sum_{t=1}^{T-1} Z_{t+1} \times Z_{t+1}'$ is $q \times q$ matrix that consistently estimates the variance of Z_{t+1}. n is the number of out-of-sample forecasts. A level of α test can be conducted by rejecting the null hypothesis of equal conditional predictive ability whenever $T > \chi^2_{q,1-\alpha}$, where $\chi^2_{q,1-\alpha}$ is the $1-\alpha$ quantile of a χ^2_q distribution.

As shown in Table 6, the equal conditional predictive abilities between the term structural models and the CRR model have been rejected, indicating there are statistical significances of the superior out-of-sample performance of the term structural models over the CRR model. As we turn our interests to compare these competing term structural models, we find that Model 3, 4, 5 and 6 predicts significantly better than Model 1 and 2. We obtain the consistent results even though the loss function is measured by square error. The results based on in-sample, out-of-sample error and significance test in Table 4, 5, and 6 show that a model with specification of time to maturity in volatility function is crucial in pricing crude oil options across time to maturity and moneyness.

Table 6 Pair of Performance Comparison

This Table tests whether one model predicts significantly better than the other model, using the test of Giacomini and White (2006). The loss function in this table is evaluated by absolute error where the absolute error on current Wednesday is determined by absolute value of the difference between market and model price where the model price is obtained from the volatility parameters on previous Wednesday. The test function h_t is chosen as $h_t = (1,\ \Delta L_t)'$. The 95% significance level of a $\chi^2_{q,1-\alpha}$ distribution with $q = 2$ degree of freedom is 5.99.

	Model 2	Model 3	Model 4	Model 5	Model 6	CRR
Model 1	5.390	14.468	11.784	5.544	7.764	29.048
Model 2		15.379	14.126	6.661	8.702	23.409
Model 3			1.864	3.097	3.944	32.609
Model 4				4.689	4.629	31.462
Model 5					0.285	28.099
Model 6						30.397

[8] h_t can be chosen by the researcher to include variables that thought to help distinguish between the forecast performance of two methods. For instance, one can consider their past relative performance such as lagged loss differences or moving average of past loss differences, or business cycle indicators.

4.4 Hedging Performance

Pricing performance is to measure whether a model is able to fit into the skew of the underlying asset distribution. To examine the pricing ability of each model, we calibrate each model with market option prices across strikes and maturities every Wednesday. Hence, it enables us to examine time series and cross sections pricing performance of each model. If a model consistently outperforms the other model, it should perform better than the other this week and the consecutive week.

Next, a model can be evaluated based on the option pricing paradigm that theoretically correct risk management emerges naturally, that the model is derived from absence of arbitrage. The risk-neutral valuation underlying the option model is derived from constructing the portfolio with a short position of call (put) and purchase (sell) a delta portion of underlying assets. In theory, to maintain the delta neutral at all time, the underlying asset price should be rebalanced at an instant of time that asset price changes. In reality, a continuous hedge is infeasible and a discrete rebalancing is normally conducted. Hence, part of hedging errors occurs due to infrequent rebalancing. The other part of hedging error is due to the model misspecification. Apart from the error of infrequent hedging, Green and Figleswi (1999) indicate that part of hedge error is due to model misspecification, deriving from incorrect assumption of the underlying asset process and incorrect input unobservable parameters.

The hedging error is determined as

$$\Delta H_{t+\Delta} = (C_{t+\Delta} - D_{t+\Delta} \times F_{t+\Delta}) - (C_t - D_t \times F_t) \times e^{r_t \times \Delta} \quad (8)$$

where C_t, D_t, and F_t is the call option value, delta, and futures price on t. Δ indicates a rebalanced interval, which is one week in this case. $\Delta H_{t+\Delta}$ is the change of delta hedge portfolio over a week. If delta computed from each option model is correct, the hedge error should be zero when rebalancing continuously. Here, we rebalance once a week until the desired period is reached. To avoid any offsetting hedging error over a long hedging period, we compute absolute hedge error and percentage absolute hedge error, expressed as,

$$AHE = \frac{1}{n} \sum_{i=1}^{n} |\Delta H_{t+i\Delta}| \quad (9)$$

$$PHE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\Delta H_{t+i\Delta}}{H_{t+(i-1)\Delta}}\right| \quad (10)$$

where AHE and PHE are average absolute error and average percentage error, respectively, for the period of n week.

Table 7 shows hedging performance of all competing models for the period of 2, 5, and 10 weeks. The major finding is that two-factor models tend to outperform one-factor models. The AHE for Model 5 and 6 for 2-week hedging period is 0.304 and 0.309, but the best performing one-factor model are Model 3 and Model 4, which produce the AHE 0.420 and 0.421, respectively. When the error is estimated with PHE, the relative difference seems not as large as AHE. In general, the difference between 2-factor model and 1-factor model is about 1 percent. The measure of PHE is so low since the hedge error is divided a large portfolio value. Similar patterns of relative performance are repeated for 2-factor and 1-factor models for 5-week and 10-week hedging period.

The relative hedging performance for 2-factor model and 1-factor models seem to be larger than the relative pricing performance of both models. These results are consistent to Gupta and Subrahmanyam (2005) and Driessen to al. (2003), showing that more complex models outperform simple models. Two-factor models are preferable over one-factor models in hedging than pricing probably because the change in term structure of futures prices generated by 2-factor models is imperfectly correlated, consistent to the term structure of volatility for crude oil price options across maturities.

Three observable patterns are worthy to note. First, the model with the volatility function incorporating time to maturity tends to perform better than the model has no this feature. As an example, the AHE of Model 3 and 4 for the 5-week hedging period is 0.356 and 0.354, compared to 0.382 and 0.392 for Model 1 and 2. Next, we see that the CRR model outperforms one-factor model, but underperform two-factor model. Finally, the results from the spectrum of the hedge period show that the longer the hedging period, the lower average hedge error for all models.

Table 7 Hedging Performance

This table presents AHE and PHE, which are average absolute hedge error and average percentage hedge error using equations (9) and (10). This is done by building a delta neutral portfolio and rebalanced weekly until the desired period is reached.

Period (weeks)	2		5		10	
	AHE	PHE	AHE	PHE	AHE	PHE
M1	0.455	0.027	0.382	0.032	0.259	0.026
M2	0.460	0.038	0.392	0.035	0.261	0.029
M3	0.420	0.023	0.356	0.029	0.239	0.021
M4	0.421	0.024	0.354	0.029	0.238	0.021
M5	0.304	0.019	0.243	0.017	0.131	0.011
M6	0.309	0.019	0.254	0.017	0.138	0.011
CRR	0.412	0.046	0.347	0.034	0.233	0.016

To break out the aggregate hedging error into different ranges of moneyness and maturities, Table 8 display AHE (absolute hedging error) for 5-week hedging period across 5 ranges of moneyness and 2 ranges of maturities. The general results show that within the category of 0-90 days, term structure models perform the best for in-the-money options whereas within the range of 91-180 days these models produce lowest errors for out-the-money options. However, the CRR model, which has not considered term structure of futures price, produces inconsistent results across moneyness and maturities, but the model performs best for in-the-money calls among all competing models.

The results in Table 7 and 8 also reveal that number of factors matter for performance of hedging for term structure model. Two-factor models, particularly model 5, generally outperform all one-factor models across most ranges of moneyness. The performance of two-factor models also depends upon the specification of volatility function. It is found that the function with time to maturity performs better than the model, which do not contain this variable. Thus, in the category of two-factor models model 5 for most moneyness series produces lower errors than Model 6, and Model 4 performs best among all competing one-factor models.

Table 8 Hedging error across Moneyness and Maturities

This table shows average absolute hedge error rebalancing for 5 weeks. The hedge error is grouped and averaged according moneyness and maturities.

	Day	Money	M1	M2	M3	M4	M5	M6	BS
Call	0-90	1.1<F/K≦1.2	0.200	0.126	0.220	0.217	0.220	0.164	0.331
		1.05<F/K≦1.1	0.448	0.472	0.399	0.414	0.378	0.394	0.482
		1<F/K≦1.05	0.353	0.358	0.307	0.308	0.314	0.338	0.318
		0.95<F/K≦1	0.400	0.406	0.372	0.367	0.348	0.362	0.365
		0.9<F/K≦0.95	0.531	0.526	0.476	0.470	0.444	0.466	0.467
		0.8<F/K≦0.9	0.578	0.564	0.497	0.496	0.446	0.475	0.448
	91-180	1.1<F/K≦1.2	0.391	0.467	0.363	0.334	0.437	0.454	0.481
		1.05<F/K≦1.1	0.502	0.456	0.469	0.461	0.451	0.454	0.508
		1<F/K≦1.05	0.450	0.456	0.429	0.433	0.417	0.426	0.416
		0.95<F/K≦1	0.408	0.413	0.387	0.390	0.384	0.389	0.382
		0.9<F/K≦0.95	0.422	0.435	0.414	0.406	0.368	0.374	0.362
		0.8<F/K≦0.9	0.380	0.410	0.350	0.350	0.318	0.329	0.307
Put	0-90	1.1<K/F≦1.2	0.383	0.385	0.364	0.361	0.293	0.307	0.303
		1.05<K/F≦1.1	0.233	0.316	0.238	0.241	0.247	0.343	0.230
		1<K/F≦1.05	0.340	0.357	0.342	0.336	0.330	0.360	0.343
		0.95<K/F≦1	0.309	0.305	0.265	0.261	0.281	0.297	0.274
		0.9<K/F≦0.95	0.366	0.394	0.347	0.346	0.344	0.360	0.368
		0.8<K/F≦0.9	0.500	0.442	0.448	0.452	0.468	0.504	0.480
	91-180	1.1<K/F≦1.2	0.435	0.477	0.324	0.293	0.266	0.291	0.277
		1.05<K/F≦1.1	0.406	0.387	0.404	0.412	0.402	0.409	0.429
		1<K/F≦1.05	0.357	0.374	0.338	0.334	0.334	0.337	0.335
		0.95<K/F≦1	0.323	0.331	0.301	0.298	0.311	0.315	0.317
		0.9<K/F≦0.95	0.296	0.292	0.267	0.264	0.260	0.269	0.280
		0.8<K/F≦0.9	0.319	0.342	0.301	0.302	0.311	0.317	0.317

5. Conclusion

Term structure models are originally used for modeling term structure of interest rates and price interest rate contingent claims. Based on the cost of carry argument, futures prices should be related with time to maturity and can be viewed as a structure and can be modeled by the term structure models. Pricing and hedging futures options with term structure models contain the stochastic futures prices and stochastic interest rates, which relaxes the assumption of the constant interest rates behind the Black-Scholes model. This paper contributes to option pricing literature by examining the term structure models in pricing and hedging futures options. Light Sweet crude oil futures options from 2007 and 2011 are selected to test several competing models in in-sample, out-of-sample, and hedging performance.

Results have shown that the term structure models incorporating stochastic interest rates reduce in-sample fitting performance between 1% and 3%. One-factor models appear to be more superior in in-sample fitting and out-of-sample prediction, whereas two-factor models are preferable for hedging. In addition, volatility function with time to maturity outperform than those models without this feature for all measures, but they tend to overfit into spurious options prices. Taken together, two-factor models with time to maturity of futures prices in volatility function perform the best among all models, but correctly specifying one-factor model may replace two-factor models with incorrect volatility function.

Although all term structure models outperform the CRR model significantly in out-of-sample prediction, but they tend to overvalue short-term ATM options and undervalued other options. For longer options, different scales of mispricing still persist across models, suggesting that these models may be mis-specified to a certain extent. However, it is too early to judge their usefulness unless full scales of tests are conducted. To reduce mispricing of term structure models, these models with volatility functions incorporating a second-order polynomial of strike price and maturity suggested by Dumas et al. (1998) may considered. We left this work for future research.

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